Monotone Class Theorem

Crosby Collins
crosbyqcollins@gmail.com

Theorem 1.1. Let Ω be a set and $\mathcal{A}$ an algebra of subsets of Ω such that $\Omega, \varnothing \in \mathcal{A}$. Then, there exists a smallest monotone class S containing $\mathcal{A}$ such that it is also the smallest σ-algebra containing $\mathcal{A}$.

Proof. Let S be the intersection of all monotone classes contiaing $\mathcal{A}$, namely the smallest monotone class containing $\mathcal{A}$. Clearly, since σ-algebras are closed under countable unions and intersections, and in particular, closed under increasing and decreasing countable unions and intersections respectively, σ-algebras are monotone classes. Thus, we have that $\sigma(\mathcal{A}) \supseteq S$. From here, it suffices to show that S is closed under finite intersections and complementation.

Define the set

$$\mathcal{C}(A) =: \{B \in S : A \in S, B \cup A \in S\}.$$

Thus, we have that $\mathcal{C}(A) \supseteq \mathcal{A}$, since $\mathcal{A}$ is an algebra. Define an increasing sequence $B_i \subset \mathcal{C}(A)$. Since S is a monotone class and $A \cup B_i \uparrow$ in S

$$\bigcup_{i=1}^{\infty} A \cup B_i \in S,$$

by definition. Similarly, for a decreasing sequence of sets B_i,

$$\bigcap_{i=1}^{\infty} A \cup B_i \in S.$$

Now, we conclude that $\mathcal{C}(A)$ is a monotone class and that $\mathcal{C}(A) = S$. Therefore, S is closed under finite unions.

Now, we prove the latter requirement. Define

$$\mathcal{N} =: \{B \in S : B^c \in S\}.$$

Similar to before, we have that $N \supseteq \mathcal{A}$ since algebras are closed under relative complementation. If $B_i \subset \mathcal{N}$ is increasing, then the sequence $B_i^c \downarrow$ and

$$\bigcap_{i=1}^{\infty} B_i^c \in S.$$

On the contrary, for a decreasing set B_i^c, we have

$$\bigcup_{i=1}^{\infty} B_i^c \in S.$$

We see that $\mathcal{N} = S$ and thus, $\sigma(\mathcal{A}) \subseteq S$. $\square$