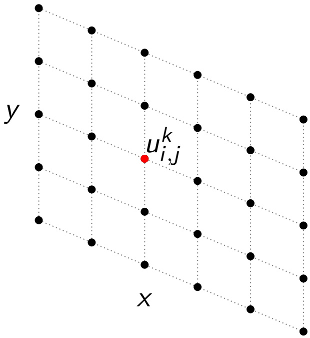
Discretize the Domain



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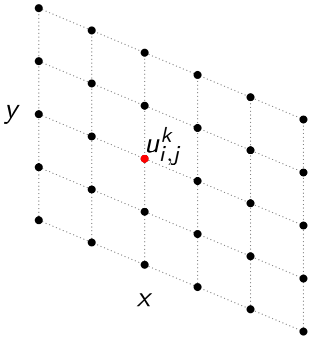
- $u_{i,j}^k$ represents the concentration $u(x_i, y_j, t_k)$



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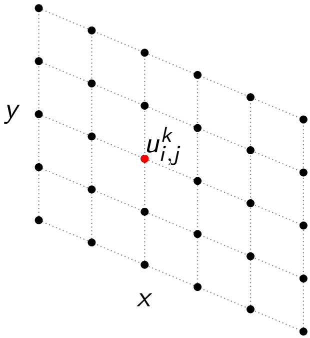

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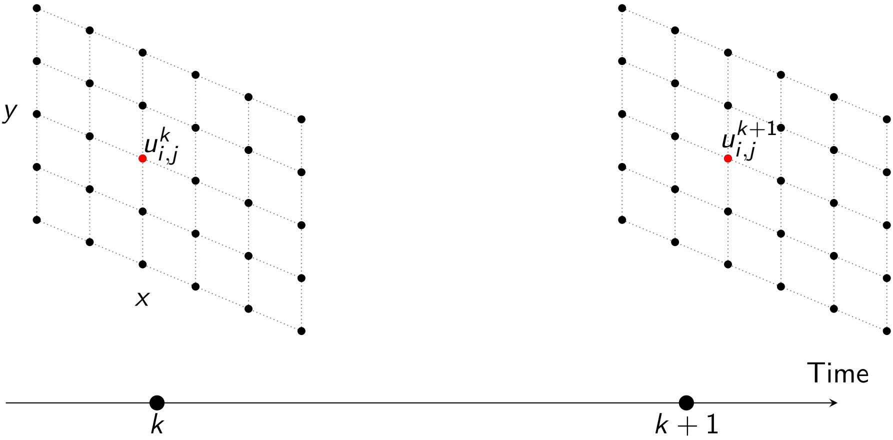

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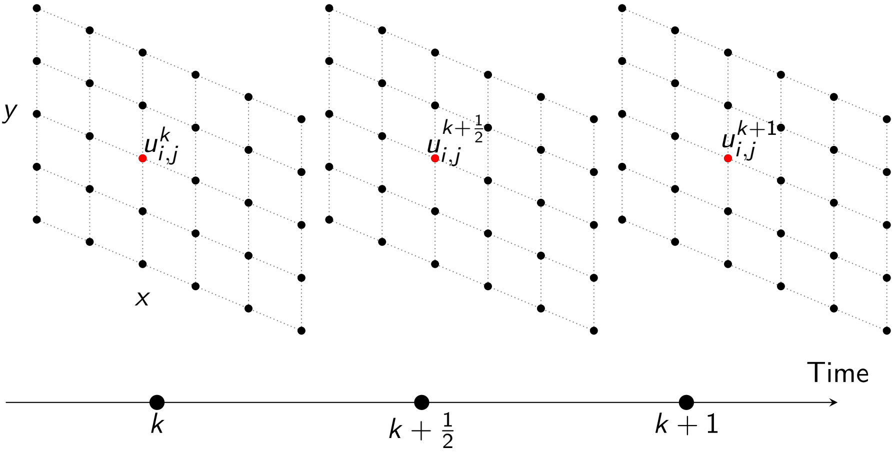

Simulationg point-like sinks and sources

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \sum_{b=1}^{N} K_{out}(c_b)\delta(r_b - r) - K_{in}(u)\sum_{b=1}^{N}\delta(r_b - r)$$

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We can discretize the first half-step (x implicit, y explicit) using the following scheme:

$$\frac{u_{i,j}^{k+\frac{1}{2}} - u_{i,j}^{k}}{\Delta t/2} = D\left(\frac{u_{i+1,j}^{k+\frac{1}{2}} - 2u_{i,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k}}{\Delta y^2}\right) + S^k$$

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And the second half-step (x explicit, y implicit) using the following scheme:

$$\frac{u_{i,j}^{k+1} - u_{i,j}^{k+\frac{1}{2}}}{\Delta t/2} = D\left(\frac{u_{i+1,j}^{k+\frac{1}{2}} - 2u_{i,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{k+1} - 2u_{i,j}^{k+1} + u_{i,j-1}^{k+1}}{\Delta y^2}\right) + S^{k+\frac{1}{2}}$$

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Where S is the sources and sinks term:

$$S^k = \sum_{b=1}^{N} K_{out}(c_b)\delta_\varepsilon(r_b - r_{i,j}) - K_{in}(u_{i,j}^k)\sum_{b=1}^{N}\delta_\varepsilon(r_b - r_{i,j})$$

Let's start with the first half-step:

$$\frac{u_{i,j}^{k+\frac{1}{2}} - u_{i,j}^{k}}{\Delta t/2} = D\left(\frac{u_{i+1,j}^{k+\frac{1}{2}} - 2u_{i,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k}}{\Delta y^2}\right) + S^k$$

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To simplify, we notice that $\Delta x = \Delta y$, and we can define $\alpha = \frac{D\Delta t}{2\Delta x^2}$ to get:

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Now we can rearrange the equation to isolate the unknown terms $u_{i,j}^{k+\frac{1}{2}}$ on the left-hand side:

$$\boxed{-\alpha u_{i-1,j}^{k+\frac{1}{2}} + (1+2\alpha)u_{i,j}^{k+\frac{1}{2}} - \alpha u_{i+1,j}^{k+\frac{1}{2}} = \alpha u_{i,j-1}^{k} + (1-2\alpha)u_{i,j}^{k} + \alpha u_{i,j+1}^{k} + \frac{\Delta t}{2}S^k}$$

We know all the terms in right-hand side of the equation (blue), but we need to compute the left-hand side (red).

First half-step (x implicit, y explicit)

$$\boxed{-\alpha u_{i-1,j}^{k+\frac{1}{2}} + (1+2\alpha) u_{i,j}^{k+\frac{1}{2}} - \alpha u_{i+1,j}^{k+\frac{1}{2}} = \alpha u_{i,j-1}^{k} + (1-2\alpha) u_{i,j}^{k} + \alpha u_{i,j+1}^{k} + \frac{\Delta t}{2} S^k}$$

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Let's look around $i = 3$ and $j = 2$:

First half-step (x implicit, y explicit)

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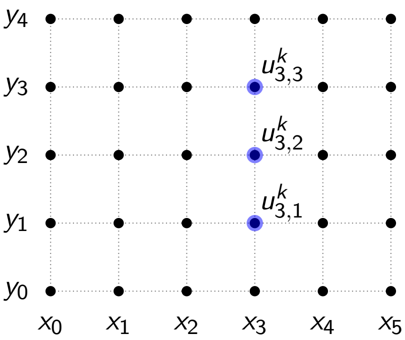

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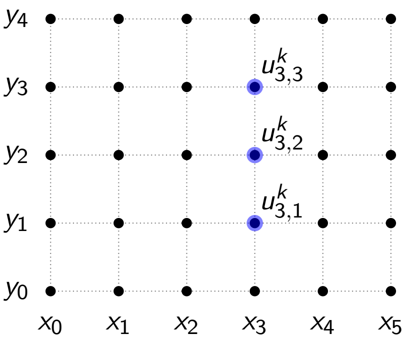
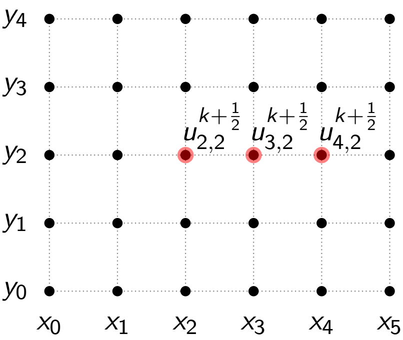
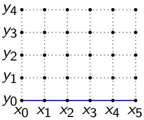




Let's fix $j = 0$ and write the equations for $0 \leq i \leq 5$.

k



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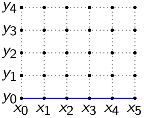
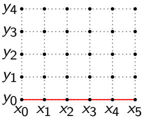


$k + \frac{1}{2}$



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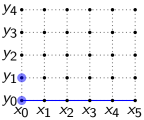

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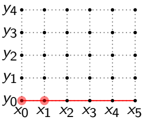

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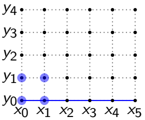

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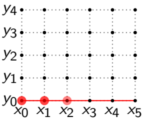

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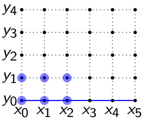

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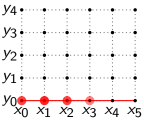

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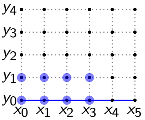

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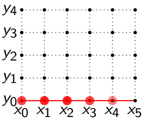

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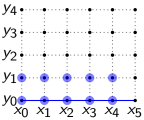
k



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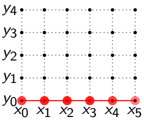

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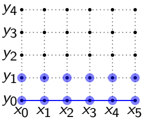
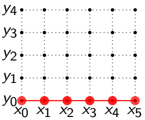
k



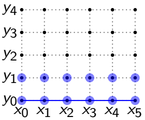
$k + \frac{1}{2}$



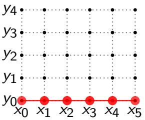
Let's fix $j = 0$ and write the equations for $0 \leq i \leq 5$.

$$-\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha)u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} = \alpha u_{1,-1}^{k} + (1-2\alpha)u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{2,-1}^{k} + (1-2\alpha)u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{2,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,0}^{k+\frac{1}{2}} - \alpha u_{4,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha)u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{3,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,0}^{k+\frac{1}{2}} - \alpha u_{5,0}^{k+\frac{1}{2}} = \alpha u_{4,-1}^{k} + (1-2\alpha)u_{4,0}^{k} + \alpha u_{4,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{4,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,0}^{k+\frac{1}{2}} - \alpha u_{6,0}^{k+\frac{1}{2}} = \alpha u_{5,-1}^{k} + (1-2\alpha)u_{5,0}^{k} + \alpha u_{5,1}^{k} + \frac{\Delta t}{2} S^k$$

k



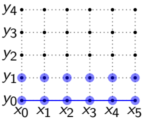
$k + \frac{1}{2}$



Let's fix $j = 0$ and write the equations for $0 \le i \le 5$.

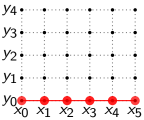
$$
\begin{aligned}
-\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} &= \alpha u_{0,-1}^{k} + (1-2\alpha)u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} &= \alpha u_{1,-1}^{k} + (1-2\alpha)u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} &= \alpha u_{2,-1}^{k} + (1-2\alpha)u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{2,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,0}^{k+\frac{1}{2}} - \alpha u_{4,0}^{k+\frac{1}{2}} &= \alpha u_{3,-1}^{k} + (1-2\alpha)u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{3,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,0}^{k+\frac{1}{2}} - \alpha u_{5,0}^{k+\frac{1}{2}} &= \alpha u_{4,-1}^{k} + (1-2\alpha)u_{4,0}^{k} + \alpha u_{4,1}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{4,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,0}^{k+\frac{1}{2}} - \alpha u_{6,0}^{k+\frac{1}{2}} &= \alpha u_{5,-1}^{k} + (1-2\alpha)u_{5,0}^{k} + \alpha u_{5,1}^{k} + \frac{\Delta t}{2} S^k
\end{aligned}
$$

k



Which can be written as a matrix:

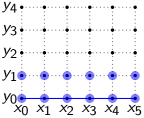
$k + \frac{1}{2}$



Let's fix $j = 0$ and write the equations for $0 \le i \le 5$.

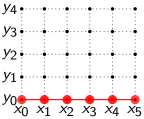
$$-\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha)u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} = \alpha u_{1,-1}^{k} + (1-2\alpha)u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{2,-1}^{k} + (1-2\alpha)u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{2,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,0}^{k+\frac{1}{2}} - \alpha u_{4,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha)u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{3,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,0}^{k+\frac{1}{2}} - \alpha u_{5,0}^{k+\frac{1}{2}} = \alpha u_{4,-1}^{k} + (1-2\alpha)u_{4,0}^{k} + \alpha u_{4,1}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{4,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,0}^{k+\frac{1}{2}} - \alpha u_{6,0}^{k+\frac{1}{2}} = \alpha u_{5,-1}^{k} + (1-2\alpha)u_{5,0}^{k} + \alpha u_{5,1}^{k} + \frac{\Delta t}{2} S^k$$

k



Which can be written as a matrix:

$k + \frac{1}{2}$



$$\begin{bmatrix} -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \end{bmatrix} \begin{bmatrix} u_{-1,0}^{k+\frac{1}{2}} \\ u_{0,0}^{k+\frac{1}{2}} \\ u_{1,0}^{k+\frac{1}{2}} \\ u_{2,0}^{k+\frac{1}{2}} \\ u_{3,0}^{k+\frac{1}{2}} \\ u_{4,0}^{k+\frac{1}{2}} \\ u_{5,0}^{k+\frac{1}{2}} \\ u_{6,0}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \\ b_{4,0} \\ b_{5,0} \end{bmatrix}$$

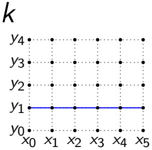
$$
\begin{bmatrix}
-\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha
\end{bmatrix}
\begin{bmatrix}
u_{-1,0}^{k+\frac{1}{2}} \\
u_{0,0}^{k+\frac{1}{2}} \\
u_{1,0}^{k+\frac{1}{2}} \\
u_{2,0}^{k+\frac{1}{2}} \\
u_{3,0}^{k+\frac{1}{2}} \\
u_{4,0}^{k+\frac{1}{2}} \\
u_{5,0}^{k+\frac{1}{2}} \\
u_{6,0}^{k+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
b_{0,0} \\
b_{1,0} \\
b_{2,0} \\
b_{3,0} \\
b_{4,0} \\
b_{5,0}
\end{bmatrix}
$$

$$
\begin{bmatrix}
-\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha
\end{bmatrix}
\begin{bmatrix}
u_{-1,0}^{k+\frac{1}{2}} \\
u_{0,0}^{k+\frac{1}{2}} \\
u_{1,0}^{k+\frac{1}{2}} \\
u_{2,0}^{k+\frac{1}{2}} \\
u_{3,0}^{k+\frac{1}{2}} \\
u_{4,0}^{k+\frac{1}{2}} \\
u_{5,0}^{k+\frac{1}{2}} \\
u_{6,0}^{k+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \\ b_{4,0} \\ b_{5,0}
\end{bmatrix}
$$

We know the values on the boundaries, so we don't need to include them in the matrix.

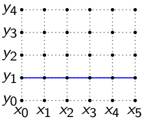
$$
\begin{bmatrix}
1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\
-\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\
0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\
0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\
0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\
0 & 0 & 0 & 0 & -\alpha & 1+\alpha
\end{bmatrix}
\begin{bmatrix}
u_{0,0}^{k+\frac{1}{2}} \\
u_{1,0}^{k+\frac{1}{2}} \\
u_{2,0}^{k+\frac{1}{2}} \\
u_{3,0}^{k+\frac{1}{2}} \\
u_{4,0}^{k+\frac{1}{2}} \\
u_{5,0}^{k+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \\ b_{4,0} \\ b_{5,0}
\end{bmatrix}
$$

Let's fix $j = 1$ and write the equations for $0 \leq i \leq 5$.

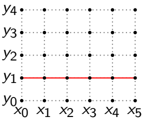


Let's fix $j = 1$ and write the equations for $0 \le i \le 5$.

k



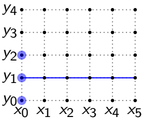
$k + \frac{1}{2}$



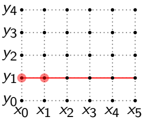
Let's fix $j = 1$ and write the equations for $0 \le i \le 5$.

$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k$$

k



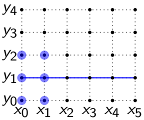
$k + \frac{1}{2}$



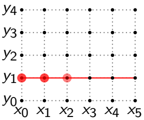
Let's fix $j = 1$ and write the equations for $0 \le i \le 5$.

$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha) u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha) u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k$$

k



$k + \frac{1}{2}$



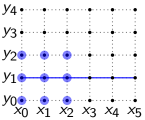
Let's fix $j = 1$ and write the equations for $0 \leq i \leq 5$.

$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k$$

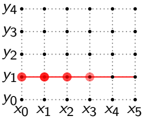
$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k$$

$$-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^k$$

k



$k + \frac{1}{2}$



Let's fix $j = 1$ and write the equations for $0 \le i \le 5$.

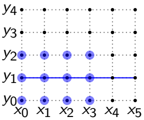
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k$$

$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k$$

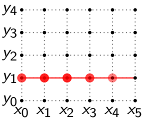
$$-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^k$$

$$-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2} S^k$$

k



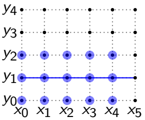
$k + \frac{1}{2}$



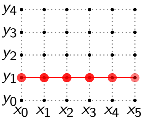
Let's fix $j = 1$ and write the equations for $0 \le i \le 5$.

$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} = \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2} S^k$$

k



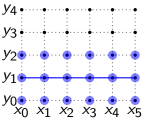
$k + \frac{1}{2}$



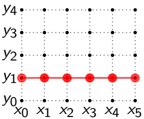
Let's fix $j = 1$ and write the equations for $0 \leq i \leq 5$.

$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} = \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2} S^k$$
$$-\alpha u_{4,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,1}^{k+\frac{1}{2}} - \alpha u_{6,1}^{k+\frac{1}{2}} = \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2} S^k$$

k



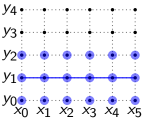
$k + \frac{1}{2}$



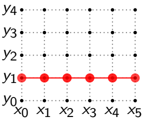
Let's fix $j = 1$ and write the equations for $0 \le i \le 5$.

$$
\begin{aligned}
-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} &= \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} &= \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} &= \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} &= \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} &= \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{4,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,1}^{k+\frac{1}{2}} - \alpha u_{6,1}^{k+\frac{1}{2}} &= \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2} S^k
\end{aligned}
$$

k



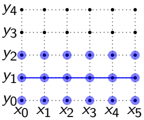
$k + \frac{1}{2}$



Let's fix $j = 1$ and write the equations for $0 \leq i \leq 5$.

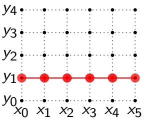
$$
\begin{aligned}
-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} &= \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} &= \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} &= \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} &= \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} &= \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{4,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,1}^{k+\frac{1}{2}} - \alpha u_{6,1}^{k+\frac{1}{2}} &= \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2} S^k
\end{aligned}
$$

k



Which can be written as a matrix:

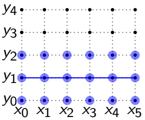
$k + \frac{1}{2}$



Let's fix $j = 1$ and write the equations for $0 \leq i \leq 5$.

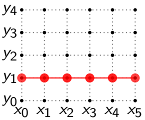
$$
\begin{aligned}
-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} &= \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} &= \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} &= \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} &= \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} &= \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2} S^k \\
-\alpha u_{4,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,1}^{k+\frac{1}{2}} - \alpha u_{6,1}^{k+\frac{1}{2}} &= \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2} S^k
\end{aligned}
$$

k



Which can be written as a matrix:

$k + \frac{1}{2}$



$$
\begin{bmatrix}
-\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha
\end{bmatrix}
\begin{bmatrix}
u_{-1,1}^{k+\frac{1}{2}} \\ u_{0,1}^{k+\frac{1}{2}} \\ u_{1,1}^{k+\frac{1}{2}} \\ u_{2,1}^{k+\frac{1}{2}} \\ u_{3,1}^{k+\frac{1}{2}} \\ u_{4,1}^{k+\frac{1}{2}} \\ u_{5,1}^{k+\frac{1}{2}} \\ u_{6,1}^{k+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \\ b_{5,1}
\end{bmatrix}
$$

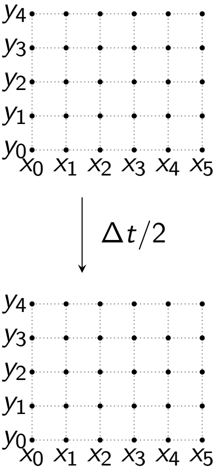
$$\begin{bmatrix} -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \end{bmatrix} \begin{bmatrix} u_{-1,1}^{k+\frac{1}{2}} \\ u_{0,1}^{k+\frac{1}{2}} \\ u_{1,1}^{k+\frac{1}{2}} \\ u_{2,1}^{k+\frac{1}{2}} \\ u_{3,1}^{k+\frac{1}{2}} \\ u_{4,1}^{k+\frac{1}{2}} \\ u_{5,1}^{k+\frac{1}{2}} \\ u_{6,1}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \\ b_{5,1} \end{bmatrix}$$

$$
\begin{bmatrix}
-\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha
\end{bmatrix}
\begin{bmatrix}
u_{-1,1}^{k+\frac{1}{2}} \\
u_{0,1}^{k+\frac{1}{2}} \\
u_{1,1}^{k+\frac{1}{2}} \\
u_{2,1}^{k+\frac{1}{2}} \\
u_{3,1}^{k+\frac{1}{2}} \\
u_{4,1}^{k+\frac{1}{2}} \\
u_{5,1}^{k+\frac{1}{2}} \\
u_{6,1}^{k+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \\ b_{5,1}
\end{bmatrix}
$$

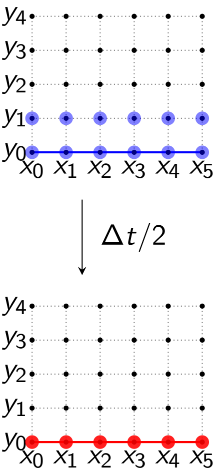
We know the values on the boundaries, so we don't need to include them in the matrix.

$$
\begin{bmatrix}
1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\
-\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\
0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\
0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\
0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\
0 & 0 & 0 & 0 & -\alpha & 1+\alpha
\end{bmatrix}
\begin{bmatrix}
u_{0,1}^{k+\frac{1}{2}} \\
u_{1,1}^{k+\frac{1}{2}} \\
u_{2,1}^{k+\frac{1}{2}} \\
u_{3,1}^{k+\frac{1}{2}} \\
u_{4,1}^{k+\frac{1}{2}} \\
u_{5,1}^{k+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \\ b_{5,1}
\end{bmatrix}
$$

We get one system of equations for each j value.

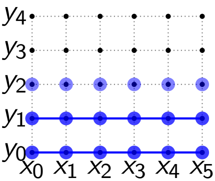


We get one system of equations for each j value.



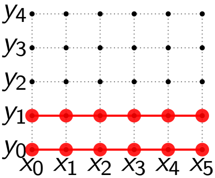
$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{0,0}^{k+\frac{1}{2}} \\ u_{1,0}^{k+\frac{1}{2}} \\ u_{2,0}^{k+\frac{1}{2}} \\ u_{3,0}^{k+\frac{1}{2}} \\ u_{4,0}^{k+\frac{1}{2}} \\ u_{5,0}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \\ b_{4,0} \\ b_{5,0} \end{bmatrix}$$

We get one system of equations for each j value.



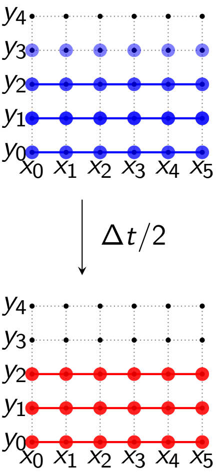
$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{0,0}^{k+\frac{1}{2}} \\ u_{1,0}^{k+\frac{1}{2}} \\ u_{2,0}^{k+\frac{1}{2}} \\ u_{3,0}^{k+\frac{1}{2}} \\ u_{4,0}^{k+\frac{1}{2}} \\ u_{5,0}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \\ b_{4,0} \\ b_{5,0} \end{bmatrix}$$

$$\Big\downarrow \Delta t/2$$



$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{0,1}^{k+\frac{1}{2}} \\ u_{1,1}^{k+\frac{1}{2}} \\ u_{2,1}^{k+\frac{1}{2}} \\ u_{3,1}^{k+\frac{1}{2}} \\ u_{4,1}^{k+\frac{1}{2}} \\ u_{5,1}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \\ b_{5,1} \end{bmatrix}$$

We get one system of equations for each j value.



$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{0,0}^{k+\frac{1}{2}} \\ u_{1,0}^{k+\frac{1}{2}} \\ u_{2,0}^{k+\frac{1}{2}} \\ u_{3,0}^{k+\frac{1}{2}} \\ u_{4,0}^{k+\frac{1}{2}} \\ u_{5,0}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \\ b_{4,0} \\ b_{5,0} \end{bmatrix}$$

$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{0,1}^{k+\frac{1}{2}} \\ u_{1,1}^{k+\frac{1}{2}} \\ u_{2,1}^{k+\frac{1}{2}} \\ u_{3,1}^{k+\frac{1}{2}} \\ u_{4,1}^{k+\frac{1}{2}} \\ u_{5,1}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \\ b_{5,1} \end{bmatrix}$$

$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{0,2}^{k+\frac{1}{2}} \\ u_{1,2}^{k+\frac{1}{2}} \\ u_{2,2}^{k+\frac{1}{2}} \\ u_{3,2}^{k+\frac{1}{2}} \\ u_{4,2}^{k+\frac{1}{2}} \\ u_{5,2}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,2} \\ b_{1,2} \\ b_{2,2} \\ b_{3,2} \\ b_{4,2} \\ b_{5,2} \end{bmatrix}$$

Each tridiagonal system can be solved using Thomas algorithm.