Exercise Set 6

1. *Consistency of the LS method for FIR models* (Problem 6.3 of [1])
Consider an FIR model

$$y_t = b_1 u_{t-1} + \cdots + b_{n_b} u_{t-n_b} + v_t$$

where $\{u_t\}$ is a persistently exciting input of order n_b, and independent of $\{v_t\}$. Show that the parameters $b_1, \ldots, b_{n_b}$ can be consistently estimated using the Least Squares method even in the presence of colored noise $\{v_t\}$.

2. *Estimating the AR part of an ARMA model* (Problem 7E.1 of [2])
A method to estimate the AR part has been given as follows. Let

$$\hat{R}_y^N(\tau) := \frac{1}{N} \sum_{t=\tau}^{N} y_t y_{t-\tau}$$

Then solve for $\hat{a}_i^N$ from

$$\hat{R}_y^N(\tau) + a_1 \hat{R}_y^N(\tau - 1) + \cdots + a_{n_a} \hat{R}_y^N(\tau - n_a) = 0, \qquad \tau = n_c + 1, \ldots, n_c + n_a$$

Show that this is (essentially) an application of the IV method using specific instruments. Which ones?

3. *Maximum likelihood for disturbances with exponential distribution* (Problem 6.10 of [1])
Consider the model

$$y_t = -a y_{t-1} + b u_{t-1} + w_t$$

where $\{w_t\}$ is a sequence of independent and identically distributed random variables with probability density function (where $\mu > 0$ is known)

$$P\{x; \mu\} = \begin{cases} \mu \exp(-\mu x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and $\{u_t\}$ is an input signal, independent of $\{w_t\}$. Design an ML method to estimate a and b.

Advanced Questions

1A. *The Steiglitz-McBride method* (Problem 7.22 of [3])
Consider the output error model

$$y_t = \frac{B(q^{-1})}{A(q^{-1})} u_t + \varepsilon_t$$

where $A(q^{-1})$ and $B(q^{-1})$ are polynomials in q^{-1} of degree n. The following iterative scheme is based on successive linear least squares fits for determining A and B

$$(\hat{A}_{k+1}, \hat{B}_{k+1}) = \arg\min_{(A,B)} \sum_{t=1}^{N} \left[A(q^{-1}) \left\{ \frac{1}{\hat{A}_k(q^{-1})} y_t \right\} - B(q^{-1}) \left\{ \frac{1}{\hat{A}_k(q^{-1})} u_t \right\} \right]^2 \qquad (1)$$

Assume that the data satisfy

$$y_t = \frac{B_0(q^{-1})}{A_0(q^{-1})} u_t + v_t$$

where $A_0(q^{-1})$ and $B_0(q^{-1})$ are coprime and of degree n, $\{u_t\}$ is persistently exciting of order $2n$ and $\{v_t\}$ is a stationary disturbance that is independent of the input $\{u_t\}$. Consider the asymptotic case where N, the number of data samples, tends to infinity.

(a) Assume that $\{v_t\}$ is white. Show that the only possible stationary solution of (1) is given by $A(q^{-1}) = A_0(q^{-1})$ and $B(q^{-1}) = B_0(q^{-1})$.

(b) Assume that $\{v_t\}$ is colored noise. Show that $A(q^{-1}) = A_0(q^{-1})$, $B(q^{-1}) = B_0(q^{-1})$ is in general not a possible stationary solution to (1).

Hint. Analyze this method in a similar way as in the bias of the LS method.

2A. *Consistency and uniform convergence* (Problem 8D.1 of [2])
Show that if

$$\sup_{-1 \le x \le 1} \left| f_N(x) - \bar{f}(x) \right| \to 0, \qquad N \to \infty$$

where $f_N, \bar{f} : [-1,1] \to \mathbb{R}$ (for $N \in \mathbb{N}$) are continuous, and

$$x_N = \arg\min_{-1 \le x \le 1} f_N(x)$$

then

$$x_N \to \arg\min_{-1 \le x \le 1} \bar{f}(x), \qquad N \to \infty$$

Hint. Let $\varepsilon > 0$ be arbitrary and set $X^* := \{x \in [-1,1] : \left| x - x^* \right| < \varepsilon$ for some $x^* \in \arg\min_{-1 \le \tilde{x} \le 1} \bar{f}(x)\}$. Next choose a number $\delta > 0$ so that $\inf_{x \in [-1,1] - X^*} \bar{f}(x) \ge \min_{-1 \le x \le 1} \bar{f}(x) + \delta$. Then choose N_0 such that $\left| f_N(x) - \bar{f}(x) \right| < \delta/3$ for all $N \ge N_0$ and all $x \in [-1,1]$. Deduce that $x_N \in X^*$ for all $N \ge N_0$.

References
[1] R. Johansson. *System Modeling and Identification*. Prentice-Hall, 1993.
[2] L. Ljung. *System Identification: Theory for the User*, 2nd Edition. Prentice-Hall, 1999.
[3] T. Söderström and P. Stoica. *System Identification*. Prentice-Hall, 1989.