CHAPTER 2: PROBABILITY

Counting Methods (Section 2.2, page 62)

Permutations: order matters

In how many ways can you order the letters A, B, C?

ABC, ACB, BAC, BCA, CAB, CBA $\Rightarrow 3! = 3(2)(1) = 6$

— The number of permutations of n objects is $n!$

Five lifeguards are available for duty one Saturday afternoon. There are three lifeguard stations. In how many ways can three lifeguards be chosen and ordered among the stations?

$$\left.\begin{array}{l} \text{5 ways to choose } 1^{st} \text{ L.G} \\ \text{4 ways to choose } 2^{nd} \text{ L.G} \\ \text{3 ways to choose } 3^{rd} \text{ L.G} \end{array}\right\} \therefore 5 \times 4 \times 3 = 60$$

— The number of permutations of k objects chosen from a group of n is $\frac{n!}{(n-k)!}$

Combinations: order doesn't matter

Let's look at the lifeguard question again. If I denote the 5 people as A, B, C, D, E, how do I get 60 permutations?

ABC	ACD	ABE	ADE	ABD	ACE	BCD	BDE	BCE	CDE
ACB									
BAC									
BCA									
CAB									
CBA									

permutation: $\frac{n!}{(n-k)!} = \frac{5!}{(5-3)!} = 60$

Now, if order doesn't matter, there are only 10

combinations: $\binom{n}{k} = \binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$

— The number of combinations of k objects chosen from a group of n objects is $\binom{n}{k}$ = n "choose" k = $\frac{n!}{k!(n-k)!}$

$_nC_r$ ↑ Calculator Button

At a certain event, 30 people attend, and 5 will be chosen at random to receive door prizes. The prizes are all the same, so the order in which the people are chosen does not matter. How many different groups of five people can be chosen?

$\left.\begin{array}{l} n=30 \\ k=5 \end{array}\right\} \Rightarrow$ 30 "choose" 5 $\Rightarrow \binom{30}{5} = \frac{30!}{5!(30-5)!} = 142506$

Conditional Probability and Independence (Section 2.3, page 69)

Conditional Probability

The term **conditional probability** comes from the fact that sometimes, the probability that A occurs depends heavily on whether B has occurred. At other times, when the occurrence or nonoccurrence of B has no effect whatsoever on the probability that A occurs, we say that A and B are **independent events**.

Definition:

Let A and B be two events with $P(B) > 0$. The **conditional probability of A occurring given that B has already occurred** is denoted by $P(A \mid B)$ and can be calculated from the formula:

$$P(A \mid B) = \frac{P(A and B)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

Similarly, Let A and B be two events with $P(A) > 0$. The **conditional probability of B occurring given that A has already occurred** is denoted by $P(B \mid A)$ and can be calculated from the formula:

$$P(B \mid A) = \frac{P(A and B)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

Example (revisited): In a process that manufactures aluminum cans, the probability that a can has a flaw on its side is 0.02, the probability that a can has a flaw on the top is 0.03, and the probability that a can has a flaw on both the side and the top is 0.01.

Given: $P(S)=0.02$, $P(T)=0.03$, $P(S\cap T)=0.01$

(a) What is the probability that a can will have a flaw on the side, given that is has a flaw on top?

$$P(s\mid t)=\frac{P(s\cap t)}{P(t)}=\frac{0.01}{0.03}\approx 0.33$$

(b) What is the probability that a can will have a flaw on the top, given that it has a flaw on the side?

$$P(t\mid s)=\frac{P(t\cap s)}{P(s)}=\frac{0.01}{0.02}=0.5$$

Note: This definition implies that for any events A and B

$$P(A\cap B)=P(A\mid B)P(B)=P(B\mid A)P(A)$$

This statement is called the Multiplication Rule.

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}$$

$$\Longrightarrow P(A\cap B)=P(A\mid B)\cdot P(B)$$

by re-arranging the addition rule

Also: $P(A\cap B)=P(A)+P(B)-P(A\cup B)$

Example: Suppose two cards are selected at random from a deck one at a time and without replacement. Find the probability that

(a) an ace is drawn first and a king is drawn second.

$$P(A\cap K)=P(A)\cdot P(K\mid A)=\left(\frac{4}{52}\right)\left(\frac{4}{51}\right)=0.00603$$

prob. of Ace prob. of King given 1st was an Ace

(b) Two queens are drawn.

↳ a queen is drawn and another queen

$$P(Q_1\cap Q_2)=P(Q_1)\cdot P(Q_2\mid Q_1)=\left(\frac{4}{52}\right)\left(\frac{3}{51}\right)=0.0045$$

prob. of 1st queen prob. of 2nd queen given 1st was a queen

Complete part (a) and (b) with replacement:

$$P(A \cap K) = P(A) \cdot P(K|A) = \left(\frac{4}{52}\right)\left(\frac{4}{52}\right) = 0.0059$$

$$P(Q_1 \cap Q_2) = P(Q_1) \cdot P(Q_1|Q_2) = \left(\frac{4}{52}\right)\left(\frac{4}{52}\right) = 0.0059$$

Example: In a large insurance agency, 60% of the customers have automobile insurance, 40% of the customers have homeowners insurance and 75% of the customers have one type or the other.

(a) Find the proportion of customers with both types of insurance. $\Rightarrow P(A \cap H) = ?$

auto and homeowner insurance

Given: $P(A) = 0.6$, $P(H) = 0.4$, $P(A \cup H) = 0.75$

(can't use this: $P(A \cap H) = P(H) \cdot P(A|H)$, since not enough info. given)

Recall: $P(A \cup H) = P(A) + P(H) - P(A \cap H)$

$$\therefore P(A \cap H) = P(A) + P(H) - P(A \cup H) = 0.6 + 0.4 - 0.75 = 0.25$$

(b) Find the probability that a customer has homeowner insurance given that he has automobile insurance.

$$P(H|A) = \frac{P(H \cap A)}{P(A)} = \frac{0.25}{0.6} = 0.416$$

Example (revisited): A sample of 1000 persons screened for a certain disease is distributed according to height and disease status resulting from a clinical exam as follows:

		DISEASE STATUS				
		None	Mild	Moderate	Severe	Totals
	Tall	122	78	139	61	400
HEIGHT	Medium	74	51	90	35	250
	Short	104	71	121	54	350
		300	200	350	150	1000

(a) A person is sampled at random from this population. What is the probability that he/she has a moderate disease, given the person is tall?

Let m be moderate

T be tall

$$P(m \mid T) = \frac{P(m \cap T)}{P(T)} = \frac{139/10000}{400/10000} = 0.3475$$

must re-word this so that it states one event given another event

(b) Given the person is short, find the probability that he/she has no disease.

Let N be no disease

Let S be short

$$P(N \mid S) = \frac{P(N \cap S)}{P(S)} = \frac{104/10000}{350/10000} = 29.7\%$$

(c) Find the probability that the person has no disease.

$$P(N) = \frac{300}{1000} = 30\%$$

Notice that prob. of No disease is 30%;

prob. of No disease given short height is 29.7%

The additional information of short height affects the prob. of no disease. (slighting since prob. went down)

This means that "short" & "no disease" are NOT independent events.

If those 2 prob. were exactly the same, then the 2 events would be independent.

Independent Events

Conditional probability is used when the likelihood of occurrence of an event depends on whether or not another event occurs. At the other end of the spectrum are events that do not impose such restrictions on each other's chances of occurring.

Definition:

Two events, A and B, are **independent events** if the probability of each event remains the same whether or not the other occurs. In this case,

$$P(A/B) = P(A) \text{ or, equivalently, } P(B/A) = P(B)$$

If either $P(A) = 0$ or $P(B) = 0$, then A and B are independent.

Also, if A and B are independent events, then

$$P(A \cap B) = P(A)P(B)$$

Example: A group of 200 voters in a certain electoral poll are classified according to gender on one hand and candidate preference on the other.

	A	B	C	Total
Male	20	36	24	80
Female	40	44	36	120
Total	60	80	60	200

The probability a voter favors candidate A given that the voter is a male is

$$P(A\,|\,M) = \frac{P(A \cap M)}{P(M)} = \frac{20/200}{80/200} = \frac{20}{80} = 0.25$$

Notice that the probability a randomly chosen voter favors A is

P(A)= 60/200 = .30

While 30% of this group of voters favor candidate A, a different percentage of the male voters (namely 25%) favor candidate A. Thus knowledge of the gender of the voter is useful and leads us to update our probability of candidate A preference.

The events A = "a voter favors candidate A" and M = "a voter is a male" are called DEPENDENT EVENTS because P(A|M)≠P(A).

Now look at the probability a voter favors candidate C given that the voter is a male.

$$P(C \mid M) = \frac{P(C \cap M)}{P(M)} = \frac{24}{80} = 0.3$$

Notice that the probability a voter favors candidate C is

P(C) = 60/200 = .30

Now in this case knowledge of the gender of the voter gives us no additional information regarding preference for candidate C, and does not lead to an updating of this probability (while 30% of this group of voters favor candidate C, the same percentage of the male voters also favor candidate C). The events C = "a voter favors candidate C" and M="a voter is a male" are called **INDEPENDENT EVENTS** because P(C|M) = P(C).

There are a number of equivalent ways of defining independence.

Remember: two events A and B are said to be **INDEPENDENT** if any of the following statements hold:

(a) P(A|B) = P(A)

(b) P(B|A) = P(B)

(c) P(A and B) = P(A)P(B).

independent events formula

Statement (c) is the MULTIPLICATION RULE FOR INDEPENDENT EVENTS.

Example: Suppose A is the event an adult is a male and B is the event an adult favors capital punishment. Further suppose that in a certain country, 50% of the adults are male, 80% of the adults favor capital punishment, and 40% of the adults are male and favor capital punishment. For this country, answer the following:

(a) Are the events A and B independent? Explain.

show either (a), (b) or (c) above, easiest to show (c)

Given: $P(A)=0.5$, $P(B)=0.8$, $P(A\cap B)=0.4$

(c) $P(A\cap B) = P(A)\cdot P(B)$

since $P(A\cap B)=0.4$, $P(A)\cdot P(B) = 0.5(0.8) = 0.4$

$\therefore P(A\cap B) = P(A)\cdot P(B)$

$\therefore$ Yes, A and B are independent.

(b) Are the events A and B mutually exclusive? Explain.

No! since $P(A\cap B)=0.4 \neq 0$

for M.E, $P(A\cap B)$ must equal 0

Example: A vehicle contains two engines, a main engine and a backup. The engine component fails only if both engines fail. The probability that the main engine fails is 0.05, and the probability that the backup engine fails is 0.10. Assume that the main and backup engines function independently. What is the probability that the engine component fails?

can use independent formula

let m be main, B be back up $\Rightarrow P(m)=0.05$, $P(B)=0.1$

$$P(\text{failure}) = P(m\cap B) = P(m)\cdot P(B)$$
$$= (0.05)(0.1) = 0.005$$

Example: A system contains two components, A and B. Both components must function for the system to work. The probability that component A fails is 0.08, and the probability that component B fails is 0.05. Assume the two components function independently. What is the probability that the system functions?

$$P(\text{sys. function}) = P(A\cap B)$$
$$= P(A)\cdot P(B)$$
$$= (1-0.08)(1-0.05)$$
$$= 0.874$$

Example: An unfair coin has a probability of 0.7 of coming up "heads" when tossed. Suppose it is tossed in such a way that the outcome of each toss is independent of the outcome of any other toss.

(a) If the coin is tossed n=2 times, find the probability of obtaining two heads.

$$P(H_1 \cap H_2) = P(H_1) \cdot P(H_2) = (0.7)(0.7) = 0.49$$

(b) If the coin is tossed n = 3 times, find the probability of obtaining two heads followed by a tail.

$$P(H_1 \cap H_2 \cap T) = P(H_1) \cdot P(H_2) \cdot P(T)$$
$$= (0.7)(0.7)(1-0.7) = 0.147$$

Note: independent events $\Rightarrow P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdots\cdots P(A_n) = \prod_{i=1}^{n} P(A_i)$

Example: Given $P(A) = 0.7$, $P(B) = 0.4$, $P(AorB) = 0.82$

(a) Find $P(AandB)$

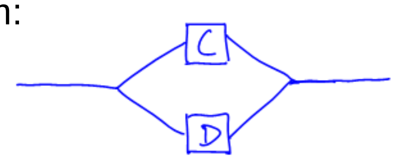
$$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= 0.7 + 0.4 - 0.82$$
$$= 0.28$$

(b) Find $P(A/B)$

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.28}{0.4} = 0.7$$

(c) Are A & B independent? Yes ! since $P(A|B) = P(A) = 0.7$

Example: A system contains two components, C and D, connected in parallel as shown in the following diagram:

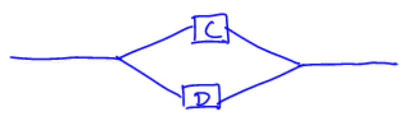


The system will function if either C or D functions. The probability that C functions is 0.90, and the probability that D functions is 0.85. Assume C and D function independently. Find the probability that the system functions.

↳ so, can use the independent formula; otherwise, no!

$$
\begin{aligned}
P(\text{sys. function}) &= P(C \cup D) \\
&= P(C) + P(D) - P(C \cap D) \quad \text{independent} \\
&= 0.9 + 0.85 - P(C) \cdot P(D) \\
&= 0.9 + 0.85 - (0.9)(0.85) \\
&= 0.985
\end{aligned}
$$

Example: A system contains two components, C and D, connected in parallel as shown in the following diagram:



Assume C and D function independently. For the system to function, either C or D must function.

a) If the probability that C fails is 0.08 and the probability that D fails is 0.12, find the probability that the system functions.

$$
\begin{aligned}
\underbrace{P(\text{sys. functions})}_{P(C \cup D)} &= P(C) + P(D) - P(C \cap D) \quad \text{independent} \\
&= P(C) + P(D) - P(C) \cdot P(D) \\
&= (1-0.08) + (1-0.12) - (1-0.08)(1-0.12) = 0.9904
\end{aligned}
$$

Or:

$$
\begin{aligned}
\underbrace{P(\text{sys. functions})}_{P(C \cup D)} &= 1 - P(\text{sys. fails}) \\
&= 1 - P(C' \cap D') = 1 - P(C') \cdot P(D') = 1 - (0.08)(0.12) \\
&= 0.9904
\end{aligned}
$$

b) If both C and D have probability *p* of failing, what must the value of *p* be so that the probability that the system functions is 0.99?

$$P(\text{sys. functions}) = 1 - P(C' \cap D')$$

$$\Longrightarrow \quad 0.99 = 1 - [\underbrace{P(C')}_{p} \cdot \underbrace{P(D')}_{p}] \Rightarrow 0.99 = 1 - p^2 \Rightarrow p = 0.1$$

c) If three components are connected in parallel, function independently, and each has probability p of failing, what must the value of p be so that the probability that the system functions is 0.99?

$$\Rightarrow 0.99 = 1 - P(C' \cap D' \cap E')$$

$$\Rightarrow 0.99 = 1 - [\underbrace{P(C')}_{p} \cdot \underbrace{P(D')}_{p} \cdot \underbrace{P(E')}_{p}]$$

$$\Rightarrow 0.99 = 1 - p^3$$

$$\therefore p = 0.2154$$

Test your knowledge:

1. A study has been done to determine whether or not a certain drug leads to an improvement in symptoms for patients with a particular medical condition. The results are shown in the following table:

	Improvement	No Improvement	Total
Drug	270	530	800
No Drug	120	280	400
Total	390	810	1200

Based on this table, what is the probability that a patient shows improvement if it is known that the patient was given the drug?

a) 0.225
b) 0.325
c) 0.3375
d) 0.6667
d) none of the above

$$P(I|D) = \frac{P(I \cap D)}{P(D)} = \frac{270/1200}{800/1200} = 0.3375$$

2. A box contains four red, two white, and three green marbles, all of which are the same size. Two marbles are selected one after the other from the box, without replacement. What is the probability that the marbles are the same colour?

a) 0.014
b) 0.25
c) 0.28
d) 0.67
e) none of the above

$$P(R_1 \cap R_2) + P(W_1 \cap W_2) + P(G_1 \cap G_2)$$

$$= \left(\frac{4}{9}\right)\left(\frac{3}{8}\right) + \left(\frac{2}{9}\right)\left(\frac{1}{8}\right) + \left(\frac{3}{9}\right)\left(\frac{2}{8}\right)$$

$$= \frac{1}{6} + \frac{1}{36} + \frac{1}{12} = \frac{5}{18} \approx 0.28$$

3. A company has three plants at which it produces a certain item. 30% are produced at Plant A, 50% at Plant B, and 20% at Plant C. Suppose the 1%, 4% and 3% of the items produced at Plants A, B, and C respectively are defective. If an item is selected at random from all those produced, what is the probability that the item was produced at Plant B and is defective?

a) 0.04
b) 0.02
c) 0.2
d) none of the above

$$P(B \cap D) = P(B) \cdot P(D \mid B)$$
$$= (0.5)(0.04)$$
$$= 0.02$$