Exercise Set 10

1. *Misspecification of the model structure and prediction ability* (Problem 11.6 of [1])
Consider a first-order moving average process

$$y_t = w_t + cw_{t-1}, \qquad E\{w_t^2\} = \lambda^2, \qquad \{w_t\} \text{ white noise}$$

Assume that this process is identified by the least squares method as an autorregressive process

$$y_t + a_1 y_{t-1} + \cdots + a_n y_{t-n} = \varepsilon_t$$

Consider the asymptotic case ($N \to \infty$).

(a) Find the prediction error variance $E\{\varepsilon_t^2\}$ for the model when $n = 1, 2, 3$. Compare with the optimal prediction error variance based on the true system. Generalize this comparison to an arbitrary value of n.
Hint. The variance corresponding to the model is given by

$$E\{\varepsilon_t^2\} = \min_{a_1,\ldots,a_n} E\{[y_t + a_1 y_{t-1} + \cdots + a_n y_{t-n}]^2\}$$

(b) By what percentage is the prediction error variance deteriorated for $n = 1, 2, 3$ (as compared to the optimal value) in the following two cases?

Case I: $c = 0.5$
Case II: $c = 1.0$

2. *On testing cross-correlations between residuals and input* (Problem 11.9 of [1])
It is shown in Example 11.2 of [1] that

$$\sqrt{N}\, x_\tau \xrightarrow[N\to\infty]{d} \mathcal{N}(0,1)$$

where

$$x_\tau = \frac{\hat{r}_{\varepsilon u}(\tau)}{[\hat{r}_\varepsilon(0)\hat{r}_u(0)]^{1/2}}, \qquad \hat{r}_{\varepsilon u}(\tau) = \frac{1}{N}\sum_{t=1-\min(0,\tau)}^{N-\max(\tau,0)} \varepsilon_{t+\tau} u_t, \qquad \hat{r}_u(0) = \frac{1}{N}\sum_{t=1}^{N} u_t^2, \qquad \hat{r}_\varepsilon(0) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon_t^2$$

Hence, for every τ it holds asymptotically with 95 percent probability that $|x_\tau| \le 1.96/\sqrt{N}$. By analogy with equation (11.9) of [1], it may be tempting to define and use the following test quantity:

$$y := N\sum_{k=1}^{M} x_{\tau+k}^2 = \frac{N}{\hat{r}_\varepsilon(0)\hat{r}_u(0)}\sum_{k=1}^{M} \hat{r}_{\varepsilon u}^2(\tau+k)$$

instead of equation (11.18) of [1]. Compare the test quantities y above and equation (11.18) of [1]. Evaluate their means.

References
[1] T. Söderström and P. Stoica. *System Identification*. Prentice-Hall, 1989.