EXTENSION OF LEMMA 2.1 OF LJUNG

CRISTIAN R. ROJAS

In order to solve problem 6G.1 of [1], it would be convenient to use a result similar to Lemma 2.1 of [1], which could establish the weak convergence of $|S_N(\omega)|^2$ (instead of its expected value) to $\Phi_s(\omega)$ with probability 1 as $N \to \infty$, or in symbols:

$$|S_N(\omega)|^2 \xrightarrow[w]{} \Phi_s(\omega) \qquad a.s. \text{ for } N \to \infty \tag{1}$$

which means that

$$\int_{-\pi}^{\pi} |S_N(\omega)|^2 \Psi(\omega) d\omega \xrightarrow[N\to\infty]{a.s.} \int_{-\pi}^{\pi} \Phi_s(\omega)\Psi(\omega) d\omega \tag{2}$$

for all sufficiently smooth functions $\Psi(\omega)$.

In fact, this is the kind of result required to establish equation (6.47) of [1], which motivates the use of windows in the Blackman-Tukey estimators. The purpose of this note is to establish (1) by means of Theorem 2.3 of [1].

Theorem 1. Let $\{s(t)\}$ be a quasi-stationary signal with spectrum $\Phi_s(\omega)$ such that

$$s(t) - E\{s(t)\} = v(t) = \sum_{k=0}^{\infty} h_t(k) e(t-k) = H_t(q) e(t) \tag{3}$$

where $\{e(t)\}$ is a sequence of independent random variables with zero mean, variance λ_e, and bounded moments of order 4, and where the family of filters $\{H_t(q) :\ t \in \mathbb{N}\}$ is uniformly stable [1, page 27]. Assume that the covariance function of $\{s(t)\}$, $R_s(\tau) := \bar{E}\{s(t)s(t-\tau)\}$, satisfies

$$\sum_{\tau=-\infty}^{\infty} |R_s(\tau)| < \infty \tag{4}$$

Let

$$S_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} s(t) e^{-it\omega} \tag{5}$$

and let $\Psi : [-\pi, \pi] \to \mathbb{R}$ be a bounded function with Fourier coefficients $\{a_\tau\}$ such that

$$\sum_{\tau=-\infty}^{\infty} |a_k| < \infty \tag{6}$$

Then (1) holds.

Proof. This proof resembles the one of Lemma 2.1 of [1]. From (5) we have

$$
\begin{aligned}
\left|S_N(\omega)\right|^2 &= \frac{1}{N}\sum_{k=1}^{N}\sum_{l=1}^{N} s(k)s(l)e^{i\omega(k-l)} \\
&= [l-k=\tau] \\
&= \frac{1}{N}\sum_{\tau=1-N}^{N-1}\sum_{l=\max\{1,\tau+1\}}^{\min\{N,N+\tau\}} s(l)s(l-\tau)e^{-i\omega\tau} \\
&= \sum_{\tau=1-N}^{N-1}\hat{R}_N(\tau)e^{-i\omega\tau}
\end{aligned}
\tag{7}
$$

where

$$
\hat{R}_N(\tau) := \frac{1}{N}\sum_{l=\max\{1,\tau+1\}}^{\min\{N,N+\tau\}} s(l)s(l-\tau) \tag{8}
$$

Then, if we multiply (7) by $\Psi(\omega)$ and integrate on ω over $[-\pi,\pi]$, we obtain

$$
\begin{aligned}
\int_{-\pi}^{\pi}\left|S_N(\omega)\right|^2\Psi(\omega)d\omega &= \int_{-\pi}^{\pi}\sum_{\tau=1-N}^{N-1}\hat{R}_N(\tau)\Psi(\omega)e^{-i\omega\tau}d\omega \\
&= \sum_{\tau=1-N}^{N-1}\hat{R}_N(\tau)\int_{-\pi}^{\pi}\Psi(\omega)e^{-i\omega\tau}d\omega \\
&= \sum_{\tau=1-N}^{N-1}\hat{R}_N(\tau)a_\tau
\end{aligned}
\tag{9}
$$

Here the interchange of summation and integration is allowed since the sum is finite (it has $2N-1$ terms). Similarly,

$$
\begin{aligned}
\int_{-\pi}^{\pi}\Phi_s(\omega)\Psi(\omega)d\omega &= \int_{-\pi}^{\pi}\sum_{\tau=-\infty}^{\infty}R_s(\tau)e^{-i\tau\omega}\Psi(\omega)d\omega \\
&= \sum_{\tau=-\infty}^{\infty}R_s(\tau)\int_{-\pi}^{\pi}\Psi(\omega)e^{-i\omega\tau}d\omega \\
&= \sum_{\tau=-\infty}^{\infty}R_s(\tau)a_\tau
\end{aligned}
\tag{10}
$$

In (10) the interchange of summation and integration is allowed by Fubini's Theorem [2] because $\{R_s(\tau)\}$ is absolutely summable (by the assumptions of the Theorem), and $\Psi(\omega)$ is bounded.

Then, subtracting (10) from (9) we have

$$
\begin{aligned}
\int_{-\pi}^{\pi}\left|S_N(\omega)\right|^2\Psi(\omega)d\omega - \int_{-\pi}^{\pi}\Phi_s(\omega)\Psi(\omega)d\omega &= \sum_{\tau=1-N}^{N-1}\hat{R}_N(\tau)a_\tau - \sum_{\tau=-\infty}^{\infty}R_s(\tau)a_\tau \\
&= \sum_{\tau=1-N}^{N-1}a_\tau[\hat{R}_N(\tau)-R_s(\tau)] - \sum_{|\tau|\geq N}a_\tau R_s(\tau)
\end{aligned}
\tag{11}
$$

To show that this expression goes to 0 as $N\to\infty$, the idea is to use Problem 2D.5 of [1]. However, we still need to show that $\hat{R}_N(\tau)\xrightarrow{a.s.}R_s(\tau)$. To this end, notice that from (8)

$$
\hat{R}_N(\tau) = \frac{\min\{N,N+\tau\}}{N}\left[\frac{1}{\min\{N,N+\tau\}}\sum_{l=1}^{\min\{N,N+\tau\}}s(l)s(l-\tau) - \frac{1}{\min\{N,N+\tau\}}\sum_{l=1}^{\max\{1,\tau+1\}}s(l)s(l-\tau)\right] \tag{12}
$$

For fixed τ, the factor outside the brackets tends to 1 as $N\to\infty$. The first term inside the brackets tends a.s. to $R_s(\tau)$ by Theorem 2.3 of [1] and the assumptions of this Theorem. Finally, the second term in the brackets always converges to 0 as $N\to\infty$, since the sum does not depend on N (for a fixed τ).

The last condition that we need in order to use Problem 2D.5 of [1] is that $R_s(\tau)$ should be bounded (in τ). However, this condition follows from the definition of quasi-stationarity of $\{s(t)\}$. This concludes the proof. ■

Remark. Actually the conditions stated in Problem 2D.5 of [1] do not imply its conclusion. An extra condition, such as

$$|b_N(\tau)| \le C, \qquad \text{for all } N, \tau \in \mathbb{N} \tag{13}$$

is required. In terms of the notation of Theorem 1, this requires to show that $\{\hat{R}_N(\tau)\}$ is a.s. uniformly bounded in both N and τ. One way to do this is to notice that $\hat{R}_N(|\tau|)$, like a usual covariance sequence, is positive semidefinite, i.e., for all $x_1, \dots, x_n \in \mathbb{R}$ we have that

$$\sum_{i,k=1}^{n} x_i x_k \hat{R}_N(|i-k|) \ge 0 \tag{14}$$

In fact,

$$\begin{aligned}
\sum_{i,k=1}^{n} x_i x_k \hat{R}_N(|i-k|) &= \frac{1}{N} \sum_{i,k=1}^{n} \sum_{l=1}^{N} x_i x_k s(l) s(l-|i-k|) \\
&= [x_1 \ \cdots \ x_n] \begin{bmatrix} \hat{R}_N(0) & \cdots & \hat{R}_N(n-1) \\ \vdots & \ddots & \vdots \\ \hat{R}_N(1-n) & \cdots & \hat{R}_N(0) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= [x_1 \ \cdots \ x_n] \left\{ \sum_{l=1}^{N} \begin{bmatrix} s(l)s(l) & \cdots & s(l)s(l-n+1) \\ \vdots & \ddots & \vdots \\ s(l)s(l-n+1) & \cdots & s(l)s(l) \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= [x_1 \ \cdots \ x_n] \left\{ \sum_{l=1-n}^{N} \begin{bmatrix} s(l) \\ \vdots \\ s(l+n) \end{bmatrix} \begin{bmatrix} s(l) & \cdots & s(l+n) \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \sum_{l=1-n}^{N} [x_1 s(l) + \cdots x_n s(l+n)]^2 \\
&\ge 0
\end{aligned} \tag{15}$$

where, to simplify notation we have assumed $s(t) = 0$ outside $[1, \dots, N]$ (for a fixed N). From (15) we can deduce that $\hat{R}_N(0) \ge |\hat{R}_N(\tau)|$ (e.g. choose $x_1 = 1$, $x_\tau = \pm 1$ and $x_i = 0$ otherwise). Since we have shown that $\hat{R}_N(0) \xrightarrow[N\to\infty]{a.s.} R_s(0)$, then $\{\hat{R}_N(0)\}$ is a bounded sequence a.s., hence $\{\hat{R}_N(\tau)\}$ is uniformly bounded. □

References

[1] L. Ljung. *System Identification: Theory for the User*. 2nd Edition. Prentice-Hall, 1999.
[2] R. Bartle. *Elements of Integration*. John Wiley & Sons, 1966.