More on Measures

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1 Jordan Measure

Definition 1.1. Let $E \subset \mathbb{R}^n$ be a bounded subset. Recall that an elementary set is a subset of euclidean n-space such that it can be expressed as the union of finitely many rectangles.

- We define the *Jordan inner measure* $m_{*,(J)}(E)$ as

$$m_{*,(J)}(E) =: \sup_{A \subset E;\, A \text{ elementary}} m(A).$$

- We define the *Jordan outer measure* $m^{*,(J)}(E)$ as

$$m^{*,(J)}(E) =: \inf_{B \supset E;\, B \text{ elementary}} m(B).$$

- If a set's Jordan inner and outer measures are equivilant, then it is called *Jordan measurable* and we define its *Jordan measure* m^n by

$$m^n(E) =: m^{*,(J)}(E) = m_{*,(J)}(E).$$

  Note that by $m(A)$ and $m(B)$, we refer to the elementary measure of the elementary sets A and B respectively.

In a moment, we will see the connection between the Jordan Measure and the Riemann integral (note that the definitions of Riemann and Darboux integration can be extended to higher dimensions however, this is not neccesary for our purposes). The standard definition of the Riemann Integral is rather awkward to use in practice hence, we will construct the Riemann Integral in terms of upper and lower Darboux Integrals. Define a *piecewise constant function* $f : [a, b] \to \mathbb{R}$ as one that is locally constant, that is there exists a collection of intervals $I_1, \ldots, I_n$ of $[a, b]$ such that $f \equiv c_i$ for some $c_i \in \mathbb{R}$ on each interval I_i. The piecwise integral of such a function f is

$$\text{p.c} \int_a^b f =: \sum_{i=1}^{\infty} c_i |I_i|.$$

Proposition 1.2. *The above definition of the piecewise constant integral is independent of the choice of partition.*

Proof. Let $P = (x_0, x_1, \ldots, x_{n-2}, x_{n-1})$ and $P' = (x'_0, x'_1, \ldots, x'_{m-2}, x'_{m-1})$ be partitions of the interval $I = [x_0, x_{n-1}] = [a, b]$. Let $A_i = (x_{i-1}, x_i]$ for $0 \leq i \leq n$ and $B_j = (x_{j-1}, x_j]$ for $0 \leq j \leq m$. Then, there exists two sequences $\{\alpha_i\}$ and $\{\beta_j\}$ such that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} = \sum_{j=1}^{m} \beta_j \chi_{B_j},$$

where χ_{A_i} and χ_{B_j} both represent a sequence of *characteristic functions*[1] defined by

$$\chi_A =: \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

Observe, also, that

$$\bigcup_{i,j} A_i \cap B_j = \bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j = I,$$

[1] Charateristic functions are also called indicator functions and are sometimes denoted $\mathbf{1}_A$. As the name implies, the function is just indicating to us whether something is in a given set. This is convenient when we want to focus in on a particular set and nullify anything outside of it.

and $\alpha_i = \beta_j$ on $A_i \cap B_j$ by the definition of f. Now,

$$\begin{aligned}
\text{p.c} \int_a^b f &= \sum_{i=0}^n \alpha_i |A_i| = \sum_{i=0}^n \alpha_i |A_i \cap I| \\
&= \sum_{i=0}^n \alpha_i |A_i \cap \bigcup_j B_j| = \sum_{i=0}^n \alpha_i (\sum_{j=0}^m |A_i \cap B_j|) \\
&= \sum_{i=0}^n \sum_{j=0}^m \alpha_i |A_i \cap B_j| = \sum_{i=0}^n \sum_{j=0}^m \beta_j |A_i \cap B_j| \\
&= \sum_{j=0}^m \sum_{i=0}^n \beta_j |A_i \cap B_j| = \sum_{j=0}^m \beta_j |\bigcup_i A_i \cap B_j| \\
&= \sum_{j=0}^m \beta_j |I \cap B_j| = \sum_{j=0}^m \beta_j |B_j|.
\end{aligned}$$

Thus, we conclude that the piecewise integral is independent of partitions. □

Definition 1.3. Let $f : [a, b] \to \mathbb{R}$ be bounded, then we define its *upper Darboux Integral* as

$$\underline{\int_a^b} f(x)\ dx =: \sup_{g \geq f, \text{piecwise constant}} \text{p.c} \int_a^b g(x)\ dx,$$

where g is piecewise bounded above f. Then as well, we define its *lower Darboux Integral* by

$$\overline{\int_a^b} f(x)\ dx =: \inf_{h \geq f, \text{piecwise constant}} \text{p.c} \int_a^b h(x)\ dx.$$

If these two integrals are equal, then we say that f is *Darboux Integrable.*

Proposition 1.4. *A function is Riemann Integrable if and only if it is Darboux Integrable, in which case they are equivalent.*

Proof. If f is an arbitrary fixed function on $[a, b]$, then it's Riemann sum is sandwiched between two Darboux sums (Darboux Integrals). Therefore, Darboux integrability implies Riemann integrability for fixed f. Contrarily, if $f : [a, b] \to \mathbb{R}$ is bounded, as proposed in the hypothesis, and $\mathcal{P}$ is a partition of $[a, b]$, then for some $\varepsilon > 0$ there exists two taggings of the partition $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$\mathcal{R}(f, \mathcal{P}_1) \leq L(f, \mathcal{P}) + \varepsilon \quad \text{and} \quad U(f, \mathcal{P}) - \varepsilon \leq \mathcal{R}(f, \mathcal{P}_2).$$

where $U(\cdot, \cdot)$ and $L(\cdot, \cdot)$ represent the upper and lower Darboux sums respectively and $\mathcal{R}(\cdot, \cdot)$ represents a Riemann sum. □

Observation 1.5. As you will recall, the integral over some interval of a function can be interpreted as the are between the function and the x-axis which naturally, extends to the two dimensional Jordan measure. Take a bounded function $f : [a, b] \to \mathbb{R}$ and define the sets $E_+ =: \{(x, t) \mid x \in [a, b]; 0 \leq t \leq f\}$ and $E_- =: \{(x, t) \mid x \in [a, b]; f(x) \leq t \leq 0\}$. Then,

$$\int_a^b f(x)\ dx = m^2(E^+) - m^2(E_-).$$

Of course, this follows from the fact that $f = f^+ - f^-$.

2 Lebesgue Measure

Observation 2.1. Notice that the outer Jordan measure has the property of finite additivity, namely it can be written as

$$m^{*,(J)}(E) = \inf_{\bigcup_{i=1}^n B_i \supset E; \{B_i\}_{i=1}^n \text{ rectangles}} |B|_1 + \cdots + |B_n|.$$

If we replace the finite union of boxes with a countable union, we attain the Lebesgue Outer Measure.

Definition 2.2. We define the *Lebesgue Outer Measure*[2] $\lambda^* : E \to \overline{\mathbb{R}}_+$ where E is a subset of euclidean n-space by

$$\lambda^*(E) =: \inf_{\bigcup_{j=1}^\infty R_j \supset E; R_j \text{ boxes}} \sum_{j=1}^{\infty} |R_j|.$$

A subset $E \subset \mathbb{R}^n$ is *Lebesgue Measurable* if for an arbitrary $\varepsilon > 0$, there exists an open set $\mathcal{O} \supseteq E$ such that

$$\lambda^*(\mathcal{O} - E) \leq \varepsilon$$

For a measurable subset $E \subset \mathbb{R}^n$, its lebesgue measure $\lambda : E \to \overline{\mathbb{R}}$ is defined by

$$\lambda(E) =: \lambda^*(E).$$

Proposition 2.3. (Outer Measure Properties)

(i) (Null Set) $\lambda^*(\varnothing) = 0$;

(ii) (Monotonicity) *If* $E \subset F \subset \mathbb{R}^n$, *then* $\lambda^*(E) \leq \lambda^*(F)$;

[2]The Lebesgue Outer measure is sometimes called the exterior measure

(iii) (Countable sub-additivity) *If* $\{E_j\}_{j\in\mathbb{N}}$ *is a countable collection of subsets of* $\mathbb{R}^n$, *then*

$$\lambda^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \lambda^*(E_j).$$

Proof. Properties (i) and (ii) follow from the definition of the Lebesgue Outer Measure since all collections of rectangles R_j cover the empty set and any cover of F covers E. Now, onto property (iii). The proof for $\lambda^*(E_j) = \infty$ is trivial so we assume $\lambda^*(E_j) < \infty$. If $\varepsilon > 0$, we have a countable collection of coverings of $E_j \subset \bigcup_{j=1}^{\infty} R_{ij}$ by rectangles such that

$$\sum_{j=1}^{\infty} \lambda(R_{ij}) \leq \lambda^*(E) + \frac{\varepsilon}{2^i}.$$

Thus, R_j is a countable covering of

$$E =: \bigcup_{i=1}^{\infty} E_i,$$

and therefore

$$\lambda^*(E) \leq \sum_{i,j}^{\infty} \lambda(R_{ij}) \leq \sum_{i=1}^{\infty} \left\{\lambda^*(E) + \frac{\varepsilon}{2}\right\} = \sum_{i=1}^{\infty} \lambda^*(E_i) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, our condition is satisfied. $\square$

Remark 2.4. The reason we require two definitions is that neither are perfect:

1. The Lebesgue Outer Measure is defined for all subsets of $\mathbb{R}^n$, but it is not additive as stated above.

2. The Lebesgue Measure is a measure, but it is not defined for all subsets of $\mathbb{R}^n$. It is, however, defined for all open subsets of $\mathbb{R}^n$ (Excersise 1.1).