Mathematics Analysis - Lecture 1

Hai Zhang

Main Items in Syllabus

- Instructor: Prof. Hai Zhang
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- Email: haizhang@ust.hk
- Office Hour: 10am - 11am, Tue / Thu or by appointment
- Pre-requisite: MATH 1014 / 1018 / 1020 / 1024
- All the course related materials will be posted on Canvas.

Grading Scheme

- Homework: 15 %
- Midterm Exam: 10 % (take-home exam)
- Final Exam: 75 % (2 hours. Will be arranged by ARRO)

Course Overview

- Algebra - matrix, quadratic equations
- Geometry - planes, curves, surfaces
- Number Theory - integers, prime numbers
- Calculus - compute derivatives and integrals / applications
- Analysis - A rigorous foundation of playing with infinity
    - Rigorous definition of limit, continuity, differentiation and integration

What is $\sqrt{2}$?

Definition

Solution to $x^2 = 2$, $\sqrt{2} \approx 1.41421356237\ldots$ (approximate value of $\sqrt{2}$, not real $\sqrt{2}$)

What is $\sqrt{2}$?

- History: In 1800 - 1600 BC, Babylonian provided an approximation, $\sqrt{2} \approx 1.41421356237\ldots$
- Story: In 5 BC, Hippasus discovered that $\sqrt{2}$ is an irrational number. But Pythagoras, believed in the absoluteness of numbers, did not accept the fact and sentenced Hippasus to death.

What is $\sqrt{2}$?

We need the concept of limit (or infinity) to understand the precise meaning of $\sqrt{2}$. For rational number, the concept of limit (infinity) is NOT needed.

Definition with Sequence

$$\sqrt{2} = \lim_{n\to\infty} a_n$$

where $a_1 = 1, a_2 = \frac{a_1}{2} + \frac{1}{a_1}, a_3 = \frac{a_2}{2} + \frac{1}{a_2}, \ldots, a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$

Question

What are real numbers?

What is the meaning of $\lim_{n\to\infty} a_n = A$?

Example

Intuitively, a_n tends to A as n tends to ∞, or a_n is close to A as desired when n is sufficiently large.

What is the meaning of $\lim_{n\to\infty} a_n = A$?

- Questions:
  - How large is sufficiently large?
    $n > 10000$? $n > 10^{10}$? $n > 100^{100}$?
  - How close is close?
    $|a_n - A| < 10^{-3}$? $|a_n - A| < 10^{-10}$?
- One need precise definition to avoid ambiguity!
- We will introduce $\epsilon - \delta$ language to make thing rigurous.

What is continuity?

Example

Huetistically, a function $f(x)$ is continuous at x_0 if $f(x)$ may be close to $f(x_0)$ as desired as x is sufficiently close to x_0.

Rigurous definition using $\epsilon - \delta$ language

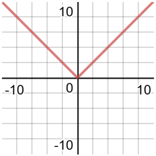
$$\lim_{x \to x_0} f(x) = f(x_0)$$

What is derivative, or “speed”, “tangent line”?

Remark

differentiability $\neq$ continuity

There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$, which is not differentiable everywhere.



Mathematical Statement

Example

2 is / is not an even number.

Remark: A statement may be false.

Example

$p : x \geq 0$

Notation: use p or q to denote a statement.

Quantifiers

- for all, for any, for every
  - Notation: $\forall$
  - $\forall x$: for all x
- there exists, there is (at least one)
  - Notation: $\exists$

Quantified Statement: statements involving quantifiers

Example

$\forall x > 0$, x has a square root.

Example

$\exists x > 0$, x does not have a square root.

Example

$\forall x > 0$, $\exists y > 0$ such that $x = y^2$.

Negating a Statement: taking the opposite of a statement

Let p be a statement, then the negation of p is denoted by $\sim p$.

Example

$p : x \geq 0, \sim p : x < 0$

Rules of Negation

- Rule 1: $\sim(\sim p) = p$

Example

$p : x > 0$, $\sim p : x \leq 0$, $\sim(\sim p) : x > 0$

Rules of Negation

- Rule 2: $\sim (p \textit{ and } q) = \sim p \textit{ or } \sim q$

Example

$p : x > 0$, $q : x < 1$, $p \textit{ and } q : 0 < x < 1$

$\sim (p \textit{ and } q) = \sim (0 < x < 1) = \underbrace{x \leq 0}_{\sim p} \textit{ or } \underbrace{x \geq 1}_{\sim q}$

Rules of Negation

- Rule 3: $\sim (p \text{ or } q) = (\sim p) \text{ and } (\sim q)$

Example

$p : x > 0,\ q : x < 1,\ p \text{ or } q : \quad \underbrace{x > 0 \text{ or } x < 1}_{x \text{ can be any number in between}}$

$\sim (p \text{ or } q) = \sim (x > 0 \text{ or } x < 1) = \underbrace{x \leq 0 \text{ and } x \geq 1}_{x \text{ can not be a number}}$

Exercise

$p : x < 0,\ q : x > 1,\ \sim (p \text{ or } q) \stackrel{?}{=} \sim p \text{ and } \sim q$

Rules of Negation

- Rule 4: $\sim (\forall x, p) = \exists x, \sim p$

Example

$$\forall x > 0, \underbrace{2x > 0}_{p} \qquad \text{(T)}$$

$$\text{the opposite is: } \exists x > 0, 2x \leq 0 \qquad \text{(F)}$$

Rules of Negation

- Rule 5: $\sim (\exists x, p) = \forall x, \sim p$

Example

$$\exists x > 0, \underbrace{x \text{ has a square root.}}_{p} \qquad \text{(T)}$$

$$\text{the opposite is: } \forall x > 0, x \text{ has no square root.} \qquad \text{(F)}$$

Rules of Negation

- Rule 6: $\sim (\forall x, \underbrace{\exists y, p}_{q}) = \exists x, \forall y, \sim p$

Proof.

$$
\begin{aligned}
\sim (\forall x, q) &= \exists x, \sim q && \text{(Rule 4)} \\
&= \exists x, \sim (\exists y, p) \\
&= \exists x, \forall y, \sim p && \text{(Rule 5)}
\end{aligned}
$$

□

Rules of Negation

- Rule 6 (cont' d)

Example

$$\forall x > 0, \exists y > 0 \ s.t. \ x = y^2 \qquad \text{(T)}$$
$$\text{opposite: } \exists x > 0, \forall y > 0, x \neq y^2 \qquad \text{(F)}$$

Remark

If statement p is true, then $\sim p$ is False.

Condition Statements (If-then Statements)

Notation

$$p \Rightarrow q \text{ means } \begin{cases} \text{If } p, \text{ then } q \\ p \text{ imples } q \\ p \text{ is sufficient for } q \\ q \text{ is necessary for } p \end{cases}$$

Example

$$\text{If } \underbrace{x > 0}_{p}, \text{ then } \underbrace{x = |x|}_{q} \tag{T}$$

Condition Statements (If-then Statements)

Rule

$$p \Rightarrow q = \sim p \text{ or } q \qquad (\bigstar)$$

Example

If $x > 0$, then $|x| = x$

$$\underbrace{(x > 0)}_{p} \Rightarrow \underbrace{(|x| = x)}_{q} \qquad (\mathrm{T})$$

$$\underbrace{(x \leq 0)}_{\sim p} \text{ or } \underbrace{(|x| = x)}_{q} \qquad (\mathrm{T})$$

Rules of Negation (cont' d)

- Rule 7: $\sim(p \Rightarrow q) = p \text{ and } \sim q$

Proof.

$$\begin{aligned} \sim(p \Rightarrow q) &= \sim(\sim p \text{ or } q) \\ &= \sim(\sim p) \text{ and } \sim q \\ &= p \text{ and } \sim q \end{aligned}$$

$\square$

Example

$$x > 0 \Rightarrow x = |x| \quad \text{(T)}$$

$$\text{opposite: } x > 0 \text{ and } x \neq |x| \quad \text{(F)}$$

Converse / Contrapositive Statement

For the statement "If p, then q" or $p \Rightarrow q$

- Its converse is "If q, then p" or $q \Rightarrow p$
- Its Contrapositive is "If $\sim q$, then $\sim p$" or $\sim q \Rightarrow \sim p$

Converse / Contrapositive Statement

Example

$$\begin{aligned} \text{Statement: } & \text{If } x = -3, \text{ then } x^2 = 9 \\ \text{Converse: } & \text{If } x^2 = 9, \text{ then } x = -3 \\ \text{Contrapositive: } & \text{If } x^2 \neq 9, \text{ then } x \neq -3 \end{aligned}$$

Example

$$\begin{aligned} \text{Statement: } & \text{If } x = -3, \text{ then } 2x = -6 \\ \text{Converse: } & \text{If } 2x = -6, \text{ then } x = -3 \\ \text{Contrapositive: } & \text{If } 2x \neq -6, \text{ then } x \neq -3 \end{aligned}$$

Converse / Contrapositive Statement

Remark

Contrapositive = Statement

Remark

If "$p \Rightarrow q$" and "$q \Rightarrow p$" both are true, we will write "$p \Leftrightarrow q$", and say that p if and only if q, or use abbreviation "p iff q".

Remark

$$\forall\alpha, \forall\beta = \forall\beta, \forall\alpha$$
$$\exists\alpha, \exists\beta = \exists\beta, \exists\alpha$$
$$\forall\alpha, \exists\beta = \exists\beta, \forall\alpha$$

Example

$$(\forall\alpha > 0, \exists\beta, \alpha\beta > 0) \ \textit{True}$$
$$\neq (\exists\beta, \forall\alpha, \alpha\beta > 0) \ \textit{False}$$

Exercise: Negate Each of the Following

- If $\triangle ABC$ is a right triangle, then $a^2 + b^2 = c^2$
- $\forall \epsilon > 0$, $\exists \delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$