Problem 138: Enter the Matrix

Difficulty: Medium

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Originally Published: Code Quest 2021

Problem Background

In the fields of science, engineering, and technology, many products and projects that we deal with are highly dependent on specific mathematical operations to be able to function properly. Matrices are an excellent example of this; they have applications in computer graphics, aerospace engineering, and quantum mechanics. A matrix is a rectangular array of numbers, typically shown with two large square brackets:

$$M = \begin{bmatrix} m_{11} & \cdots & m_{n1} \\ \vdots & \ddots & \vdots \\ m_{1p} & \cdots & m_{np} \end{bmatrix}$$

Problem Description

Lockheed Martin is working with the United States Navy to test a new submersible drive system. The system involves two engines, which each have a different task in orienting the submarine. During the testing process, the submarine performs two maneuvers repeatedly, which place different demands on the two engines, as shown in this table:

Fuel Consumption	Maneuver 1	Maneuver 2
Engine 1	2 barrels	5 barrels
Engine 2	3 barrels	2 barrels

Based on the total amount of fuel consumption by each engine, we need to determine how many times each engine was fired. We can solve this problem using matrices.

We'll need three matrices to hold the information we need. First, our solution, the number of times each engine fired. This will be held in Matrix E:

$$E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$$

Second, the table above can be represented as Matrix C, showing how much fuel is consumed by each engine during each maneuver:

$$C = \begin{bmatrix} C_{1,1} & C_{2,1} \\ C_{1,2} & C_{2,2} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$$

Finally, we need to know the total amount of fuel used during each maneuver. We'll keep this information in Matrix F.

$$F = [F_1 \quad F_2]$$

If we were dealing with a single engine and a single maneuver, multiplying the number of times the engine was fired by its fuel consumption rate would tell us the total amount of fuel used. The advantage of using matrices to hold all of this information is that this same logic holds true when working with multiple engines and maneuvers:

$$E \times C = F$$

$$[E_1 \quad E_2] \times \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix} = [F_1 \quad F_2]$$

Let's try an example. During one test, 49 barrels of fuel were spent on Maneuver 1 (F_1), and 73 barrels were used on Maneuver 2 (F_2). How many times was each engine fired?

In a normal algebra problem, we would divide F by C to determine the value of E. However, matrices can't be divided! Instead, we can invert a matrix, then multiply that inverse by a matrix to get the same result.

To calculate the inverse of a matrix, you can use the following formula:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$X^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{(ad - bc)} & \frac{-b}{(ad - bc)} \\ \frac{-c}{(ad - bc)} & \frac{a}{(ad - bc)} \end{bmatrix}$$

In short, we build a new matrix based on the values within the original; then each of those new values is divided by the original matrix's ***determinant***. In order to be inverted, a matrix must be square, and it must have a non-zero determinant (since you can't divide by zero). For this problem, we will be inverting matrix C, which is square; in this example, C's determinant is $|C| = (2 \times 2) - (5 \times 3) = 4 - 15 = -11$, which is non-zero, so C can be inverted. For all test cases, C will have a non-zero determinant.

Once we invert C, we can multiply C^{-1} by F to determine the values in E. Using our example above:

$$C^{-1} = \begin{bmatrix} \frac{2}{-11} & \frac{-5}{-11} \\ \frac{-3}{-11} & \frac{2}{-11} \end{bmatrix} = \begin{bmatrix} -.1818 & .4545 \\ .2727 & -.1818 \end{bmatrix}$$

$$E = F \times C^{-1}$$

$$E = [49 \quad 73] \times \begin{bmatrix} -.1818 & .4545 \\ .2727 & -.1818 \end{bmatrix}$$

$$E = \left[\left({C^{-1}}_{1,1} F_1 + {C^{-1}}_{1,2} F_2 \right) \quad \left({C^{-1}}_{2,1} F_1 + {C^{-1}}_{2,2} F_2 \right) \right]$$

$$E = \left[\left((-.1818 \times 49) + (.2727 \times 73) \right) \quad \left((.4545 \times 49) + (-.1818 \times 73) \right) \right]$$

$$E = [10.9989 \quad 8.9991] \approx [11 \quad 9]$$

And we have our answer! Since $E = [11 \quad 9]$ (after rounding to the nearest whole number), we can see that engine 1 was fired 11 times, and engine 2 was fired 9 times during the course of the test.

In order to automate this testing routine, your team has been asked to write a program that can calculate the values in E given the values of C and F.

Sample Input

The first line of your program's input, received from the standard input channel, will contain a positive integer representing the number of test cases. Each test case will include three lines, containing the information listed below. On each line, values are separated by spaces. All values will be positive integers.

- The first line will contain the values for $C_{1,1}$ and $C_{2,1}$; the fuel consumed by Engine 1 during Maneuver 1 and Maneuver 2, respectively.
- The second line will contain the values for $C_{1,2}$ and $C_{2,2}$; the fuel consumed by Engine 2 during Maneuver 1 and Maneuver 2, respectively.
- The third line will contain the values for F_1 and F_2; the total fuel consumed by both engines during Maneuver 1 and Maneuver 2, respectively.

```
2
2 5
3 2
49 73
7 3
4 5
193 155
```

Sample Output

For each test case, your program must print a single line containing the values for E_1 and E_2. These represent the number of times Engines 1 and 2 were fired during the test, respectively. Both values should be rounded to the nearest integer and be separated by spaces.

```
11 9
15 22
```