TV-L1 Denoising via ADMM

TV-L1 regularization denoising formulation is as follows:

$$\min_u \|f-u\|_1 + \lambda\|\nabla u\|_1$$

*** ADMM form :** $\min_{u,z,y}\|z\|_1 + \lambda\|y\|, \;\; s.t \;\; z = u - f, \; y = \nabla u$

*** Augmented Lagrangian :** $L_\rho(u,z,y,a,b) = \|z\|_1 + \lambda\|y\|_1 + \frac{\beta}{2}\|z-u+f+\frac{a}{\beta}\|^2 + \frac{\alpha}{2}\|y-\nabla u+\frac{b}{\alpha}\|^2$

*** Parameter update**

1. u-update

$$\frac{\partial L_\rho(u^k,z^k,y^k,a^k,b^k)}{\partial u} = -\beta(z^k-u^k+f+\frac{a^k}{\beta}) - \alpha\nabla^T(y^k-\nabla u^k+\frac{b^k}{\alpha}) = 0$$

$$= (z^k-u^k+f+\frac{a^k}{\beta}) + \frac{\alpha}{\beta}\nabla^T(y^k-\nabla u^k+\frac{b^k}{\alpha}) = 0$$

$$= (z^k+f+\frac{a^k}{\beta}) + \frac{\alpha}{\beta}\nabla^T(y^k+\frac{b^k}{\alpha}) = u^k + \frac{\alpha}{\beta}\nabla^2 u^k$$

-> Fourier Transform : $F(z^k+f+\frac{a^k}{\beta}) + \frac{\alpha}{\beta}F(\nabla^T(y^k+\frac{b^k}{\alpha})) = F(u^k+\frac{\alpha}{\beta}\nabla^2 u^k)$

$$\therefore u^{k+1} = F^{-1}\left(\frac{F(z^k+f+\frac{a^k}{\beta}) + \frac{\alpha}{\beta}F(\nabla^T(y^k+\frac{b^k}{\alpha}))}{1+F(\frac{\alpha}{\beta}\nabla^2)}\right)$$

2. z-update

$$(1)\ \text{if}\ z>0 : \frac{\partial L_\rho(u^{k+1},z^k,y^k,a^k,b^k)}{\partial z} = 1 + \beta(z^k-u^{k+1}+f+\frac{a^k}{\beta}) = 0$$

$$\rightarrow \; z^{k+1} = -\frac{1}{\beta} + u^{k+1} - f - \frac{a^k}{\beta} = v_1 - \frac{1}{\beta}, \;\; v_1 = u^{k+1} - f - \frac{a^k}{\beta}$$

$$(2)\ \text{if}\ z<0 : \frac{\partial L_\rho(u^{k+1},z^k,y^k,a^k,b^k)}{\partial z} = -1 + \beta(z^k-u^{k+1}+f+\frac{a^k}{\beta}) = 0$$

$$\rightarrow \; z^{k+1} = \frac{1}{\beta} + u^{k+1} - f - \frac{a^k}{\beta} = v_1 + \frac{1}{\beta}, \;\; v_1 = u^{k+1} - f - \frac{a^k}{\beta}$$

$$(3)\ \text{if}\ z=0$$

$$(1)+(2)+(3): \;\; \therefore z^{k+1} = \max(|v_1|-\frac{1}{\beta},0)sign(v_1), \;\; v_1 = u^{k+1} - f - \frac{a}{\beta}$$

3. y-update

$$(1)\ \text{if}\ y>0 : \frac{\partial L_\rho(u^{k+1},z^{k+1},y^k,a^k,b^k)}{\partial y} = \lambda + \alpha(y^k - \nabla u^{k+1} + \frac{b^k}{\alpha}) = 0$$

$$\rightarrow \; y^{k+1} = -\frac{\lambda}{\alpha} + \nabla u^{k+1} - \frac{b^k}{\alpha} = v_2 - \frac{\lambda}{\alpha}, \;\; v_2 = \nabla u^{k+1} - \frac{b^k}{\alpha}$$

$$(2)\ \text{if } y < 0: \frac{\partial L_\rho(u^{k+1}, z^{k+1}, y^k, a^k, b^k)}{\partial y} = -\lambda + \alpha(y^k - \nabla u^{k+1} + \frac{b^k}{\alpha}) = 0$$

$$\rightarrow\ y^{k+1} = \frac{\lambda}{\alpha} + \nabla u^{k+1} - \frac{b^k}{\alpha} = v_2 + \frac{\lambda}{\alpha},\quad v_2 = \nabla u^{k+1} - \frac{b^k}{\alpha}$$

$$(3)\ \text{if } y = 0$$

$$(1)+(2)+(3):\ \therefore y^{k+1} = \max(|v_2| - \frac{\lambda}{\alpha}, 0)\, sign(v_2),\quad v_2 = \nabla u^{k+1} - \frac{b^k}{\alpha}$$

4. a-update

$$a^{k+1} = a^k + \beta(z^{k+1} - u^{k+1} + f)$$

5. b-update

$$b^{k+1} = b^k + \alpha(y^{k+1} - \nabla u^{k+1})$$

6. α-update

$$\alpha^{k+1} = \alpha^k + 0.07$$

7. β-update

$$\beta^{k+1} = \beta^k + 0.07$$