1. X și Y variabile aleatoare discrete

$E[X^2], E[Y^2] < \infty$

a) $VAR[X+Y] = VAR[X] + VAR[Y] + 2Cov[X,Y]$

$VAR[X] = E[X^2] - (E[X])^2$

$Cov[X,Y] = E[XY] - E[X]E[Y]$

$VAR[X+Y] = E[(X+Y)^2] - (E[X+Y])^2 =$

$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 =$

$= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 + 2E[X]E[Y] + (E[Y])^2 =$

$= (E[X^2] - (E[X])^2) + (E[Y^2] - (E[Y])^2) + 2E[XY] - 2E[X]E[Y] =$

$= VAR[X] + VAR[Y] + 2(E[XY] - E[X]E[Y]) =$

$= VAR[X] + VAR[Y] + 2Cov[X,Y]$

b) $\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{VAR[X]\,VAR[Y]}}$

$VAR[X+Y] = VAR[X] + VAR[Y] + 2Cov[X,Y]$

$VAR[X+Y], VAR[X], VAR[Y] \geq 0 \rightarrow$

$2Cov[X,Y] \geq -VAR[X] - VAR[Y]$

$Cov[X,Y] \geq -\frac{VAR[X]}{2} - \frac{VAR[Y]}{2} \quad \Big| : \sqrt{VAR[X] \cdot VAR[X]}$

$\frac{Cov[X,Y]}{\sqrt{VAR[X]\,VAR[X]}} \geq -\frac{1}{2}\frac{VAR[X]}{\sqrt{VAR[X]\,VAR[Y]}} - \frac{1}{2}\frac{VAR[Y]}{\sqrt{VAR[X]\,VAR[Y]}}$

$\frac{Cov[X,Y]}{\sqrt{VAR[X]\,VAR[Y]}} \geq -\frac{1}{2}\frac{\sqrt{VAR[X]}}{\sqrt{VAR[Y]}} - \frac{1}{2}\frac{\sqrt{VAR[Y]}}{\sqrt{VAR[X]}} \geq -1$

$-1 \leq \frac{Cov[X,Y]}{\sqrt{VAR[X]\,VAR[Y]}} \leq 1$

2. $X \sim Hypergeom(N, K, n)$

$$P(X=x) = \frac{C_K^x \cdot C_{N-K}^{n-x}}{C_N^n}$$

$$E[X] = \sum_x x \cdot P(X=x) \qquad VAR[X] = E[X^2] - (E[X])^2$$

$$E[X] = \sum_x x \cdot \frac{C_K^x C_{N-K}^{n-x}}{C_N^n}$$

$$E[X^2] = \sum_x x^2 \cdot \frac{C_K^x \cdot C_{N-K}^{n-x}}{C_N^n}$$

$$VAR[X] = n \cdot \frac{C_K^1}{C_N^1} \cdot \frac{C_{N-K}^{n-1}}{C_N^{n-1}} \cdot \frac{N-n}{N-1}$$