a) $f:\mathbb{R}^4 \to \mathbb{R}$, $f(x_1, x_2, x_3, x_4) = x_1 + x_2$.

Să se determine dimensiunea

$\dim_{\mathbb{R}} \operatorname{Ker} f$, precum și matricea funcționalei

în reperul $B = \{(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\}$.

<u>Sol.</u>

$$\operatorname{Ker} f \overset{def}{=} \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,/\, f(x_1, x_2, x_3, x_4) = 0\}$$

$f(x_1, x_2, x_3, x_4) = 0 \iff x_1 + x_2 = 0$

minor pp.

$A = (\boxed{1} \ 1 \ 0 \ 0)$

rang $A = 1 <$ nr nec $4 \Rightarrow$

sistem compatibil $4-1 = 3$ nedeterminat.

$x_1 =$ nec pp.

x_2, x_3, x_4 nec sec.

$$\Rightarrow \left.\begin{array}{l} x_1 = -x_2 \\ x_3 \in \mathbb{R} \\ x_4 \in \mathbb{R} \end{array}\right| \Rightarrow (x_1, x_2, x_3, x_4) =$$

$= (-x_2, x_2, x_3, x_4) \Rightarrow \operatorname{Ker} f = \{(-x_2, x_2, x_3, x_4)$

$= \{(-x_2, x_2, 0, 0) + (0, 0, x_3, 0) + (0, 0, 0, x_4) /$

$$= \text{Span}_{\mathbb{R}} \underbrace{\{ v_1 = (-1,1,0,0),\ v_2 = (0,0,1,0),\ v_3 = (0,0,0,1) \}}_{\text{sistem de generatori}}$$

$$\{v_1, v_2, v_3\} \overset{\text{bază}}{\subset} \text{Ker } f$$

$$\dim_{\mathbb{R}} \text{Ker } f = 3$$

$$\dim_{\mathbb{R}} \text{Im} f + \dim_{\mathbb{R}} \text{Ker } f = \dim_{\mathbb{R}} \mathbb{R}^4 \Rightarrow \dim_{\mathbb{R}} \text{Im} f = 1.$$

<u>matricea functionale</u>

$$[f]_B^{1} = \begin{pmatrix} f(1,0,0,0) & f(1,1,0,0) & f(1,1,1,0) & f(1,1,1,1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 2 & 2 \end{pmatrix}$$

$\mid x_2, x_3, x_4 \in \mathbb{R} \}$

$z'(x) = 4y(x) + 3z(x)$

Sol

E_1 $A = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$

E_2 Val proprii Λ : (sau $\sqrt{\lambda} = \dots$)

$\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} -\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = 0 \iff -\lambda(3-\lambda) - 4 = 0$

$\iff \lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda_1 = \frac{3 + \sqrt{25}}{2} = 4 \Rightarrow m_{a\lambda_1} = 1.$ $\quad \to 3^2 + 4^2 = 25$

$\Rightarrow \lambda_2 = \frac{3-5}{2} = -1. \quad m_{a\lambda_2} = 1.$

$\Rightarrow \Lambda = \{4, -1\}.$

E_3 subspații proprii

Pentru $\lambda_1 = 4$ căutăm $v_1 = (a, b)$ din

$\underbrace{(A - \lambda_1 I_2)} v_1 = (0, 0)$

$\iff \begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (0, 0) \iff$

$\iff -4a + b = 0 \Rightarrow \boxed{b = 4a} \Rightarrow$

$v_1 = (a, b) = (a, 4a) = a(1, 4)$

$\Rightarrow X_{\lambda_1} = \operatorname{span}_{\mathbb{R}} \{ v_{\lambda_1} = (1,4) \} \Rightarrow m_{g_{\lambda_1}} = \dim_{\mathbb{R}} X_{\lambda_1} = 1$

$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow p + q = 0 \Rightarrow p = -q \Rightarrow$

$v_2 = (-q, q) = q(-1, 1)$

$\Rightarrow X_{\lambda_2} = \operatorname{span}_{\mathbb{R}} \{ \underbrace{(-1,1)}_{v_{\lambda_2}} \} \Rightarrow m_{g_{\lambda_2}} = \dim_{\mathbb{R}} X_{\lambda_2} = 1.$

E_4: Comparăm

$$\left.\begin{array}{l} m_{a_{\lambda_1}} = m_{g_{\lambda_1}} \\ m_{a_{\lambda_2}} = m_{g_{\lambda_2}} \end{array}\right| \Rightarrow A \text{ diagonalizabilă}$$

$$\boxed{\Rightarrow \exists\, C = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \text{ a.î. } D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} = C^{-1} A \cdot C}$$

$\underline{\underline{E5}}$ Efectuăm schimbarea de variabila

$W = C \cdot u$; $W = (y, z)$; $u = (u_1, u_2)$

sistemul se scrie echivalent

$\boxed{W' = A \cdot W} = (C \cdot u)' = A\, C \cdot u \;/\cdot C^{-1} \Leftrightarrow u' = \underbrace{C^{-1} A \cdot C}_{D}$

$\Leftrightarrow u' = D \cdot u \Leftrightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Leftrightarrow$

$\left| u_1' = 4 u_1 \right| : u_1 \neq 0 \Rightarrow (\ln u_1)' \ldots$

$\ln u_2 = -x \Rightarrow \boxed{u_2 = e^{-x} \cdot C_2.}$

Revenim la schimbarea de variabilă

$$W = C \cdot u \Leftrightarrow \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} e^{4x} c_1 \\ e^{-x} c_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} y(x) = c_1 e^{4x} - e^{-x} c_2 \\ z(x) = c_1 4 e^{4x} + e^{-x} c_2 \end{cases}$$

pt soluția generală

$$\varphi(x) = C_1 \begin{pmatrix} e^{4x} \\ 4e^{4x} \end{pmatrix} + C_2 \begin{pmatrix} -e^{-x} \\ e^{-x} \end{pmatrix}$$

$C_1, C_2 \in \mathbb{R}$, $\varphi(x) = \begin{pmatrix} y(x) \\ z(x) \end{pmatrix}$

$\varphi_1(x) = \begin{pmatrix} e^{4x} \\ 4e^{4x} \end{pmatrix}$ și $\varphi_2(x) = \begin{pmatrix} -e^{-x} \\ e^{-x} \end{pmatrix}$

$\rightarrow$ sunt soluții particulare