Exercise Set 9

1. *Parameter covariance when using the wrong noise covariance matrix* (Problem 9E.4 of [1])
Consider the notation of [1]. Apply the multivariable expression (9.47) of [1] to the quadratic criterion (7.27) of [1]: $l(\varepsilon) = (1/2)\varepsilon^T \Lambda^{-1}\varepsilon$, assuming $\Lambda_0 = \bar{E}\{e_0(t)e_0^T(t)\}$. Let $P_\theta(\Lambda)$ denote the resulting covariance matrix, and show that

$$P_\theta(\Lambda) = [E\{\psi\Lambda^{-1}\psi^T\}]^{-1}[E\{\psi\Lambda^{-1}\Lambda_0\Lambda^{-1}\psi^T\}][E\{\psi\Lambda^{-1}\psi^T\}]^{-1}$$

Use Problem 7D.8 of [1] to show that $P_\theta(\Lambda) \geq P_\theta(\Lambda_0)$ for all symmetric positive definite Λ.

2. *Asymptotic variance when G and H have shared parameters*
Consider the model structure

$$y_t = G(q;\alpha,\beta)u_t + H(q;\beta)w_t \tag{1}$$

where $\{w_t\}$ is zero mean white noise, $\{u_t\}$ is a persistently exciting input independent of $\{w_t\}$, and (α,β) are parameters to be estimated using PEM. Assume that $S \in \mathcal{M}$, so that there are constants (α_0,β_0) such that $G_0(q) = G(q;\alpha_0,\beta_0)$ and $H_0(q) = H(q;\beta_0)$. Denote the parameter covariance matrix of (1) as P_1.
(a) Show, using the geometric approach of [2], that the parameter covariance P_2 of the model

$$y_t = G(q;\alpha,\beta_0)u_t + H(q;\beta)w_t$$

satisfies $P_2 \leq P_1$.
(b) Consider the particular case of (1) given by

$$G(q;\alpha,\beta) = (\alpha+\beta)q^{-1}$$
$$H(q;\beta) = 1 + \beta q^{-1}$$

Show that for this case, $P_2 = P_1$. Is the result intuitively obvious? Explain.

3. (**Advanced**) *Derivation of the Barankin bound* (Problems 2.4.18 and 2.4.19 of [3])
Let $P_Y\{y;\theta\}$ be the probability density function of Y, given θ. Let H be an arbitrary random variable that is independent of Y defined so that $\theta + H$ ranges over all possible values of θ. Assume that $P_{H_1}\{h\}$ and $P_{H_2}\{h\}$ are two arbitrary probability density functions for H. Assuming that $\hat{\theta}(Y)$ is an unbiased estimator of θ, we have

$$\int[\hat{\theta}(y) - (\theta+h)]P_Y\{y;\theta+h\}\,dy = 0$$

Multiplying by $P_{H_i}\{h\}$ and integrating over h, we obtain

$$\int dh P_{H_i}\{h\}\int[\hat{\theta}(y) - (\theta+h)]P_Y\{y;\theta+h\}\,dy = 0, \qquad i = 1,2 \tag{2}$$

(a) Show that

$$\mathrm{var}\{\hat{\theta}(Y)\} \geq \frac{[E_{H_1}\{H\} - E_{H_1}\{H\}]^2}{\int\left[\frac{\left(\int[P_{H_1}\{h\} - P_{H_2}\{h\}]P_Y\{y;\theta+h\}\,dh\right)^2}{P_Y\{y;\theta\}}\right]dy} \tag{3}$$

for every $P_{H_1}\{h\}$ and $P_{H_2}\{h\}$. Observe that since this is true for all $P_{H_1}\{h\}$ and $P_{H_2}\{h\}$, we may write

$$\operatorname{var}\{\hat{\theta}(Y)\} \geq \sup_{P_{H_1}, P_{H_2}} \frac{[E_{H_1}\{H\} - E_{H_1}\{H\}]^2}{\int \left[\frac{\left(\int [P_{H_1}\{h\} - P_{H_2}\{h\}] P_Y\{y;\theta + h\}\, dh \right)^2}{P_Y\{y;\theta\}} \right] dy}$$

Hint. To establish (3), take the difference between the two expressions in (2), and use Cauchy-Schwarz inequality. You will probably have to exchange some limits of integration (assume that regularity conditions apply).

Remark. Observe that this bound does not require any regularity conditions. Barankin (1949) has shown that this is actually the greatest lower bound on the variance of an unbiased estimator.

(b) We now consider two special cases:

(b.1) First, let $P_{H_2}\{h\} = \delta(h)$. What is the resulting bound?

(b.2) Second, let $P_{H_1}\{h\} = \delta(h - h_0)$, where $h_0 \neq 0$. Show that

$$\operatorname{var}\{\hat{\theta}(Y)\} \geq \left(\inf_{h_0 \neq 0} \frac{1}{h_0^2} \left[\int \frac{(P_Y\{y;\theta + h_0\})^2}{P_Y\{y;\theta\}} dy - 1 \right] \right)^{-1}$$

The infimum being over all $h_0 \neq 0$ such that $P_Y\{y;\theta\} = 0$ implies that $P_Y\{y;\theta + h_0\} = 0$.

(b.3) Show that the bound given in (b.2) is always as good as the Cramér-Rao inequality when the latter applies.

Hint. Since h_0 can be chosen (almost) arbitrarily, expand $P_Y\{y;\theta + h_0\}$ in a Taylor series around $h_0 = 0$ and assume that regularity conditions apply.

References

[1] L. Ljung. *System Identification: Theory for the User*, 2nd Edition. Prentice-Hall, 1999.

[2] J. Mårtensson. *Geometric Analysis of Stochastic Model Errors in System Identification*. PhD Thesis, KTH, 2007.

[3] H. L. Van Trees. *Detection, Estimation, and Modulation Theory, Part I*. John Wiley & Sons, 2001.