THE MAX-MAX PROBLEM

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During the last lecture we required the following result involving iterative maximizations:

Theorem 1. Let X and Y be arbitrary sets, and consider the function $f : X \times Y \to \mathbb{R}$. Then

$$\boxed{\max_{x\in X, y\in Y} f(x,y) = \max_{x\in X}\left[\max_{y\in Y} f(x,y)\right]} \qquad (1)$$

assuming that all the maxima are well defined.

Proof. Equality (1) will be proven as a combination of two inequalities:

1. $\max_{x\in X, y\in Y} f(x,y) \geq \max_{x\in X}\left[\max_{y\in Y} f(x,y)\right]$

   To establish this inequality, notice that by definition

   $$\max_{\tilde{x}\in X, \tilde{y}\in Y} f(\tilde{x},\tilde{y}) \geq f(x,y), \qquad \text{for every } x \in X,\ y \in Y \qquad (2)$$

   Thus, we can maximize both sides of (2) with respect to $y \in Y$, which gives

   $$\max_{\tilde{x}\in X, \tilde{y}\in Y} f(\tilde{x},\tilde{y}) \geq \max_{y\in Y} f(x,y), \qquad \text{for every } x \in X$$

   and then with respect to $x \in X$, to obtain

   $$\max_{\tilde{x}\in X, \tilde{y}\in Y} f(\tilde{x},\tilde{y}) \geq \max_{x\in X}\left[\max_{y\in Y} f(x,y)\right]$$

2. $\max_{x\in X, y\in Y} f(x,y) \leq \max_{x\in X}\left[\max_{y\in Y} f(x,y)\right]$

   Let $x^* \in X$ and $y^* \in Y$ be such that

   $$f(x^*, y^*) = \max_{x\in X, y\in Y} f(x,y)$$

   Then, since

   $$f(x,y) \leq \max_{\tilde{y}\in Y} f(x,\tilde{y}), \qquad \text{for every } x \in X,\ y \in Y$$

   we have that

   $$f(x^*, y) \leq \max_{\tilde{y}\in Y} f(x^*,\tilde{y}) \leq \max_{\tilde{x}\in X}\left[\max_{\tilde{y}\in Y} f(\tilde{x},\tilde{y})\right], \qquad \text{for every } y \in Y$$

   Therefore,

   $$\max_{x\in X, y\in Y} f(x,y) = f(x^*, y^*) \leq \max_{\tilde{x}\in X}\left[\max_{\tilde{y}\in Y} f(\tilde{x},\tilde{y})\right]$$

This concludes the proof. ■

The previous result can be further generalized to cases where the maxima do not exist, and to more general sets than $X \times Y$. To this end, let Ω be a constraint set for (x, y), and define

$X := \{x : (x, y) \in \Omega \;\text{ for some }\; y\}$
$Y_x := \{y : (x, y) \in \Omega\}$

Then we have the following result:

Theorem 2. Let $f : \Omega \to \mathbb{R}$. Then

$$\boxed{\sup_{(x,y)\in\Omega} f(x, y) = \sup_{x\in X}\left[\sup_{y\in Y_x} f(x, y)\right]} \tag{3}$$

Proof. As in the proof of Theorem 1, equality (3) is established by proving two inequalities:

1. $\sup_{(x,y)\in\Omega} f(x, y) \ge \sup_{x\in X}\left[\sup_{y\in Y_x} f(x, y)\right]$

   For this proof we need to be careful about X and Y_x. Notice that

   $$\sup_{(\tilde{x},\tilde{y})\in\Omega} f(\tilde{x}, \tilde{y}) \ge f(x, y), \qquad \text{for every } x \in X \text{ , } y \in Y_x \tag{4}$$

   Thus, for a fixed $x \in X$ we can take the supremum of both sides of (4) with respect to $y \in Y_x$, giving

   $$\sup_{(\tilde{x},\tilde{y})\in\Omega} f(\tilde{x}, \tilde{y}) \ge \sup_{y\in Y_x} f(x, y), \qquad \text{for every } x \in X$$

   and then with respect to $x \in X$, to obtain

   $$\sup_{(\tilde{x},\tilde{y})\in\Omega} f(\tilde{x}, \tilde{y}) \ge \sup_{x\in X}\left[\sup_{y\in Y_x} f(x, y)\right]$$

2. $\sup_{(x,y)\in\Omega} f(x, y) \le \sup_{x\in X}\left[\sup_{y\in Y_x} f(x, y)\right]$

   This inequality is a bit trickier because the supremum of a function is not always achieved, but we can get as close to its value as we want. Take an $\varepsilon > 0$. By definition of the supremum, there exists $(x^*, y^*) \in \Omega$ such that

   $$\sup_{(x,y)\in\Omega} f(x, y) - \varepsilon < f(x^*, y^*) \le \sup_{(x,y)\in\Omega} f(x, y)$$

   Then, since

   $$f(x, y) \le \sup_{\tilde{y}\in Y_x} f(x, \tilde{y}), \qquad \text{for every } x \in X \text{ , } y \in Y_x$$

   we have that

   $$f(x^*, y) \le \sup_{\tilde{y}\in Y_x} f(x^*, \tilde{y}) \le \sup_{x\in X}\left[\sup_{y\in Y_x} f(x, y)\right], \qquad \text{for every } y \in Y_{x^*}$$

   Therefore,

   $$\sup_{(x,y)\in\Omega} f(x, y) < f(x^*, y^*) + \varepsilon \le \sup_{x\in X}\left[\sup_{y\in Y_x} f(x, y)\right] + \varepsilon \tag{5}$$

As $\varepsilon > 0$ was arbitrary, we can make $\varepsilon \to 0_+$ in (5), which gives the desired inequality. ■