Mathematics Analysis - Lecture 2

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Notation

- Mathematical statement: p, q
- Qualifiers
  - $\forall$ for all, for any, for every
  - $\exists$ there exists, there is (at least one)
- The opposite statement of p: $\sim p$

Rules of negation

- $\sim(\sim p) = p$
- $\sim(p \text{ and } q) = \sim p \text{ or } \sim q$
- $\sim(p \text{ or } q) = \sim p \text{ and } \sim q$
- $\sim(\forall x, p) = \exists x, \sim p$
- $\sim(\exists y, p) = \forall y, \sim p$
- $\sim(\forall x, \exists y, p) = \exists x, \forall y, \sim p$

Conditional statement

Notation

$$p \Rightarrow q \begin{cases} \text{If } p, \text{ then } q \\ p \text{ implies } q \\ p \text{ is sufficient for } q \\ q \text{ is necessary for } p \end{cases}$$

Rules

$$p \Rightarrow q = \sim p \text{ or } q \qquad (\bigstar)$$

$$\sim(p \Rightarrow q) = p \text{ and } \sim q$$

Converse / contrapositive statement

For the statement "If p then q" or $(p \Rightarrow q)$,

- Its converse is "If q then p" or $(q \Rightarrow p)$
- Its contrapositive is "If $\sim q$ then $\sim p$" or $(\sim q \Rightarrow \sim p)$

Converse / contrapositive statement

Example

$$\text{Statement: } x = -3 \Rightarrow x^2 = 9 \qquad \text{(T)}$$

$$\text{Converse: } x^2 = 9 \Rightarrow x = -3 \qquad \text{(F)}$$

$$\text{Contrapositive: } x^2 \neq 9 \Rightarrow x \neq -3 \qquad \text{(T)}$$

Converse / contrapositive statement

Example

$$\text{Statement: } x = -3 \Rightarrow 2x = -6 \qquad \text{(T)}$$

$$\text{Converse: } 2x = -6 \Rightarrow x = -3 \qquad \text{(T)}$$

$$\text{Contrapositive: } 2x \neq -6 \Rightarrow x \neq -3 \qquad \text{(T)}$$

Converse / contrapositive statement

Remark

$$\text{contrapositive} = \text{statement}$$

(Foundation of proof by contrapositive)

Proof.

$$\begin{aligned} \sim q \Rightarrow \sim p &= \sim(\sim q) \text{ or } \sim p && (\text{Rule } \bigstar) \\ &= q \text{ or } \sim p \\ &= \sim p \text{ or } q \\ &= p \Rightarrow q \end{aligned}$$

$\square$

Converse / contrapositive statement

Remark

If both $p \Rightarrow q$ and $q \Rightarrow p$ are true, we will write $p \Leftrightarrow q$, and say that "p if and only if q" or use abbreviation "p iff q".

Converse / contrapositive statement

Remark

$$\forall\alpha, \forall\beta = \forall\beta, \forall\alpha$$
$$\exists\alpha, \exists\beta = \exists\beta, \exists\alpha$$
$$\forall\alpha, \exists\beta \neq \exists\beta, \forall\alpha$$

Example

$$\forall\alpha, \exists\beta > 0, \alpha + \beta > 0 \qquad \text{(T)}$$
$$\exists\beta > 0, \forall\alpha, \alpha + \beta > 0 \qquad \text{(F)}$$

Exercise: Negate each of the following

1. If $\triangle ABC$ is a right triangle, then $a^2 + b^2 = c^2$
2. $\forall \epsilon > 0, \exists \delta > 0,$ s.t. $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

Solution

1. $\sim(p \Rightarrow q) = p$ and $\sim q$
   $\triangle ABC$ is a right triangle, and $a^2 + b^2 = c^2$

2. $\sim(\forall\epsilon, \exists\delta, p \Rightarrow q) = \exists\epsilon, \forall\delta, \sim(p \Rightarrow q)$
   $\exists\epsilon > 0, \forall\delta > 0, 0 < |x - x_0| < \delta$ and $|f(x) - L| \geq \epsilon$

Set Theory

Modern theory founded by G. Cantor in 1870's to resolve paradoxes in naive set theory.

Definition

A set is a collection of math "objects" (usually numbers, functions, ordered pair of numbers, ...).
The objecst in the set are the elements of the set.

Notation

We write $x \in S$ to say that x is an element of set S.
We write $x \overline{\in} S$ to say that x is NOT an element of set S.

Example

Let $\mathbb{Z}$ be the set of all integers, then $-1 \in \mathbb{Z}$, $2 \in \mathbb{Z}$, but $\sqrt{2} \overline{\in} \mathbb{Z}$.

Definition

- A set is finite if it has finitely many elements.
- A set is infinite if it has infinitely many elements.
- The empty set is the set having no element and it is denoted by $\varnothing$

Common sets in math

$\mathbb{N}$ the set of all positive integers

$\mathbb{Z}$ the set of all integers

$\mathbb{Q}$ the set of all rational numbers

$\mathbb{R}$ the set of all real numbers

$\mathbb{C}$ the set of all complex numbers

List all elements

$$S = \{1, 2, 3\}$$
$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$
$$\varnothing = \{\ \}$$
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Write the form of the elements

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z},\ n \in \mathbb{N} \right\}$$

$$\mathbb{R} = \{ x : x \text{ is a real number } \}$$

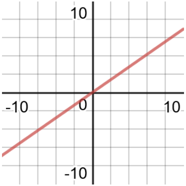
$$\mathbb{C} = \left\{ x + iy : x \in \mathbb{R},\ y \in \mathbb{R},\ i = \sqrt{-1} \right\}$$

$$[a, b] = \{ x : x \in \mathbb{R} \text{ and } a \leq x \leq b \}$$

$$(a, b) = \{ x : x \in \mathbb{R} \text{ and } a < x < b \}$$

Describe a line

Let l_m be a straight line with equation $y = mx$ in the xy-plane, i.e. the set of ordered pairs (x, y) satisfying the equation $y = mx$.



We can describe l_m by

$$l_m = \{\, (x, y) : x, y \in \mathbb{R} \text{ and } y = mx \,\}$$

$$\text{or } l_m = \{\, (x, mx) : x \in \mathbb{R} \,\}$$

Set Relations

Let A, B be two sets.

Def

We say that A is a subset of B is every element of A is also an element of B. In this case, we write

$$A \subseteq B$$

Using math notations,

$$A \subseteq B \Leftrightarrow (\forall x \in A, x \in B)$$

Remark

$$\varnothing \subseteq S \text{ for any set } S$$

Set Relations

Def

We say that $A = B$ iff

$$A \subseteq B \text{ and } B \subseteq A$$

Using math notations,

$$A = B \Leftrightarrow (A \subseteq B \text{ and } B \subseteq A) \Leftrightarrow (x \in A \Leftrightarrow x \in B)$$

Set Relations

Def

A is a proper subset of B if $A \subseteq B$ and $A \neq B$. We write

$$A \subset B$$

Set Relations

Example

Let

$$A = \{1, 2\}, B = \{1, 2, 3\}, C = \{1, 1, 2, 3\}$$

then

$$A \subset B = C$$

Remark

Repeated elements counts only once in the set, e.g.

$$\{1, 1, 1\} = \{1\}$$

If $X \subseteq Y$, the number of elements of $X \leq$ the number of elements of Y.

Power Set

Let S be a set, the power set of S is the set of all subsets of S. It is denoted by $P(S)$ or 2^S.

Example

$S = \varnothing$, then $\varnothing$ is the only subset of S, so

$$P(\varnothing) = 2^{\varnothing} = \{\varnothing\} \neq \varnothing$$

Note that $\varnothing$ has no element, but $2^{\varnothing}$ has one element.

Power Set

Example

$S = \{x\}$, then $\varnothing, \{x\} \subseteq S$, so

$$P(S) = 2^S = \{\varnothing, \{x\}\}$$

$S = \{x, y\}$, then $\varnothing, \{x\}, \{y\}, \{x, y\} \subseteq S$, so

$$P(S) = 2^S = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$$

Remark

If S has n elements, then $P(S)$ has 2^n elements.

Set Union

Let $A, B, \ldots$ be sets. Their union is

$$\begin{aligned} A \cup B \cup \ldots &= \{\, x : x \text{ is an element in at least one of the sets } A, B, \ldots \,\} \\ &= \{\, x : x \in A \text{ or } x \in B \text{ or } \ \ldots \,\} \end{aligned}$$

Example

$$\{\, p, q \,\} \cup \{\, r \,\} = \{\, p, q, r \,\}$$
$$\{\, x, y, z \,\} \cup \{\, v, w, x, y \,\} = \{\, x, y, z, v, w \,\}$$
$$\mathbb{R} \cup \mathbb{Q} = \mathbb{R}, \mathbb{N} \cup \mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$$
$$S \cup \varnothing = S$$

Set Intersection

The intersection of $A, B, \ldots$ is

$$\begin{aligned} A \cap B \cap \ldots &= \{\, x : x \text{ is an element in every one of the sets } A, B, \ldots \,\} \\ &= \{\, x : x \in A \text{ and } x \in B \text{ and } \ldots \,\} \end{aligned}$$

Example

$$\{\, p, q \,\} \cap \{\, r \,\} = \varnothing$$
$$\{\, x, y, z \,\} \cap \{\, v, w, x, y, z \,\} = \{\, x, y, z \,\}$$
$$\{\, x, y, z \,\} \cap \{\, v, w, x, y, z \,\} \cap \{\, u, v, w, x \,\} = \{\, x \,\}$$
$$\mathbb{R} \cap \mathbb{Q} \cap [0, 1] = \{\, x : x \in \mathbb{Q},\ 0 \le x \le 1 \,\}$$
$$S \cap \varnothing = \varnothing$$

Cartesian Product

The Cartesian product of $A, B, C, \dots$ is

$$A \times B \times C \times \cdots = \{ (a, b, c, \dots) : a \in A,\ b \in B,\ c \in \mathbb{C} \dots \}$$

Example

$$\mathbb{R} \times \mathbb{R} = \{ (x, y) : x, y \in \mathbb{R} \} = \mathbb{R}^2$$
$$\mathbb{N} \times \mathbb{Z} \times \{ 0, 1 \} = \{ (x, y, z) : x \in \mathbb{N},\ y \in \mathbb{Z},\ z = 0 \text{ or } 1 \}$$

Remark

$$S \times \varnothing = \{ (x, y) : x \in S,\ y \in \varnothing \} = \varnothing = \varnothing \times S$$
$$\text{If } A \neq B \text{ then } A \times B \neq B \times A$$

Complement

The complement of B in A is

$$A \setminus B = \{\, x : x \in A,\ x \overline{\in} B \,\}$$

Example

$$\mathbb{R} \setminus \mathbb{Q} = \{\, x : x \text{ is an irrational number } \}$$
$$\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) = \{\, (u, v) : u \in \mathbb{Q},\ v \text{ is irrational } \}$$
$$S \setminus \varnothing = S, \varnothing \setminus S = \varnothing$$

Disjoint Sets

We say sets $A, B, C, \ldots$ are disjoint if

$$A \cap B \cap C \cap \cdots = \varnothing$$

We say sets $A, B, C, \ldots$ are mutually disjoint if the intersection of each two of the sets is empty.

Example

$$A = \{x, y\}, B = \{y, z\}, C = \{z, x\}$$

then $A \cap B \cap C = \varnothing$, A, B, C are disjoint. But NOT mutually disjoint since $A \cap B \neq \varnothing$.

Some Notations

n is a positive number

$$S_1 \cup S_2 \cup \cdots \cup S_n = \bigcup_{k=1}^{n} S_k$$

$$S_1 \cap S_2 \cap \cdots \cap S_n = \bigcap_{k=1}^{n} S_k$$

$$S_1 \times S_2 \times \cdots \times S_n = \bigtimes_{k=1}^{n} S_k$$

$$S_1 \cup S_2 \cup S_3 \cup \ldots = \bigcup_{k=1}^{\infty} S_k \text{ or } \bigcup_{k \in \mathbb{N}} S_k$$

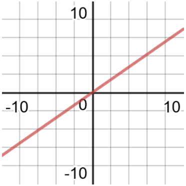
Some Notations

Example

If $m \in \mathbb{R}$, let l_m be the line $y = mx$ in the x-y plane. Then

$$\bigcup_{m \in \mathbb{R}} l_m = \mathbb{R}^2 \setminus \{ (0, y) : y \neq 0 \}$$

$$\bigcap_{m \in \mathbb{R}} l_m = \{ (0, 0) \}$$



Example 1

If $A \subseteq B$ and $C \subseteq D$, prove that $A \cap C \subseteq B \cap D$.

Idea: cccording to the definition of "$\subseteq$", we need to check that $\forall x \in A \cap C, x \in B \cap D$.

Proof.

$\forall x \in A \cap C$, $x \in A$ and $x \in C$.
Sicne $A \subseteq B$, $C \subseteq D$, we have $x \in B$ and $x \in D$ (according to the definition of "*subseteq*").
Hence $x \in B \cap D$ (according to the definition of "*cap*") □

Example 2

Prove that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

Strategy: to get "$=$", check "$\subseteq$" and "$\supseteq$"

Proof.

Step 1, we show that $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.
$\forall x \in (A \cup B) \setminus C$, we have

$$x \in A \cup B \text{ and } x \overline{\in} C. \qquad \text{(by the definition of "}\setminus\text{")}$$

$$\because x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B$$

$$\therefore (x \in A \text{ and } x \overline{\in} C) or (x \in B \text{ and } x \overline{\in} C)$$

$$\therefore x \in A \setminus C \text{ or } x \in B \setminus C$$

$$\therefore x \in (A \setminus C) \cup (B \setminus C)$$

□

Example 2 cont'd

Using math symbols:

$$
\begin{aligned}
\forall x \in (A \cup B) \setminus C &\Rightarrow (x \in A \cup B) \text{ and } (x \overline{\in} C) \\
&\Rightarrow (x \in A \text{ or } x \in B) \text{ and } x \overline{\in} C \\
&\Rightarrow (x \in A \text{ and } x \overline{\in} C) \text{ or } (x \in B \text{ and } x \overline{\in} C) \\
&\Rightarrow x \in A \setminus C \text{ or } x \in B \setminus C \\
&\Rightarrow x \in (A \setminus C) \cup (B \setminus C)
\end{aligned}
$$

Example 2 cont'd

Proof.

Step 2, we show that $(A \cup B) \setminus C \supseteq (A \setminus C) \cup (B \setminus C)$.
$\forall x \in (A \setminus C) \cup (B \setminus C)$, we have

$$x \in A \setminus C \text{ or } x \in B \setminus C$$

or equivalently

$$x \in A \text{ and } x \overline{\in} C, \text{ or } x \in B \text{ and } x \overline{\in} C$$
$$\therefore x \in A \cup B \text{ and } x \overline{\in} C$$
$$\therefore x \in (A \cup B) \setminus C$$

$\square$

Example 2 cont'd

Actually, we can prove "$=$" in one step

Proof.

$$\begin{aligned} x \in (A \cup B \setminus C) &\Leftrightarrow (x \in A \cup B) \text{ and } (x \overline{\in} C) \\ &\Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } x \overline{\in} C \\ &\Leftrightarrow (x \in A \text{ and } x \overline{\in} C) \text{ or } (x \in B \text{ and } x \overline{\in} C) \\ &\Leftrightarrow x \in A \setminus C \text{ or } x \in B \setminus C \\ &\Leftrightarrow x \in (A \setminus C) \cup (B \setminus C) \end{aligned}$$

$\square$

Functions

Definition

- A function (or map or mapping) f from a set A to a set B (denoted by $f : A \to B$) is a rule of assigning to every $a \in A$ exactly one $b \in B$.
- This b (denoted by $f(a)$) is called the value of f at a. A is called the domain of f, and B is called the codomain of f.
- $f(A) = \{ y : y = f(x) \text{ for some } x \in A \}$ is called the range (or image) of f.
- We may say that f is a function on A or f is a B-valued function.
- The set $\{ (x, f(x)) : x \in A \}$ is called the graph of f. Two functions are equal iff their graphs are the same.

Functions

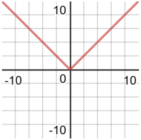
Example

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\text{or } f(x) = |x|$$

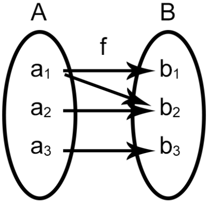
The domain of f is $\mathbb{R}$. The range of f is $[0, \infty)$. The graph of f is $\{ (x, |x|) : x \in \mathbb{R} \}$.



Functions

Example

f is not a map



Types of Functions

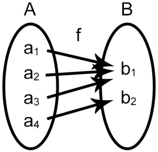
Definition

- The identity function on a set S, denoted by $I_S : S \to S$, is given by $I(x) = x$ for any $x \in S$.
- Let $f : A \to B, g : B \to C$ be functions. The composition of g by f is denoted by $g \circ f : A \to C$ and is given by $g \circ f(x) = g(f(x))$ for any $x \in A$.
- Let $f : A \to B$ be a function and $C \subseteq A$. The restriction of f to C is denoted by $f|_C : C \to B$ and is given by $f|_C(x) = f(x)$ for any $x \in C$.

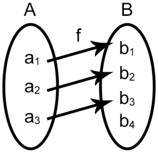
Types of Functions

Definition

- $f : A \to B$ is surjective (or onto) iff $f(A) = B$.



- $f : A \to B$ is injective (or one-to-one) iff $f(x) = f(y) \Rightarrow x = y$.



Types of Functions

Definition

- For an injective function $f : A \to B$, the inverse function of f is denoted by $f^{-1} : f(A) \to A$ and is given by $f^{-1}(y) = x$ where x is such that $x = f(y)$.
- $f : A \to B$ is bijective (or a one-to-one correspondence) iff f is injective and surjective.

Example

Show that $f : [0,1] \to [3,4]$ defined by $f(x) = x^3 + 3$ is a bijective.

Proof.

Step 1, we show that f is injective.

If $f(x) = f(y)$ for $x, y \in [0,1]$, then $x^3 + 3 = y^3 + 3$, $x^3 = y^3$.

Since $x, y > 0 \Rightarrow x = y$, this shows that f is injective.

Step 2, we show that f is surjective.

$\forall y \in [3,4]$, let $f(x) = x^3 + 3 = y$.

Then $x^3 = y - 3, x = \sqrt[3]{y-3} \in [0,1]$.

This shows that f is surjective. $\square$