Compressible flows: important definitions

Cristopher Morales Ubal

c.m.ubal@gmail.com

January 20, 2025

1 Definitions

1. Total Pressure:

$$p_{\text{total}} = p \cdot \left(1 + \frac{\gamma}{2} \cdot \text{Ma}^2\right) \tag{1}$$

2. Total Temperature:

$$T_{\text{total}} = T \cdot \left(1 + \frac{(\gamma - 1)}{2} \cdot \text{Ma}^2\right) \tag{2}$$

3. Mach Number:

$$\text{Ma} = \frac{\|\vec{u}\|}{c} \tag{3}$$

4. Speed of sound:

$$c = \sqrt{\gamma R T} \tag{4}$$

5. Ratio of specific heats:

$$\gamma = \frac{c_p}{c_v} \tag{5}$$

6. Entropy of a mixture:

$$s = \sum_{i=1}^{N_s} \left[s_i^0 + \int_{T_0}^{T} \frac{c_{p,i}}{T} \mathrm{dT} \right] Y_i - \frac{R}{M} \ln\left(\frac{p}{p_0}\right) \tag{6}$$

7. Total chemical energy of a mixture:

$$e = \int_{T_0}^{T} c_v \mathrm{dT} - R T_0 + \sum_{i=1}^{N_s} h_{f,i}^0 Y_i + \frac{1}{2} \vec{u} \cdot \vec{u} \tag{7}$$

8. Static energy of a mixture:

$$e_s = \int_{T_0}^{T} c_v \mathrm{dT} - R T_0 = \int_{T_0}^{T} c_p \mathrm{dT} - R T \tag{8}$$

2 Energy equation for multi-component flows

For compressible multicomponent flows, we have the following energy equation: For reacting flows, we have the energy equation for the total non-chemical energy E [1] given by:

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho \vec{u} H) = \dot{\omega}_T + \nabla \cdot (\kappa \nabla T) - \nabla \cdot \left(\rho \sum_{i=1}^{N} h_{s,i} Y_i \vec{V}_i \right) + \nabla \cdot (\boldsymbol{\tau} \cdot \vec{u}) \tag{9}$$

where H is the total non-chemical enthalpy:

$$H = E + \frac{p}{\rho} \tag{10}$$

and E is the total non-chemical energy:

$$E = e_s + \frac{1}{2}\vec{u} \cdot \vec{u} = \int_{T_0}^{T} c_p \mathrm{dT} - RT + \frac{1}{2}\vec{u} \cdot \vec{u} \tag{11}$$

The conservative variable for compressible solvers are usually:

$$\vec{U} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \tag{12}$$

then, the working variables $\vec{V}$ can be written in terms of the conservative variables as follows:

$$\vec{V} = \begin{pmatrix} \rho \\ u \\ v \\ w \\ E \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2/U_1 \\ U_3/U_1 \\ U_4/U_1 \\ U_5/U_1 \end{pmatrix} \tag{13}$$

3 Heat flux jacobian

For implicit solver, the heat flux Jacobians computations are needed. Following the approach explained in [2], we can compute the heat flux jacobian as follows:

$$\frac{\partial \overline{F}^v \left(\vec{U}, \nabla \vec{U} \right)}{\partial \vec{U}} = \frac{\partial \overline{F}^v (\vec{V}, \nabla \vec{V})}{\partial \vec{V}} \cdot \frac{\partial \vec{V}}{\partial \vec{U}} \tag{14}$$

For the heat flux in the energy equation (9), we have $\vec{U} = (U_1, U_2, U_3, U_4, U_5)^T$, and $V = (T, \rho)$. From (11), we can rewrite it in terms of the conservative variables as follows:

$$\int_{T_0}^{T} c_p \mathrm{dT} - RT = \frac{U_5}{U_1} - \frac{1}{2U_1^2} \left(U_2^2 + U_3^2 + U_4^2 \right) \tag{15}$$

Then, the temperature is implicitly defined as a function of the conservative variables $\vec{U}$. Hence, we can compute the partial derivative of T with respect to the conservative variables as follows:

$$\frac{\partial T}{\partial U_1} = \frac{1}{c_v}\left(-\frac{U_5}{U_1^2} + \frac{\left(U_2^2 + U_3^2 + U_4^2\right)}{U_1^3}\right) \tag{16}$$

$$\frac{\partial T}{\partial U_2} = \frac{1}{c_v} \cdot -\frac{U_2}{U_1^2} \tag{17}$$

$$\frac{\partial T}{\partial U_3} = \frac{1}{c_v} \cdot -\frac{U_3}{U_1^2} \tag{18}$$

$$\frac{\partial T}{\partial U_4} = \frac{1}{c_v} \cdot -\frac{U_4}{U_1^2} \tag{19}$$

$$\frac{\partial T}{\partial U_4} = \frac{1}{c_v} \cdot \frac{1}{U_1} \tag{20}$$

Likewise, we have:

$$\frac{\partial \rho}{\partial U_1} = 1\ ,\ \frac{\partial \rho}{\partial U_i} = 0\ ,\ \forall i = 2, 3, 4, 5 \tag{21}$$

References

[1] T. Poinsot **and** D. Veynante. *Theoretical and Numerical Combustion*. Edwards, 2005. ISBN: 9781930217102. URL: `https://books.google.nl/books?id=cqFDkeVABYoC`.

[2] Enrico Rinaldi, Rene Pecnik **and** Piero Colonna. "Exact Jacobians for implicit Navier–Stokes simulations of equilibrium real gas flows". **in***Journal of Computational Physics*: 270 (2014), **pages** 459–477. ISSN: 0021-9991. DOI: `https://doi.org/10.1016/j.jcp.2014.03.058`. URL: `https://www.sciencedirect.com/science/article/pii/S0021999114002459`.