05.02.2020

Pregătire analiză

Determinați $\underline{\lim}$ și $\overline{\lim}\, x_n$ și precizați dacă $\exists \lim\limits_{n\to\infty} x_n$, unde

$$x_n = (-1)^n \cdot \underbrace{\frac{n}{2n+1}}_{\to \frac{1}{2}} + \underbrace{\frac{n^2}{2n^2+3}}_{\to \frac{1}{2}} \cdot \sin\frac{n\pi}{2} \quad \forall n \in \mathbb{N}^*$$

$$\boxed{\begin{array}{l} \sin(n\pi) = 0 \\ \cos(n\pi) = (-1)^n \end{array}}$$

Fie $a_n = \frac{n}{2n+1}$ $\quad a_n \longrightarrow \frac{1}{2}$

$b_n = \frac{n^2}{2n^2+3}$ $\quad b_n \longrightarrow \frac{1}{2}$ $\quad \forall n \in \mathbb{N}^*$

$$x_{4k} = (-1)^{4k} \cdot a_{4k} + b_{4k} \cdot \underbrace{\sin\frac{4k\pi}{2}}_{0} \xrightarrow[k\to\infty]{} \frac{1}{2}$$

$$x_{4k+1} = (-1)^{4k+1} \cdot a_{4k+1} + b_{4k+1} \cdot \sin\frac{4k\pi+\pi}{2} \xrightarrow[k\to\infty]{} 0$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

$$x_{4k+2} = (-1)^{4k+2} \cdot a_{4k+2} + b_{4k+2} \cdot \sin\frac{4k\pi+2\pi}{2} \xrightarrow[k\to\infty]{} \frac{1}{2}$$

$$x_{4k+3} = (-1)^{4k+3} \cdot a_{4k+3} + b_{4k+3} \cdot \sin\frac{4k\pi+3\pi}{2} \xrightarrow[k\to\infty]{} -\frac{1}{2} - \frac{1}{2} = -1$$

$\mathbb{N}^* = 4\mathbb{N}^* \cup (4\mathbb{N}^*+1) \cup (4\mathbb{N}^*+2) \cup (4\mathbb{N}^*+3) \cup \{1,2,3\}$

$\Rightarrow \mathcal{L}((x_n)_n) = \{\frac{1}{2}, 0, -1\}$

$$\left.\begin{array}{l} \overline{\lim}\, x_n = \frac{1}{2} \\ \underline{\lim}\, x_n = -1 \end{array}\right\} \Rightarrow \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow \nexists \lim_{n\to\infty} x_n.$$

2) Fie $a, b \in \mathbb{R}$, $a<b$ și $C([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ cont.} \}$.
Fie $d_1 : C([a,b]) \times C([a,b]) \to \mathbb{R}$, $d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx$.

Arătați că d_1 este metrică pe $C([a,b])$.

0) $d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx \geq 0 \ \forall f,g \in C([a,b])$.

1) $d_1(f,g)=0 \Leftrightarrow \int_a^b |f(x)-g(x)|\,dx = 0 \Leftrightarrow |f(x)-g(x)| = 0 \ \forall x \in [a,b]$
$\Leftrightarrow f(x) = g(x) \ \forall x \in [a,b] \Leftrightarrow f = g$.

2) $d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx = \int_a^b |g(x)-f(x)|\,dx = d_1(g,f)$.

3) $d_1(f,g) \leq d_1(f,h) + d_1(h,g)$

$d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx = \int_a^b |\underline{f(x)-h(x)}+\underline{h(x)-g(x)}|\,dx \leq$

$\leq \int_a^b |f(x)-h(x)|\,dx + \int_a^b |h(x)-g(x)|\,dx =$

$= d_1(f,h) + d_1(h,g)$.

3) Fie $(X = C([0,1]), d_1)$, d_1 dat la ex 2, $(f_n)_n \subset C([0,1])$,
$f_n(x) = \frac{1}{1+nx}$ $\forall n \in \mathbb{N}^*$ și $f \in X$, $f(x) = 0$.

Arătați că $f_n \xrightarrow[n\to\infty]{d_1} f$.

$d_1(f_n, f) \xrightarrow[n\to\infty]{} 0$

$d_1(f_n,f) = \int_0^1 |f_n(x) - f(x)|\,dx = \int_0^1 \left|\frac{1}{1+nx} - 0\right| dx =$

$= \int_0^1 \frac{1}{1+nx} = \frac{\ln(1+nx)}{n}\Big|_0^1 = \frac{\ln(1+n)}{n}$

$\lim_{n\to\infty} \frac{\ln(1+n)}{n} = \lim_{x\to\infty} \frac{\ln(1+x)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0$

Deci, $d_1(f_n, f) \xrightarrow[n\to\infty]{} 0$, i.e.

4. Studiați convergența seriilor.

a) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^3+n^2}}, x>0$

Fie $x_n = \frac{x^n}{\sqrt{n^3+n^2}} \quad \forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{x^{n+1}\, x}{\sqrt{(n+1)^3+(n+1)^2}} \cdot \frac{\sqrt{n^3+n^2}}{\cancel{x^n}} = \lim_{n\to\infty} \frac{x\cdot\sqrt{n^3+n^2}}{\sqrt{(n+1)^3+(n+1)^2}} = x$$

cf. crit. raportului avem:

1) dacă $x<1$, seria conv.

2) dacă $x>1$, seria div.

3) dacă $x=1$, crit. nu decide

Pt. $x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+n^2}}$

Fie $a_n = \frac{1}{\sqrt{n^3+n^2}}$

$b_n = \frac{1}{\sqrt{n^3}} \quad \forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\sqrt{n^3}}{\sqrt{n^3+n^2}} = 1 \in (0,\infty)$$

cf. crit. de comparație cu lim, $\sum a_n \sim \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ conv.

(serie. arm. gen cu $\alpha = \frac{3}{2}$)

$\Rightarrow \sum a_n$ conv.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,(n+3)!}{(2n+1)!\cdot x^n}, \ x>0$

$x_n = \dfrac{n!\,(n+3)!}{(2n+1)!\cdot x^n} \quad \forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}\,\cancel{(n+4)!}^{\,n+4}}{\underbrace{\cancel{(2n+3)!}}_{(2n+2)(2n+3)}\cdot \cancel{x^{n+1}}_{\,x}} \cdot \frac{\cancel{(2n+1)!}\cdot \cancel{x^n}}{\cancel{n!}\,\cancel{(n+3)!}} = \frac{1}{4x}$$

Cf. criteriului raportului avem:

1) dacă $\frac{1}{4x} < 1$ (i.e. $x > \frac{1}{4}$), seria conv.

2) dacă $\frac{1}{4x} > 1$ (i.e. $x < \frac{1}{4}$), seria div.

3) dacă $\frac{1}{4x} = 1$ (i.e. $x = \frac{1}{4}$), crit. nu decide.

Pt. $x = \frac{1}{4}$, seria devine $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,(n+3)!\ 4^n}{(2n+1)!}$

Fie $x_n = \dfrac{n!\,(n+3)!\cdot 4^n}{(2n+1)!} \quad \forall n \in \mathbb{N}^*$.

R-D

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)\cdot\frac{1}{4}}{(n+1)(n+4)} - 1\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+20n+16} - 1\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{-10n-10}{4n^2+20n+16}\right) = -\frac{10}{4} = -\frac{5}{2} < 1 \Rightarrow \text{seria div.}$$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^2+6n+11}\cdot x^n, \ x>0$

$x_n = \dfrac{n+2}{n^2+6n+11}\cdot x^n \quad \forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{n+3}{(n+1)^2+6n+17}\cdot \cancel{x^{n+1}}^{\,x} \cdot \frac{n^2+6n+11}{n+2}\cdot \cancel{x^n} = x$$

cf. crit. rap. avem:

1) dacă $x<1$, seria conv.

2) dacă $x>1$, seria div

3) dacă $x=1$, crit. nu decide

Pt. $x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{n+2}{n^2+6n+11}$

Fie $a_n = \frac{n+2}{n^2+6n+11}, \forall n \in \mathbb{N}^*$

$b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2n}{n^2+6n+11} = 1 \in (0,\infty)$

cf. crit. de comparație cu lim. $\Rightarrow \sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ div

(serie arm. gen cu $\alpha=1$).

d) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{C_{2n}^n}}, \; x>0.$

$C_{2n}^n = \frac{(2n)!}{n! \cdot n!}$

$\lim_{n\to\infty} \frac{\cancel{(2n+2)!}^{(2n+1)(2n+2)}}{\underbrace{\cancel{(n+1)!}}_{n+1} \cdot \underbrace{\cancel{(n+1)!}}_{n+1}} \cdot \frac{\cancel{n! \cdot n!}}{\cancel{(2n)!}} = 4$

cf. crit. radicalului pt. șiruri: $\lim \sqrt[n]{C_{2n}^n} = 4$

Fie $x_n = \frac{x^n}{\sqrt[n]{C_{2n}^n}} \; \forall n \in \mathbb{N}^*.$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{x^{n+1}}\, x}{\sqrt[n+1]{C_{2n+2}^{n+1}}} \cdot \frac{\sqrt[n]{C_{2n}^n}}{\cancel{x^n}} = x \cdot \frac{4}{4} = x.$

$\ln |x|$.

$\ln |x|$

Cf. crit. raportului avem:

1) dacă $x<1$, seria conv.

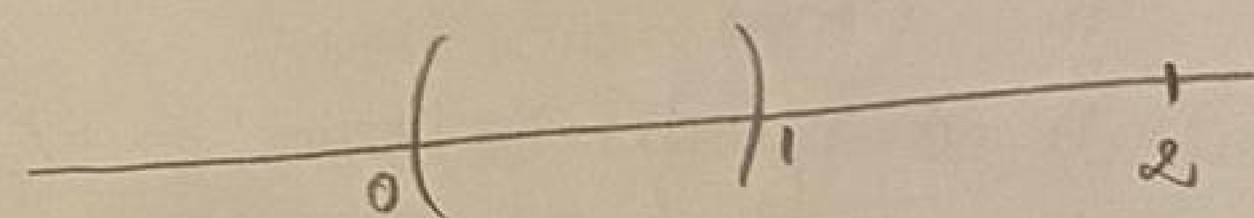
2) dacă $x>1$, seria div

3) dacă $x=1$, crit. nu decide.

Pt. $x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{C_{2n}^{n}}}$

$\lim_{n\to\infty} \frac{1}{\sqrt[n]{C_{2n}^{n}}} = \frac{1}{4} \neq 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{C_{2n}^{n}}}$ e. div.

6. $A = [\mathbb{Q} \cap (0,1)] \cup \{2\}$



$\mathring{A} = \emptyset$

$\bar{A} = [0,1] \cup \{2\}$.

$A' = [0,1]$.

$Fr(A) = \bar{A} \setminus \mathring{A} = [0,1] \cup \{2\}$

$Iz(A) = \bar{A} \setminus A' = \{2\}$

Studiați convergența simplă și uniformă pt. șirul de funcții

$f_n : [1,2] \to \mathbb{R},\ f_n(x) = \frac{nx}{1+nx}\ \forall n \in \mathbb{N}^*$.

<u>C.S.</u>

Fie $x \in [1,2]$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx} = 1 \Rightarrow f_n \xrightarrow{s} f$, unde $f : [1,2] \to \mathbb{R}$
$f(x) = 1$.

<u>C.U.</u>

$$\sup_{x\in[1,2]} |f_n(x) - f(x)| = \sup_{x\in[1,2]} \left| \frac{nx}{1+nx} - 1 \right| = \sup_{x\in[1,2]} \left| \frac{\cancel{nx} - 1 - \cancel{nx}}{1+nx} \right| =$$

$$= \sup_{x\in[1,2]} \left| \frac{-1}{1+nx} \right| = \sup_{x\in[1,2]} \frac{1}{1+nx} \leq \frac{1}{1+n} \longrightarrow 0$$

7. a) Studiați continuitatea lui f

b) $\frac{\partial f}{\partial x}$

c) Studiați derivabilitatea lui f în $(0,0)$, unde $f: \mathbb{R}^2 \to \mathbb{R}^2$,

~~d)~~

$$f(x,y) = \begin{cases} \frac{x^3y^3}{\sqrt{x^6+y^6}} ; & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$$

a) f cont. pe $\mathbb{R}^2 \setminus \{(0,0)\}$.

Studiem cont. lui f în $(0,0)$.

Fie $x,y \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$|f(x,y) - f(0,0)| = \left| \frac{x^3y^3}{\sqrt{x^6+y^6}} - 0 \right| = \frac{|x^3y^3|}{\sqrt{x^6+y^6}} = |x^3| \cdot \frac{|y^3|}{\sqrt{x^6+y^6}} <$$

$\sqrt{y^6} = |y^3|$

≤ 1 $(\sqrt{x^6+y^6} \geq$

$$< |x^3| \xrightarrow[(x,y)\to(0,0)]{} 0$$

b) Fie $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2y^3\sqrt{x^6+y^6} - x^3y^3 \frac{1}{2\sqrt{x^6+y^6}} \cdot 6x^5}{x^6+y^6}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot e_1) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0.$$

c) $$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot e_2) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Dacă f ar fi deriv. în $(0,0)$. atunci $f'(0,0): \mathbb{R}^2 \to \mathbb{R}$,

$$f'(0,0)(u,v) = \left(\left(\underset{\overset{\shortparallel}{0}}{\frac{\partial f}{\partial x}(0,0)} \quad \underset{\overset{\shortparallel}{0}}{\frac{\partial f}{\partial y}(0,0)} \right) \begin{pmatrix} u \\ v \end{pmatrix} \right)^t = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f'(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^3y^3}{\sqrt{x^6+y^6}} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^3y^3}{\sqrt{x^6+y^6}\cdot\sqrt{x^2+y^2}}$$

Fie $(x,y) \neq (0,0)$

$$\left| \frac{x^3y^3}{\sqrt{x^6+y^6}\cdot\sqrt{x^2+y^2}} - 0 \right| = \frac{|x^3y^3|}{\sqrt{x^6+y^6}\cdot\sqrt{x^2+y^2}} = |x^2| \cdot \overset{\leq 1}{\frac{|y^3|}{\sqrt{x^6+y^6}}} \cdot \overset{\leq 1}{\frac{|x|}{\sqrt{x^2+y^2}}} \leq$$

$$\leq |x^2| \xrightarrow[(x,y)\to(0,0)]{} 0.$$