# TECHNIQUES OF INTEGRATION

## *Integration by Parts (Section 3.1; Book 2)*

**Recall:** So far, we have learned how to integrate some basic functions.

e.g. $\int x(x+1)dx$ → Expand

e.g. $\int xe^{x^2}dx$ → U-Sub ($x^2$)

Now we will continue to investigate more advanced integration techniques where our previous methods won't work.

e.g. $\int x\sin x dx$ ?

**Recall:** You may remember that *u*-substitution came about based on undoing the chain rule. Similarly, we can use the product rule for differentiation to derive a useful rule for integration. $\frac{d}{dx}(x\sin x) =$ product rule $x\cos x + \sin x \cdot 1$ {can't diff each}

The product rule states that, for *f*, *g* differentiable,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Int. both sides: $f(x)g(x) = \int \boxed{f(x)g'(x)\,dx} + \int g(x)f'(x)\,dx$

deriv. just cancel

**Integration by Parts Formula:** re-arrange

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$u = f(x)$
$v = g(x)$

or, alternatively

$$\int udv = uv - \int vdu$$

should be easier

**Example:** $\int \underset{dv}{\overset{u}{x}\sin x dx}$

$\int u\,dv = uv - \int v\,du$

let $u = x$   Derivative $\frac{du}{dx} = 1$ → $du = dx$

$dv = \sin x\,dx$   Anti-deriv. Integrate ← $v = -\cos x$

$= x(-\cos x) - \int(-\cos x)dx$

$= -x\cos x + \int \cos x\,dx$

$= -\cos x + \sin x + C$

**Question:** What if we had chosen $u$ and $dv$ differently in the above example?

let $u = \sin x$ $\quad dv = x\,dx$

$\hookrightarrow du = \cos x\,dx$ $\quad v = \frac{1}{2}x^2$

$\rightarrow uv - \int v\,du$

$\int x \sin x\,dx = (\sin x)(\frac{1}{2}x^2) - \int \frac{1}{2}x^2 \cos x\,dx$

now it is harder than the original question X

**Question:** How do we choose $u$ and $dv$?

- The new integral $\int v\,du$ should be easier than the original
- You have to be able to integrate $dv$ to obtain $v$
- L I A T E tells you, from left to right, what function get priority for being u

log Inverse trig Algebraic {root function, polys, ....} Trig Exp

**Examples:** $\int \underset{A}{x} \underset{T}{\sec^2 x}\,dx$ $\quad \int \underset{A}{x}\,\underset{E}{3^x}\,dx$ $\quad \int \frac{\overset{L}{\ln x}}{\underset{A}{\sqrt{x}}}\,dx$

$u =$ x $\quad u =$ x $\quad u =$ ln x

$dv =$ $\sec^2 x\,dx$ $\quad dv =$ $3^x\,dx$ $\quad dv =$ $\frac{1}{\sqrt{x}}\,dx$

**Example:** $\int \underset{E}{x^7}\,\underset{L}{\ln x}\,dx$

$\int u\,dv = uv - \int v\,du$ $\quad \frac{d}{dx}\ln x = \frac{1}{x}$

$u = \ln x$ $\quad dv = x^7\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{1}{8}x^8$

$= (\ln x)(\frac{1}{8}x^8) - \int \frac{1}{8}\cancel{x^8}^{7}\,\frac{1}{\cancel{x}}\,dx$

$= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7\,dx$

$= \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C$ $\quad$ ($1/8 \cdot 1/8$)

**Question:** What about definite integrals?

**Integration by Parts for Definite Integrals:**

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

**Example:** If the previous question had been $\int_2^5 x^7 \ln x\,dx$, in the 1st step we'd get:

$$\ln x \cdot \frac{1}{8}x^8 \Big|_2^5 - \int_2^5 \frac{1}{8}x^8 \cdot \frac{1}{x}\,dx$$

Now let's go on to study some more complicated examples of applying integration by parts:

Sometimes, you have to apply integration by parts more than once.

example

$t^9 e^{-t}$ then 9 times

**Application:** If the rate of change of medication in the bloodstream is $\frac{dA}{dt} = t^2 \cdot e^{-t}$, what is the net change in the amount of medication from time $t=0$ to $t=1$?

Integrate

2

$$\int_0^1 t^2 e^{-t}\,dt$$

$S: \int u\,dv = uv - \int v\,du$

$u = t^2$ $dv = e^{-t}\,dt$

$du = 2t\,dt$ $v = -e^{-t}$

undo chain

$$= t^2 \cdot (-e^{-t})\Big|_0^1 - \int_0^1 -e^{-t} \cdot 2t\,dt$$

just simplify it

$$= -t^2 e^{-t}\Big|_0^1 + 2\boxed{\int_0^1 t e^{-t}\,dt}$$

$u = t$ $dv = e^{-t}\,dt$

$du = dt$ $v = -e^{-t}$

$$= -t^2 e^{-t}\Big|_0^1 + 2\,t(-e^{-t})\Big|_0^1 - \int_0^1 -e^{-t}\,dt$$

$$= -t^2 e^{-t}\Big|_0^1 - 2te^{-t}\Big|_0^1 + 2\int_0^1 e^{-t}\,dt$$

$$= -t^2 e^{-t}\Big|_0^1 - 2te^{-t}\Big|_0^1 - 2e^{-t}\Big|_0^1$$

$$= 0.16$$

$\therefore$ net change of 0.16

Sometimes, you can apply integration by parts even though you're only integrating a single function.

**Example:** $\int \ln x \, dx$

L A

$\int u\,dv = uv - \int v\,du$

$u = \ln x$  $dv = dx$

$du = \frac{1}{x}dx$  $v = x$

$= \ln x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} dx$

$= x\ln x - \int dx$

$= x\ln x - x + C$

Occasionally, you have to "go in circles".

**Example:** $\int e^x \sin x \, dx$

E T

$\int u\,dv = uv - \int v\,du$

$u = \sin x$  $dv = e^x dx$

$du = \cos x \, dx$  $v = e^x$

$= \sin x \cdot e^x - \int e^x \cos x \, dx$

not easier nor harder

$u = \cos x$  $dv = e^x dx$

$du = -\sin x \, dx$  $v = e^x$

$= \sin x \cdot e^x - \left[\cos x \cdot e^x - \int e^x - \sin x \, dx\right]$

$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$\therefore \int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

divide by 1/2

$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$

$\int e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C$