11.12.2023

Tema 4

1. $\mathbb{E}[X^2], \mathbb{E}[Y^2] < +\infty$

a) $\mathbb{V}ar[X+Y] = \mathbb{V}ar[X] + \mathbb{V}ar[Y] + 2\,Cov[X,Y]$

b) coeficientul de corelație a celor două variabile este un nr. din intervalul $[-1,1]$

a) $Cov[X,Y] := \mathbb{E}[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))] = \mathbb{E}[X\cdot Y] - \mathbb{E}[X]\cdot\mathbb{E}[Y]$

$\mathbb{V}ar(X] + \mathbb{V}ar[Y] = \mathbb{E}[X^2] - (\mathbb{E}(X])^2 + \mathbb{E}[Y^2] - (\mathbb{E}(Y))^2$

$\mathbb{V}ar[X+Y] = \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 = \mathbb{E}[X^2+2XY+Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 -$

$-2\mathbb{E}[X]\cdot\mathbb{E}[Y]$

$\mathbb{V}ar[X+Y] = \mathbb{V}ar(X] + \mathbb{V}ar[Y] + 2Cov[X,Y]$

$\Leftrightarrow \mathbb{E}[X^2+2XY+Y^2] - \cancel{\mathbb{E}[X]^2} - \cancel{\mathbb{E}[Y]^2} - 2\mathbb{E}[X]\mathbb{E}[Y] = 2(\mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y]) + \mathbb{E}[Y^2]$

$+ \mathbb{E}[X^2] - \cancel{\mathbb{E}[X]^2} - \cancel{\mathbb{E}[Y]^2}$

$$\boxed{\mathbb{E}[X^2+2XY+Y^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY]}$$ ✓

b) <u>Inegalitatea Cauchy-Schwarz</u>: $\mathbb{E}[XY]^2 \le \mathbb{E}[X^2]\mathbb{E}[Y^2]$

$\rho(X,Y) = \frac{Cov[X,Y]}{\sigma_X\cdot\sigma_Y} = \frac{Cov[X,Y]}{\sqrt{\mathbb{V}ar[X]\cdot\mathbb{V}ar[Y]}} \in [-1,1] \Rightarrow |Cov[X,Y]| \le \sqrt{\mathbb{V}ar[X]\cdot\mathbb{V}ar[Y]}$

$\Rightarrow (Cov[X,Y])^2 \le \mathbb{V}ar[X]\cdot\mathbb{V}ar(Y] \Rightarrow \mathbb{E}[(X-\mathbb{E}(X))\cdot(Y-\mathbb{E}(Y))]^2 \le \mathbb{V}ar[X]\cdot\mathbb{V}ar[Y]$

știm că din Cauchy $\mathbb{E}[(X-\mathbb{E}(X))\cdot(Y-\mathbb{E}(Y))]^2 \le \mathbb{E}[(X-\mathbb{E}(X))^2]\cdot\mathbb{E}[(Y-\mathbb{E}(Y))^2] = \mathbb{V}ar[X]\cdot\mathbb{V}ar[Y]$

$\Rightarrow \rho(X,Y) \in [-1,1]$

2. Calculați varianța unei variabile aleatoare distribuite hypergeometric

$X \sim Hypergeom(N, K, n)$

$$\cancel{k(k-1)} \cdot \frac{K!}{\cancel{k!}(K-k)!} = \frac{(k-2)! \cdot k \cdot (k-1)}{(k-2)!} \qquad \frac{N!}{n!(N-n)!} \rightarrow$$

$$E[X] = \frac{nK}{N}$$

$$E[X^2] = \sum k^2 \cdot \frac{C_K^k \cdot C_{N-K}^{n-k}}{C_N^n} = \sum k(k-1) \cdot \frac{C_K^k \cdot C_{N-K}^{n-k}}{C_N^n} + \underbrace{\sum k \cdot \frac{C_K^k \cdot C_{N-K}^{n-k}}{C_N^n}}_{E[X]}$$

$$= \sum \frac{C_{K-2}^{k-2} \cdot C_{N-K}^{n-k}}{C_N^n} \cdot K(K-1) + \frac{nK}{N}$$

$$= \sum \frac{C_{K-2}^{k-2} \cdot C_{N-K}^{n-k}}{C_{N-2}^{n-2}} \cdot \frac{K(K-1) \cdot n(n-1)}{N(N-1)} + \frac{nK}{N}$$

$$= \frac{nK(N-1) + K(K-1) \cdot n(n-1)}{N(N-1)} = \frac{nK(N - \cancel{1} + Kn - n - K + \cancel{1})}{N(N-1)}$$

$$= \frac{nK(N - n - K + Kn)}{N(N-1)}$$

$$Var[X] = \frac{nK}{N}\left(\frac{N-n-K+Kn}{(N-1)} - \frac{nK}{N}\right) = \frac{nK}{N}\left(\frac{N^2 - Nn - NK + \cancel{NKn} + nK - \cancel{Nnk}}{N(N-1)}\right)$$

$$= \frac{nK}{N}\left(\frac{N^2 - Nn - KN + nK}{N(N-1)}\right)$$

$$= \frac{nK}{N} \cdot \frac{N(N-K) + n(K-N)}{N(N-1)}$$

$$= \boxed{\frac{nK}{N} \cdot \frac{(N-n)(N-K)}{N(N-1)}}$$