SILENT SURFACE SOURCES FOR THE HELMHOLTZ EQUATION AND DECOMPOSITION OF L^2 VECTOR FIELDS

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Abstract. We shall discuss an inverse problem where the underlying model is related to sources generated by currents on an anisotropic layer. This problem is a generalization of another motivated by the recovering of magnetization distribution in a rock sample from outer measurements of the generated static magnetic field. The original problem can be formulated as inverse source problem for the Laplace equation [5] with sources being the divergence of the magnetization whereas the generalization comes from taking the Helmholtz equation. Either inverse problem is non uniquely solvable with a kernel of infinite dimension. We shall present a decomposition of the space of sources that will allow us to discuss constraints that may restore uniqueness.

1. Introduction

Inverse source problems in linear inverse scattering are routinely encountered in medical imaging, for instance in ultrasonic imaging, microwave imaging and still other methods that use waves, or multi-modality imaging techniques like pohotoacoustics where an electromagnetic wave is used to locally heat a tissue which generates an acoustic wave which is measured. One typically wants to identify the source and its intensity from these measurements, because this gives information on the local properties of the medium. In this paper, we concentrate on source terms in divergence form, which arise naturally when modeling anisotropy in the medium response. The corresponding inverse problems are extremely ill-posed, as the forward operator is not even injective and therefore a fundamental uncertainty attaches to the solution, which is dissolved only upon making extra-assumptions. The aim of the paper is to contribute bringing out the structure of this uncertainty, in the special case where the source is supported on a thin shell (modeled by a surface). In fact, we will rather consider a similar but simpler (scalar) problem originating from acoustics, where an incident wave u_0 is sent onto a thin shell and one seeks to identify an anisotropic inclusion. The acoustic scattering wave satisfies the equation

$$\nabla \cdot (I + \epsilon A_0)\nabla u + k^2 u = f,$$

where A_0 is the material anisotropy matrix, ϵ measures the thickness of the layer and f is the acoustic source. Then, at least formally,

$$u = u_0 + \epsilon\tilde{u} + o(\epsilon)$$

where u_0 is the incident wave satisfying $\Delta u_0 + k^2 u_0 = f$, and the scattered field can be approximated to the first order by $\epsilon\tilde{u}$ where

$$\Delta\tilde{u} + k^2\tilde{u} = \nabla \cdot M, \tag{1.1}$$

with $M := -A_0\nabla u_0$. The left hand side of (1.1) is the Helmholtz operator with wave number k, while the right hand side is a source term in divergence form carried by a surface $\Sigma \subset \mathbb{R}^3$. The inverse problem associated with (1.1) typically consists in recovering M ($= A_0\nabla u_0$ in the example above), from the knowledge of $\nabla\tilde{u}$ off the surface. When M is such that the field $\nabla\tilde{u}$ vanishes inside (resp. outside) Σ, one says it is *silent* inside (resp. outside). The existence of non trivial silent fields results in the non-injectivity of the forward operator and is one big issue facing such problems.

Now, when Σ is a compact, connected Lipschitz surface and $k = 0$, so that the left hand side of (1.1) reduces to the ordinary Laplacian, then a direct sum decomposition of $\mathbb{R}^3$-valued vecor fields from $L^2(\Sigma)$ into a component silent inside, a component silent outside, and a tangent divergence-free term (which is silent on both sides) was obtained in [3]. In the present paper we generalize such a decomposition to the case of nonzero k and show that a fourth, finite-dimensional summand is needed in general. Moreover, the description of the summands allows one to ansatz the form of the solutions one seeks, and brings structure to the inverse problem.

One word about techniques is in order. We rely on a mixture of the classical theory, that introduces fundamental tools like Calderón projectors, and of the theory of singular integrals on Lipschitz sutfaces that provides us with the necessary material to handle low (L^2) regularity. While it is possible to derive corresponding results for vector fields M whose tangential and normal components belong to $H^{1/2}(\Sigma)$, the authors feel that the L^2 theory is both natural because L^2 vector fields can be defined independently of the normal frame (that depends in turn of the embedding $\Sigma \to \mathbb{R}^3$) and are better suited for applications .

2. Preliminaries and Notation

If V is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$ we denote by V^* its dual, and given a linear map $\mathcal{G} : V \longrightarrow W$ we put $\mathcal{G}^* : W^* \to V^*$ to mean the adjoint map. Also, $\operatorname{Ker}\mathcal{G}$ will denote its kernel and $\operatorname{Im}\mathcal{G}$, its image. For $v \in V$ and $\omega \in V^*$, let $\langle \omega, v\rangle$ indicate the value of ω at v and, if V is equipped with a conjugation ($v \mapsto \overline{v}$), we denote the sesquilinear form on $V^* \times V$ by $\langle\!\langle \omega, v\rangle\!\rangle := \overline{\langle \omega, \overline{v}\rangle}$. For a Hilbert space V, we will denote its inner product for $v, u \in V$ as $\langle\!\langle v, u\rangle\!\rangle_V$ and we will take the convention that these inner products are linear on the second entry; also, we will identify V with V^* by the linear isometry $v \mapsto \langle\!\langle \overline{v}, \cdot\rangle\!\rangle_V$. When $\Omega \subset \mathbb{R}^n$ is open, we set $C^\infty(\Omega)$ to be the space of infinitely differentiable functions on Ω, and $C_c^\infty(\Omega)$ the subspace of those having compact support. We use the notation of $\mathcal{E}(\Omega)$ for the set $C^\infty(\Omega)$ together with its usual topology of uniform convergence of derivatives on compact sets, and we will denote its dual by $\mathcal{E}^*(\Omega)$. Similarly, the notation $\mathcal{D}(\Omega)$ stands for the space $C_c^\infty(\Omega)$ equiped with the usual inductive topology of subspaces with support in a fixed compact set [14, Chapter I, Section 2], and then $\mathcal{D}^*(\Omega)$ is the space of distributions on Ω. When $\omega \in \mathcal{D}^*(\mathbb{R}^n)$ we let $\operatorname{supp}(\omega)$ denote its support and we write ∂_j for the (distributional) partial derivative with respect to the j-th coordinate in $\mathbb{R}^n$.

For $p \geq 1$ and Q a Borel set in $\mathbb{R}^3$ with ρ a Borel measure on Q, we let $L^p(Q, \rho)$ denote the familiar Lebesgue space on Q of p-summable functions (essentially bounded if $p = \infty$). When ρ is Lebesgue measure on $\mathbb{R}^n$, we simply write $L^p(Q)$. For $\Omega \subset \mathbb{R}^3$ an open set and $s \in \mathbb{R}$, let $H^s(\Omega)$ denote the Bessel potential space of order s (with index 2); the latter consists of restrictions to Ω of Bessel potentials on $\mathbb{R}^3$, see [12, ch. 3]. Recall that $H^0(\Omega) = L^2(\Omega)$ and $H^t(\Omega) \subset H^s(\Omega)$ for $s < t$. Given a closed set $\Sigma \subset \mathbb{R}^3$ and $s \in \mathbb{R}$, let

$$H^s_\Sigma := \{\omega \in H^s(\mathbb{R}^3) \ : \ \operatorname{supp}(\omega) \subset \Sigma\}.$$

A Lipschitz domain in Ω is one whose boundary is locally the graph of a Lipschitz function. Recall that for a Lipschitz domain Ω, $H^s(\Omega)^* = H^{-s}_{\overline{\Omega}}$ and $(H^s_{\overline{\Omega}})^* = H^{-s}(\Omega)$; see [12, thms 3.29 & 3.30]. In the case that Ω is bounded and $s \geq 0$, we will also define

$$H^s_{\mathrm{loc}}(\mathbb{R}^3 \smallsetminus \overline{\Omega}) := \{\omega \in \mathcal{D}^*(\mathbb{R}^3 \smallsetminus \overline{\Omega}) \ : \ \omega|_{\mathbb{B}_r \smallsetminus \overline{\Omega}} \in H^s(\mathbb{B}_r \smallsetminus \overline{\Omega}), \text{ for each } r > 0 \text{ such that } \overline{\Omega} \subset \mathbb{B}_r\},$$

where $\mathbb{B}_r \subset \mathbb{R}^3$ denotes the open ball of radius r around 0.

For a compact Lipschitz surface $M \subset \mathbb{R}^3$, we will let σ denote the surface measure restricted to M, and we will just write $L^2(M)$ for $L^2(M, \sigma)$. Also, for $n \geq 1$ and $\phi, \tilde{\phi} \in L^2(M)^n$, we will let

$$\langle \phi, \tilde{\phi}\rangle_{L^2(M)^n} := \int_M \phi \cdot \tilde{\phi} \, \mathrm{d}\sigma \qquad \text{and} \qquad \langle\!\langle \phi, \tilde{\phi}\rangle\!\rangle_{L^2(M)^n} := \int_M \overline{\phi} \cdot \tilde{\phi} \, \mathrm{d}\sigma,$$

where $\overline{\phi}$ denotes the complex conjugate of ϕ. For the rest of the definitions we will fix a particular compact Lipschitz surface $M \subset \mathbb{R}^3$ with atlas $\{(\theta_j, U_j)\}_{j\in I}$. Given a point $x \in U_i$ such that θ_i is differentiable at x and θ_j^{-1} is differentiable at $\theta_j(x)$, which happens for σ-a.e. $x \in M$, let $T_x M \subset \mathbb{R}^3$ denote the tangent space of M at x. For a function $f : M \longrightarrow \mathbb{C}$ and a point $x \in U_i$ such that $f \circ \theta_j^{-1}$ is differentiable at $\theta_j(x)$, we let $\nabla_{\mathrm{T}} f(x) \in T_x M$ denote the surface gradient of f at the point x. Note that if the function $f : M \longrightarrow \mathbb{C}$ is Lipschitz then, for σ-a.e. $x \in M$, $\nabla_{\mathrm{T}} f(x)$ is well defined. We will let $\operatorname{Lip}(M)$ denote the set of Lipschitz functions on M, together the topology induced by the norm $\|f\|_\infty + \|\nabla_{\mathrm{T}} f\|_\infty$. For $s \in [-1,1]$, let

$$H^s(M) := \left\{\psi \in \mathrm{Lip}(M)^* : \text{ for every } j \in I, \text{ the map } \psi^{\theta_j}(g) := \langle \psi, g \circ \theta_i \rangle \text{ belongs to } H^s(\theta_j(U_j))\right\}.$$

Again, for $s < t$ we have, $H^t(M) \subset H^s(M)$ and $H^0(M) = L^2(M)$. For the usual topology in $H^s(M)$, the sequence $\{\psi_n\}_n \subset H^s(M)$ converges to $\psi \in H^s(M)$ if and only if, $\psi_n^{\theta_j} \to \psi^{\theta_j}$ in $H^s(\theta_j(U_j))$ for any $j \in I$. We have as well that this convergence is independent of the atlas and the $H^s(M)$ are Hilbert spaces. In particular, $L^2(M)$ and $H^0(M)$ have the same topology. What is more, for any $s \in [-1,1]$, $H^{-s}(M)$ can be identified with the dual of $H^s(M)$. Note that $\mathrm{Lip}(M)$ is dense on $H^s(M)$ for any $s \in [-1,1]$.

We will be using results from [9] that use a more general definition for the space $H^1(M)$ (what they call $W^{1,2}(M)$), which they took from [8] and coincides with the one we have just described.

We will say that $\boldsymbol{f} \in \mathrm{Lip}(M)^3$ belongs to $\mathrm{Lip}_T(M)$ if for every $x \in M$, $\boldsymbol{f}(x) \in T_x M$.

For a $\boldsymbol{\phi} \in L^2(M)^3$, if the image of $\boldsymbol{\phi}$ belongs to $T_x M$ for σ-a.e. $x \in M$, define by duality the surface divergence of ϕ, denoted by $\nabla_{\mathrm{T}}\cdot\boldsymbol{\phi}$; i.e. for $f \in \mathrm{Lip}(M)$,

$$\langle \nabla_{\mathrm{T}}\cdot\boldsymbol{\phi}, f\rangle := -\langle \boldsymbol{\phi}, \nabla_{\mathrm{T}} f\rangle_{L^2(M)^3}.$$

Notice that here $\nabla_{\mathrm{T}}\cdot\boldsymbol{\phi} \in H^{-1}(M)$. We analogously define, for a $\phi \in L^2(M)$, the weak tangential gradient of ϕ, which we will denote by $\nabla_{\mathrm{T}}\phi$. By density it is satisfied for $\varphi \in H^1(M)$, $\boldsymbol{\phi} \in L^2(M)^3$, $\phi \in L^2(M)$ and $\boldsymbol{\varphi} \in H^1(M)^3$,

$$\langle \nabla_{\mathrm{T}}\cdot\boldsymbol{\phi}, \varphi\rangle := -\langle \boldsymbol{\phi}, \nabla_{\mathrm{T}}\varphi\rangle_{L^2(M)^3}, \quad \text{and} \quad \langle \nabla_{\mathrm{T}}\cdot\boldsymbol{\varphi}, \phi\rangle_{L^2(M)^3} := -\langle \boldsymbol{\varphi}, \nabla\phi\rangle.$$

In this paper, we will use Ω_+ to denote a bounded Lipschitz domain, which during most of the paper, will be fixed. It should be note that we follow the choice of signs of [9], which differs from other references we use such as [12]. We let $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega_+}$ but just write $\partial\Omega$ for the boundary of $\Omega_\pm$. Note that $\partial\Omega$ is a Lipschitz surface.

Using [9, Theorem 4.3.6] with Lemma A.1 we get that

$$H^1(\partial\Omega) = \left\{\varphi \in L^2(\partial\Omega) : \nabla_{\mathrm{T}}\varphi \in L^2(\partial\Omega)^3\right\}.$$

Note that, for $\varphi, \tilde{\varphi} \in H^1(\partial\Omega)$, we have that $\nabla_{\mathrm{T}}\cdot\nabla_{\mathrm{T}}\varphi \in H^{-1}(\partial\Omega) \equiv H^1(\partial\Omega)^*$ and $\langle \nabla_{\mathrm{T}}\cdot\nabla_{\mathrm{T}}\varphi, \tilde{\varphi}\rangle = -\langle \nabla_{\mathrm{T}}\varphi, \nabla_{\mathrm{T}}\tilde{\varphi}\rangle_{L^2(\partial\Omega)^3}$. We will let $\Delta_{\mathrm{T}} := \nabla_{\mathrm{T}}\cdot\nabla_{\mathrm{T}}$, that is, the Laplace-Beltrami operator. We will also use the Hermitian form

$$\langle\!\langle \varphi, \tilde{\varphi}\rangle\!\rangle_{H^1(\partial\Omega)} := \langle\!\langle \varphi, \tilde{\varphi}\rangle\!\rangle_{L^2(\partial\Omega)} + \langle\!\langle \nabla_{\mathrm{T}}\varphi, \nabla_{\mathrm{T}}\tilde{\varphi}\rangle\!\rangle_{L^2(\partial\Omega)^3},$$

which generates the usual topology on $H^1(\partial\Omega)$. We will denote by $\|\cdot\|_{H^1(\partial\Omega)}$ the norm generated by $\langle\!\langle \cdot,\cdot\rangle\!\rangle_{H^1(\partial\Omega)}$. Also, we will denote the dual Hermitian product by $\langle\!\langle \cdot,\cdot\rangle\!\rangle_{H^{-1}(\partial\Omega)}$ and by $\|\cdot\|_{H^{-1}(\partial\Omega)}$ its corresponding norm. In [9] they use a different norm for this space which is equivalent to ours.

We will denote the classical trace in $\Omega_\pm$ by $\gamma^\pm : H^1(\Omega_\pm) \longrightarrow H^{1/2}(\partial\Omega)$, which is a bounded linear operator. For a $\phi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ if $\gamma^+\phi = \gamma^-\phi$, we will write $\gamma\phi := \gamma^\pm\phi$.

We will also use the non-tangential limits of of function; Given a $\alpha > 0$, let

$$\mathfrak{C}^\pm_\alpha(x) := \{y \in \Omega_\pm \ : \ |x-y| \le (\alpha+1)\mathrm{dist}(y,\partial\Omega)\}$$

where dist is the standard Euclidean distance between a point and a set. Take any measurable function ψ on $\Omega_\pm$ and a $x \in \partial\Omega$ and let

$$\gamma^\pm_\alpha\psi(x) := \lim_{\substack{y \to x \\ y \in \mathfrak{C}^\pm_\alpha(x)}} \psi(y),$$

when this limit exists. By [9, Proposition 3.3.1], $x \in \overline{\mathfrak{C}^\pm_\alpha(x)}$ for σ-a.e. $x \in \partial\Omega$, which makes this definitions meaningful. If these limits are well defined and their values are independent of $\alpha > 0$, we will say that the non-tangential limit of ψ from $\Omega_\pm$ exists at x. In the case that the non-tangential limit of ψ exists for σ-a.e. $x \in \partial\Omega$, with a slight abuse of notation, we will denote the resulting function by $\gamma^\pm\psi$, and, if we also have that $\gamma^+\psi = \gamma^-\psi$, we will denote this function by $\gamma\psi$. This use of notation is justified from the fact that Ω_+ is a Lipschitz domain and thus the traces coincide with their respective non-tangential limits for H^1_{loc}

functions, in the case that such limits exists (see for example B.1). Note that the restriction of γ, mapping $\gamma : \mathcal{E}(\mathbb{R}^3) \longrightarrow \mathrm{Lip}(\partial\Omega)$, is continuous with the respect to the topologies of $\mathcal{E}(\mathbb{R}^3)$ and $\mathrm{Lip}(\partial\Omega)$.

3. Statement of the problem and layer potentials

3.1. **Statement of the problem.** We fix throughout $k \geq 0$ and some bounded Lipschitz domain $\Omega_+ \subset \mathbb{R}^3$, with boundary $\partial\Omega$, surface measure σ and outward-pointing unit normal $\boldsymbol{\nu}(x)$ at σ-a.e. $x \in \partial\Omega$. Set $G(x) := -\frac{e^{ik|x|}}{4\pi|x|}$, which is a fundamental solution the Helmholtz equation. We will use $\mathcal{G}$ to denote its potential operator, that is:

$$\begin{array}{cccc} \mathcal{G} & : & \mathcal{E}^*(\mathbb{R}^n) & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & & d & \mapsto & G * d. \end{array}$$

By [15, Theorem 27.6], the map $\mathcal{G}$ is continuous and injective. For $\boldsymbol{M} \in L^2(\partial\Omega)^3$, we write $\boldsymbol{M} = \boldsymbol{\nu} M_{\boldsymbol{\nu}} + \boldsymbol{M}_T$ with $M_{\boldsymbol{\nu}} := \boldsymbol{M}\cdot\boldsymbol{\nu}$ and $\boldsymbol{M}_T := \boldsymbol{M} - \boldsymbol{\nu} M_{\boldsymbol{\nu}}$. Clearly, $\boldsymbol{M}_T(x)$ belongs to $T_x\partial\Omega$ for σ-a.e. $x \in \partial\Omega$, therefore one can define $\nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T \in H^{-1}(\partial\Omega)$. We then introduce the forward operator:

$$\begin{array}{cccc} \mathcal{F} & : & L^2(\Sigma)^3 & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & & \boldsymbol{M} & \mapsto & \mathcal{G}[\nabla\cdot(\boldsymbol{M}\sigma)], \end{array}$$

where by $\boldsymbol{M}\sigma$ we mean the measure on $\mathbb{R}^3$ such that $\mathrm{d}(\boldsymbol{M}\sigma) = \boldsymbol{M}\mathrm{d}\sigma$ and $\nabla\cdot(\boldsymbol{M}\sigma)$ is taking the weak divergence of $\boldsymbol{M}\sigma$ in $\mathbb{R}^3$. Note that $\mathcal{F}(\boldsymbol{M}) = \nabla G * (\boldsymbol{M}\sigma)$ and thus, $\mathcal{F}(\boldsymbol{M})$ is a locally integrable function on $\mathbb{R}^3 \setminus \Sigma$ which is real analytic there. If we set $u = \mathcal{F}(\boldsymbol{M})$, then u satisfies the Helmholtz equation:

$$\Delta u + k^2 u = \nabla\cdot(\boldsymbol{M}\sigma), \tag{3.1}$$

as well as the Sommerfeld radiation condition:

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial|x|} - ik \right) u(x) = 0. \tag{3.2}$$

This implies in particular that the kernel of $\mathcal{F}$ consist precisely of those $\boldsymbol{M} \in L^2(\Sigma)^3$ such that $\nabla\cdot(\boldsymbol{M}\sigma) = 0$. Further, since $\mathcal{F}(\boldsymbol{M})$ is locally integrable we have the following equivalences,

$$\mathcal{F}(\boldsymbol{M})(x) = 0 \text{ for all } x \in \mathbb{R}^3 \setminus \Sigma \quad \text{if and only if} \quad \boldsymbol{M} \in \operatorname{Ker}\mathcal{F} \quad \text{if and only if} \quad \nabla\cdot(\boldsymbol{M}\sigma) = 0. \tag{3.3}$$

We will say that $\boldsymbol{M}$ is *silent inside* (resp. *silent outside*) if $(\mathcal{F}(\boldsymbol{M}))|_{\Omega_+} = 0$ (resp. $(\mathcal{F}(\boldsymbol{M}))|_{\Omega_-} = 0$). When $\boldsymbol{M}$ is both silent inside and silent outside, we will say that it is *silent everywhere* and, if it is neither silent inside nor silent outside we will say that it is *silent nowhere*.

Note that u and $\boldsymbol{M}$ satisfy (3.1) and (3.2) if and only if $u = \mathcal{F}(\boldsymbol{M})$.

3.2. **Layer potential and Green identities in Sobolev spaces.** Below we recall classical facts from boundary integral equations that we will be using (see for example [12]).

We write $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ for the coordinates of the unit outer normal of $\partial\Omega$. For $u \in H^1(\Omega_\pm)$ such that $\Delta u \in L^2(\Omega_\pm)$, we let $\partial^+_{\boldsymbol{\nu}} u \in H^{-1/2}(\partial\Omega)$ and $\partial^-_{\boldsymbol{\nu}} u \in H^{-1/2}(\partial\Omega)$ be the interior and exterior normal derivatives respectively. Note that they coincide with the co-normal derivatives for the Helmholtz differential operator. As with the trace, for a $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, if $\partial^-_{\boldsymbol{\nu}} u = \partial^+_{\boldsymbol{\nu}} u$ we will simply write $\partial_{\boldsymbol{\nu}} u := \partial^\pm_{\boldsymbol{\nu}} u$. As a remark, they satisfy,

$$\partial^+_{\boldsymbol{\nu}} u = \boldsymbol{\nu}\cdot\gamma^+(\nabla u) \text{ for } u \in H^2(\Omega_+) \quad \text{and} \quad \partial^-_{\boldsymbol{\nu}} u = \boldsymbol{\nu}\cdot\gamma^-(\nabla u) \text{ for } u \in H^2_{\mathrm{loc}}(\Omega_-).$$

Also, as it is the case for γ, the restriction of $\partial_{\boldsymbol{\nu}}$ mapping $\partial_{\boldsymbol{\nu}} : \mathcal{E}(\mathbb{R}^3) \longrightarrow \mathrm{Lip}(\partial\Omega)$ is continuous with the respective topologies of $\mathcal{E}(\mathbb{R}^3)$ and $\mathrm{Lip}(\partial\Omega)$.

We will denote the single and double layer potentials associated to (3.1) by SL and DL. Recall that $SL = \mathcal{G}\circ\gamma^*$ and $DL = \mathcal{G}\circ\partial^*_{\boldsymbol{\nu}}$, and both are continuous from $\mathrm{Lip}(\partial\Omega)^*$ to $\mathcal{D}^*(\mathbb{R}^3)$. In particular, we have for $x \in \mathbb{R}^3 \setminus \partial\Omega$ and $\phi \in L^2(\partial\Omega)$,

$$SL\phi(x) = \int_{\partial\Omega} G(x-y)\phi(y)\mathrm{d}\sigma(y), \quad DL\phi(x) = \int_{\partial\Omega} \partial_{\boldsymbol{\nu},y} G(x-y)\phi(y)\mathrm{d}\sigma(y),$$

where $\partial_{\boldsymbol{\nu},y}$ is the normal derivative with respect to the variable y. We also get the mapping properties:

$$SL : H^{-1/2}(\partial\Omega) \longrightarrow H^1_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{and} \quad DL : H^{1/2}(\partial\Omega) \longrightarrow H^1_{\mathrm{loc}}(\Omega_\pm). \tag{3.4}$$

Recall the three *Green Identities;* for $u, v \in H^1(\Omega_\pm)$, with $\Delta u \in L^2(\Omega_\pm)$

$$\langle\!\langle \nabla u, \nabla v \rangle\!\rangle_{L^2(\Omega_\pm)^3} = \langle\!\langle \Delta u, v \rangle\!\rangle_{L^2(\Omega_\pm)} + \langle\!\langle \partial^\pm_{\boldsymbol{\nu}} u, \gamma^\pm v \rangle\!\rangle \tag{3.5a}$$

if, further, $\Delta v \in L^2(\Omega_\pm)$

$$\left\langle\!\!\left\langle \Delta u + k^2 u, v \right\rangle\!\!\right\rangle_{L^2(\Omega_\pm)} - \left\langle\!\!\left\langle u, \Delta v + k^2 v \right\rangle\!\!\right\rangle_{L^2(\Omega_\pm)} = \langle\!\langle \gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} v \rangle\!\rangle - \langle\!\langle \partial^\pm_{\boldsymbol{\nu}} u, \gamma^\pm v \rangle\!\rangle \tag{3.5b}$$

and, for $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, with $u|_{\Omega_\pm} \in H^1_{\mathrm{loc}}(\Omega_\pm)$, satisfying (3.2), and

$$\Delta u|_{\Omega_\pm} + k^2 u|_{\Omega_\pm} = 0 \;\text{ in }\; \Omega_\pm,$$

we get

$$u = DL(\gamma^+ u - \gamma^- u) - SL(\partial^+_{\boldsymbol{\nu}} u - \partial^-_{\boldsymbol{\nu}} u). \tag{3.5c}$$

Recall the following well know boundary version of the layer potentials

$$S : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega) \quad \text{and} \quad K : H^{1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega),$$

which are well defined, bounded and satisfy the integral representations, for $f \in \mathrm{Lip}(\partial\Omega)$,

$$Sf(x) = \int_{\partial D} G(x-y) f(y) \mathrm{d}\sigma(y), \quad Kf(x) = \text{p.v.} \int_{\partial D} \partial_{\boldsymbol{\nu},y} G(x-y) f(y) \mathrm{d}\sigma(y), \tag{3.6}$$

as well the jump relations, taking $\phi \in H^{1/2}(\partial\Omega)$ and $\psi \in H^{-1/2}(\partial\Omega)$,

$$(SL\psi)|_{\partial\Omega} = S\psi \quad \text{and} \quad \gamma^\pm(DL\phi) = \left(\pm\frac{1}{2} Id + K\right)\phi, \tag{3.7}$$

where Id represents the identity operator. Further, there are jump relations for the normal derivative of SL,

$$\partial^\pm_{\boldsymbol{\nu}}(SL\phi) = \left(\mp\frac{1}{2} Id + K^*\right)\phi,$$

and the following operator is well defined and bounded,

$$T := \partial_{\boldsymbol{\nu}} DL : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega).$$

Both this operator and S are self-adjoint.

Finally, we recall the *Calderón projectors*; let $P^\pm : H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ be defined by matrix multiplication as

$$P^\pm(\phi, \psi) := \begin{pmatrix} \frac{1}{2} Id \pm K & \mp S \\ \pm T & \frac{1}{2} Id \mp K^* \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{3.8}$$

which are projections that satisfy $P^+ + P^- = Id$ and, for $(\phi, \psi) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$,

$$\text{if } \; u = -DL(\phi) + SL(\psi) \; \text{ then } \; (\gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} u) = \mp P^\pm(\phi, \psi) \; \text{ and } \; \Delta u + k^2 u = 0 \text{ on } \Omega_\pm. \tag{3.9}$$

We will let $P^\pm_j(\phi, \psi)$ denote the j-th component of $P^\pm(\phi, \psi)$, for $j = 1, 2$.

Finally, using (3.5c), for $u \in H^1_{\mathrm{loc}}(\Omega_\pm)$,

$$\Delta u + k^2 u = 0 \quad \text{implies that} \quad u = DL(\pm\gamma^\pm u) - SL(\pm\partial^\pm_{\boldsymbol{\nu}} u) \text{ and } (\gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} u) = P^\pm(\gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} u). \tag{3.10}$$

3.3. **Extension of boundary operators.** For our results in this paper we will need to extend the domain of definition of the boundary operators we have just recall. We will base the extensions on the work of [9]. It should be noted that the results we will be referencing from [9] are made for the case $k = 0$ but can be adapted with the use of [9, Lemma 6.4.2], proven in [7]. For convenience of the reader, proofs for the most of this adaptation can be found in Appendix A.

Thanks to [9, Corollary 3.6.3, Proposition 3.6.4 and Proposition 3.3.2], we can extend the definition of the boundary versions of the singular and double layer operators to bounded linear operators with mapping properties,

$$S : H^{-1}(\partial\Omega) \longrightarrow L^2(\partial\Omega) \quad \text{and} \quad K : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega).$$

Furthermore, if we restrict S to $L^2(\partial\Omega)$ and K to $H^1(\partial\Omega)$ we get that following mapping properties,

$$S : L^2(\partial\Omega) \longrightarrow H^1(\partial\Omega) \quad \text{and} \quad K : H^1(\partial\Omega) \longrightarrow H^1(\partial\Omega), \tag{3.11}$$

which define bounded linear operators. As a small abuse of notation, we will write the dual of $K|_{H^1(\partial\Omega)}$ as K^*.

Remark 1. The integral representations, (3.6) are still valid for functions in $L^2(\partial\Omega)$ and, taking $\phi \in L^2(\partial\Omega)$ and $\psi \in H^{-1}(\partial\Omega)$, we still get the jump relations in (3.7), though in this case using the non-tangential limits.

Similarly as we did with S and K, by [9, Theorem 3.2.8, Proposition 3.6.2] and equation (3.5b), if we restrict T to $H^1(\partial\Omega)$ we get that following mapping property,

$$T : H^1(\partial\Omega) \longrightarrow L^2(\partial\Omega),$$

which also which defines bounded linear operator. Now, based on the fact that T on $H^{1/2}(\partial\Omega)$ is self-adjoint, we use $(T|_{H^1(\partial\Omega)})^*$ to extend T to a mapping

$$T : L^2(\partial\Omega) \longrightarrow H^{-1}(\partial\Omega).$$

Remark 2. We should note that [9, Proposition 3.6.2], further says that the linear operators $\varphi \mapsto \gamma^{\pm}(\nabla DL\varphi)$, defined $H^1(\partial\Omega) \longrightarrow L^2(\partial\Omega)^3$, are bounded.

Finally, we extend the Calderón projectors using the extended versions of S, K, K^* and T to get a bounded operator, which by density is still a projection, and has the mapping property

$$P^{\pm} : L^2(\partial\Omega) \times H^{-1}(\partial\Omega) \longrightarrow L^2(\partial\Omega) \times H^{-1}(\partial\Omega).$$

For convenience, we will define the bounded linear operator,

$$\begin{array}{cccc} \widetilde{\mathcal{F}} & : & L^2(\partial\Omega) \times H^{-1}(\partial\Omega) & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & & (\phi, \psi) & \mapsto & -DL(\phi) + SL(\psi), \end{array}$$

based on the fact that we can write, for $x \notin \partial\Omega$,

$$\mathcal{F}(\boldsymbol{M})(x) = -DL(M_{\boldsymbol{\nu}})(x) + SL(\nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T)(x).$$

Note that the image of $\widetilde{\mathcal{F}}$ is a subset both of $C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$ and, by [9, equation (3.6.15)], of $L^1_{\mathrm{loc}}(\mathbb{R}^3)$.

Then, regarding the extensions of $\gamma^{\pm}$ and $\partial^{\pm}_{\boldsymbol{\nu}}$, we will limit ourselves for distributions u of the form $\widetilde{\mathcal{F}}(\phi, \psi)$, for $(\phi, \psi) \in L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$. First note that for such u,

$$\Delta u + k^2 u = 0 \text{ on } \Omega_{\pm}.$$

Next, by remark 1, the non-tangential limits $\gamma^{\pm} u$ are well defined and belong to $L^2(\partial\Omega)$. They also satisfy

$$\gamma^{\pm} u = \left(\mp\frac{1}{2} Id - K\right)(\tilde{\phi}) + S(\tilde{\psi}) = \mp P^{\pm}_1(\tilde{\phi}, \tilde{\psi}), \tag{3.12}$$

for any $(\tilde{\phi}, \tilde{\psi}) \in L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$ such that $u = \widetilde{\mathcal{F}}(\tilde{\phi}, \tilde{\psi})$.

For the extension of $\partial^{\pm}_{\boldsymbol{\nu}}$ we will use the following lemma.

Lemma 3.1. *Take* $(\phi,\psi) \in L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$ *and let* $u = \widetilde{\mathcal{F}}(\phi,\psi)$. *Then,*

$$u|_{\Omega_\pm} = 0 \quad \text{if and only if} \quad P^\pm(\phi,\psi) = 0.$$

Proof. First assume that $(\phi,\psi) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. Thus, if $u|_{\Omega_\pm} = 0$, using the classical definitions, $\gamma^\pm u = 0$ and $\partial^\pm_{\boldsymbol{\nu}} u = 0$ then, (3.10) give us the first implication.

Assume now that $(\phi,\psi) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ and $P^\pm(\phi,\psi) = 0$. Then $\gamma^\pm u = 0$ and by (3.4), $u \in H^1_{\mathrm{loc}}(\Omega_\pm)$. Also, u is a solution of the Dirichlet problem,

$$\begin{aligned} \Delta u + k^2 u = 0 \quad &\text{on } \Omega_\pm \\ \gamma^\pm u = 0.& \end{aligned} \tag{3.13}$$

If $P^-(\phi,\psi) = 0$, then u also satisfies the Sommerfeld radiation condition, which implies that $u|_{\Omega_-} = 0$ (see for example [12, Theorem 9.10]). If, on the other hand $P^+(\phi,\psi) = 0$, since u is a solution of (3.13), we get that $u|_{\Omega_+}$ can be extended by zero to a solution of the homogeneous Helmholtz equation on $\mathbb{R}^3$ and thus the unique continuation principle implies that $u|_{\Omega_+} = 0$.

Next, we will take any $(\phi,\psi) \in L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$. We will start by assuming that $P^\pm(\phi,\psi) = 0$. By [9, equations (3.6.13) and (3.6.16)], there exist a sequence, $\{(\phi_n,\psi_n)\}_n \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ such that $(\phi_n,\psi_n) \to (\phi,\psi)$ in $L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$. Then $P^\mp(\phi,\psi) = (\phi,\psi)$ and so, by continuity of $P^\mp$, it follows that $P^\mp(\phi_n,\psi_n) \to (\phi,\psi)$ in $L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$ as well. On the other hand, $P^\mp(\phi_n,\psi_n) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, and the equality $P^\pm P^\mp(\phi_n,\psi_n) = 0$ is satisfied. Hence, we can use the result of this lemma for the case $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ and we get that $\tilde{u}_n := \widetilde{\mathcal{F}}(P^\mp(\phi_n,\psi_n))|_{\Omega_\pm} = 0$. Finally, noticing that the function $\widetilde{\mathcal{F}}(P^\mp(\cdot,\cdot))|_{\Omega_\pm}$ from $L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$ to $\mathcal{D}^*(\mathbb{R}^3)$ is continuous, we get that $\tilde{u}_n \to u|_{\Omega_\pm}$ in $\mathcal{D}^*(\mathbb{R}^3)$. Therefore, $u|_{\Omega_\pm} = 0$.

We will prove the other implication only for the case corresponding to Ω_+, the case for Ω_- is analogous. Assume then that $u|_{\Omega_+} = 0$. Let $(\phi^\pm,\psi^\pm) := P^\mp(\phi,\psi)$, and $u^\pm := \widetilde{\mathcal{F}}(\phi^\pm,\psi^\pm)$. Thus, by the proven implication, $u^\pm|_{\Omega_\pm} = 0$ and by linearity $u = u^+ + u^-$. However,

$$0 = u|_{\Omega_+} = u^+|_{\Omega_+} + u^-|_{\Omega_+} = u^-|_{\Omega_+},$$

and thus, since u^- is locally integrable, $u^- = 0$ and $u = u^+$. Then, by equation (3.12), $\phi = \phi^+$ and $\phi^- = 0$. Hence, ψ^- belongs to the kernel of SL, therefore to the kernel of γ^*. Then, ψ^- must be zero, that is, $P^+(\phi,\psi) = 0$. ☐

Note that now, for $(\phi,\psi), (\tilde{\phi},\tilde{\psi}) \in L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$, such that $u^\pm = \widetilde{\mathcal{F}}(\phi,\psi)|_{\Omega_\pm} = \widetilde{\mathcal{F}}(\tilde{\phi},\tilde{\psi})|_{\Omega_\pm}$, it is clear that $P^\pm(\phi,\psi) = P^\pm(\tilde{\phi},\tilde{\psi})$. Hence, based on (3.9), we define

$$\partial^\pm_{\boldsymbol{\nu}} u^\pm := \mp P^\pm_2(\phi,\psi) = -T(\phi) + \left(\mp\frac{1}{2} + K^*\right)(\psi),$$

which extends the classical definition of normal derivative, and thus we have

$$\mp P^\pm(\gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} u) = (\gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} u) = \mp P^\pm(\phi,\psi) \quad \text{and} \quad u = \widetilde{\mathcal{F}}(\gamma^\pm u, \partial^\pm_{\boldsymbol{\nu}} u). \tag{3.14}$$

Remark 3. Using again [9, Theorem 3.2.8, Proposition 3.6.2] and equation (3.5b), as well as [9, Proposition 6.3.1] we get for any $u = \widetilde{\mathcal{F}}(\varphi,\phi)$ with $(\varphi,\phi) \in H^1(\partial\Omega) \times L^2(\partial\Omega)$, that

$$\partial^\pm_{\boldsymbol{\nu}} u = \gamma^\pm(\nabla u) \cdot \boldsymbol{\nu} \quad \text{and} \quad \gamma^\pm(\nabla u) = \partial^\pm_{\boldsymbol{\nu}} u \, \boldsymbol{\nu} + \nabla_{\mathrm{T}} \gamma^\pm u.$$

Remark 4. Note that, for any $\boldsymbol{M} \in L^2(\partial\Omega)^3$, we can write $\mathcal{F}(\boldsymbol{M}) = \sum_j \partial_j SL(M_j)$, and hence using [9, Theorem 4.3.6 and Proposition 6.3.3] we get that $\mathcal{F}(\boldsymbol{M}) \in L^3_{\mathrm{loc}}(\mathbb{R}^3)^3 \subset L^2_{\mathrm{loc}}(\mathbb{R}^3)^3$.

4. Decomposition of $L^2(\partial\Omega)^3$

We will start by introducing the spaces which we will use for decomposing $L^2(\partial\Omega)^3$. First define

$$\mathcal{M}_0 := \{\boldsymbol{M} \in L^2(\partial\Omega)^3 \ : \ \boldsymbol{M} \text{ is silent everywhere } \}$$

and let $\mathcal{M}_0^{\perp}$ denote the perpendicular subspace to $\mathcal{M}_0$ in $L^2(\partial\Omega)^3$. Then define the following subspaces of $\mathcal{M}_0^{\perp}$:

$$\mathcal{M}_- = \{\boldsymbol{M} \in \mathcal{M}_0^{\perp} \ : \ \boldsymbol{M} \text{ is silent outside } \},$$
$$\mathcal{M}_+ = \{\boldsymbol{M} \in \mathcal{M}_0^{\perp} \ : \ \boldsymbol{M} \text{ is silent inside } \}.$$

Remark 5. It follows that $\mathcal{M}_+ \cap \mathcal{M}_- = \{0\}$ since their intersection must be silent everywhere and both this spaces belong to $\mathcal{M}_0^{\perp}$. Also, thanks to lemma 3.1,

$$\mathcal{M}_0 \coloneqq \{\boldsymbol{M} \in L^2(\partial\Omega)^3 \ : \ \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T = 0 \text{ and } M_{\boldsymbol{\nu}} = 0\}$$
$$\mathcal{M}_{\pm} = \{\boldsymbol{M} \in \mathcal{M}_0^{\perp} \ : \ P^{\pm}(M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T) = (0,0)\}$$

and, by Lemma B.4,

$$\mathcal{M}_0^{\perp} = \{\boldsymbol{M} \in L^2(\partial\Omega)^3 \ : \ \boldsymbol{M}_T = \nabla_{\mathrm{T}} U_{\boldsymbol{M}_T}, \text{ for some } U_{\boldsymbol{M}_T} \in H^1(\Gamma)\}.$$

$\mathcal{M}_-$, $\mathcal{M}_+$ and $\mathcal{M}_0$ are not enough to decompose $L^2(\partial\Omega)^3$ in its entirety, that is, there exists a bounded Lipschitz domain Ω_+ such that for there is some $\boldsymbol{M} \in L^2(\partial\Omega)^3 \smallsetminus (\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$ which is thus silent nowhere. In the end of this section we will give a description of the space perpendicular to $(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$ in $L^2(\partial\Omega)^3$ with respect to the L^2 scalar product. First, we will introduce a space $\mathcal{M}_{\boldsymbol{\nu}} \subset L^2(\partial\Omega)^3$, whose elements are functions which are purely normal to $\partial\Omega$ and satisfies

$$\mathcal{M}_{\boldsymbol{\nu}} \oplus \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0 = L^2(\partial\Omega)^3.$$

For convenience, let $\Gamma \coloneqq \partial\Omega$ and let $\{\Gamma_j\}_{j\in J}$ be the family of connected components of Γ. The fact that Ω_+ is a bounded Lipschitz domain implies that J must be finite and each Γ_j has a positive and finite area (see for example Lemma B.6). Also, we can index the connected components of Ω_- by Ω_-^j for $j \in J$ and assume that;

- $J = \{1, ..., n_\Gamma\}$, so that, n_Γ would be the number of connected components of Γ,
- Ω_-^1 is unbounded,
- for each $j > 1$, the set Ω_-^j is bounded,
- and for all $j \in J$, the set Γ_j is the boundary of Ω_-^j.

For a $\Sigma \subset \Gamma$ we will let 1_Σ denote the characteristic function of Σ in Γ. Also, for a vector space V and family of vectors $\{v_\ell\}_{\ell\in L} \subset V$, we will let $\langle v_\ell\rangle_{\ell\in L}$ denote the linear span of $\{v_\ell\}_{\ell\in L}$ in V. In order to study the dimension of $\mathcal{M}_{\boldsymbol{\nu}}$, we introduce the space $\mathcal{U} \coloneqq \{1_{\Gamma_j}\}_{j\in J} \subset H^1(\Gamma)$, and the spaces $\mathcal{N}_{\pm}$ defined on the lemma below:

Lemma 4.1. *The following subspaces of* $H^{1/2}(\Gamma)$,

$$\mathcal{N}_{\pm}^1 : = \{ \ \gamma^{\pm}u \ : \ u \in H^1_{\mathrm{loc}}(\Omega_{\pm}) \textit{ satisfies } (3.2), \ \Delta u + k^2 u = 0 \textit{ on } \Omega_{\pm}, \textit{ and } \partial^{\pm}_{\boldsymbol{\nu}} u = 0 \textit{ on } \Gamma \ \}$$
$$\mathcal{N}_{\pm}^2 : = \{ \ \phi \in H^{1/2}(\Gamma) \ : \ P^{\pm}(\phi, 0) = (\phi, 0) \ \}$$
$$\mathcal{N}_{\pm}^3 : = \{ \ \phi \in H^{1/2}(\Gamma) \ : \ \phi\boldsymbol{\nu} \in \mathcal{M}_{\mp} \ \},$$

are all the same and will be denoted by $\mathcal{N}_{\pm}$. *Also, these spaces are finite dimensional and* $\mathcal{N}_+ \cap \mathcal{N}_- = \{0\}$.

Proof. Note that, by Remark 5, $\mathcal{N}_{\pm}^2 = \mathcal{N}_{\pm}^3$. Take a $\gamma^{\pm}u \in \mathcal{N}_{\pm}^1$. Then, by implication (3.10), we have $(\gamma^{\pm}u, 0) = (\gamma^{\pm}u, \partial^{\pm}_{\boldsymbol{\nu}}u) = P^{\pm}(\gamma^{\pm}u, \partial^{\pm}_{\boldsymbol{\nu}}u) = P^{\pm}(\gamma^{\pm}u, 0)$ and thus $\gamma^{\pm}u \in \mathcal{N}_{\pm}^2$. If, on the other hand, we have that $\phi \in \mathcal{N}_{\pm}^2$, letting $u = -DL(\mp\phi)$, then, by equation (3.4) we get that $u|_{\Omega_{\pm}} \in H^1_{\mathrm{loc}}(\Omega_{\pm})$. Also, by implication (3.9), it follows that $\Delta u + k^2 u = 0$ on $\Omega_{\pm}$, and $(\gamma^{\pm}u, \partial^{\pm}_{\boldsymbol{\nu}}u) = \mp P^{\pm}(\mp\phi, 0) = (\phi, 0)$. Hence, $\phi \in \mathcal{N}_{\pm}^1$ and therefore, $\mathcal{N}_{\pm}^1 = \mathcal{N}_{\pm}^2$.

If Ω_1 is connected then, by uniqueness of the exterior Neumann problem for the Helmholtz equation, when equation (3.2) is satisfied, we obtain $\{0\} = \mathcal{N}_-^1 = \mathcal{N}_-$. Otherwise, both $\Omega_{\pm} \smallsetminus \Omega_-^1$ are bounded and there exists Neumann eigen-values, $\{\xi_j^{\pm}\}_{j=1}^{\infty}$, with $0 \le \xi_1^{\pm} \le \xi_2^{\pm} \le \cdots$, and $\xi_j^{\pm} \to \infty$ as $j \to \infty$, and that have corresponding eigen-functions $\{u_j\}_{j=1}^{\infty} \subset H^1(\Omega_{\pm} \smallsetminus \Omega_-^1)$, satisfying

$$\begin{cases} -\Delta u_j = \xi_j^{\pm} u_j & \text{in } \Omega_{\pm} \smallsetminus \Omega_-^1 \\ \partial^{\pm}_{\boldsymbol{\nu}} u_j = 0 & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where the $\{u_j\}_{j=1}^{\infty}$ are not identically zero and form a complete orthonormal system in $L^2(\Omega_\pm)$. The fact that $\mathcal{N}_\pm$ is finite dimensional comes from the fact that if $\phi \in \mathcal{N}_\pm$, then there exists finitely many $j > 0$ such that $k^2 = \xi_j^\pm$ and ϕ is a linear combination of the corresponding $\gamma^\pm u_j$. This implies that the dimension of $\mathcal{N}_\pm$ is finite. Finally, the fact $\mathcal{N}_+ \cap \mathcal{N}_- = \{0\}$ comes the definition of $\mathcal{N}_\pm^3$ and Remark 5. □

Then, we will fix an orthonormal basis, with respect to the L^2-metric, say $\{\upsilon_j\}_{j\in J}$ of $\mathcal{U}$ such that a subset of this basis is a basis of $\mathcal{U} \cap (\mathcal{N}_+ \oplus \mathcal{N}_-)$ and we will let

$$\tilde{J} := \{j \in J \ : \ \upsilon_j \notin \mathcal{N}_+ \oplus \mathcal{N}_-\};$$

we let $\tilde{n}_\Gamma$ denote the cardinality of $\tilde{J}$.

For each $j \in \tilde{J}$, since J is finite and $\mathcal{N}_+ \oplus \mathcal{N}_-$ is finite dimensional, the subspace of $L^2(\Gamma)$ defined as $V_j := \left(\mathcal{N}_+ \oplus \mathcal{N}_- \oplus \langle \upsilon_\ell \rangle_{\ell \neq j}^{\ell \in \tilde{J}}\right)^\perp$ is non-empty. Thus, we can take a non-trivial $\tilde{\Lambda}_j \in V_j$ such that $\upsilon_j - \tilde{\Lambda}_j \in \mathcal{N}_+ \oplus \mathcal{N}_- \oplus \langle \upsilon_\ell \rangle_{\ell \neq j}^{\ell \in \tilde{J}}$. Then, $\Lambda_j := \tilde{\Lambda}_j / \|\tilde{\Lambda}_j\|_{L^2(\Gamma)}^2$ satisfies, $\langle\!\langle \Lambda_j, \upsilon_l \rangle\!\rangle_{L^2(\Gamma)} = \delta_l^j$ for $l \in J$, and $\langle\!\langle \Lambda_j, \phi \rangle\!\rangle_{L^2(\Gamma)} = 0$ for each $\phi \in \mathcal{N}_+ \oplus \mathcal{N}_-$. Therefore, by Lemma 4.1 and Fredholm alternative, (see for example the discussion at the beginning the section *Uniqueness and Existence of Solutions* of [12, Chapter 9]) we can introduce for each $j \in \tilde{J}$, the function $u_j^+ \in H^1(\Omega_+)$ verifying

$$\begin{cases} \Delta u_j^+ + k^2 u_j^+ = 0 & \text{in } \Omega_+ \\ \partial_{\boldsymbol{\nu}}^+ u_j^+ = \Lambda_j & \text{on } \Gamma, \end{cases}$$

and, for each $j \in \tilde{J}$, we can take $u_j^- \in H^1_{\text{loc}}(\Omega_-)$ verifying

$$\begin{cases} \Delta u_j^- + k^2 u_j^- = 0 & \text{in } \Omega_- \\ \partial_{\boldsymbol{\nu}}^- u_j^- = \Lambda_j & \text{on } \Gamma, \end{cases}$$

together with the Sommerfeld radiation condition, (3.2). Then define the following functions that belong to $H^{1/2}(\Gamma)$,

$$\phi_j^+ := \gamma^- u_j^-, \quad \phi_j^- := \gamma^+ u_j^+ \quad \text{and} \quad \phi_j := \phi_j^- - \phi_j^+.$$

Then, we will define $\mathcal{M}_{\boldsymbol{\nu}} := \langle \phi_j \boldsymbol{\nu} \rangle_{j \in \tilde{J}}$.

Theorem 4.2. *We have the decomposition,*

$$L^2(\partial\Omega)^3 = \mathcal{M}_{\boldsymbol{\nu}} \oplus \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0, \tag{4.2}$$

where ⊕ denotes direct sum. Furthermore,

$$(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^\perp = \langle [\nabla \mathcal{F}(\upsilon_j \boldsymbol{\nu})]|_\Gamma \rangle_{j \in \tilde{J}}. \tag{4.3}$$

In particular, if $k = 0$, then $\mathcal{M}_{\boldsymbol{\nu}} = \{0\}$ and thus,

$$L^2(\partial\Omega)^3 = \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0. \tag{4.4}$$

On the other hand, if k^2 is not an eigen-value for the problems in (4.1), then

$$\operatorname{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0) = \dim(\mathcal{M}_{\boldsymbol{\nu}}) = n_\Gamma.$$

Proof. We will first show that the $\phi_j \boldsymbol{\nu}$, which clearly are in $\mathcal{M}_0^\perp$, do not belong to $\mathcal{M}_- \oplus \mathcal{M}_+$ and they are indeed linearly independent. Assume for a contradiction that there exists $\boldsymbol{M}^+ \in \mathcal{M}_+$ and $\boldsymbol{M}^- \in \mathcal{M}_-$ such that $\phi_j \boldsymbol{\nu} = \boldsymbol{M}^+ + \boldsymbol{M}^-$. By equation (3.10),

$$P^+(\phi_j^+, \Lambda_j) = 0 \quad \text{and} \quad P^-(\phi_j^-, \Lambda_j) = 0.$$

Then, $P^-(\phi_j, 0) = P^-(\phi_j^-, \Lambda_j) - P^-(\phi_j^+, \Lambda_j) = -(\phi_j^+, \Lambda_j)$, however, by the definitions of $\mathcal{M}_+$ and $\mathcal{M}_-$,

$$P^-(\phi_j, 0) = P^-((M_{\boldsymbol{\nu}}^-, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^-) + (M_{\boldsymbol{\nu}}^+, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^+)) = (M_{\boldsymbol{\nu}}^+, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^+),$$

which is not possible since $\langle\!\langle \Lambda_j, \upsilon_j \rangle\!\rangle_{L^2(\Gamma)} = 1$ but $\langle\!\langle \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T^+, \upsilon_i \rangle\!\rangle_{L^2(\Gamma)} = 0$. Therefore $\phi_j \boldsymbol{\nu} \notin \mathcal{M}_- \oplus \mathcal{M}_+$.

Now, taking a family of complex numbers $\{a_\ell\}_{\ell\in\tilde{J}}$ such that $\sum_\ell a_\ell\phi_\ell = 0$, we get that $0 = P^-(\sum_\ell a_\ell\phi_\ell, 0) = -\sum_\ell a_j(\phi_\ell^+, \Lambda_\ell)$. In particular, for any $j \in \tilde{J}$, $0 = \langle\!\langle \sum_\ell a_\ell\Lambda_\ell, \upsilon_j\rangle\!\rangle_{L^2(\Gamma)} = a_j$ and thus, the ϕ_j are linearly independent.

Note that, since $\mathcal{M}_{\boldsymbol{\nu}} \cap (\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0) = \{0\}$, we then have the inequality

$$\tilde{n}_\Gamma = \dim \mathcal{M}_{\boldsymbol{\nu}} \le \operatorname{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0), \tag{4.5}$$

and thus, for equation (4.2) to hold it is only necessary to show that $\tilde{n}_\Gamma \ge \operatorname{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$.

Define the following linear operators

$$\begin{array}{cccc} \pi & : & \mathcal{M}_0^\perp & \longrightarrow & L^2(\Gamma) \times H^{-1}(\Gamma) \\ & & \boldsymbol{M} & \mapsto & (M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T) \end{array} \quad \text{and} \quad \begin{array}{cccc} \eta & : & H^{-1}(\Gamma) & \longrightarrow & \mathbb{C}^{n_\Gamma} \\ & & \psi & \mapsto & \left(\langle\psi, 1_{\Gamma_j}\rangle\right)_j. \end{array}$$

Then, by Lemma B.4, given a $(\phi,\psi) \in L^2(\Gamma)\times H^{-1}(\Gamma)$, we get the equivalence;

$$(\phi,\psi) \in \operatorname{Im}\pi \quad \text{if and only if} \quad \psi \in \operatorname{Ker}\eta. \tag{4.6}$$

By Remark 5 and the fact that the projections $P^\pm$ satisfy $P^+ + P^- = Id$, given $\boldsymbol{M} \in \mathcal{M}_0^\perp$, it follows that

$$\boldsymbol{M} \in \mathcal{M}_\pm \quad \text{if and only if} \quad P^\mp(\pi(\boldsymbol{M})) \in \operatorname{Im}\pi \text{ and } \pi(\boldsymbol{M}) \in \operatorname{Im}(P^\mp\pi). \tag{4.7}$$

Also, using again the fact that $P^+ + P^- = Id$, we obtain that, for any $\boldsymbol{M} \in \mathcal{M}_0^\perp$,

$$P^+(\pi(\boldsymbol{M})) \in \operatorname{Im}\pi \quad \text{if and only if} \quad P^-(\pi(\boldsymbol{M})) \in \operatorname{Im}\pi.$$

Hence, by equivalence (4.6), the following four inclusions are equivalent for any $\boldsymbol{M} \in \mathcal{M}_0^\perp$,

$$P_2^+(\pi(\boldsymbol{M})) \in \operatorname{Ker}\eta, \quad P^+(\pi(\boldsymbol{M})) \in \operatorname{Im}\pi, \quad P^-(\pi(\boldsymbol{M})) \in \operatorname{Im}\pi, \quad \text{and} \quad P_2^-(\pi(\boldsymbol{M})) \in \operatorname{Ker}\eta.$$

Thus, we can define

$$\begin{aligned} \Pi := \operatorname{Ker}(\eta P_2^+\pi) &= \{\boldsymbol{M} \in \mathcal{M}_0^\perp \ : \ P^+(\pi(\boldsymbol{M})) \in \operatorname{Im}\pi\} \\ = \operatorname{Ker}(\eta P_2^-\pi) &= \{\boldsymbol{M} \in \mathcal{M}_0^\perp \ : \ P^-(\pi(\boldsymbol{M})) \in \operatorname{Im}\pi\}. \end{aligned}$$

Then, $\mathcal{M}_+ \oplus \mathcal{M}_- \subset \Pi$, hence equivalence (4.7) implies that that $\pi(\mathcal{M}_\pm) = [P^\mp \circ \pi](\Pi)$. Thus $\pi(\mathcal{M}_+ \oplus \mathcal{M}_-) = \pi(\Pi)$ and, by injectiveness of π it follows that $\mathcal{M}_+ \oplus \mathcal{M}_- = \Pi$.

For V a close subspace of $\mathcal{M}_0^\perp$, with the topology from $L^2(\Gamma)^3$, let $V^{\perp_0}$, denote the close subspace of $\mathcal{M}_0^\perp$ which is perpendicular to V and such that $V \oplus V^{\perp_0} = \mathcal{M}_0^\perp$. Now, since the $\eta P_2^\pm\pi$ are continuous on the topology of $\mathcal{M}_0^\perp$ as a subspace of $L^2(\Gamma)^3$, we get

$$(\mathcal{M}_+ \oplus \mathcal{M}_- \oplus \mathcal{M}_0)^\perp = (\mathcal{M}_+ \oplus \mathcal{M}_-)^{\perp_0} = \Pi^{\perp_0} = (\operatorname{Ker}(\eta P_2^\pm\pi))^{\perp_0} = \overline{\operatorname{Im}(\pi^*(P_2^\pm)^*\eta^*)} = \operatorname{Im}(\pi^*(P_2^\pm)^*\eta^*),$$

where the last equality comes from the fact that $\operatorname{Im}(\pi^*(P_2^\pm)^*\eta^*)$ is finite dimensional and thus, closed on $\mathcal{M}_0^\perp$. Now, taking a $\boldsymbol{c} \in \mathbb{C}$, any $\psi \in H^{-1}(\Gamma)$, a pair $(\phi,\varphi) \in L^2(\Gamma)\times H^1(\Gamma)$ and any $\boldsymbol{M} \in \mathcal{M}_0^\perp$ we have,

$$\langle\!\langle \eta^*\boldsymbol{c}, \psi\rangle\!\rangle = \sum_j \overline{c_j}\langle\psi, 1_{\Gamma_j}\rangle = \left\langle \sum_j \overline{c_j} 1_{\Gamma_j},\ \psi \right\rangle = \left\langle\!\!\left\langle \sum_j c_j 1_{\Gamma_j},\ \psi \right\rangle\!\!\right\rangle,$$

$$\langle\!\langle\ \pi^*(\phi,\varphi),\ \boldsymbol{M}\ \rangle\!\rangle = \langle\!\langle\ (\phi,\varphi),\ (M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T)\ \rangle\!\rangle = \langle\!\langle \phi,\ M_{\boldsymbol{\nu}}\rangle\!\rangle - \langle\!\langle \nabla_{\mathrm{T}}\varphi,\ \boldsymbol{M}_T\rangle\!\rangle = \langle\!\langle\ \phi\boldsymbol{\nu} - \nabla_{\mathrm{T}}\varphi,\ \boldsymbol{M}\ \rangle\!\rangle_{L^2(M)^3}, \tag{4.8}$$

and, by the fact that $T^* = T$ on $H^{1/2}(\Gamma)$ and by Remark 3,

$$\begin{aligned} \mp\pi^*(P_2^\pm)^*\eta^*(\boldsymbol{c}) &= \mp\sum_{j\in J} c_j\left[\pm(T1_{\gamma_j})\boldsymbol{\nu} - \nabla_{\mathrm{T}}\left(\frac{1}{2}1_{\gamma_j} \mp K1_{\gamma_j}\right)\right] \\ &= \sum_{j\in J} c_j\left[-(T1_{\gamma_j})\boldsymbol{\nu} - \nabla_{\mathrm{T}}(K1_{\gamma_j})\right] \\ &= \sum_{j\in J} c_j\left[-(T1_{\gamma_j})\boldsymbol{\nu} - \nabla_{\mathrm{T}}\left(\pm\frac{1}{2}1_{\gamma_j} + K1_{\gamma_j}\right)\right] = \gamma^\pm\left(\nabla\mathcal{F}\left(\sum_{j\in J} c_j 1_{\gamma_j}\boldsymbol{\nu}\right)\right). \end{aligned}$$

Then, the $[\nabla\mathcal{F}(\upsilon_j\boldsymbol{\nu})]|_\Gamma$ are well defined and belong to $L^2(\Gamma)^3$ for any $j\in J$. This also shows, in light of Lemma 4.1, that for any $j\notin\tilde{J}$ it follows that $[\nabla\mathcal{F}(\upsilon_j\boldsymbol{\nu})]|_\Gamma = 0$. Hence, equation (4.3) is satisfied and thus,

$$\tilde{n}_\Gamma \geq \dim\left((\mathcal{M}_-\oplus\mathcal{M}_+\oplus\mathcal{M}_0)^\perp\right) = \operatorname{codim}\left(\mathcal{M}_-\oplus\mathcal{M}_+\oplus\mathcal{M}_0\right).$$

Therefore, by equation (4.5), it follows that $\tilde{n}_\Gamma = \operatorname{codim}\left(\mathcal{M}_-\oplus\mathcal{M}_+\oplus\mathcal{M}_0\right)$, then equation (4.2) holds and the set $\{[\nabla\mathcal{F}(\upsilon_j\boldsymbol{\nu})]|_\Gamma\}_{j\in\tilde{J}}$ is linearly independent.

Clearly, if k^2 is not an eigen-value for the problem (4.1), then $\tilde{J}=J$, and hence,

$$n_\Gamma = \tilde{n}_\Gamma = \dim(\mathcal{M}_{\boldsymbol{\nu}}) = \operatorname{codim}\left(\mathcal{M}_-\oplus\mathcal{M}_+\oplus\mathcal{M}_0\right).$$

Finally, in the case $k=0$, by noticing that

$$\mathcal{F}(1_{\Gamma_j}\boldsymbol{\nu}) = \begin{cases} -\chi_{\mathbb{R}^3\setminus\overline{\Omega_-^j}} & \text{if } \ j=1 \\ \chi_{\Omega_-^j} & \text{otherwise,}\end{cases}$$

it follows that $[\nabla\mathcal{F}(\upsilon_j\boldsymbol{\nu})]|_\Gamma = 0$, for every $j\in J$, then $\tilde{n}_\Gamma = 0$, $\mathcal{M}_{\boldsymbol{\nu}}=\{0\}$, and thus equation (4.4) is satisfied. □

To finish this section we will find a characterization for the spaces $\mathcal{M}_\pm^\perp$.

Corollary 4.3.

$$\left(\mathcal{M}_\pm\oplus\mathcal{M}_0\right)^\perp = \overline{\left\{\ \gamma^\mp\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi)\right)\ :\ (\varphi,\phi)\in H^1(\Gamma)\times L^2(\Gamma)\ \right\}}.$$

Proof. Take η, π and Π, as defined of the proof of Theorem 4.2 and recall that $\left[P^\mp\circ\pi\right](\Pi) = \pi(\mathcal{M}_\pm)\subset\pi(\Pi)$. So, defining the bijective operator

$$\begin{array}{cccc} \pi_\Pi & : & \mathcal{M}_0^\perp & \longrightarrow & \pi(\Pi) \\ & & \boldsymbol{M} & \mapsto & \pi(\boldsymbol{M}),\end{array}$$

it follows that $\pi_\Pi(\mathcal{M}_\pm) = \operatorname{Im}\left(P^\mp\pi_\Pi\right)$ which implies that,

$$\mathcal{M}_\pm = \operatorname{Im}\left(\pi_\Pi^{-1}P^\mp\pi_\Pi\right)$$

For a V, subspace of Π, let $V^{\perp_\Pi}$, denote the close subspace of Π perpendicular to V and such that $V\oplus V^{\perp_\Pi}=\Pi$. Noting that the $\pi_\Pi^{-1}P^\mp\pi_\Pi$ are projections, we get that $\mathcal{M}_\pm = \operatorname{Ker}\left(\pi_\Pi^{-1}P^\pm\pi_\Pi\right)$ and thus,

$$\mathcal{M}_\pm^{\perp_\Pi} = \operatorname{Im}\overline{\left[\pi_\Pi^*(P^\pm)^*\left(\pi_\Pi^{-1}\right)^*\right]}. \tag{4.9}$$

Now, given $\boldsymbol{M}\in\Pi=\Pi^*$, $\phi\in L^2(\Gamma)$ and $\psi\in\operatorname{Ker}\eta$, recalling the equivalence (4.6) and using Remark 5 and Lemma B.4,

$$\begin{aligned}\left\langle\!\!\left\langle\ \left(\pi_\Pi^{-1}\right)^*\boldsymbol{M},\ (\phi,\psi)\ \right\rangle\!\!\right\rangle &= \left\langle\!\!\left\langle\ \boldsymbol{M},\ \pi_\Pi^{-1}(\phi,\psi)\ \right\rangle\!\!\right\rangle = \langle\!\langle\ \boldsymbol{M},\ \phi\boldsymbol{\nu}+\nabla_{\mathrm{T}}\varphi_\psi\ \rangle\!\rangle \\ &= \langle\!\langle\ M_{\boldsymbol{\nu}},\ \phi\ \rangle\!\rangle - \langle\!\langle\ \nabla_{\mathrm{T}}\cdot U_{\boldsymbol{M}_T},\ \nabla_{\mathrm{T}}\varphi_\psi\ \rangle\!\rangle = \langle\!\langle\ M_{\boldsymbol{\nu}},\ \phi\ \rangle\!\rangle - \langle\!\langle\ U_{\boldsymbol{M}_T},\ \psi\ \rangle\!\rangle,\end{aligned}$$

Thus, $\left(\pi_\Pi^{-1}\right)^*\boldsymbol{M} = (M_{\boldsymbol{\nu}},-U_{\boldsymbol{M}_T})\in L^2(\Gamma)\times H^1(\Gamma)$. Then, from equation (4.9),

$$\mathcal{M}_\pm^{\perp_\Pi}\subset\overline{\left[\pi_\Pi^*\circ(P^\pm)^*\right](L^2(\Gamma)\times H^1(\Gamma))}. \tag{4.10}$$

Now, as $T=T^*$ on $H^{1/2}(\Gamma)\supset H^1(\Gamma)$ and $S=S^*$ on $H^{-1/2}(\Gamma)\supset L^2(\Gamma)$ we get for $(\phi,\varphi)\in L^2(\Gamma)\times H^1(\Gamma)$, that

$$(P^\pm)^*(\phi,\varphi) = \begin{pmatrix}\frac{1}{2}Id\pm K^* & \pm T \\ \mp S & \frac{1}{2}Id\mp K\end{pmatrix}\begin{pmatrix}\phi\\ \varphi\end{pmatrix}.$$

Hence, using equation (4.8) and Remark 3, we obtain for $(\phi,\varphi)\in L^2(\Gamma)\times H^1(\Gamma)$,

$$\begin{aligned}
\pm\left[\pi_\Pi^*\circ(P^\pm)^*\right](\phi,-\varphi) &= \pi_\Pi^*\left(\left(\pm\frac{1}{2}Id+K^*\right)\phi - T\varphi,\ -S\phi+\left(\mp\frac{1}{2}Id+K\right)\varphi\right)\\
&= \left(-T\varphi+\left(\pm\frac{1}{2}+K^*\right)\phi\right)\boldsymbol{\nu}\ +\ \nabla_{\mathrm{T}}\left(-\left(\mp\frac{1}{2}Id+K\right)\varphi+S\phi\right)\\
&= \partial_{\boldsymbol{\nu}}^{\mp}\left(-DL(\varphi)+SL(\phi)\right)\ +\ \nabla_{\mathrm{T}}\gamma^{\mp}\left(-DL(\varphi)+SL(\phi)\right)\\
&= \gamma^{\mp}\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi)\right).
\end{aligned}$$

Then, noting that $(\mathcal{M}_\pm\oplus\mathcal{M}_0)^\perp=\mathcal{M}_\pm^{\perp\Pi}\oplus(\Pi\oplus\mathcal{M}_0)^\perp=\mathcal{M}_\pm^{\perp\Pi}\oplus(\mathcal{M}_+\oplus\mathcal{M}_-\oplus\mathcal{M}_0)^\perp$, and recalling equations (4.10) and (4.3), we have the inclusion

$$(\mathcal{M}_\pm\oplus\mathcal{M}_0)^\perp\subset\overline{\left\{\ \gamma^{\mp}\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi)\right)\ :\ (\varphi,\phi)\in H^1(\Gamma)\times L^2(\Gamma)\ \right\}}.$$

Thus, to finish the proof it only remains to show the inclusion on the opposite direction for the equation above. Take any $(\varphi,\phi)\in H^1(\Gamma)\times L^2(\Gamma)$ and let $\boldsymbol{M}:=\gamma^{\mp}\left(\nabla\widetilde{\mathcal{F}}(\varphi,\phi)\right)$ and $w=\widetilde{\mathcal{F}}(\varphi,\phi)$. First note that using Remark 5 and the fact that $\boldsymbol{M}_T$ is a tangential gradient, it follows that $\boldsymbol{M}\perp\mathcal{M}_0$. Next, let $\boldsymbol{M}^\pm\in\mathcal{M}_\pm$, $w^\pm=\mathcal{F}(\boldsymbol{M}^\pm)$ and note that, by implication 3.9 and Remark 5,

$$\pm(\gamma^{\mp}w^\pm,\partial_{\boldsymbol{\nu}}^{\mp}w^\pm)=P^{\mp}(M_{\boldsymbol{\nu}}^\pm,\nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T^\pm)=(M_{\boldsymbol{\nu}}^\pm,\nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T^\pm).$$

Then, using Lemma B.2,

$$\begin{aligned}
\langle\!\langle\boldsymbol{M},\boldsymbol{M}^\pm\rangle\!\rangle_{L^2(\Gamma)} &= \langle\!\langle\nabla_{\mathrm{T}}(\gamma^{\mp}w),\ \boldsymbol{M}_T^\pm\rangle\!\rangle+\langle\!\langle\partial_{\boldsymbol{\nu}}^{\mp}w,\ M_{\boldsymbol{\nu}}^\pm\rangle\!\rangle=-\langle\!\langle\gamma^{\mp}w,\ \nabla_{\mathrm{T}}\cdot\boldsymbol{M}_T^\pm\rangle\!\rangle+\langle\!\langle\partial_{\boldsymbol{\nu}}^{\mp}w,\ M_{\boldsymbol{\nu}}^\pm\rangle\!\rangle\\
&= \mp\langle\!\langle\gamma^{\mp}w,\ \partial_{\boldsymbol{\nu}}^{\mp}w^\pm\rangle\!\rangle\pm\langle\!\langle\partial_{\boldsymbol{\nu}}^{\mp}w,\ \gamma^{\mp}w^\pm\rangle\!\rangle=0.
\end{aligned}$$

Therefore $\boldsymbol{M}\perp\mathcal{M}_\pm$ as well and since $(\mathcal{M}_\pm\oplus\mathcal{M}_0)^\perp$ is closed, the corollary follows. ☐

4.1. **Spherical case.** In this subsection we will assume that $\Gamma=\mathbb{S}^2$, the unit sphere on $\mathbb{R}^3$ and that $k>0$. In this case, some calculations from the previous subsection can be made explicit using the Addition Theorem.

Recall that, if we let P_n^m denote the associated Legendre function of order m, then the following define a complete orthonormal system in $L^2(\mathbb{S})$ [6, Theorem 2.8] and a complete orthogonal system in $H^1(\mathbb{S})$ [13, Theorem 2.4.4]:

$$Y_n^m(x):=\sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\ P_n^{|m|}(\cos\theta)\ e^{im\varphi}\qquad\text{for } m=-n,\dots,n,\ \text{ and } n=0,1,2,\dots,$$

where $x=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$. Note that $\overline{(Y_n^m(x))}=Y_n^{-m}(x)$. These functions also satisfy

$$\Delta_{\mathrm{T}}Y_n^m=-n(n+1)\ Y_n^m. \tag{4.11}$$

Note that this implies that $\langle\!\langle Y_n^m,Y_n^m\rangle\!\rangle_{H^1(\mathbb{S})}=1+n(n+1)$. Given $\phi\in L^2(\mathbb{S})$ and $\boldsymbol{M}\in L^2(\mathbb{S})^3$ define the coefficients:

$$c_n^m(\phi):=\langle\!\langle Y_n^m,\phi\rangle\!\rangle_{L^2(\mathbb{S})}=\frac{\langle\!\langle Y_n^m,\phi\rangle\!\rangle_{H^1(\mathbb{S})}}{\langle\!\langle Y_n^m,Y_n^m\rangle\!\rangle_{H^1(\mathbb{S})}}\qquad\text{for } m=-n,\dots,n,\ \text{ and } n=0,1,2,\dots,$$

and, for $m=-n,\dots,n$ and $n=1,2,3,\dots$,

$$g_n^m(\boldsymbol{M}):=\frac{\langle\!\langle\nabla_{\mathrm{T}}Y_n^m,\boldsymbol{M}\rangle\!\rangle_{L^2(\mathbb{S})^3}}{n(n+1)},\qquad r_n^m(\boldsymbol{M}):=\frac{\langle\!\langle\boldsymbol{\nu}\times\nabla_{\mathrm{T}}Y_n^m,\boldsymbol{M}\rangle\!\rangle_{L^2(\mathbb{S})^3}}{n(n+1)},$$

$$g_0^0(\boldsymbol{M}):=0\qquad\text{and}\qquad r_0^0(\boldsymbol{M}):=0.$$

Then, $\phi=\sum c_n^m(\phi)Y_n^m$ in $L^2(\mathbb{S})$. Note that, for any n and m, $g_n^m(\boldsymbol{M})=g_n^m(\boldsymbol{M}_T)$ and $r_n^m(\boldsymbol{M})=r_n^m(\boldsymbol{M}_T)$. Additionally, if $u\in H^1(\mathbb{S})$ then $u=\sum c_n^m(u)Y_n^m$ in $H^1(\mathbb{S})$ and we have:

$$g_n^m(\nabla_{\mathrm{T}}u)=c_n^m(u),\quad r_n^m(\nabla_{\mathrm{T}}u)=0,\quad g_n^m(\boldsymbol{\nu}\times\nabla_{\mathrm{T}}u)=0\quad\text{and}\quad r_n^m(\boldsymbol{\nu}\times\nabla_{\mathrm{T}}u)=c_n^m(u).$$

By Hodge decomposition, there exist $u, v \in H^1(\mathbb{S})$ such that $\boldsymbol{M}_T = \nabla_{\mathrm{T}} u + \boldsymbol{\nu} \times \nabla_{\mathrm{T}} v$, and hence,

$$\begin{aligned}\boldsymbol{M}_T &= \nabla_{\mathrm{T}} \sum c_n^m(u) Y_n^m + \boldsymbol{\nu} \times \nabla_{\mathrm{T}} \sum c_n^m(v) Y_n^m \\ &= \sum g_n^m(\boldsymbol{M}) \nabla_{\mathrm{T}} Y_n^m + \sum r_n^m(\boldsymbol{M})(\boldsymbol{\nu} \times \nabla_{\mathrm{T}} Y_n^m)\end{aligned}$$

in $L^2(\mathbb{S})^3$. Therefore, $\nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T = -n(n+1) \sum g_n^m(\boldsymbol{M}) Y_n^m$ in $H^{-1}(\mathbb{S})$.

Further define, for a non-negative integer n, $h_n^{(1)}$ as the spherical Hankel function of the first kind of order n, and let j_n denote the spherical Bessel function of order n. Then, since we are assuming that $k \neq 0$, we have the following Addition Theorem [6, Theorem 2.11]

$$G(x-y) = -ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k|x|) Y_n^m\left(\frac{x}{|x|}\right) j_n(k|y|) \overline{\left(Y_n^m\left(\frac{y}{|y|}\right)\right)} \quad \text{for } |x| > |y|,$$

where the series and its term by term first derivatives with respect to $|x|$ and $|y|$ are absolutely and uniformly convergent on compact subsets of $|x| > |y|$.

Then, using Fubini-Tonelli theorem we obtain:

$$SL(Y_n^m)(x) = \begin{cases} -ik\ h_n^{(1)}(k|x|)\ j_n(k)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| > 1 \\ -ik\ h_n^{(1)}(k)\ j_n(k|x|)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| < 1 \end{cases},$$

$$DL(Y_n^m)(x) = \begin{cases} -ik^2\ h_n^{(1)}(k|x|)\ j_n'(k)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| > 1 \\ -ik^2\ (h_n^{(1)})'(k)\ j_n(k|x|)\ Y_n^m\left(\frac{x}{|x|}\right) & \text{for } |x| < 1 \end{cases},$$

thus,

$$\begin{aligned} K(Y_n^m) &= -\frac{1}{2} ik^2 \left(h_n^{(1)}(k) j_n'(k) + (h_n^{(1)})'(k)\ j_n(k)\right) Y_n^m \qquad & T(Y_n^m) &= -ik^3\ (h_n^{(1)})'(k)\ j_n'(k)\ Y_n^m \\ S(Y_n^m) &= -ik\ h_n^{(1)}(k)\ j_n(k)\ Y_n^m & K^*(Y_n^m) &= K(Y_n^m), \end{aligned}$$

Since K, T, S, K^* and $\nabla_{\mathrm{T}} \cdot : L^2(\mathbb{S})^3 \longrightarrow H^{-1}(\mathbb{S})$ are continuous, and $h_n^{(1)}(k) j_n'(k) - (h_n^{(1)})'(k)\ j_n(k) = 1/(ik^2)$, using (4.11) we get

$$\begin{aligned} P^-(M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T) &= \Bigg(\sum_{n,m} \left[(1/2 - K(Y_n^m)) c_n^m(M_{\boldsymbol{\nu}}) \ + \ S(Y_n^m)(-n(n+1) g_n^m(\boldsymbol{M}_T)) \right] Y_n^m, \\ &\qquad \sum_{n,m} \left[-T(Y_n^m) c_n^m(M_{\boldsymbol{\nu}}) \ + \ (1/2 + K(Y_n^m))(-n(n+1) g_n^m(\boldsymbol{M}_T)) \right] Y_n^m \Bigg) \\ &= \Bigg(\sum_{n,m} ik h_n^{(1)}(k) \left[k j_n'(k) c_n^m(M_{\boldsymbol{\nu}}) + j_n(k) n(n+1) g_n^m(\boldsymbol{M}_T) \right] Y_n^m, \\ &\qquad \sum_{n,m} ik^2 (h_n^{(1)})'(k) \left[k j_n'(k) c_n^m(M_{\boldsymbol{\nu}}) + j_n(k) n(n+1) g_n^m(\boldsymbol{M}_T) \right] Y_n^m \Bigg), \end{aligned}$$

and

$$\begin{aligned} P^+(M_{\boldsymbol{\nu}}, \nabla_{\mathrm{T}} \cdot \boldsymbol{M}_T) &= \Bigg(\sum_{n,m} -ik j_n(k) \left[k (h_n^{(1)})'(k) c_n^m(M_{\boldsymbol{\nu}}) + h_n^{(1)}(k) n(n+1) g_n^m(\boldsymbol{M}_T) \right] Y_n^m, \\ &\qquad \sum_{n,m} -ik^2 j_n'(k) \left[k (h_n^{(1)})'(k) c_n^m(M_{\boldsymbol{\nu}}) + h_n^{(1)}(k) n(n+1) g_n^m(\boldsymbol{M}_T) \right] Y_n^m \Bigg), \end{aligned}$$

Recall that for all n, $h_n^{(1)}(k) \neq 0 \neq (h_n^{(1)})'(k)$ for k is real and positive. Therefore,

$$\begin{aligned} \mathcal{M}_- = \{ \boldsymbol{M} \in L^2(\mathbb{S})^3 :\ & k j_n'(k) c_n^m(M_{\boldsymbol{\nu}}) = -j_n(k)\ n(n+1) g_n^m(\boldsymbol{M}_T) \\ & \text{for } m = -n, \ldots, n, \text{ and } n = 0, 1, 2, \ldots, \}, \end{aligned} \tag{4.12}$$

and

$$\mathcal{M}_+ = \{\boldsymbol{M} \in L^2(\mathbb{S})^3 :\ k(h_n^{(1)})'(k)\ c_n^m(M_{\boldsymbol{\nu}}) = -h_n^{(1)}(k)\ n(n+1) g_n^m(\boldsymbol{M}_T) \\ \text{for } m = -n, ..., n, \text{ and } n = 0, 1, 2, ..., \text{ such that } j_n(k) \neq 0 \text{ or } j_n'(k) \neq 0\}. \tag{4.13}$$

Since $j_0(k) = \sin(k)/k$, for no real k we get $j_0(k) = 0 = j_0'(k)$. Hence, for a $\boldsymbol{M} \in \mathcal{M}_+ + \mathcal{M}_-$, if $j_0'(k) \neq 0$, we have that $c_0^0(M_{\boldsymbol{\nu}}) = 0$. Thus, $\langle\!\langle \boldsymbol{M}, Y_0^0 \boldsymbol{\nu} \rangle\!\rangle_{L^2(M)^3} = 0$. Otherwise, when $j_0'(k) = 0$ and $k > 0$, we have $P^-(Y_0^0, 0) = (0, 0)$. Therefore, using Theorem 4.2 we get the following result.

Theorem 4.4. *For a $k > 0$, if $j_0'(k) \neq 0$ (which happens a.e.), then*

$$(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^\perp = \{\boldsymbol{M} \in L^2(\mathbb{S})^3 \ : \ \boldsymbol{M}_T = 0 \text{ and } M_{\boldsymbol{\nu}} \text{ is constant}\},$$

on the other hand, if $j_0'(k) = 0$, which happens for example when $k = 0$, then

$$L^2(\mathbb{S})^3 = \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0.$$

Appendix A. Adaptation of results from [9]

The statements in this section are either adaptations to the case $k > 0$, of directly taken from [9]. For each of them, we write in parenthesis where in [9] they can be found. For convenience, during this section we will denote the function G and the operators SL, DL, S, K, K^* and T by G_k, SL_k, DL_k, S_k, K_k, K_k^* and T_k respectively.

Recall $\mathfrak{C}_\alpha^\pm(x)$ from the definition of the non-tangential limit. For a vector valued measurable function $\boldsymbol{\psi}$ on $\Omega_\pm$, we define the function $\mathfrak{N}_\alpha^\pm \boldsymbol{\psi}$, on $\partial\Omega$, such that, for $x \in \partial\Omega$,

$$\mathfrak{N}_\alpha^\pm \boldsymbol{\psi}(x) := \sup\left\{|\boldsymbol{\psi}(y)| \ : \ y \in \mathfrak{C}_\alpha^\pm(x)\right\},$$

taking the convention that $\mathfrak{N}_\alpha^\pm \boldsymbol{\psi}(x) = 0$ when $\mathfrak{C}_\alpha^\pm(x) = \varnothing$.

In [9, section 3.6], the Sobolev space $L_1^2(\partial\Omega, d\sigma)$ is defined as the subspace of $L^2(\partial\Omega)$ comprised of those functions φ such that $|\langle\ \varphi\ ,\ \nu_j\gamma(\partial_l f) - \nu_l\gamma(\partial_j f)\ \rangle_{L^2(\partial\Omega)}| \leq C\|f_{|\partial\Omega}\|_{L^2(\partial\Omega)}$ for all $f \in C^1(\mathbb{R}^3)$, any $l, j \in \{1, 2, 3\}$ and some constant $C = C(\varphi)$, with ν_j to mean the j-th coordinate of the unit normal field on $\partial\Omega$. That is, if one puts as in [9] $\partial_{\tau_{l,j}} f := \nu_j\gamma(\partial_l f) - \nu_l\gamma(\partial_j f)$ for $f \in C^1(\mathbb{R}^3)$ then $\partial_{\tau_{l,j}} f$ depends only on the restriction $f_{|\partial\Omega}$ and members of $L_1^2(\partial\Omega)$ are those $\varphi \in L^2(\partial\Omega)$ whose distributional $\partial_{\tau_{l,j}}\varphi$ is an $L^2(\partial\Omega)$-function for each j, l. To justify quoting certain results from [9], we will show in the next lemma that this definition agrees with the one of the Sobolev space $H^1(\partial\Omega)$ made in Section 2.

Lemma A.1. *Given $j, l \in \{1, 2, 3\}$, one can define a bounded linear operator $\partial_{\tau_{i,j}} : H^1(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ on letting, for any $\varphi \in H^1(\partial\Omega)$ and $f \in C^1(\mathbb{R}^3)$:*

$$\langle\ \partial_{\tau_{j,l}}\varphi,\ \gamma(f)\ \rangle := -\ \langle\ \varphi\ ,\ \nu_j\gamma(\partial_l f) - \nu_l\gamma(\partial_j f)\ \rangle_{L^2(\partial\Omega)}.$$

Proof. Note that a tangent vector field on $\partial\Omega$ can be regarded as a 1-form, defined by taking the scalar product in the tangent space at regular points. For $\{(\theta_j, U_j)\}_{j \in I}$ (I finite) a Lipschitz atlas on $\partial\Omega$, we say that a k-form ω is of L^2-class (here $k \in \{0, 1, 2\}$) if its expression in local coordinates (pullback of ω under the Lipschitz map θ_j^{-1}), say

$$(\theta_j^{-1})^*(\omega)(y) = \sum_{i_1 < i_2, \cdots, < i_k} a_{i_1, \cdots, i_k}^{\{\phi_j\}}(y)\ dy_{i_1} \wedge \cdots \wedge dy_{i_k}$$

has coefficients $a_{i_1, \cdots, i_k}^{\{\phi_j\}}$ that are L^2 functions on $\theta_j(U_j)$. This notion is independent of the atlas. Now, for $f \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$(\partial_{\tau_{2,3}} f\ ,\ \partial_{\tau_{3,1}} f\ ,\ \partial_{\tau_{1,2}} f)^t = \nabla f \times \nu \tag{A.1}$$

where "×" indicates the vector product and the superscript "t" means "transpose". Thus, observing that $\nu = \partial_{y_1}\theta_j^{-1} \times \partial_{y_2}\theta_j^{-1} / |\partial_{y_1}\theta_j^{-1} \times \partial_{y_2}\theta_j^{-1}|$ on $\theta_j(U_j)$, we get from the double vector product formula that the 1-form defined by (A.1) is given in local coordinates (y_1, y_2) on $\theta_j(U_j)$ by

$$\Big(g_{1,1}\partial_{y_2}(f\circ\theta_j^{-1}) - g_{2,1}\partial_{y_1}(f\circ\theta_j^{-1})\Big)dy_1 + \Big(g_{1,2}\partial_{y_2}(f\circ\theta_j^{-1}) - g_{2,2}\partial_{y_1}(f\circ\theta_j^{-1})\Big)dy_2 \tag{A.2}$$

where (g_{i_1,i_2}) is the metric tensor (the Gram matrix of $\partial_{y_1}\theta_j^{-1}\partial_{y_1}\theta_j^{-1}$, $\partial_{y_2}\theta_j^{-1}$). Since the latter is uniformly boundedly invertible on compact manifold that are local Lipschitz graphs, the fact that (A.2) is of L^2-class amounts to say that $\nabla\psi \circ \theta_i^{-1}$ lies in $(L^2(\theta_j(U_j)))^2$-norm of . By density of traces of $C_c^\infty(\mathbb{R}^3)$-functions in $L^2(\partial\Omega$, we conclude what we want.

∎

Then, we have a lemma that was just stated on [9] since it was proven in [7]. However, we add a proof for convenience of the reader.

Lemma A.2 (Lemma 6.4.2)**.** *For each fixed $R > 0$ and $k > 0$, there exists a constant $C > 0$ such that, for $1 \le j \le 3$ the following estimates are uniformly satisfied for $0 < |x| < R$:*

$$|G_k(x) - G_0(x)| \le C$$
$$|\partial_j G_k(x) - \partial_j G_0(x)| \le C$$
$$|\partial_\ell\partial_j G(x) - \partial_\ell\partial_j G_0(x)|\ |x| \le C$$

Proof. Since $G_k - G_0$ is $C^\infty(\mathbb{R}^3 \setminus \{0\})$, it is enough to show that the lim sup when $x \to 0$ in all of the left hand sides of the equations of the lemma are bounded by a constant depending only on k:

$$\limsup_{x\to 0} |G_k(x) - G_0(x)| = \lim_{x\to 0} \frac{\left|-1 + e^{ik|x|}\right|}{4\pi|x|} = \frac{k}{4\pi},$$

and

$$\limsup_{x\to 0} |\partial_j G_k(x) - \partial_j G_0(x)| = \limsup_{x\to 0} \left| x_j \frac{e^{ik|x|}k|x| + ie^{ik|x|} - i}{4\pi|x|^3} \right| \le \lim_{x\to 0} \frac{\left|x_j e^{ik|x|}k|x| + ie^{ik|x|} - i\right|}{4\pi|x|^2} = \frac{k^2}{8\pi}.$$

$$\begin{aligned}
\limsup_{x\to 0} |\partial_j\partial_j G(x) - \partial_j\partial_j G_0(x)||x| &= \limsup_{x\to 0} \frac{\left|e^{ik|x|}\left(ik|x|^3 - |x|^2 - k^2x_j^2|x|^2 - 3ikx_j^2|x| + 3x_j^2\right) + |x|^2 - 3x_j^2\right|}{4\pi|x|^4} \\
\le \limsup_{x\to 0} \frac{\left|e^{ik|x|}\left(ik|x|^3 - |x|^2\right) + |x|^2\right|}{4\pi|x|^4} &+ \limsup_{x\to 0} \frac{\left|e^{ik|x|}\left(-k^2x_j^2|x|^2 - 3ikx_j^2|x| + 3x_j^2\right) - 3x_j^2\right|}{4\pi|x|^4} \\
\le \lim_{x\to 0} \frac{\left|e^{ik|x|}\left(ik|x| - 1\right) + 1\right|}{4\pi|x|^2} &+ \lim_{x\to 0} \frac{\left|e^{ik|x|}\left(-k^2|x|^2 - 3ik|x| + 3\right) - 3\right|}{4\pi|x|^2} = \frac{k^2}{4\pi},
\end{aligned}$$

and, for $j \neq \ell$,

$$\begin{aligned}
\limsup_{x\to 0} |\partial_\ell\partial_j G(x) - \partial_\ell\partial_j G_0(x)||x| &= \limsup_{x\to 0} \frac{\left|x_j x_\ell \left(3 + e^{ik|x|}(k^2|x|^2 + 3ik|x| - 3)\right)\right|}{4\pi|x|^4} \\
&\le \lim_{x\to 0} \frac{\left|3 + e^{ik|x|}(k^2|x|^2 + 3ik|x| - 3)\right|}{4\pi|x|^2} = \frac{k^2}{8\pi}.
\end{aligned}$$

∎

Then, we continue with a generalization of a relatively basic result that is just partly stated on [9] and whose proof, for the $k = 0$ case, can be found as part of [4, Theorem 4.5.].

Lemma A.3 (Partly stated on equation (3.6.27) and Corollary 3.6.3)**.** *Given a $\phi \in L^2(\partial\Omega)$, it is satisfied in the non-tangential sense that $\gamma^{\pm} SL_k\phi = S_k\phi$, σ-a.e. and, for every $\alpha > 0$, there exists a constant $\tilde{C}_\alpha$ such that $\|\mathfrak{N}^{\pm}_\alpha(SL_k\phi)\|_2 \le \tilde{C}_\alpha\|\phi\|_2$. Also, the left equation of (3.6) is satisfied for $f = \phi$, σ-a.e. and we have the mapping property,*

$$S_k : L^2(\partial\Omega) \longrightarrow H^1(\partial\Omega).$$

Proof. Note that for any $x \in \partial\Omega$ and $\phi \in L^2(\partial\Omega)$, using the $k = 0$ result,

$$\left|\int_{\partial\Omega} G_k(x-y)\phi(y)\mathrm{d}\sigma(y)\right| \le \int_{\partial\Omega} |G_k(x-y)|\ |\phi(y)|\ \mathrm{d}\sigma(y) = \int_{\partial\Omega} G_0(x-y)\ |\phi(y)|\ \mathrm{d}\sigma(y) = S_0|\phi|(x),$$

and thus, we have that in general, for σ-a.e. $x \in \partial\Omega$, the integral in the left equation of (3.6) defines a bounded linear operator from $L^2(\partial\Omega)$ to itself. Let's call this operator $\tilde{S}_k$. Now, notice the following facts; $\mathrm{Lip}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$; both $\mathrm{Lip}(\partial\Omega)$ and $L^2(\partial\Omega)$ are dense in $H^{-1/2}(\partial\Omega)$; S_k and $\tilde{S}_k$ coincide in $\mathrm{Lip}(\partial\Omega)$; and the image of $\mathrm{Lip}(\partial\Omega)$ over $\tilde{S}_k$ belongs to $L^2(\partial\Omega)$. Then, S_k and $\tilde{S}_k$ must also coincide in $L^2(\partial\Omega)$. Thus, as a small abuse of notation we will refer to $\tilde{S}_k$ as simply S_k. Next, if $\|\phi\|_2 = 1$, and we take C from Lemma A.2

$$\begin{aligned}
\|\nabla_{\mathrm{T}} S_k\phi - \nabla_{\mathrm{T}} S_0\phi\|_2 &= \limsup_{\substack{\boldsymbol{f} \in \mathrm{Lip}_T(\partial\Omega)\\ \|\boldsymbol{f}\|_\infty \le 1}} \int\limits_{\partial\Omega} \left(\int\limits_{\partial\Omega} (G_k - G_0)(x-y)\ \phi(y)\ \mathrm{d}\sigma(y) \right) \nabla_{\mathrm{T}}\cdot\boldsymbol{f}(x)\ \mathrm{d}\sigma(x) \\
&= \limsup_{\substack{\boldsymbol{f} \in \mathrm{Lip}_T(\partial\Omega)\\ \|\boldsymbol{f}\|_\infty \le 1}} \int\limits_{\partial\Omega} \left(\int\limits_{\partial\Omega} (\nabla G_k - \nabla G_0)(x-y)\cdot\boldsymbol{f}(x)\ \mathrm{d}\sigma(x) \right) \phi(y)\ \mathrm{d}\sigma(y) \\
&= \int\limits_{\partial\Omega} \left(\int\limits_{\partial\Omega} |(\nabla G_k - \nabla G_0)(x-y)|\ \mathrm{d}\sigma(x) \right) |\phi(y)|\ \mathrm{d}\sigma(y) \\
&\le \sqrt{3}C\sigma(\partial\Omega)\|\phi\|_1 \le \sqrt{3}C\sigma(\partial\Omega)(\|\phi\|_2^2 + \sigma(\partial\Omega)) = \sqrt{3}C\sigma(\partial\Omega)(1 + \sigma(\partial\Omega)),
\end{aligned}$$

Then, as $S_0 : L^2(\partial\Omega) \longrightarrow H^1(\partial\Omega)$ is bounded and $\|S_k\phi\|_{H^1(\partial\Omega)} \le \|S_k\phi - S_0\phi\|_{H^1(\partial\Omega)} + \|S_0\phi\|_{H^1(\partial\Omega)}$, we obtain that S_k is also a bounded linear operator from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$.

Take $\alpha > 0$, $x \in \partial\Omega$ and $y \in \mathfrak{C}^{\pm}_\alpha(x)$. Then for any $z \in \partial\Omega$

$$|y - z| \ge \mathrm{dist}(y, \partial\Omega) \ge \frac{|x-y|}{\alpha+1} \quad \text{so,} \quad |y-z|(\alpha+2) \ge |x-y| + |y-z| \ge |x-z|. \tag{A.3}$$

Thus,

$$|SL_k\phi(y)| = \left|\int_{\partial\Omega} G_k(y-z)\phi(z)\mathrm{d}\sigma(z)\right| \le \int_{\partial\Omega} \frac{(\alpha+2)|\phi(z)|}{4\pi|x-z|}\mathrm{d}\sigma(z) = (\alpha+2)S_0|\phi|(x).$$

Hence, we can use Dominated convergence and the result for $k = 0$, to obtain for σ-a.e. $x \in \partial\Omega$, that it is satisfied in the non-tangential sense $\gamma^{\pm} SL_k\phi = S_k\phi$. Also, taking $\tilde{C}_\alpha$ to be the operator norm of S_0 times $\alpha + 2$ we obtain that $\|\mathfrak{N}^{\pm}_\alpha(SL_k\phi)\|_2 \le \tilde{C}_\alpha\|\phi\|_2$. □

Proposition A.4 (Propositions 3.3.2 and 3.6.2)**.** *Take a $\phi \in L^2(\partial\Omega)$. For $f = \phi$, the principal value of equation (3.6) exists for σ-a.e. $x \in \partial\Omega$ and it can be used to extend the operator K_k to*

$$K_k : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$$

which is bounded. Furthermore, the right equation of (3.7) is satisfied in the non-tangential limit sense and for every $\alpha > 0$, we have that $\|\mathfrak{N}^{\pm}_\alpha(DL_k\phi)\|_2 \le \tilde{C}_\alpha\|\phi\|_2$ for some $\tilde{C}_\alpha > 0$ depending only on $\partial\Omega$, k and α.

Proof. By [9, Proposition 3.3.2] the result is valid for $k=0$. Take any $\phi \in L^2(\partial\Omega)$ and a $x \in \partial\Omega$ such that $K_0\phi(x)$ is well defined. Define for any $\varepsilon > 0$

$$p_k(\varepsilon) := \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \partial_{\boldsymbol{\nu},y} G_k(x-y)\ \phi(y)\ \mathrm{d}\sigma(y) = -\int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} (\nabla G_k)(x-y)\cdot\boldsymbol{\nu}(y)\ \phi(y)\ \mathrm{d}\sigma(y).$$

Then, $p_0(\varepsilon) \to K_0\phi(x)$ as $\varepsilon \to 0$. Thus, for any sequence $(\varepsilon_n)_{n=1}^\infty$ such that $\varepsilon_n \to 0$ as $n \to \infty$, the sequence $(p_0(\varepsilon_n))_{n=1}^\infty$ is Cauchy. Hence, showing that the principal value of equation (3.6) exists for x is equivalent to showing that the sequence $(p_k(\varepsilon_n))_{n=1}^\infty$ is Cauchy as well. Take $m > n$ and see that,

$$\begin{aligned}
|p_k(\varepsilon_n) - p_k(\varepsilon_m)| &\le |p_k(\varepsilon_n) - p_k(\varepsilon_m) - p_0(\varepsilon_n) + p_0(\varepsilon_m)| \ + \ |p_0(\varepsilon_n) - p_0(\varepsilon_m)| \\
&\le \int_{\substack{y \in \partial\Omega \\ \varepsilon_n > |x-y| > \varepsilon_m}} |(\nabla G_k - \nabla G_0)(x-y)|\ |\phi(y)|\ \mathrm{d}\sigma(y) \ + \ |p_0(\varepsilon_n) - p_0(\varepsilon_m)| \\
&\le \int_{\substack{y \in \partial\Omega \\ \varepsilon_n > |x-y| > \varepsilon_m}} \sqrt{3}C\ |\phi(y)|\ \mathrm{d}\sigma(y) \ + \ |p_0(\varepsilon_n) - p_0(\varepsilon_m)|
\end{aligned}$$

where the constant C, taken from Lemma A.2, depends only on k and the size of the bounded set $\partial\Omega$. Thus, the integrability of ϕ and the fact that $(p_0(\varepsilon_n))_{n=1}^\infty$ is Cauchy imply that $(p_k(\varepsilon_n))_{n=1}^\infty$ is Cauchy as well. Since the result is valid for $k=0$, the value $K_0\phi(x)$ is well defined for σ-a.e. $x \in \partial\Omega$. Then, $\tilde{K}_k\phi(x) := \lim_{\varepsilon\to 0} p_k(\varepsilon)$ is also well defined for σ-a.e. $x \in \partial\Omega$.

Note that $\tilde{K}_k$ defines a linear operator on $L^2(\partial\Omega)$. Take now any $\phi \in L^2(\partial\Omega)$ with $\|\phi\|_2 = 1$. Then using, Fatou's lemma we get

$$\begin{aligned}
\left\|\tilde{K}_k\phi - K_0\phi\right\|_2^2 &\le \liminf_{\varepsilon\to 0} \int_{x\in\partial\Omega} \left| \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} (\nabla G_k - \nabla G_0)(x-y)\cdot\boldsymbol{\nu}(y)\ \phi(y)\ \mathrm{d}\sigma(y) \right|^2 \mathrm{d}\sigma(x) \\
&\le \liminf_{\varepsilon\to 0} \int_{x\in\partial\Omega} \left(\int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} |(\nabla G_k - \nabla G_0)(x-y)|\ |\phi(y)|\ \mathrm{d}\sigma(y) \right)^2 \mathrm{d}\sigma(x) \\
&\le 3C^2\sigma(\partial\Omega)\|\phi\|_1^2 \le 3C^2\sigma(\partial\Omega)(\|\phi\|_2^2 + \sigma(\partial\Omega))^2 = 3C^2\sigma(\partial\Omega)(1+\sigma(\partial\Omega))^2,
\end{aligned}$$

with the same constant C as before. Then, as K_0 is bounded and $\|\tilde{K}_k\phi\|_2 \le \|\tilde{K}_k\phi - K_0\phi\|_2 + \|K_0\phi\|_2$, we obtain that $\tilde{K}_k$ is bounded as well and has $L^2(\partial\Omega)$ as its image. Now, using an argument analogous to the one in Lemma A.3 for $\tilde{S}_k$, we can show that $\tilde{K}_k$ coincides with K_k in $L^2(\partial\Omega)$ and thus, as a small abuse of notation we will refer to $\tilde{K}_k$ as just K_k.

Fix a $\phi \in L^2(\partial\Omega)$ such that $\|\phi\|_2 = 1$. By [9, equation (3.3.6)], for any $\alpha > 0$ there exists a constant C_α such that

$$\|\mathfrak{N}_\alpha^\pm(DL_0\phi)\|_2 \le C_\alpha.$$

On the other hand,

$$\begin{aligned}
|\mathfrak{N}_\alpha^\pm(DL_k\phi)(x) - \mathfrak{N}_\alpha^\pm(DL_0\phi)(x)| &\le \mathfrak{N}_\alpha^\pm(DL_k\phi - DL_0\phi)(x) \\
&= \sup_{z\in\mathfrak{C}_\alpha^\pm(x)} \left| \int_{\partial\Omega} (\nabla G_k - \nabla G_0)(z-y)\cdot\boldsymbol{\nu}(y)\ \phi(y)\ \mathrm{d}\sigma(y) \right| \\
&\le \sup_{z\in\mathfrak{C}_\alpha^\pm(x)} \int_{\partial\Omega} |(\nabla G_k - \nabla G_0)(z-y)|\ |\phi(y)|\ \mathrm{d}\sigma(y) \\
&\le \sqrt{3}C\|\phi\|_1 \le \sqrt{3}C(1+\sigma(\partial\Omega)).
\end{aligned}$$

Then,

$$\begin{aligned}\|\mathfrak{N}_\alpha^\pm(DL_k\phi)\|_2 &\le \|\mathfrak{N}_\alpha^\pm(DL_k\phi)-\mathfrak{N}_\alpha^\pm(DL_0\phi)\|_2+\|\mathfrak{N}_\alpha^\pm(DL_0\phi)\|_2\\ &\le \sqrt{3\sigma(\partial\Omega)}C(1+\sigma(\partial\Omega))+C_\alpha =: \tilde{C}_\alpha.\end{aligned}$$

Therefore, for a general $\phi\in L^2(\partial\Omega)$ we get

$$\|\mathfrak{N}_\alpha^\pm(DL_k\phi)\|_2 \le \tilde{C}_\alpha\|\phi\|_2. \tag{A.4}$$

With a slightly modified argument to the one of the proof of [9, Proposition 3.3.2], it follows that for all $f\in \mathrm{Lip}(\partial\Omega)$, the non-tangential limit $\gamma^\pm DL_k f$ exists and satisfies the right equation of (3.7).

Take now any $\phi\in L^2(\partial\Omega)$ and, using the the density of $\mathrm{Lip}(\partial\Omega)$ in $L^2(\partial\Omega)$, take a sequence $(f_n)_n\subset \mathrm{Lip}(\partial\Omega)$ that converges to ϕ in $L^2(\partial\Omega)$. Then define, for any measurable function ψ on $\Omega_\pm$ and for any $x\in\partial\Omega$ such that $x\in\overline{\mathfrak{C}_\alpha^\pm(x)}$ (which by [9, Proposition 3.3.1], happens for σ-a.e. $x\in\partial\Omega$),

$$\gamma^\pm_{\alpha,\inf}\psi(x) := \liminf_{\substack{y\to x\\ y\in\mathfrak{C}_\alpha^\pm(x)}}\ \psi(x)\quad\text{ and }\quad \gamma^\pm_{\alpha,\sup}\psi(x) := \limsup_{\substack{y\to x\\ y\in\mathfrak{C}_\alpha^\pm(x)}}\ \psi(x),$$

and denote the resulting function on $\partial\Omega$ by $\gamma^\pm_{\alpha,\inf}\psi$ and $\gamma^\pm_{\alpha,\sup}\psi$, respectively. Then, using (A.4)

$$\begin{aligned}\|\gamma^\pm_{\alpha,\inf}DL_k\phi-\gamma^\pm DL_kf_n\|_2 = \|\gamma^\pm_{\alpha,\inf}DL_k\phi-\gamma^\pm_{\alpha,\sup}DL_kf_n\|_2 &\le \|\gamma^\pm_{\alpha,\inf}DL_k(\phi-f_n)\|_2\\ &\le \|\mathfrak{N}_\alpha^\pm DL_k(\phi-f_n)\|_2\le\tilde{C}_\alpha\|(\phi-f_n)\|_2\end{aligned}$$

and

$$\begin{aligned}\|\gamma^\pm_{\alpha,\sup}DL_k\phi-\gamma^\pm DL_kf_n\|_2 = \|\gamma^\pm_{\alpha,\sup}DL_k\phi-\gamma^\pm_{\alpha,\sup}DL_kf_n\|_2 &\le \|\gamma^\pm_{\alpha,\sup}DL_k(\phi-f_n)\|_2\\ &\le \|\mathfrak{N}_\alpha^\pm DL_k(\phi-f_n)\|_2\le\tilde{C}_\alpha\|(\phi-f_n)\|_2.\end{aligned}$$

This implies, by the convergence of $(f_n)_n$ to ϕ in $L^2(\partial\Omega)$, that for any $\alpha>0$ it is satisfied that $\gamma^\pm_{\alpha,\inf}\psi(x)=\gamma^\pm_{\alpha,\sup}\psi(x)$ for σ-a.e. $x\in\partial\Omega$. Hence, for any $\alpha>0$ the limit $\gamma^\pm_\alpha DL\phi(x)$ exists for σ-a.e. $x\in\partial\Omega$. Next, note that for any $x\in\partial\Omega$ and $\alpha>\beta>0$, if $\gamma^\pm_\alpha DL\phi(x)$ exists then $\gamma^\pm_\beta DL\phi(x)$ also exists and is equal to $\gamma^\pm_\alpha DL\phi(x)$. Thus, by taking a sequence of $\alpha_n\to\infty$, we obtain that for σ-a.e. $x\in\partial\Omega$, the non-tangential limit $\gamma^\pm DL\phi(x)$ exists.

Finally, by Lemma B.1, the non-tangential limit $\gamma^\pm DL\phi(x)$ is equal to the classical trace and therefore, by the density of $\mathrm{Lip}(\partial\Omega)$ in $L^2(\partial\Omega)$, the continuity of operator $K_k : L^2(\partial\Omega)\longrightarrow L^2(\partial\Omega)$ and $K_k : H^{1/2}(\partial\Omega)\longrightarrow H^{1/2}(\partial\Omega)$, we obtain that $\gamma^\pm DL\phi(x)$ satisfies the right equation of (3.7) in the non-tangential sense. ☐

Proposition A.5 (Proposition 3.6.2). *For each $\varphi\in H^1(\partial\Omega)$, the non-tangential limit $\gamma^\pm\partial_j DL_k\varphi$ exists σ-a.e. on $\partial\Omega$, for each $j=1,2,3$. Also, $\tilde{C}_\alpha>0$ can be taken such that,*

$$\|\mathfrak{N}_\alpha^\pm(\nabla DL_k\varphi)\|_2\le\tilde{C}_\alpha\|\varphi\|_{H^1(\partial\Omega)}. \tag{A.5}$$

Finally, the restriction of K_k to $H^1(\partial\Omega)$ is bounded as an operator on $H^1(\partial\Omega)$ and we get the mapping property,

$$K_k : H^1(\partial\Omega)\longrightarrow H^1(\partial\Omega).$$

Proof. Adapting the proof of [9, Proposition 3.6.2], take any $x\in\Omega_\pm$ and $j=1,2,3$. Then,

$$\begin{aligned}\partial_jDL_k\varphi(x) &= -\int\limits_{\partial\Omega}\sum_{l=1}^3[\partial_j\partial_lG_k](x-y)\nu_l(y)\varphi(y)\mathrm{d}\sigma(y)\\ &= \int\limits_{\partial\Omega}\varphi(y)\left(k^2G_k(x-y)\nu_j(y)+\sum_{l\ne j}[\partial_l\partial_lG_k](x-y)\nu_j(y)-[\partial_j\partial_lG_k](x-y)\nu_l(y)\right)\mathrm{d}\sigma(y)\\ &= k^2SL_k(\varphi\nu_j)+\sum_{l\ne j}\int_{\partial\Omega}\partial_{\tau_{j,l}}\varphi(y)\,\partial_lG_k(x-y)\,\mathrm{d}\sigma(y),\end{aligned} \tag{A.6}$$

where the second inequality uses the fact that $\Delta G + k^2 G = 0$ on $\mathbb{R}^3 \setminus \{0\}$ and the third uses Lemma A.1. The first term in (A.6) is only weakly singular and can be handled as in Lemma A.3. As for the second term, recalling that the result is known for the case $k = 0$ [16, Lemma 5.7], we are left to prove: (i) the existence of the nontangential limit a.e. on $\partial\Omega$ and (ii) the domination of the L^2-norm of the nontangential maximal function by $C\|\varphi\|_{H^1(\partial\Omega)}$, this time for the quantity

$$\sum_{l \neq j} \int_{\partial\Omega} \partial_{\tau_{j,l}} \varphi(y) \Big(\partial_l G_k(x-y) - \partial_l G_0(x-y) \Big) \, \mathrm{d}\sigma(y).$$

Now, both (i) and (ii) follow by dominated convergence from the second inequality in Lemma A.2. □

Appendix B. Known Statements

In this appendix we state an proof a couple of lemmas that are clearly known but haven't yet found on the literature. Once the proper references are found this proofs will be removed.

Here, we will let λ_3 denote the 3-dimensional Lebesgue measure.

For the following Lemma, we will distinguish the classical taken trace from $\Omega_\pm$, denoted by $\gamma^\pm$, from the non-tangential limit from $\Omega_\pm$, which we will denote for any $u \in H^1_{\mathrm{loc}}(\Omega_\pm)$ and any $x \in \partial\Omega$ for which it is defined by $\gamma^\pm_{NT} u(x)$. Also, if $\gamma^\pm_{NT} u(x)$ exists σ-a.e. $x \in \partial\Omega$, we will denote the corresponding σ-measurable function by $\gamma^\pm_{NT} u$.

Lemma B.1. *For any $u \in H^1_{\mathrm{loc}}(\Omega_\pm)$, if $\gamma^\pm_{NT} u$ is well defined, then $\gamma^\pm_{NT} u(x) = \gamma^\pm u(x)$ for σ-a.e. $x \in \partial\Omega$.*

Proof. Lipschitz domains are extension domains and are non-thin at every boundary point, whereas Sobolev functions are finely continuous at quasi every point. See for instance [1, Section 6.1 and Theorem 6.4.5.]. □

Lemma B.2. *Take, for $l = 1, 2$, $(\phi^l, \psi^l) \in L^2(\Gamma) \times H^{-1}(\Gamma)$ and let $w_l = \mathcal{F}(\phi^l, \psi^l)$. Then,*

$$\langle\!\langle \gamma^\pm w_1, \partial^\pm_{\boldsymbol{\nu}} w_2 \rangle\!\rangle - \langle\!\langle \partial^\pm_{\boldsymbol{\nu}} w_1, \gamma^\pm w_2 \rangle\!\rangle = 0.$$

Proof. We will prove the equality only for the exterior traces, the interior case is analogous. Recall that $(\gamma^- w_l, \partial^-_{\boldsymbol{\nu}} w_l) = P^-(\phi^l, \psi^l)$ and that the Calderón projections are continuous on $L^2(\Gamma) \times H^{-1}(\Gamma)$. Also note that the space $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is dense in $L^2(\Gamma) \times H^{-1}(\Gamma)$ [9, equations (3.6.13) and (3.6.16)], and so, we can assume without loss of generality that $(\phi^l, \psi^l) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

Take a real-valued, non-negative compactly supported smooth function u such that $u \equiv 1$ on a open neighbourhood of $\overline{\Omega_+}$. Hence, by using (3.4), (3.5b), some cancellations, and the definition of locally integrable functions as distributions,

$$\begin{aligned}
\langle\!\langle \gamma^- w_1,\ \partial^-_{\boldsymbol{\nu}} w_2 \rangle\!\rangle - \langle\!\langle \partial^-_{\boldsymbol{\nu}} w_1,\ \gamma^- w_2 \rangle\!\rangle &= \langle\!\langle \gamma^-(w_1 u),\ \partial^-_{\boldsymbol{\nu}}(w_2 u) \rangle\!\rangle - \langle\!\langle \partial^-_{\boldsymbol{\nu}}(w_1 u),\ \gamma^-(w_2 u) \rangle\!\rangle \\
&= \Big\langle\!\!\Big\langle \Delta(w_1 u) + k^2 w_1 u,\ w_2 u \Big\rangle\!\!\Big\rangle_{L^2(\Omega_-)} - \Big\langle\!\!\Big\langle w_1 u,\ \Delta(w_2 u) + k^2 w_2 u \Big\rangle\!\!\Big\rangle_{L^2(\Omega_-)} \\
&= \int_{\Omega_-} (u \nabla u) \cdot (w_2 \nabla \overline{w_1} - \overline{w_1} \nabla w_2) \, \mathrm{d}\lambda_3 \\
&= \langle w_2 \nabla \overline{w_1} - \overline{w_1} \nabla w_2,\ \nabla(u^2/2) \rangle \\
&= -\langle w_2 \Delta \overline{w_1} - \overline{w_1} \Delta w_2,\ u^2/2 \rangle \\
&= -\langle w_2(-k^2 \overline{w_1}) - \overline{w_1}(-k^2 w_2),\ u^2/2 \rangle = 0.
\end{aligned}$$

□

Lemma B.3. *Let $\Omega_+ \subset \mathbb{R}^3$ be a bounded Lipschitz domain, take $(\phi, \psi) \in L^2(\partial\Omega) \times H^{-1}(\partial\Omega)$ and let $u = \widetilde{\mathcal{F}}(\phi, \psi)$. If $\gamma^+ u \in H^1(\partial\Omega)$ then $u \in H^{3/2}(\Omega_+)$, and if $\gamma^- u \in H^1(\partial\Omega)$ then $u \in H^{3/2}_{\mathrm{loc}}(\Omega_-)$.*

Proof. We will only prove the statement for the exterior case since the proof for the interior one is analogous and simpler. Let $\mathbb{B} \subset \mathbb{R}^3$ be an open ball around 0 that contains $\overline{\Omega_+}$ and let $u' = u|_{\mathbb{B} \setminus \overline{\Omega_+}}$ which is integrable by remark 4. By [10, Theorem B], there is a $u \in H^{3/2}(\mathbb{B} \setminus \overline{\Omega_+})$ such that $\Delta u = -k^2 u'$ and $\gamma_{\mathbb{B} \setminus \overline{\Omega_+}} u = 0$.

Note that $\gamma_{\mathbb{B}\setminus\overline{\Omega_+}}u' \in H^1(\partial(\mathbb{B}\setminus\overline{\Omega_+}))$ since $\gamma^- u \in H^1(\partial\Omega)$ and u is analytic on Ω_-. Now, by [11, Remark (b)], there is a harmonic function $v \in H^{3/2}(\mathbb{B}\setminus\overline{\Omega_+})$ with $\gamma_{\mathbb{B}\setminus\overline{\Omega_+}}v = \gamma_{\mathbb{B}\setminus\overline{\Omega_+}}u' \in H^1(\partial(\mathbb{B}\setminus\overline{\Omega_+}))$. As $u' - u - v$ is harmonic with $\gamma_{\mathbb{B}\setminus\overline{\Omega_+}}(u' - u - v) = 0$, the Strong Maximum principle implies that $u' - u - v = 0$ and thus $u' = u + v \in H^{3/2}(\mathbb{B}\setminus\overline{\Omega_+})$. Finally, since $\mathbb{B}$ is any open ball containing $\overline{\Omega_+}$, we get that $u \in H^{3/2}_{\mathrm{loc}}(\Omega_-)$. □

Lemma B.4. *Let $\Gamma \subset \mathbb{R}^3$ be the boundary of a bounded Lipschitz domain and let $\{\Gamma_j\}_{j\in J}$ be its connected components. If $\psi \in H^{-1}(\Gamma)$ is such that for every $j \in J$, $\langle\psi, 1_{\Gamma_j}\rangle = 0$, then there exists a $\varphi_\psi \in H^1(\Gamma)$ such that $\Delta_{\mathrm{T}}\varphi_\psi = \psi$.*

Proof. Let Z denote the space $\{\varphi \in H^1(\Gamma) : \text{for every } j \in J,\ \langle\varphi, 1_{\Gamma_j}\rangle = 0\}$ together with the inner product $\langle\!\langle\varphi,\tilde\varphi\rangle\!\rangle_Z := \langle\!\langle\nabla_{\mathrm{T}}\varphi, \nabla_{\mathrm{T}}\tilde\varphi\rangle\!\rangle_{L^2(\Gamma)^3}$. By Poincaré inequality it follows that Z is a Hilbert space. Take a $\psi \in H^{-1}(\Gamma)$ such that for every $j \in J$, $\langle\psi, 1_{\Gamma_j}\rangle = 0$. By using Poincaré inequality again, the function $\varphi \mapsto -\langle\overline{\psi}, \varphi\rangle$ belongs to the dual of Z. Thus there exists a $\varphi_\psi \in Z$ such that, for every $\varphi \in Z$, $\langle\psi,\varphi\rangle = -\langle\!\langle\overline{\varphi_\psi}, \varphi\rangle\!\rangle_Z$. Take now any $\varphi \in H^1(\Gamma)$ and let, for any $j \in J$, $\alpha_j = \sigma(\Gamma_j)^{-1}\langle\varphi, 1_{\Gamma_j}\rangle$. Then,

$$\begin{aligned}\langle\psi,\varphi\rangle &= \left\langle\psi\ ,\ \varphi - \sum_{j\in J}\alpha_j 1_{\Gamma_j} + \sum_{j\in J}\alpha_j 1_{\Gamma_j}\right\rangle = \left\langle\psi\ ,\ \varphi - \sum_{j\in J}\alpha_j 1_{\Gamma_j}\right\rangle = -\left\langle\!\!\left\langle\overline{\varphi_\psi}\ ,\ \varphi - \sum_{j\in J}\alpha_j 1_{\Gamma_j}\right\rangle\!\!\right\rangle_Z \\ &= -\left\langle\nabla_{\mathrm{T}}\varphi_\psi\ ,\ \nabla_{\mathrm{T}}\left(\varphi - \sum_{j\in J}\alpha_j 1_{\Gamma_j}\right)\right\rangle_{L^2(\Gamma)^3} = -\langle\nabla_{\mathrm{T}}\varphi_\psi, \nabla_{\mathrm{T}}\varphi\rangle_{L^2(\Gamma)^3} = \langle\Delta_{\mathrm{T}}\varphi_\psi, \varphi\rangle\end{aligned}$$

and hence $\Delta_{\mathrm{T}}\varphi_\psi = \psi$. □

The following lemma is a direct consequence of *Alexander duality* [2], and was proven by J. W. Alexander on the final remarks of that same paper.

Lemma B.5. *Take a connected hypersurface $\Gamma \subset \mathbb{R}^{n+1}$ which is compact as a topological space.*

Then the set $\mathbb{R}^{n+1}\setminus\Gamma$ has two connected components; one bounded, which we will denote by $\mathrm{Int}(\Gamma)$, and another unbounded, which we will denote by $\mathrm{Ext}(\Gamma)$.

Furthermore, $\partial(\mathrm{Int}(\Gamma)) = \Gamma = \partial(\mathrm{Ext}(\Gamma))$.

We say that a set $\Gamma \subset \mathbb{R}^{n+1}$ is *locally a Lipschitz graph* if for every $x \in \Gamma$ there exists an open ball $\mathbb{B} \subset \mathbb{R}^{n+1}$, a $h > 0$, a hyperplane $H \subset \mathbb{R}^{n+1}$ passing through s and with a normal unit vector $\boldsymbol{\nu}$, and a real-valued Lipschitz continuous function g on H such that the set defined as

$$C := \{x + t\boldsymbol{\nu}\ :\ x \in \mathbb{B}\cap H,\ -h < t < h\},$$

satisfies:

$$C \cap \Gamma = \{x + t\boldsymbol{\nu}\ :\ x \in \mathbb{B}\cap H,\ t = g(x)\}.$$

Lemma B.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded Lipschitz domain. Then, $\partial\Omega$ has finitely many connected components, say $\Gamma_1, ..., \Gamma_l$, each of which is locally a Lipschitz graph in $\mathbb{R}^{n+1}$.*

Moreover, the connected components of $\mathbb{R}^{n+1}\setminus\overline{\Omega}$ consist of l Lipschitz domains $O_1, ..., O_l$, and with a suitable ordering $O_1 = \mathrm{Ext}(\Gamma_1)$ while $O_j = \mathrm{Int}(\Gamma_j)$ for $j \neq 1$.

Proof. The connected components $\Omega \subset \mathbb{R}^{n+1}$ are finite in number; otherwise indeed, there would exist a sequence Ω_k of such components, with $\Omega_k \cap \Omega_j = \{\}$ for $k \neq j$. Then, we could construct a sequence $x_k \in \Omega_k$ such that x_k remains at bounded distance from $\partial\Omega_k \subset \partial\Omega$, hence x_k would be bounded and extracting a subsequence if necessary we might assume that x_k converges in $\mathbb{R}^{n+1}$ to some y. However, this is impossible for y cannot lie in Ω since the connected components of the latter are open, nor can it lie in $\mathbb{R}^{n+1}\setminus\overline{\Omega}$, and it cannot belong to $\partial\Omega$ either because, as $\partial\Omega$ is a compact Lipschitz manifold which is locally a Lipschitz graph, each $x \in \partial\Omega$ has a neighborhood whose intersections with both Ω and $\partial\Omega$ are connected. Consequently, by compactness, $\partial\Omega$ has finitely many connected components, say $\Gamma_1, ..., \Gamma_l$, and each Γ_j is locally a Lipschitz graph in $\mathbb{R}^{n+1}$.

As Ω is connected by assumption, for each $j \in \{1,...,l\}$ one of the following is true; either $\Omega \subset \mathrm{Int}(\Gamma_j)$, so that $\overline{\Omega} \subset \overline{\mathrm{Int}(\Gamma_j)}$ and then, using Lemma B.5, $\mathrm{Ext}(\Gamma_j) \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}$; or else $\Omega \subset \mathrm{Ext}(\Gamma_j)$ and then, analogously, $\mathrm{Int}(\Gamma_j) \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}$. Since there is exactly one unbounded connected component of $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, say O_1, it must contain $\mathrm{Ext}(\Gamma_j)$ for all j such that $\Omega \subset \mathrm{Int}(\Gamma_j)$; let us enumerate these j as $j_1,...,j_m$. For $1 \le i,k \le m$, it holds that $\mathrm{Int}(\Gamma_{j_i}) \cap \mathrm{Int}(\Gamma_{j_k}) \neq \varnothing$ because Ω lies in this intersection, and since the Γ_j are disjoint one of these interiors is included in the other, say $\mathrm{Int}(\Gamma_{j_i}) \subset \mathrm{Int}(\Gamma_{j_k})$. But if $j_i \neq j_k$, then $\Gamma_{j_k} \subset \mathrm{Ext}(\Gamma_{j_i})$ and the latter is contained in O_1, a contradiction. Consequently, $m = 1$ and Ω lies interior to exactly one of the Γ_j, say Γ_1. Necessarily then, $O_1 = \mathrm{Ext}(\Gamma_1)$ because O_1 cannot strictly contain $\mathrm{Ext}(\Gamma_1)$ without containing a point of Γ_1, which is impossible. Likewise, $\Omega \subset \mathrm{Ext}(\Gamma_j)$ for $j \neq 1$ and then $\mathrm{Int}(\Gamma_j)$ is a connected component of $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Next, the closure of every bounded connected component of $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ must meet some Γ_j, and necessarily $j \neq 1$ for each point of Γ_1 has a neighborhood included in $\overline{O_1} \cup \Omega$, by the local Lipschitz graph property. Hence, this connected component meets $\mathrm{Int}(\Gamma_j)$ for some $j \neq 1$, therefore it must coincide with $\mathrm{Int}(\Gamma_j)$. Finally, due to Lemma B.5 and the definition of locally Lipschitz graphs, for each $j \in \{1,...,l\}$ both $\mathrm{Int}(\Gamma_j)$ and $\mathrm{Ext}(\Gamma_j)$ are Lipschitz domains. □

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