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Computation of the Generalized Inverse of a Polynomial Matrix and Applications

N. P. Karampetakis
Department of Mathematical Sciences
Loughborough University of Technology
Loughborough, Leics LE11 3TU
United Kingdom

Submitted by Richard A. Brualdi

ABSTRACT

The computation of the generalized inverse of a constant matrix is utilized in finding the generalized inverse and its Laurent expansion of a nonregular polynomial matrix. The proposed algorithm constitutes a generalization of the algorithm proposed by Fragulis et al. for regular polynomial matrices and gives rise to numerous applications in multivariable system analysis. © *Elsevier Science Inc., 1997*

I. INTRODUCTION

Consider the polynomial matrix

$$A(s) = A_q s^q + A_{q-1} s^{q-1} + \cdots + A_1 s + A_0 \in \Re[s]^{n \times m} \tag{1.1}$$

with $A_i \in \Re^{n \times m}$, $i \in q$, and n not necessary equal to m. The problem of the investigation of the generalized inverse of a polynomial matrix has been the concern of many scientists [1, 2, 4, 6, 13–15, 17] because of the large number of its implications in multivariable system analysis, i.e., computation of the transfer function matrix of a system [4, 6], inverse systems, [9–12], solutions of systems (see Section IV), controllability and observability matri-

LINEAR ALGEBRA AND ITS APPLICATIONS 252:35–60 (1997)

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ces of general polynomial matrix descriptions (PMDs) [3], and solution of diophantique equations (see Section IV), which gives rise to numerous applications [8], etc.

In the case where ($n = m$, $q = 0$, and $\det[A(s)] \neq 0$), i.e., $A(s) \equiv A_0 \in \Re^{n \times n}$ with $\det[A_0] \neq 0$, the above problem has been investigated by Faddeev and Faddeeva [2] and Zadeh and Desoer [17], while in the case where ($n = m$, $q = 1$, and $\det[A(s)] \neq 0$) and ($n = m$, $q \in N$, and $\det[A(s)] \neq 0$) the solution proposed by the above authors has been extended by Mertzios [13] and Fragulis et al. [4], respectively. It was shown by the above authors that an interesting application of the determination of the inverse of $A(s)$ is the computation of the transfer function matrix of a linear, time invariant, multivariable system.

In the nonregular case of constant matrices $A(s) \equiv A_0 \in \Re^{n \times m}$ with n not necessarily equal to m, the *generalized inverse* of A_0 has been defined by Penrose [15] and a numerical algorithm for the computation of this matrix was later given by Decell [1]. However an open question still remains for the computation of the generalized inverse of $A(s)$ in (1.1). The answer to this question is the subject of this article.

More analytically in Section II we present some preliminary results concerning the definition of the generalized inverse of a constant matrix and its computation [1]. In Section III we determine a two-dimensional recursive algorithm which computes the generalized inverse of $A(s)$ in (1.1), in terms of its coefficient matrices $A_i \in \Re^{n \times m}$, $i \in q$, while in Section IV we present three applications of the generalized inverse: (a) the computation of the right (left) inverse of a rational matrix, (b) the investigation of the solution of an AR-representation, and (c) the solution in a specific feedback compensation problem. All the above implications are illustrated via examples. Finally in Section V we evaluate the Laurent expansion of the generalized inverse.

II. BACKGROUND MATERIAL

Some preliminary results concerning the definition of the generalized inverse of a constant matrix and its computation are necessary for what follows and we shall present it in the sequel.

DEFINITION 2.1 [15]. For every matrix $A \in \Re^{n \times m}$, a unique matrix $A^{\dagger} \in \Re^{m \times n}$, which is called *generalized inverse*, exists satisfying

(i) $AA^{\dagger}A = A$,
(ii) $A^{\dagger}AA^{\dagger} = A^{\dagger}$,
(iii) $(AA^{\dagger})^{\mathrm{T}} = AA^{\dagger}$,
(iv) $(A^{\dagger}A)^{\mathrm{T}} = A^{\dagger}A$,

where A^{T} denotes the transpose of A. If A is complex, then A^T denotes the conjugate transpose of A. In the special case that the matrix A is square nonsingular matrix, the generalized inverse of A is simply its inverse, i.e., $A^{\dagger} = A^{-1}$. In cases where there exists a matrix A^{g_1} which satisfies only the first condition is called {1}-inverse. {1}-inverses are not unique and play a fundamental role in the solution of polynomial diophantique equations as we see in Section IV.

In an analogous way we define the generalized inverse $A(s)^{\dagger} \in \mathfrak{R}(s)^{m \times n}$ of a rational matrix $A(s) \in \mathfrak{R}(s)^{n \times m}$ as the matrix which satisfies properties (i)–(iv) of Definition 2.1. Consider now the following theorem:

THEOREM 2.2 [1]. *Let $A \in \mathfrak{R}^{n \times m}$ and*

$$a(s) = \det\left[sI_n - AA^{\mathrm{T}}\right] = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \quad \text{with } a_0 = 1 \tag{2.1}$$

be the characteristic polynomial of AA^{T}. If $k \neq 0$ is the largest integer such that $a_k \neq 0$, then the generalized inverse of A is given by

$$A^{\dagger} = -a_k^{-1}A^{\mathrm{T}}\left[(AA^{\mathrm{T}})^{k-1} + a_1(AA^{\mathrm{T}})^{k-2} + \cdots + a_{k-1}I_n\right]. \tag{2.2}$$

If $k = 0$ is the largest integer such that $a_k \neq 0$, then $A^{\dagger} = 0$.

ALGORITHM 2.3. (Computation of the Generalized Inverse of a Constant Matrix $A \in \mathfrak{R}^{n \times m}$).

Step 1: Let $A \in \mathfrak{R}^{n \times m}$. Consider the sequences $\{a_0, a_1, \ldots, a_n\}$ and $\{B_0, B_1, \ldots, B_n\}$ constructed in the following way

$$\begin{array}{lll} A_0 = 0 & a_0 = 1 & B_0 = I_n \\ A_1 = (AA^{\mathrm{T}})B_0 & a_1 = -\dfrac{\operatorname{trace}[A_1]}{1} & B_1 = A_1 + a_1 I_n \\ \cdots & \cdots & \cdots \\ A_n = (AA^{\mathrm{T}})B_{n-1} & a_n = -\dfrac{\operatorname{trace}[A_n]}{n} & B_n = A_n + a_n I_n. \end{array} \tag{2.3}$$

Step 2: If $k \neq 0$ is the largest integer such that $a_k \neq 0$, then the generalized inverse $A^\dagger$ of A is given by

$$A^\dagger = -a_k^{-1}A^{\mathrm{T}}\left[(AA^{\mathrm{T}})^{k-1} + a_1(AA^{\mathrm{T}})^{k-2} + \cdots + a_{k-1}I_n\right] = -a_k^{-1}A^{\mathrm{T}}B_{k-1}; \tag{2.4}$$

else ($k = 0$ is the largest integer such that $a_k \neq 0$) $A^\dagger = 0$.

III. GENERALIZED INVERSE OF A POLYNOMIAL MATRIX

Consider now the polynomial matrix

$$A(s) = A_q s^q + A_{q-1}s^{q-1} + \cdots + A_1 s + A_0 \in \Re[s]^{n\times m} \tag{3.1}$$

and therefore the transpose of $A(s)$ is

$$A(s)^{\mathrm{T}} = A_q^{\mathrm{T}} s^q + A_{q-1}^{\mathrm{T}} s^{q-1} + \cdots + A_1^{\mathrm{T}} s + A_0^{\mathrm{T}} \in \Re[s]^{m\times n}. \tag{3.2}$$

Following similar lines with Theorem 2.2 we can easily obtain the following

THEOREM 3.1. *Let $A(s) \in \Re[s]^{n\times m}$ as in (3.1) and*

$$\begin{aligned} a(z,s) &= \det\left[zI_n - A(s)A(s)^T\right] \\ &= a_0(s)z^n + a_1(s)z^{n-1} + \cdots + a_{n-1}(s)z + a_n(s), \end{aligned} \tag{3.3}$$

$a_0(s) = 1$, be the characteristic polynomial of $A(s)A(s)^{\mathrm{T}}$. Let $a_n(s) \equiv 0, \ldots, a_{k+1}(s) \equiv 0$ while $a_k(s) \neq 0$ and $\Lambda := \{s_i \in \Re : a_k(s_i) = 0\}$. Then the generalized inverse $A(s)^\dagger$ of $A(s)$ for $s \in \Re - \Lambda$ is given by

$$A(s)^\dagger = -a_k(s)^{-1}A(s)^{\mathrm{T}}\Big[\left(A(s)A(s)^{\mathrm{T}}\right)^{k-1} + a_1(s)\left(A(s)A(s)^{\mathrm{T}}\right)^{k-2} + \cdots + a_{k-1}(s)I_n\Big]. \tag{3.4a}$$

If $k = 0$ is the largest integer such that $a_k(s) \neq 0$, then $A(s)^\dagger = 0$. For those $s_i \in \Lambda$ find the largest integer $k_i < k$ such that $a_{k_i}(s_i) \neq 0$ and then the generalized inverse $A(s_i)^\dagger$ of $A(s_i)$ is given by

$$A(s_i)^\dagger = -a_{k_i}(s_i)^{-1} A(s_i)^{\mathrm{T}} \Big[\big(A(s_i) A(s_i)^{\mathrm{T}} \big)^{k_i - 1} + a_1(s_i) \big(A(s_i) A(s_i)^{\mathrm{T}} \big)^{k_i - 2} + \cdots + a_{k_i - 1}(s_i) I_n \Big]. \quad (3.4\mathrm{b})$$

Proof. The proof is exactly the same as that of Decell [1].

The computation of $a_i(s)$ and therefore of the generalized inverse $A(s)^\dagger$ of $A(s)$ follows in a similar way as Algorithm 2.3 as follows

ALGORITHM 3.2. (Computation of the Generalized Inverse $A(s)^\dagger \in \Re(s)^{m \times n}$ of $A(s) \in \Re[s]^{n \times m}$).

Step 1: Let $A(s) \in \Re[s]^{n \times m}$. Consider the sequences $\{a_0(s), a_1(s), \dots, a_n(s)\}$ and $\{B_0(s), B_1(s), \dots, B_n(s)\}$ constructed in the following way

$$\begin{aligned} A_0(s) &= 0 & a_0(s) &= 1 \\ A_1(s) &= \big(A(s) A(s)^{\mathrm{T}} \big) B_0(s) & a_1(s) &= -\frac{\operatorname{trace}[A_1(s)]}{1} \\ &\cdots & &\cdots \\ A_n(s) &= \big(A(s) A(s)^{\mathrm{T}} \big) B_{n-1}(s) & a_n(s) &= -\frac{\operatorname{trace}[A_n(s)]}{n} \end{aligned}$$

$$\begin{aligned} B_0(s) &= I_n \\ B_1(s) &= A_1(s) + a_1(s) I_n \\ &\cdots \\ B_n(s) &= A_n(s) + a_n(s) I_n. \end{aligned} \quad (3.5)$$

Step 2: Let $a_n(s) \equiv 0, \dots, a_{k+1}(s) \equiv 0$ while $a_k(s) \neq 0$ and $\Lambda := \{s_i \in \Re : a_k(s_i) = 0\}$. Then the generalized inverse $A(s)^\dagger$ of $A(s)$ for $s \in \Re - \Lambda$

is given by

$$A(s)^{\dagger} = -a_k(s)^{-1} A(s)^{\mathrm{T}}\Big[\big(A(s)A(s)^{\mathrm{T}}\big)^{k-1} + a_1(s)\big(A(s)A(s)^{\mathrm{T}}\big)^{k-2} + \cdots + a_{k-1}(s) I_n\Big]$$

$$= -a_k(s)^{-1} A(s)^{\mathrm{T}} B_{k-1}(s). \tag{3.6a}$$

If $k = 0$ is the largest integer such that $a_k(s) \neq 0$, then $A(s)^{\dagger} = 0$. For those $s_i \in \Lambda$ find the largest integer $k_i < k$ such that $a_{k_i}(s_i) \neq 0$ and then the generalized inverse $A(s_i)^{\dagger}$ of $A(s_i)$ is given by

$$A(s_i)^{\dagger} = -a_{k_i}(s_i)^{-1} A(s_i)^{\mathrm{T}}\Big[\big(A(s_i)A(s_i)^{\mathrm{T}}\big)^{k_i-1} + a_1(s_i)\big(A(s_i)A(s_i)^{\mathrm{T}}\big)^{k_i-2} + \cdots + a_{k_i-1}(s_i) I_n\Big]$$

$$= -a_{k_i}(s_i)^{-1} A(s_i)^{\mathrm{T}} B_{k_i-1}(s_i). \tag{3.6b}$$

It is seen from (3.1), (3.2) that the greatest powers of $A_1(s) = A(s)A(s)^{\mathrm{T}}$ (and thus of the trace $a_1(s)$ of $A_1(s)$, and $B_1(s)$ are equal to $2q$. In the same way according to (3.1) and (3.2) the greatest powers of

$$A_2(s) = \big(A(s)A(s)^{\mathrm{T}}\big)B_1(s) = \big(A(s)A(s)^{\mathrm{T}}\big)^2 + a_1(s)\big(A(s)A(s)^{\mathrm{T}}\big) \tag{3.7}$$

(and thus of $a_2(s) = -\mathrm{trace}[A_2(s)]/2)$ and

$$B_2(s) = A_2(s) + a_2(s) I_n$$

$$\overset{(3.7)}{=} \big(A(s)A(s)^{\mathrm{T}}\big)^2 + a_1(s)\big(A(s)A(s)^{\mathrm{T}}\big) + a_2(s) I_n \tag{3.8}$$

are equal to $4q$. Thus it is easily seen from (3.1) and (3.5) that $a_i(s)$ and

$B_i(s)$ may be rewritten as

$$a_i(s) = \sum_{j=0}^{2iq} \hat{a}_{i,j} s^j$$

$$= -\operatorname{trace}\Big\{\Big[\big(A(s)A(s)^{\mathrm{T}}\big)^i + a_1(s)\big(A(s)A(s)^{\mathrm{T}}\big)^{i-1}$$

$$+ \cdots + a_{i-1}(s)\big(A(s)A(s)^{\mathrm{T}}\big)\Big] i\Big\} \quad (3.9)$$

with $i = 0, 1, \ldots, k$ and

$$B_i(s) = \sum_{j=0}^{2iq} B_{i,j} s^j$$

$$= \big(A(s)A(s)^{\mathrm{T}}\big)^i + a_1(s)\big(A(s)A(s)^{\mathrm{T}}\big)^{i-1} + \cdots + a_i(s) I_n, \quad (3.10)$$

with $i = 0, 1, \ldots, k - 1$, where $B_{i,j}$, $\hat{a}_{i,j}$ are constant coefficient matrices and scalar of the powers s^j. It is seen from (3.6) that for the computation of the generalized inverse of $A(s)$ we need the integer k, the quantities $a_k(s)$ and $B_{k-1}(s)$, i.e., the coefficients $\hat{a}_{k,j}$ and the coefficient matrices $B_{k-1,j}$ defined by

$$a_k(s) = \sum_{j=0}^{2kq} \hat{a}_{k,j} s^j \quad (3.11)$$

and

$$B_{k-1}(s) = \sum_{j=0}^{2(k-1)q} B_{k-1,j} s^j. \quad (3.12)$$

Taking into account that

$$A(s)A(s)^{\mathrm{T}}B_i(s) = \left(\sum_{j=0}^{q} A_j s^j\right)\left(\sum_{j=0}^{q} A_j^{\mathrm{T}} s^j\right)\left(\sum_{j=0}^{2iq} B_{i,j} s^j\right)$$

$$= \left(\sum_{j=0}^{2q}\left(\sum_{k=0}^{j} A_{j-k}A_k^{\mathrm{T}}\right)s^j\right)\left(\sum_{j=0}^{2iq} B_{i,j} s^j\right)$$

$$= \sum_{j=0}^{2(i+1)q}\left(\sum_{k=0}^{j}\left(\sum_{l=0}^{j-k}\left(A_{j-k-l}A_l^{\mathrm{T}}\right)\right)B_{i,k}\right)s^j \quad (3.13)$$

and substituting (3.9), (3.10), and (3.13) in the recursive relations (3.5), we obtain the following recursive algorithm that determine $\hat{a}_{i+1,j}$ and $B_{i+1,j}$ for $j = 0, 1, \ldots, 2(i+1)q$.

ALGORITHM 3.3. (Computation of the Generalized Inverse $A(s)^{\dagger} \in \Re(s)^{m\times n}$ of $A(s) \in \Re[s]^{n\times m}$).

Initial conditions:

$$B_{0,0} = I_n \quad (3.14)$$

Boundary conditions:

$$B_{0,j} = 0 \qquad \forall j > 0 \quad (3.15a)$$

$$B_{i,j} = 0, \qquad j = 2iq+1, 2iq+2, \ldots, 2(n-1)q \quad (3.15b)$$

Recursive relations for $a_i(s)$:

$$\hat{a}_{i+1,j} = -\frac{1}{i+1}\mathrm{trace}\left(\sum_{k=0}^{j}\left(\sum_{l=0}^{j-k}\left(A_{j-k-l}A_l^{T}\right)B_{i,k}\right)\right),$$

$$\begin{matrix} j = 0, 1, \ldots, 2(i+1)q \\ i = 0, 1, \ldots, n-1 \end{matrix} \quad (3.16)$$

Recursive relations for $B_i(s)$:

$$B_{i+1,j} = \left(\sum_{k=0}^{j} \left(\sum_{l=0}^{j-k} \left(A_{j-k-l} A_l^{\mathrm{T}} \right) B_{i,k} \right) \right) + \hat{a}_{i+1,j} I_n,$$

$$j = 0, 1, \dots, 2(i+1)q \qquad i = 0, 1, \dots, n-2 \tag{3.17}$$

Terminate:

$$\text{FIND } k: a_{k+1}(s) \equiv a_{k+2}(s) \equiv \cdots \equiv a_n(s) \equiv 0$$

$$\text{or } \hat{a}_{k+1,j} = \hat{a}_{k+2,j} = \cdots = \hat{a}_{n,j} = 0 \qquad \forall j \in N$$

then

$$B_j := B_{k-1,j}, \qquad j = 0, 1, \dots, 2(k-1)q \tag{3.18a}$$

$$\hat{a}_j := \hat{a}_{k,j}, \qquad j = 0, 1, \dots, 2kq \tag{3.18b}$$

Output:

$$A(s)^{\dagger} = -\left(\sum_{j=0}^{2kq} \hat{a}_j s^j \right)^{-1} \left(\sum_{j=0}^{q} A_j^{\mathrm{T}} s^j \right) \left(\sum_{j=0}^{2(k-1)q} B_j s^j \right)$$

$$= -\left(\sum_{j=0}^{2kq} \hat{a}_j s^j \right)^{-1} \left(\sum_{j=0}^{(2k-1)q} \sum_{l=0}^{j} \left(A_{j-l}^{\mathrm{T}} B_l \right) s^j \right). \tag{3.19}$$

It is readily seen that the generalized inversion algorithm is a two-dimensional recursive algorithm since it depends of two independent variables i, j.

REMARK 3.4. Note that (a) we use the same algorithm for those $s_i \in \Lambda$, i.e., $\Lambda := \{s_i \in \Re : a_k(s_i) = 0\}$, by finding the largest $k_i < k$ such that $a_{k_i}(s_i) \neq 0$ and (b) in case where $k = 0$ is the largest integer such that $a_k(s) \neq 0$ then $A(s)^{\dagger} = 0$.

IV. IMPLICATIONS OF THE GENERALIZED INVERSE IN LINEAR SYSTEM THEORY

An important result presented by Penrose [15], which was used later by Decell [1] to prove Theorem 3.1, will be useful in the sequel for many important problems in linear system theory.

THEOREM 4.1 [15]. *The matrix equation* $PXQ = C$ *has a solution iff* $PP^{\dagger}CQ^{\dagger}Q = C$, *in which case all the solutions are given by the formula*

$$X = P^{\dagger}CQ^{\dagger} + Y - P^{\dagger}PYQQ^{\dagger}, \tag{4.1}$$

where $P^{\dagger}$, $Q^{\dagger}$ *are the generalized inverses of* P *and* Q *respectively and* Y *is arbitrary to within having the dimension of* X.

It can be easily seen that the same theorem may be applied to polynomial matrices without any changes and thus we shall have that

THEOREM 4.2. *The polynomial matrix equation* $P(s)X(s)Q(s) = C(s)$ *has a solution iff* $P(s)P(s)^{\dagger}C(s)Q(s)^{\dagger}Q(s) = C(s)$ $\forall s \in \Re$, *in which case all the solutions are given by the formula*

$$X(s) = P(s)^{\dagger}C(s)Q(s)^{\dagger} + Y(s) - P(s)^{\dagger}P(s)Y(s)Q(s)Q(s)^{\dagger}, \tag{4.2}$$

where $P(s)^{\dagger}$, $Q(s)^{\dagger}$ *are the generalized inverses of* $P(s)$ *and* $Q(s)$ *respectively and* $Y(s)$ *is arbitrary to within having the dimension of* $X(s)$.

Proof. The proof is exactly the same as the one presented in Penrose [15].

We have to note here that the above theorem remains the same if we substitute the generalized inverses $P(s)^{\dagger}$ and $Q(s)^{\dagger}$ with the {1}-inverses $P(s)^{g_1}$ and $Q(s)^{g_1}$ respectively. In light of Algorithm 3.3 and Theorem 4.2 we may have numerous applications. However, in this article we present three only.

a. *Computation of the Right (Left) Inverse of a Polynomial Matrix*

It is known that the right (left) inverse of a polynomial matrix $A(s) \in \Re[s]^{n\times m}$ is defined as the matrix $X(s) \in \Re(s)^{m\times n}$, which satisfies the following property

$$A(s)X(s) = I_n \qquad (X(s)A(s) = I_m). \tag{4.3}$$

A direct application of Theorem 4.2 is the following

THEOREM 4.3. *A polynomial matrix* $A(s) \in \mathfrak{R}[s]^{n \times m}$ *has a right (left) inverse* $A(s)^*$ *iff* $A(s)A(s)^\dagger = I_n$ ($A(s)^\dagger A(s) = I_m$) *in which case all right (left) inverses are given by the following formula*

$$A(s)^* = A(s)^\dagger + \left(I_m - A(s)^\dagger A(s)\right)Y(s) \tag{4.4a}$$

$$\left(A(s)^* = A(s)^\dagger + Y(s)\left(I_n - A(s)A(s)^\dagger\right)\right), \tag{4.4b}$$

where $Y(s) \in \mathfrak{R}(s)^{m \times n}$ ($Y(s) \in \mathfrak{R}(s)^{n \times m}$) *is an arbitrary rational matrix to within having the dimension of* $A(s)^*$.

REMARK 4.4. The ability to compute the generalized inverse of a polynomial matrix $A(s) \in \mathfrak{R}[s]^{n \times m}$ via Algorithm 3.2 or 3.3 gives us the necessary tools for the computation of the right and left inverse of a polynomial matrix in the case where this matrix exists. The above algorithms may be used for the computation of the right and left inverse of a rational matrix also instead of a polynomial. All we have to do is to write the rational matrix $A(s) \in \mathfrak{R}(s)^{n \times m}$ as $A'(s)/d(s)$, where $d(s)$ is the greatest common multiple of all denominator polynomials of $A(s)$. Then the problem of finding the right inverse of $A(s)$ is reduced to the problem of finding a matrix $X(s)$: $A'(s)X(s) = d(s)I_n$, a problem which has a solution iff (according to Theorem 4.2) $A'(s)A'(s)^\dagger d(s) = d(s)I_n$ or equivalently iff $A'(s)A'(s)^\dagger = I_n$ $\forall s \in \mathfrak{R}$. In the case where the above condition is satisfied the right inverse $A(s)^*$ of $A(s)$ is given by the formula

$$A(s)^* = A'(s)^\dagger d(s) + \left(I_m - A'(s)^\dagger A'(s)\right)Y(s). \tag{4.5}$$

An important application of the computation of the right or left inverse of a rational matrix is the investigation of the solution of the matrix diophantique equation: Given the rational matrices $N(s)$ and $D(s)$, find $X(s)$ and $Y(s)$ of appropriate dimensions such that

$$N(s)X(s) + D(s)Y(s) = I \tag{4.6a}$$

or equivalently

$$\begin{pmatrix} N(s) & D(s) \end{pmatrix}\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = I. \tag{4.6b}$$

The investigation of the solution space of (4.6) plays an important role in problems like parametrization of stabilizing controllers, robust stabilization, disturbance rejection, reference tracking, model matching, H_2-optimal control (see survey paper of Kucera [8]).

EXAMPLE 4.5. Let

$$N(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D(s) = \begin{pmatrix} 0 \\ s \end{pmatrix}. \tag{E.1}$$

Find $X(s) \in \Re(s)^{2\times 2}$ and $Y(s) \in \Re(s)^{1\times 2}$ such that

$$N(s)X(s) + D(s)Y(s) = I_2 \tag{E.2}$$

or equivalently

$$\underbrace{\begin{pmatrix} 1 & s & 0 \\ 0 & 1 & s \end{pmatrix}}_{A(s)} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = I_2. \tag{E.3}$$

Consider the polynomial matrix $A(s)$

$$A(s) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & s \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A_1} s. \tag{E.4}$$

We have that $n = 2$, $m = 3$, and $q = 1$. We apply Algorithm 3.3 and find

$$A(s)^{\dagger} = \frac{1}{s^4 + s^2 + 1} \begin{pmatrix} s^2 + 1 & -s \\ s^3 & 1 \\ -s^2 & s + s^3 \end{pmatrix} \qquad \forall s \in \Re. \tag{E.5}$$

It is easily seen that $A(s)A(s)^{\dagger} = I_2$ and thus there exists a right inverse of the matrix $A(s)$ that is given by the formula

$$\begin{aligned} A(s)^* \equiv \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} &= A(s)^{\dagger} + \left(I_3 - A(s)^{\dagger} A(s)\right) Y(s) \\ &= \frac{1}{s^4 + s^2 + 1} \left\{ \begin{pmatrix} s^2 + 1 & -s \\ s^3 & 1 \\ -s^2 & s + s^3 \end{pmatrix} + \begin{pmatrix} s^4 & -s^3 & s^2 \\ -s^3 & s^2 & -s \\ s^2 & -s & 1 \end{pmatrix} Y(s) \right\} \\ &= \left\{ \frac{1}{s^4 + s^2 + 1} \begin{pmatrix} s^2 + 1 + s^2 a(s) & -s + s^2 b(s) \\ s^3 - s a(s) & 1 - s b(s) \\ -s^2 + a(s) & s + s^3 + b(s) \end{pmatrix} \right. \\ &\qquad \left. \text{for arbitrary } a(s), b(s) \in \Re(s) \right\}. \end{aligned} \tag{E.6}$$

Thus

$$X(s) = \frac{1}{s^4 + s^2 + 1}\begin{pmatrix} s^2 + 1 + s^2 a(s) & -s + s^2 b(s) \\ s^3 - s a(s) & 1 - s b(s) \end{pmatrix} \tag{E.7}$$

and

$$Y(s) = \frac{1}{s^4 + s^2 + 1}\left(-s^2 + a(s) \quad s + s^3 + b(s) \right). \tag{E.8}$$

For $a(s) = s^2$ and $b(s) = -s - s^3$ we have a polynomial solution of the diophantique equation (E.3)

$$X(s) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y(s) = (0 \quad 0). \tag{E.9}$$

b. *Solution of AR-Representations*

Consider the following autoregressive (AR) representation [16]

$$A(\rho)\beta(t) = 0, \tag{4.7}$$

where $\beta(t)$: $(0-, +\infty) \to \Re^m$, ρ denotes the differential operator, i.e., $\rho[\beta(t)] := d\beta(t)/dt$,

$$A(\rho) = A_q \rho^q + A_{q-1}\rho^{q-1} + \cdots + A_1 \rho + A_0 \in \Re[\rho]^{n\times m}, \tag{4.8}$$

and n not necessary equal to m. Equation (4.7) may be rewritten under Laplace transforms as

$$A(s)\hat{\beta}(s) = \left(s^{q-1}I_n \quad s^{q-2}I_n \quad \cdots \quad I_n\right) \times$$
$$\times \begin{pmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \times A_1 & A_2 & \cdots & A_q \end{pmatrix} \begin{pmatrix} \beta(0-) \\ \beta^{(1)}(0-) \\ \vdots \\ \beta^{(q-1)}(0-) \end{pmatrix} =: \hat{a}(s), \tag{4.9}$$

where $\hat{\beta}(s) = L[\beta(t)]$ (L denotes the Laplace transform). In light of Theorem 4.2 we obtain the following.

THEOREM 4.6. *The AR-representation* (4.7) *has a solution iff* $A(s)A(s)^{\dagger}\hat{a}(s) = \hat{a}(s)I_n$ $\forall s \in \mathfrak{R}$, *in which case all general solutions are given by the formula*

$$\beta(t) = L^{-1}\big[\hat{\beta}(s)\big] = L^{-1}\Big[A(s)^{\dagger}\hat{a}(s) + \big[I_m - A(s)^{\dagger}A(s)\big]y(s)\Big], \tag{4.10}$$

where $y(s)$ is arbitrary to within having the dimension of $\hat{\beta}(s)$.

EXAMPLE 4.7. Consider the following AR-representation

$$\underbrace{\begin{pmatrix} \rho & \rho^4 & \rho^2+\rho \\ 1 & \rho^3 & \rho+1 \\ 0 & \rho+1 & 0 \end{pmatrix}}_{A(\rho)} \underbrace{\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{pmatrix}}_{\beta(t)} = 0_{3,1}, \qquad \rho := d/dt. \tag{E.1}$$

We are interested in the solution of the above AR-representation under the following initial conditions:

$$\beta_2^{(1)}(0-) = 1 \qquad \text{and} \qquad \beta_i^{(j)}(0-) = 0\,\forall(i,j) \neq (2,1). \tag{E.2}$$

We have that

$$\begin{aligned} A(\rho) &= \begin{pmatrix} \rho & \rho^4 & \rho^2+\rho \\ 1 & \rho^3 & \rho+1 \\ 0 & \rho+1 & 0 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{A_0} + \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{A_1}\rho + \\ &\quad + \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_2}\rho^2 + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_3}\rho^3 + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_4}\rho^4 \end{aligned} \tag{E.3}$$

with $\operatorname{rank}_{\mathfrak{R}(s)} A(s) = 2 < 3$, $n = m = 3$, and $q = 4$. We apply Algorithm 3.3

and find that for $s \in \Re - \{-1\}$

$$A(s)^{\dagger} = \frac{1}{(s+1)^2(s^2+1)(s^2+2s+2)} \times \begin{pmatrix} -s(s+1)^2 & -(s+1)^2 & s^3(s+1)(s^2+1) \\ 0 & 0 & -(s+1)(s^2+1)(s^2+2s+2) \\ -s(s+1)^3 & -(s+1)^3 & s^3(s+1)^2(s^2+1) \end{pmatrix} \tag{E.4a}$$

for $s = -1$

$$A(-1)^{\dagger} = \begin{pmatrix} -1/4 & 1/4 & 0 \\ 1/4 & -1/4 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{E.4b}$$

Taking Laplace transforms into the AR-representation (E.1) we obtain

$$\underbrace{\begin{pmatrix} s & s^4 & s^2+s \\ 1 & s^3 & s+1 \\ 0 & s+1 & 0 \end{pmatrix}}_{A(s)} \underbrace{\begin{pmatrix} \hat{\beta}_1(s) \\ \hat{\beta}_2(s) \\ \hat{\beta}_3(s) \end{pmatrix}}_{\hat{\beta}(s)} = \underbrace{\begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix}}_{\hat{a}(s)}, \tag{E.5}$$

where $\hat{\beta}_i(s) = L[\beta_i(t)]$ (L denotes the Laplace transform). Equation (E.5) has a solution according to Theorem 4.6 iff

$$A(s)\,A(s)^{\dagger}\hat{a}(s) \equiv \hat{a}(s) \Leftrightarrow .$$

For $s \in \Re - \{-1\}$

$$\underbrace{\frac{1}{s^2+1}\begin{pmatrix} s^2 & s & 0 \\ s & 1 & 0 \\ 0 & 0 & s^2+1 \end{pmatrix}}_{A(s)\,A(s)^{\dagger}} \underbrace{\begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix}}_{\hat{a}(s)} \equiv \underbrace{\begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix}}_{\hat{a}(s)} \tag{E.6a}$$

and for $s = -1$

$$\underbrace{\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A(-1)\,A(-1)^{\dagger}} \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\hat{a}(-1)} \equiv \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\hat{a}(-1)}, \tag{E.6b}$$

which are satisfied. All the solutions of the matrix equation (E.5) are given according to Theorem 4.6 by the following formula:

For $s \in \mathfrak{R} - \{-1\}$

$$\hat{\beta}(s) = A(s)^{\dagger}\hat{a}(s) + \left(I_3 - A(s)^{\dagger}A(s)\right)y(s)$$

$$= -\frac{1}{(s+1)^2(s^2+1)(s^2+2s+2)}$$

$$\times \begin{pmatrix} -s(s+1)^2 & -(s+1)^2 & s^3(s+1)(s^2+1) \\ 0 & 0 & -(s+1)(s^2+1)(s^2+2s+2) \\ -s(s+1)^3 & -(s+1)^3 & s^3(s+1)^2(s^2+1) \end{pmatrix}$$

$$\times \begin{pmatrix} s^2 \\ s \\ 0 \end{pmatrix} + \frac{1}{s^2+2s+2}\begin{pmatrix} (s+1)^2 & 0 & -(s+1) \\ 0 & 0 & 0 \\ -(s+1) & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix}$$

$$\Leftrightarrow \hat{\beta}(s) = \frac{1}{s^2+2s+2}\begin{pmatrix} s-(s+1)a(s) \\ 0 \\ s(s+1)+a(s) \end{pmatrix} \tag{E.7a}$$

and for $s = -1$

$$\hat{\beta}(-1) = A(-1)^{\dagger}\hat{a}(-1) + \left(I_3 - A(-1)^{\dagger}A(-1)\right)Y$$

$$= \begin{pmatrix} -1/4 & 1/4 & 0 \\ 1/4 & -1/4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$\Leftrightarrow \hat{\beta}(-1) = \begin{pmatrix} -\frac{1}{2}(1-Y_1-Y_2) \\ \frac{1}{2}(1+Y_1+Y_2) \\ Y_3 \end{pmatrix}, \tag{E.7b}$$

where $a(s) := -y_1(s)(s+1) + y_3(s)$ is an arbitrary rational function. Assuming that $Y_1 = -Y_2 - 1$ and $Y_3 = a(-1)(\equiv y_3(-1))$ the solution $\hat{\beta}(s)$ is continuous at $s = -1$ and its inverse Laplace transform is

$$\beta(t) = \begin{pmatrix} e^{-t}[\cos(t) - \sin(t)] + \int_{0-}^{t} e^{-(t-\tau)}\cos(t-\tau)x(\tau)\,d\tau \\ 0 \\ \delta(t) - e^{-t}[\cos(t) + \sin(t)] + \int_{0-}^{t} e^{-(t-\tau)}\sin(t-\tau)x(\tau)\,d\tau \end{pmatrix}, \tag{E.8}$$

where $x(t)$ denotes the inverse Laplace transform of $a(s)$, i.e., $x(t) = L^{-1}[a(s)]$ and thus it is an arbitrary function.

c. *Feedback Compensation*

Consider an open loop system with transfer function matrix $G(s) \in \mathfrak{R}(s)^{n\times m}$, shown in Fig. 1.

We would like to find out when there exist an output feedback of the form

$$u(s) = -F(s)y(s) + v(s) \qquad \text{with } F(s) \in \mathfrak{R}(s)^{m\times n} \tag{4.11}$$

such that the closed loop system, shown in Fig. 2, has transfer function $H(s)$. We would like therefore to find a rational matrix $F(s) \in \mathfrak{R}(s)^{m\times n}$ which satisfies

$$H(s) = (I_n + G(s)F(s))^{-1}G(s) \Leftrightarrow G(s)F(s)H(s) = G(s) - H(s). \tag{4.12}$$

Let $G(s) = G'(s)/g(s)$, where $G'(s) \in \mathfrak{R}[s]^{n\times m}$ and $g(s)$ is the least common multiple of all the common denominators of the matrix $G(s)$. In the same way, let $H(s) = H'(s)/h(s)$, where $H'(s) \in \mathfrak{R}[s]^{m\times n}$ and $h(s)$ is the least common multiple of all the denominators of the matrix $H(s)$. Then

$$u(s) \rightarrow \boxed{G(s)} \rightarrow y(s)$$

FIG. 1. Open loop system.

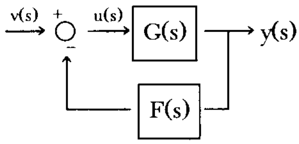

FIG. 2. Closed loop system.

Eq. (4.12) may be rewritten as

$$\frac{G'(s)}{g(s)}F(s)\frac{H'(s)}{h(s)} = \frac{G'(s)}{g(s)} - \frac{H'(s)}{h(s)} \tag{4.13}$$

$$\Leftrightarrow G'(s)F(s)H'(s) = G'(s)h(s) - H'(s)g(s).$$

In light of Theorem 4.2 we can easily obtain a necessary and sufficient condition for the existence of solution of Eq. (4.13) by the following

THEOREM 4.8. *Equation* (4.13) *has a solution iff*

$$G'(s)G'(s)^{\dagger}[G'(s)h(s) - H'(s)g(s)]H'(s)^{\dagger}H'(s)$$

$$= G'(s)h(s) - H'(s)g(s) \tag{4.14}$$

in which case all the compensators are given by

$$F(s) = G'(s)^{\dagger}[G'(s)h(s) - H'(s)g(s)]H'(s)^{\dagger} + Y(s)$$

$$- G'(s)^{\dagger}G'(s)Y(s)H'(s)H'(s)^{\dagger}, \tag{4.15}$$

where $Y(s)$ is arbitrary to within having the dimension of $F(s)$.

Proof. Let $P(s) = G'(s)$, $X(s) = F(s)$, $Q(s) = H'(s)$, and $C(s) = G'(s)h(s) - H'(s)g(s)$ in Theorem 4.2. Then the proof of Theorem 4.8 follows.

EXAMPLE 4.9. An open loop system with transfer function matrix

$$G(s) = \begin{pmatrix} 1/(s-1) & 0 & 0 \\ 0 & 1/(s-2) & 0 \end{pmatrix} \in \mathfrak{R}(s)^{2\times 3} \tag{E.1}$$

is represented in Fig. 3.

We would like to find out when there exist an output feedback of the form

$$u(s) = -F(s)y(s) + v(s) \qquad \text{with } F(s) \in \mathfrak{R}(s)^{3\times 2} \tag{E.2}$$

such that the closed loop system shown in Fig. 4 has transfer function matrix

$$H(s) = \begin{pmatrix} 0 & 1/(s+1) & 0 \\ 1/(s+1)^2 & 0 & 0 \end{pmatrix} \in \mathfrak{R}(s)^{2\times 3}. \tag{E.3}$$

It is easily seen that

$$G(s) = \frac{G'(s)}{g(s)} \qquad \text{and} \qquad H(s) = \frac{H'(s)}{h(s)}, \tag{E.4}$$

where

$$G'(s) = \begin{pmatrix} s-2 & 0 & 0 \\ 0 & s-1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{G_1'} s + \underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}}_{G_0'} \tag{E.5a}$$

$$H'(s) = \begin{pmatrix} 0 & s+1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{H_1'} s + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{H_0'} \tag{E.5b}$$

$$d(s) = (s-1)(s-2) \qquad \text{and} \qquad h(s) = (s+1)^2. \tag{E.5c}$$

$$u(s) \longrightarrow \boxed{G(s)} \longrightarrow y(s)$$

FIG. 3. Open loop system.

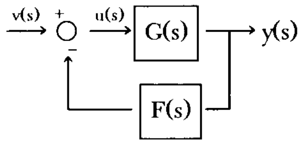

FIG. 4. Closed loop system.

It is easily seen (according to Algorithm 3.3) that the generalized inverses of $G'(s)$ and $H'(s)$ are the following

$$G'(s)^{\dagger} = \begin{pmatrix} 1/(s-2) & 0 \\ 0 & 1/(s-1) \\ 0 & 0 \end{pmatrix}$$

and
$$H'(s)^{\dagger} = \begin{pmatrix} 0 & 1 \\ 1/(s+1) & 0 \\ 0 & 0 \end{pmatrix}. \tag{E.6}$$

(We are not interested in the values of $s \in \{1, -1, 2\}$ because of the definition of $G(s)$ and $H(s)$.) A necessary and sufficient condition for the existence of $F(s)$ according to Theorem 4.8 is that

$$\begin{aligned} &G'(s)G'(s)^{\dagger}[G'(s)h(s) - H'(s)g(s)]H'(s)^{\dagger}H'(s) \\ &\quad = G'(s)h(s) - H'(s)g(s), \end{aligned} \tag{E.7}$$

which can be easily seen to be satisfied, in which case all the feedback compensators $F(s)$ which give rise to $H(s)$ are given by

$$\begin{aligned} F(s) &= G'(s)^{\dagger}[G'(s)h(s) - H'(s)g(s)]H'(s)^{\dagger} + Y(s) \\ &\quad - G'(s)^{\dagger}G'(s)Y(s)H'(s)H'(s)^{\dagger} \\ &= \left[\begin{pmatrix} \dfrac{1}{s-2} & 0 \\ 0 & \dfrac{1}{s-1} \\ 0 & 0 \end{pmatrix}\right] \left\{ \begin{pmatrix} s-2 & 0 & 0 \\ 0 & s-1 & 0 \end{pmatrix} (s+1)^2 \right. \\ &\qquad \left. - \begin{pmatrix} 0 & s+1 & 0 \\ 1 & 0 & 0 \end{pmatrix} (s-1)(s-2) \right\} \begin{pmatrix} 0 & 1 \\ \dfrac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$+ Y(s) - \begin{pmatrix} \dfrac{1}{s-2} & 0 \\ 0 & \dfrac{1}{s-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s-2 & 0 & 0 \\ 0 & s-1 & 0 \end{pmatrix}$$

$$\times Y(s) \begin{pmatrix} 0 & s+1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \dfrac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{s-2} & 0 \\ 0 & \dfrac{1}{s-1} \\ 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} (s-2)(s+1)^2 & -(s+1)(s-1)(s-2) & 0 \\ -(s-1)(s-2) & (s-1)(s+1)^2 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 1 \\ \dfrac{1}{s+1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ Y(s) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Y(s) I_2$$

$$\Leftrightarrow F(s) = \begin{pmatrix} -(s-1) & (s+1)^2 \\ (s+1) & -(s-2) \\ y_{31}(s) & y_{32}(s) \end{pmatrix}, \tag{E.8}$$

where $y_{31}(s)$, $y_{32}(s)$ are arbitrary functions of s.

V. EVALUATION OF THE LAURENT EXPANSION

It is easily seen from Section III that the generalized inverse $A(s)^{\dagger}$ of $A(s)$ is unique and is given by

$$A(s)^{\dagger} = -a_k(s)^{-1} A(s)^{\mathrm{T}} B_{k-1}(s). \tag{5.1}$$

In the sequel we shall compute the Laurent expansion of the generalized inverse $A(s)^{\dagger}$. From (5.1) we obtain that

$$a_k(s) A(s)^{\dagger} = -A(s)^{\mathrm{T}} B_{k-1}(s). \tag{5.2}$$

Let

$$A(s)^{\dagger} = H_u s^u + H_{u-1} s^{u-1} + \cdots + H_1 s + H_0 + H_{-1} \frac{1}{s} + \cdots. \tag{5.3}$$

Substituting $a_k(s)$, $A(s)^{\dagger}$, $A(s)^{\mathrm{T}}$, and $B_{k-1}(s)$ by (3.11), (5.3), (3.2), and (3.12) respectively in (5.2) we obtain

$$\begin{aligned}\left(\sum_{i=0}^{2kq} \hat{a}_{k,i} s^i\right)\left(\sum_{i=-\infty}^{u} H_i s^i\right) &= -\left(\sum_{i=0}^{q} A_i^{\mathrm{T}}\right)\left(\sum_{i=0}^{2(k-1)q} B_{k-1,i} s^i\right) \\ &\equiv \sum_{i=0}^{(2k-1)q}\left(-\sum_{j=0}^{i} A_{i-j}^{\mathrm{T}} B_{k-1,j}\right) s^i =: \sum_{i=0}^{(2k-1)q} B_i' s^i.\end{aligned} \tag{5.4}$$

Let also an integer f such that $\hat{a}_{k,f} \neq 0$ and $\hat{a}_{k,i} = 0\ \forall i > f$. Equating the coefficient matrices of each power of s in (5.4), we obtain the following relations:

$$\begin{aligned} &\hat{a}_{k,f} H_u = B_{f+u}' \\ &\hat{a}_{k,f-1} H_u + \hat{a}_{k,f} H_{u-1} = B_{f+u-1}' \\ &\text{-------------------------------} \\ &\hat{a}_{k,0} H_0 + \hat{a}_{k,1} H_{-1} + \cdots + \hat{a}_{k,f} H_{-f} = B_0' \end{aligned} \tag{5.5}$$

and

$$\sum_{i=0}^{f} \hat{a}_{k,i} H_{-i-j} = 0 \qquad \text{for } j = -1, -2, \ldots . \tag{5.6}$$

Equation (5.5) may be rewritten in matrix form as

$$\underbrace{\begin{pmatrix} \hat{a}_{k,f} I_m & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hat{a}_{k,f-1} I_m & \hat{a}_{k,f} I_m & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{k,0} I_m & \hat{a}_{k,1} I_m & \cdots & \hat{a}_{k,f} I_m & 0 & \cdots & 0 \\ 0 & \hat{a}_{k,0} I_m & \cdots & \hat{a}_{k,f-1} I_m & \hat{a}_{k,f} I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{a}_{k,0} I_m & \hat{a}_{k,1} I_m & \cdots & \hat{a}_{k,f} I_m \end{pmatrix}}_{A}$$

$$\times \underbrace{\begin{pmatrix} H_u \\ H_{u-1} \\ \vdots \\ H_{u-f} \\ \vdots \\ H_{-f+1} \\ H_{-f} \end{pmatrix}}_{H} = \underbrace{\begin{pmatrix} B'_{f+u} \\ B'_{f+u-1} \\ \vdots \\ B'_u \\ \vdots \\ B'_1 \\ B'_0 \end{pmatrix}}_{B}$$

$$\Leftrightarrow A \times H = B. \tag{5.7}$$

From our assumption that $\hat{a}_{k,f} \neq 0$ we conclude that the Toeplitz matrix A is always nonsingular. Hence we can find a unique solution $H = A^{-1}B$ that determines the first $u + f + 1$ matrices H_i, $i = -f, -f + 1, \ldots, u$. The inverse A^{-1} may be written in the form [4, 14]:

$$D = A^{-1} = \begin{pmatrix} d_0 I_m & 0 & \cdots & 0 \\ d_1 I_m & d_0 I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_{f+u} I_m & d_{f+u-1} I_m & \cdots & d_0 I_m \end{pmatrix}, \tag{5.8}$$

where

$$d_0 = \frac{1}{\hat{a}_{k,f}} \tag{5.9}$$

and

$$d_j = (-1)^j \frac{1}{\hat{a}_{k,f}} \det \begin{pmatrix} \hat{a}_{k,f-1} & \hat{a}_{k,f-2} & \cdots & \hat{a}_{k,f-j+1} & \hat{a}_{k,f-j} \\ \hat{a}_{k,f} & \hat{a}_{k,f-1} & \cdots & \hat{a}_{k,f-j} & \hat{a}_{k,f-j-1} \\ 0 & \hat{a}_{k,f} & \cdots & \hat{a}_{k,f-j-1} & \hat{a}_{k,f-j-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \hat{a}_{k,f} & \hat{a}_{k,f-1} \end{pmatrix} \tag{5.10a}$$

or equivalently

$$d_j = -\frac{1}{\hat{a}_{k,f}} \sum_{i=0}^{j-1} \left(\hat{a}_{k,f-j+i} d_i \right) \qquad \text{for } j = 1, 2, \ldots, f + u. \tag{5.10b}$$

From (5.7) and (5.8) we obtain that

$$H = A^{-1}B = DB \tag{5.11a}$$

or equivalently

$$H_i = -\sum_{j=0}^{u-i} \left(d_j B'_{f+i+j} \right) \qquad \text{for } i = -f, -f+1, \ldots, u, \tag{5.11b}$$

which gives the first $f + u + 1$ matrices H_i, $i = -f, -f + 1, \ldots, u$. From Eq. (5.6) we obtain

$$H_{-i-f} = -\frac{1}{\hat{a}_{k,f}} \left(\sum_{j=0}^{f-1} \left(\hat{a}_{k,j} H_{-i-j} \right) \right) \qquad \text{for } i = -1, -2, \ldots. \tag{5.12}$$

This equation gives rise to the values of the rest matrices H_i, $i = -f - 1$, $-f - 2 \ldots, -\infty$. The whole theory described above for the computation of the generalized inverse of $A(s)$ is summarized in the following

ALGORITHM 5.1. (Computation of the Laurent Expansion of Generalized Inverse $A(s)^{\dagger}$ of $A(s)$)

Step 1: Compute $f := \deg[a_k(s)]$.
Step 2: $d_0 = 1/\hat{a}_{k,f}$
For $i = 1$ to $f + u$

$$d_i = -\frac{1}{\hat{a}_{k,f}} \sum_{j=0}^{i-1} \left(\hat{a}_{k,f-i+j} d_j\right)$$

Next i
Step 3: For $i = -f$ to u

$$H_i = -\sum_{j=0}^{u-i} \left(d_j B'_{f+i+j}\right)$$

Next i
Step 4: For $i = -1$ to $-\infty$ step -1

$$H_{-i-f} = -\frac{1}{\hat{a}_{k,f}} \left(\sum_{j=0}^{f-1} \left(\hat{a}_{k,j} H_{-i-j}\right) \right)$$

Next i
Output: $A(s)^{\dagger} = \sum_{i=-\infty}^{u} H_i s^i$.

VI. CONCLUSIONS

A two-dimension recursive algorithm is determined for the computation of the generalized inverse $A^{\dagger}(s)$ of a polynomial matrix $A(s) = A_q s^q + \cdots + A_1 s + A_0 \in \Re[s]^{n \times m} \forall s \in \mathbb{R}$ in terms of the coefficient matrices $A_i \in \Re^{n \times m}$ and its Laurent expansion has also been evaluated. However, $A^{\dagger}(s)$ need not be the generalized inverse of $A(s)$ is s is not real. The whole theory has been illustrated via three implications in linear system theory. An implementation of all the above algorithms in the symbolic computation language of MAPLE is presented in [5]. The above results may also be extended to the multivariable polynomial matrices using the same framework of this article (see [7]) with analogous implications in analysis of multidimensional systems.

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