Nomenclature of Measure Theory

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Definition 1.1. A subset $\Sigma \subset \mathcal{P}(\Omega)$ of a set Ω is called a *σ-algebra* if

(i) $A \in \Sigma \Rightarrow A^c \in \Sigma$

(ii) If $\{A_j\}_{j\in\mathbb{N}} \subset \Sigma$ then $\bigcup_{j=1}^{\infty} \in \Sigma$

(iii) $\Omega \in \Sigma$

Consequently, the *smallest σ-algebra* containing a family $\mathcal{F} \subset \mathcal{P}(\Omega)$ is

$$\sigma(\mathcal{F}) =: \bigcap_j \Sigma_j,$$

where $\Sigma_j \supseteq \mathcal{F}$. A *measure* is a function $\mu : \Sigma \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\bigcup_{j=1}^{\infty}) = \sum_{i=1}^{\infty} \mu(A_j)$

A subset is said to be *measurable* if it is an element of a σ-algebra of the set containing it. We can equip a measure and a σ-algebra to a sample space Ω. This is called a *measure space* written as the triple (Ω, Σ, μ). A measure space is *σ-finite* if there are countably many sets $A_j \in \Sigma$ such that $\mu(A_j) < \infty$ and $\Omega = \bigcup_{j=1}^{\infty} A_j$. Given two measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ their *product σ-algebra* is

$$\Sigma_1 \times \Sigma_2 = \sigma(\{A_1 \times A_2 : A_j \in \Omega_j\}).$$

This σ-algebra has the *section property* namely, if we take an arbitrary $A \in \Sigma_1 \times \Sigma_2$ such that if we define

$$A_1(x_2) =: \{x_1 \in \Omega_1 : (x_1, x_2) \in A\} \in \Sigma_1,$$

for all x_2. An analogous property holds for $A_2(x_1) \in \Sigma_2$. Similarly, we have the unique *product measure* of the two measure spaces $\mu =: \mu_1 \times \mu_2$ with the property that

$$\mu(A_1 \times A_2) = \mu(A_1)\mu(A_2).$$

A collection of sets $\mathcal{M}$ is a *monotone class* if

(i) $A_j \in \mathcal{M}, \forall i \in \mathbb{N}$ and $A_1 \subset A_2 \subset A_3 \cdots \Longrightarrow \bigcup_j A_j \in \mathcal{M}$

(ii) $B_j \in \mathcal{M}, \forall i \in \mathbb{N}$ and $B_1 \supset B_2 \supset B_3 \cdots \Longrightarrow \bigcap_j A_j \in \mathcal{M}$

Lastly, a collection of sets $\mathcal{A}$ forms an *algebra of sets* if for two arbitrary elements of $\mathcal{A}$, their relative complements and their union are also elements of $\mathcal{A}$.