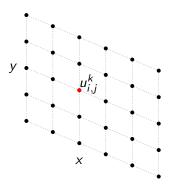
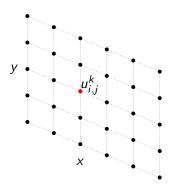


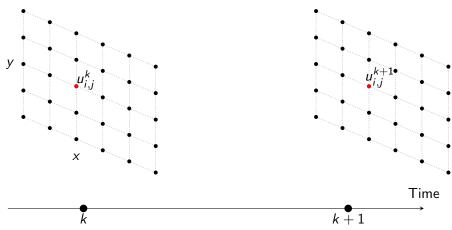
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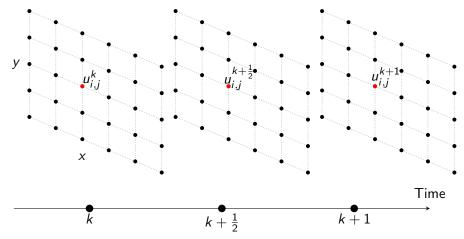
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$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \sum_{b=1}^{N} K_{out}(c_b)\delta(r_b - r) - K_{in}(u)\sum_{b=1}^{N} \delta(r_b - r)$$

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We can discretize the first half-step (x implicit, y explicit) using the following scheme:

$$\frac{u_{i,j}^{k+\frac{1}{2}} - u_{i,j}^{k}}{\Delta t/2} = D\left(\frac{u_{i+1,j}^{k+\frac{1}{2}} - 2u_{i,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+\frac{1}{2}}}{\Delta x^{2}} + \frac{u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k}}{\Delta y^{2}}\right) + S^{k}$$

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And the second half-step (x explicit, y implicit) using the following scheme:

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Where S is the sources and sinks term:

$$S^{k} = \sum_{b=1}^{N} K_{out}(c_b) \delta_{\varepsilon}(r_b - r_{i,j}) - K_{in}(u_{i,j}^{k}) \sum_{b=1}^{N} \delta_{\varepsilon}(r_b - r_{i,j})$$

Let's start with the first half-step:

$$\frac{u_{i,j}^{k+\frac{1}{2}} - u_{i,j}^{k}}{\Delta t/2} = D\left(\frac{u_{i+1,j}^{k+\frac{1}{2}} - 2u_{i,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+\frac{1}{2}}}{\Delta x^{2}} + \frac{u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k}}{\Delta y^{2}}\right) + S^{k}$$

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To simplify, we notice that $\Delta x = \Delta y$, and we can define $\alpha = \frac{D\Delta t}{2\Delta x^2}$ to get:

$$u_{i,j}^{k+\frac{1}{2}} - u_{i,j}^{k} = \alpha \left(u_{i+1,j}^{k+\frac{1}{2}} - 2u_{i,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+\frac{1}{2}} + u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k} \right) + \frac{\Delta t}{2} S^{k}$$

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Now we can rearrange the equation to isolate the unknown terms $u_{i,j}^{k+\frac{1}{2}}$ on the left-hand side:

$$\frac{-\alpha u_{i-1,j}^{k+\frac{1}{2}} + (1+2\alpha)u_{i,j}^{k+\frac{1}{2}} - \alpha u_{i+1,j}^{k+\frac{1}{2}}}{\alpha u_{i,j-1}^{k} + (1-2\alpha)u_{i,j}^{k} + \alpha u_{i,j+1}^{k} + \frac{\Delta t}{2}S^{k}}$$

We know all the terms in right-hand side of the equation (blue), but we need to compute the left-hand side (red).

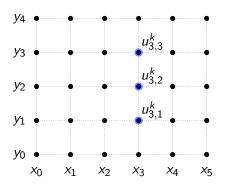
$$\frac{-\alpha u_{i-1,j}^{k+\frac{1}{2}} + (1+2\alpha)u_{i,j}^{k+\frac{1}{2}} - \alpha u_{i+1,j}^{k+\frac{1}{2}}}{\alpha u_{i,j-1}^{k} + (1-2\alpha)u_{i,j}^{k} + \alpha u_{i,j+1}^{k} + \frac{\Delta t}{2}S^{k}}$$

$$-\alpha u_{i-1,j}^{k+\frac{1}{2}} + (1+2\alpha)u_{i,j}^{k+\frac{1}{2}} - \alpha u_{i+1,j}^{k+\frac{1}{2}} = \alpha u_{i,j-1}^{k} + (1-2\alpha)u_{i,j}^{k} + \alpha u_{i,j+1}^{k} + \frac{\Delta t}{2}S^{k}$$

Let's look around i = 3 and j = 2:

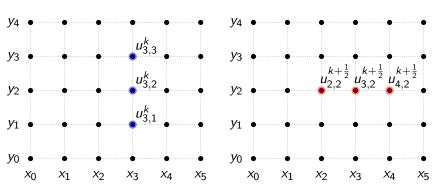
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Let's look around i = 3 and j = 2:



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Let's look around i = 3 and j = 2:



Let's fix j = 0 and write the equations for $0 \le i \le 5$.



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$$\begin{array}{c} k + \frac{1}{2} \\ & \\ y_{3} \\ & \\ y_{2} \\ & \\ y_{1} \\ & \\ y_{0} \\ & \\ y_{0} \\ & \\ x_{0} \\ & x_{1} \\ & x_{2} \\ & x_{3} \\ & x_{4} \\ & x_{5} \\ \end{array}$$

Let's fix
$$j=0$$
 and write the equations for $0 \le i \le 5$.
$$-\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha) u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha) u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2} S^{k}$$





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$$j=0$$
 and write the equations for $0 \le i \le 5$.

$$\begin{aligned} & -\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha) u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha) u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2} S^{k} \\ & -\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha) u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} = \alpha u_{1,-1}^{k} + (1-2\alpha) u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2} S^{k} \end{aligned}$$





$$k + \frac{1}{2}$$



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$$-\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha)u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2}S^{k}$$

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$$-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{2,-1}^{k} + (1-2\alpha)u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$k + \frac{1}{2}$$

Let's fix
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 and write the equations for $0 \le i \le 5$.

$$\begin{split} &-\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha) u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha) u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2} S^{k} \\ &-\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha) u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} = \alpha u_{1,-1}^{k} + (1-2\alpha) u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2} S^{k} \\ &-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha) u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{2,-1}^{k} + (1-2\alpha) u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2} S^{k} \\ &-\alpha u_{2,0}^{k+\frac{1}{2}} + (1+2\alpha) u_{3,0}^{k+\frac{1}{2}} - \alpha u_{4,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha) u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2} S^{k} \end{split}$$

$$k + \frac{1}{2}$$

$$y_4$$

$$y_3$$

$$y_2$$

$$y_1$$

$$y_2$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_5$$

$$y_4$$

$$y_5$$

$$y_5$$

$$y_5$$

$$y_7$$

$$y_8$$

$$y_8$$

$$y_8$$

$$y_8$$

$$y_8$$

Let's fix
$$j = 0$$
 and write the equations for $0 \le i \le 5$.

and write equations for
$$0 \le T \le S$$
.
$$-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha)u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} = \alpha u_{1,-1}^{k} + (1-2\alpha)u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{2,-1}^{k} + (1-2\alpha)u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{2,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,0}^{k+\frac{1}{2}} - \alpha u_{4,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha)u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{3,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,0}^{k+\frac{1}{2}} - \alpha u_{5,0}^{k+\frac{1}{2}} = \alpha u_{4,-1}^{k} + (1-2\alpha)u_{4,0}^{k} + \alpha u_{4,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$k + \frac{1}{2}$$

$$y_4$$

$$y_3$$

$$y_2$$

$$y_1$$

$$y_0$$

$$y_1$$

$$y_2$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_5$$

$$y_7$$

$$y_$$

Let's fix
$$j = 0$$
 and write the equations for $0 \le i \le 5$.

$$\begin{split} &-\alpha u_{-1,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{0,0}^{k+\frac{1}{2}}-\alpha u_{1,0}^{k+\frac{1}{2}}=\alpha u_{0,-1}^{k}+(1-2\alpha)u_{0,0}^{k}+\alpha u_{0,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{0,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{1,0}^{k+\frac{1}{2}}-\alpha u_{2,0}^{k+\frac{1}{2}}=\alpha u_{1,-1}^{k}+(1-2\alpha)u_{1,0}^{k}+\alpha u_{1,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{1,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{2,0}^{k+\frac{1}{2}}-\alpha u_{3,0}^{k+\frac{1}{2}}=\alpha u_{2,-1}^{k}+(1-2\alpha)u_{2,0}^{k}+\alpha u_{2,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{2,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{3,0}^{k+\frac{1}{2}}-\alpha u_{4,0}^{k+\frac{1}{2}}=\alpha u_{3,-1}^{k}+(1-2\alpha)u_{3,0}^{k}+\alpha u_{3,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{3,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{4,0}^{k+\frac{1}{2}}-\alpha u_{5,0}^{k+\frac{1}{2}}=\alpha u_{4,-1}^{k}+(1-2\alpha)u_{4,0}^{k}+\alpha u_{4,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{4,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{5,0}^{k+\frac{1}{2}}-\alpha u_{5,0}^{k+\frac{1}{2}}=\alpha u_{5,-1}^{k}+(1-2\alpha)u_{5,0}^{k}+\alpha u_{5,1}^{k}+\frac{\Delta t}{2}S^{k} \end{split}$$

$$k + \frac{1}{2}$$

$$y_4$$

$$y_3$$

$$y_2$$

$$y_1$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_5$$

$$y_7$$

$$y_$$

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 and write the equations for $0 \le i \le 5$.

$$\begin{split} &-\alpha u_{-1,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{0,0}^{k+\frac{1}{2}}-\alpha u_{1,0}^{k+\frac{1}{2}}=\alpha u_{0,-1}^{k}+(1-2\alpha)u_{0,0}^{k}+\alpha u_{0,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{0,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{1,0}^{k+\frac{1}{2}}-\alpha u_{2,0}^{k+\frac{1}{2}}=\alpha u_{1,-1}^{k}+(1-2\alpha)u_{1,0}^{k}+\alpha u_{1,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{1,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{2,0}^{k+\frac{1}{2}}-\alpha u_{3,0}^{k+\frac{1}{2}}=\alpha u_{2,-1}^{k}+(1-2\alpha)u_{2,0}^{k}+\alpha u_{2,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{2,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{3,0}^{k+\frac{1}{2}}-\alpha u_{4,0}^{k+\frac{1}{2}}=\alpha u_{3,-1}^{k}+(1-2\alpha)u_{3,0}^{k}+\alpha u_{3,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{3,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{4,0}^{k+\frac{1}{2}}-\alpha u_{5,0}^{k+\frac{1}{2}}=\alpha u_{4,-1}^{k}+(1-2\alpha)u_{4,0}^{k}+\alpha u_{4,1}^{k}+\frac{\Delta t}{2}S^{k}\\ &-\alpha u_{4,0}^{k+\frac{1}{2}}+(1+2\alpha)u_{5,0}^{k+\frac{1}{2}}-\alpha u_{5,0}^{k+\frac{1}{2}}=\alpha u_{5,-1}^{k}+(1-2\alpha)u_{5,0}^{k}+\alpha u_{5,1}^{k}+\frac{\Delta t}{2}S^{k} \end{split}$$

$$k + \frac{1}{2}$$

$$y_4$$

$$y_3$$

$$y_2$$

$$y_1$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_3$$

$$y_4$$

$$y_5$$

$$y_7$$

$$y_$$

Let's fix j = 0 and write the equations for $0 \le i \le 5$.

$$\begin{array}{l} -\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha)u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} = \alpha u_{1,-1}^{k} + (1-2\alpha)u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha)u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,0}^{k+\frac{1}{2}} - \alpha u_{4,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha)u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{3,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,0}^{k+\frac{1}{2}} - \alpha u_{5,0}^{k+\frac{1}{2}} = \alpha u_{4,-1}^{k} + (1-2\alpha)u_{4,0}^{k} + \alpha u_{4,1}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{4,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,0}^{k+\frac{1}{2}} - \alpha u_{6,0}^{k+\frac{1}{2}} = \alpha u_{5,-1}^{k} + (1-2\alpha)u_{5,0}^{k} + \alpha u_{5,1}^{k} + \frac{\Delta t}{2}S^{k} \end{array}$$

Which can be written as a matrix:

$$k + \frac{1}{2}$$
 y_4
 y_3
 y_2
 y_1
 y_3
 y_4
 y_5
 y_7
 $y_$

Let's fix j = 0 and write the equations for $0 \le i \le 5$.

$$-\alpha u_{-1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,0}^{k+\frac{1}{2}} - \alpha u_{1,0}^{k+\frac{1}{2}} = \alpha u_{0,-1}^{k} + (1-2\alpha)u_{0,0}^{k} + \alpha u_{0,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{0,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,0}^{k+\frac{1}{2}} - \alpha u_{2,0}^{k+\frac{1}{2}} = \alpha u_{1,-1}^{k} + (1-2\alpha)u_{1,0}^{k} + \alpha u_{1,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{2,-1}^{k} + (1-2\alpha)u_{2,0}^{k} + \alpha u_{2,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{1,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,0}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha)u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2}S^{k}$$

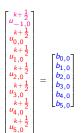
$$-\alpha u_{2,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,0}^{k+\frac{1}{2}} - \alpha u_{4,0}^{k+\frac{1}{2}} = \alpha u_{3,-1}^{k} + (1-2\alpha)u_{3,0}^{k} + \alpha u_{3,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{3,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,0}^{k+\frac{1}{2}} - \alpha u_{5,0}^{k+\frac{1}{2}} = \alpha u_{4,-1}^{k} + (1-2\alpha)u_{4,0}^{k} + \alpha u_{4,1}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{4,0}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,0}^{k+\frac{1}{2}} - \alpha u_{5,0}^{k+\frac{1}{2}} = \alpha u_{5,-1}^{k} + (1-2\alpha)u_{5,0}^{k} + \alpha u_{5,1}^{k} + \frac{\Delta t}{2}S^{k}$$

Which can be written as a matrix:

$$k + \frac{1}{2}$$



$$\begin{bmatrix} -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \end{bmatrix} \begin{bmatrix} u_{k+\frac{1}{2}}^{k+\frac{1}{2}} \\ u_{1,0}^{k} \\ k+\frac{1}{2} \\ u_{1,0}^{k} \\ u_{2,0}^{k} \\ u_{3,0}^{k} \\ u_{4,0}^{k} \\ u_{4,0}^{k} \\ u_{5,0}^{k} \end{bmatrix}$$

$$\begin{bmatrix} -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \end{bmatrix} \begin{bmatrix} u_{1,0} \\ k+\frac{1}{2} \\ u_{1,0} \\ k+\frac{1}{2} \\ u_{4,0} \\ k+\frac{1}{2} \\ u_{4,0} \\ k+\frac{1}{2} \\ u_{5,0} \\ k+\frac{1}{2} \\ u_{6,0} \end{bmatrix}$$
 by the values on the boundaries, so we don't need to include them in the matrix.

We know the values on the boundaries, so we don't need to include them in the matrix,

$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} k+\frac{1}{2} \\ u_{0,0} \\ k+\frac{1}{2} \\ u_{2,0} \\ k+\frac{1}{2} \\ u_{3,0} \\ k+\frac{1}{2} \\ u_{4,0} \\ k+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} b_{0,0} \\ b_{2,0} \\ b_{3,0} \\ b_{5,0} \end{bmatrix}$$

Let's fix j = 1 and write the equations for $0 \le i \le 5$.



Let's fix j = 1 and write the equations for $0 \le i \le 5$.



$$\begin{array}{c} k + \frac{1}{2} \\ y_4 \\ y_5 \\ y_2 \\ y_1 \\ y_0 \\ x_0 \ x_1 \ x_2 \ x_3 \ x_4 \ x_5 \end{array}$$

Let's fix
$$j=1$$
 and write the equations for $0 \leq i \leq 5$.
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^k + (1-2\alpha) u_{0,1}^k + \alpha u_{0,2}^k + \frac{\Delta t}{2} \mathsf{S}^k$$



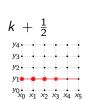
$$k + \frac{1}{2}$$
 y_4
 y_3
 y_2
 y_4
 y_3
 y_2
 y_4
 y_3
 y_4
 y_5
 y_7
 y_8
 $y_$

Let's fix
$$j=1$$
 and write the equations for $0 \le i \le 5$.
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^k + (1-2\alpha) u_{0,1}^k + \alpha u_{0,2}^k + \frac{\Delta t}{2} S^k \\ -\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^k + (1-2\alpha) u_{1,1}^k + \alpha u_{1,2}^k + \frac{\Delta t}{2} S^k$$





Let's fix
$$j=1$$
 and write the equations for $0 \le i \le 5$.
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^k + (1-2\alpha)u_{0,1}^k + \alpha u_{0,2}^k + \frac{\Delta t}{2} S^k \\ -\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^k + (1-2\alpha)u_{1,1}^k + \alpha u_{1,2}^k + \frac{\Delta t}{2} S^k \\ k \\ -\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^k + (1-2\alpha)u_{2,1}^k + \alpha u_{2,2}^k + \frac{\Delta t}{2} S^k \\ k \\ y_4 + y_4 + y_{2,2} + y_{2,2} + y_{2,3} + y_{2,4} +$$



Let's fix
$$j=1$$
 and write the equations for $0 \le i \le 5$.
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2}S^{k} \\ k \\ -\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} - \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k} + (1+2\alpha)u_{3,1}^{k} + (1+2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{3,1}^{k} + (1+2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{3,1}^{k} + (1+2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{3,1}^{k} + (1+2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{3,1}^{k} + (1+2\alpha)u_{3,2}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{3,1}^{k} + (1+2\alpha)u_{3,2}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{3,1}^{k} + (1+2\alpha)u_{3,2}^{k} + \alpha u_{3,2}^{k} + \alpha u_{3,2}^{k} + \alpha u_{3,2}^{k}$$



$$\text{Let's fix } j = 1 \text{ and write the equations for } 0 \leq i \leq 5. \\ -\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2}S^{k} \\ k \\ -\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2}S^{k} \\ -\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k} \\ y_{1} \\ y_{2} \\ y_{3} \\ y_{4} \\ y_{4} \\ y_{5} \\ y_{4} \\ y_{5} \\ y_{5} \\ y_{5} \\ y_{6} \\ y_{7} \\ y_{$$



Let's fix
$$j=1$$
 and write the equations for $0 \le i \le 5$.
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} = \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{4,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,1}^{k+\frac{1}{2}} - \alpha u_{6,1}^{k+\frac{1}{2}} = \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2}S^{k}$$



Let's fix
$$j=1$$
 and write the equations for $0 \le i \le 5$.
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} = \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2}S^{k}$$

$$-\alpha u_{4,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{5,1}^{k+\frac{1}{2}} - \alpha u_{6,1}^{k+\frac{1}{2}} = \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2}S^{k}$$



Let's fix j=1 and write the equations for $0 \le i \le 5$. $-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha)u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^{k}$ $-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha)u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^{k}$ k $-\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha)u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^{k}$ $-\alpha u_{2,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{3,1}^{k+\frac{1}{2}} - \alpha u_{4,1}^{k+\frac{1}{2}} = \alpha u_{3,0}^{k} + (1-2\alpha)u_{3,1}^{k} + \alpha u_{3,2}^{k} + \frac{\Delta t}{2} S^{k}$ $-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} = \alpha u_{4,0}^{k} + (1-2\alpha)u_{4,1}^{k} + \alpha u_{4,2}^{k} + \frac{\Delta t}{2} S^{k}$ $-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} = \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2} S^{k}$ $-\alpha u_{3,1}^{k+\frac{1}{2}} + (1+2\alpha)u_{4,1}^{k+\frac{1}{2}} - \alpha u_{5,1}^{k+\frac{1}{2}} = \alpha u_{5,0}^{k} + (1-2\alpha)u_{5,1}^{k} + \alpha u_{5,2}^{k} + \frac{\Delta t}{2} S^{k}$

Which can be written as a matrix:

$$k + \frac{1}{2}$$
 y_4
 y_3
 y_2
 y_1
 y_0
 y_1
 y_2
 y_3
 y_4
 y_5
 y_7
 $y_$

Let's fix
$$j=1$$
 and write the equations for $0 \leq i \leq 5$.
$$-\alpha u_{-1,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{0,1}^{k+\frac{1}{2}} - \alpha u_{1,1}^{k+\frac{1}{2}} = \alpha u_{0,0}^{k} + (1-2\alpha) u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{\Delta t}{2} S^{k}$$

$$\begin{array}{c} -\alpha u_{-1,1}^{-1} + (1+2\alpha) u_{0,1}^{-1}{}^{2} - \alpha u_{1,1}^{-1}{}^{2} = \alpha u_{0,0}^{k} + (1-2\alpha) u_{0,1}^{k} + \alpha u_{0,2}^{k} + \frac{2}{2} S^{k} \\ -\alpha u_{0,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{1,1}^{k+\frac{1}{2}} - \alpha u_{2,1}^{k+\frac{1}{2}} = \alpha u_{1,0}^{k} + (1-2\alpha) u_{1,1}^{k} + \alpha u_{1,2}^{k} + \frac{\Delta t}{2} S^{k} \\ \kappa \\ -\alpha u_{1,1}^{k+\frac{1}{2}} + (1+2\alpha) u_{2,1}^{k+\frac{1}{2}} - \alpha u_{3,1}^{k+\frac{1}{2}} = \alpha u_{2,0}^{k} + (1-2\alpha) u_{2,1}^{k} + \alpha u_{2,2}^{k} + \frac{\Delta t}{2} S^{k} \\ \kappa \\ \gamma_{3} \\ \gamma_{3} \\ \gamma_{4} \\ \gamma_{5} \\ \gamma_{6} \\ \gamma_{6} \\ \gamma_{6} \\ \gamma_{5} \\ \gamma$$

Which can be written as a matrix:

$$k + \frac{1}{2}$$
 y_4
 y_3
 y_2
 y_1
 y_0
 y_1
 y_2
 y_3
 y_4
 y_3
 y_4
 y_3
 y_4
 y_5
 y_5
 y_5
 y_7
 $y_$

$$\begin{bmatrix} -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \end{bmatrix}$$

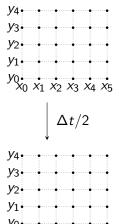
$$\begin{bmatrix} u_{-1,1} \\ u_{-1,1} \\ k+\frac{1}{2} \\ u_{0,1} \\ k+\frac{1}{2} \\ u_{1,1} \\ k+\frac{1}{2} \\ u_{2,1} \\ k+\frac{1}{2} \\ u_{3,1} \\ k+\frac{1}{2} \\ u_{4,1} \\ u_{k+\frac{1}{2}} \\ u_{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{2,1} \\ b_{3,1} \\ b_{3,1} \\ b_{5,1} \end{bmatrix}$$

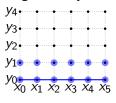
$$\begin{bmatrix} -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \end{bmatrix} \begin{bmatrix} u_{k+\frac{1}{2}}^{k+\frac{1}{2}} \\ u_{1,1}^{k+\frac{1}{2}} \\ u_{1,1}^{k+\frac{1}{2}} \\ u_{2,1}^{k+\frac{1}{2}} \\ u_{3,1}^{k+\frac{1}{2}} \\ u_{4,1}^{k+\frac{1}{2}} \\ u_{5,1}^{k+\frac{1}{2}} \\ u_{6,1}^{k+\frac{1}{2}} \end{bmatrix}$$

$$\begin{bmatrix} -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \end{bmatrix} = \begin{bmatrix} b_{0,1} \\ u_{1,1} \\ u_{2,1} \\ u_{4,1} \\ u_{4,1} \\ u_{4,1} \\ u_{5,1} \end{bmatrix}$$
 bow the values on the boundaries, so we don't need to include them in the matrix.

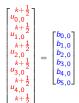
We know the values on the boundaries, so we don't need to include them in the matrix,

$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \\ \end{bmatrix} \begin{bmatrix} k+\frac{1}{2} \\ u_{0,1} \\ k+\frac{1}{2} \\ u_{2,1} \\ k+\frac{1}{2} \\ u_{3,1} \\ k+\frac{1}{2} \\ u_{4,1} \\ k+\frac{1}{2} \\ u_{4,1} \\ k+\frac{1}{2} \\ \end{bmatrix}$$

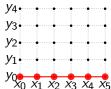


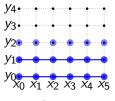


$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix}$$

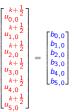


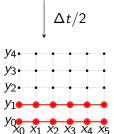






$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix}$$

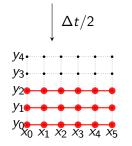




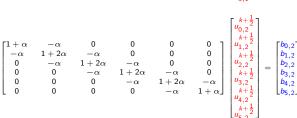
$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix}$$

$$\begin{bmatrix} k+\frac{1}{2} \\ u_{0,1} \\ k+\frac{1}{2} \\ u_{1,1} \\ k+\frac{1}{2} \\ u_{2,1} \\ k+\frac{1}{2} \\ u_{3,1} \\ k+\frac{1}{2} \\ u_{4,1} \\ k+\frac{1}{2} \\ u_{5,1} \end{bmatrix} = \begin{bmatrix} b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ b_{4,1} \\ b_{5,1} \end{bmatrix}$$

$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ \end{bmatrix}$$



$$\begin{bmatrix} 1+\alpha & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha & 0 \\ 0 & 0 & 0 & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{0,1}^{k+\frac{1}{2}} \\ u_{1,1}^{k+\frac{1}{2}} \\ u_{2,1}^{k+\frac{1}{2}} \\ u_{3,1}^{k+\frac{1}{2}} \\ u_{4,1}^{k+\frac{1}{2}} \\ u_{5,1}^{k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\ u_{4,1} \\ u_{5,1}^{k+\frac{1}{2}} \end{bmatrix}$$



 $\begin{bmatrix} u_{0,0}^{k+\frac{1}{2}} \\ k+\frac{1}{2} \end{bmatrix}$

 $b_{0,0}$

 $b_{1,0}$

 $b_{2,0}$

b3.0

 $b_{4,0}$

 $\lfloor b_{5,0} \rfloor$

 $u_{1,0}$

u_{2.0}

 $u_{3,0}$

u_{4.0}

 $k+\frac{1}{2}$

 $k + \frac{1}{2}$

 $k + \frac{1}{2}$

 $u_{5,0}^{k+\frac{1}{2}}$

