Lebesgue Integral

Crosby Collins
crosbyqcollins@gmail.com
(You are welcome to email me with questions).

Definition 1.1. The function $f : (X, \Sigma) \to (\mathbb{R}, \Sigma')$ is said to be a *measureable* $E \in \Sigma \Rightarrow f^{-1}(E) \in \Sigma'$.

Definition 1.2. A *simple* function $\varphi(x)$ is a finite linear combination of characteristic functions

$$\varphi(x) =: \sum_{i=1}^{n} a_i \chi_{A_i},$$

where

$$\chi_A(x) =: \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

Another description is that a simple function is one that takes on finitely many values in its range. One can think of such functions as a sum of finitely many rectangles. Using this idea, we define the Lebesgue integral of a simple function $\varphi(x)$ defined on the set $E \subseteq A$ as

$$\int_E \varphi(x)\ d\mu =: \sum_i c_i \mu(E \cap A).$$

We wish to extend this definition to a higher class of functions, measurable functions. We define the Lebesgue Integral of a function $f : E \to \mathbb{R}$ as

$$\int_E |f|\ d\mu =: \sup_{\substack{0 \le \varphi \le f \\ \varphi \text{ simple}}} \int_E \varphi(x)\ d\mu. \tag{1}$$

Basically, what we are doing here is approximating the function f with simple functions, the integrals of which by definition can be expressed in terms of the measure. Taking the limit of these integrals, we obtain our desired result.

We define the space of *Lebesgue integrable* functions L^1 as all $f : E \to \mathbb{R}$ such that

$$\int_E |f| \; d\mu < \infty, \tag{2}$$

or equivalently, as it follows from the Monotone Convergence Theorem, such that

(i) $\sum_{j=1}^{\infty} \int_{\mathbb{R}} |\varphi_j| < \infty$

(ii) $f(x) = \sum_{j=1}^{\infty} \varphi_j(x)$ a.e

where $\{\varphi_j\}_{j \in \mathbb{N}}$ is a series of simple functions and by a.e, we mean *almost everywhere* (up to a set of measure zero). The L^1 norm is defined as

$$\|f\|_{L^1(\mathbb{R}^n)} =: \int_{\mathbb{R}^n} |f| \; d\mu.$$

Remark 1.3. In the definitions (1) and (2) above, we included absolute value bars in the integrand on the RHS. The justification for this is that

$$f \in L^1 \iff |f| \in L^1,$$

for measurable f. We can see this by partitioning f into its positive and negative parts f^-, f^+. Recall that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. By definition, the positive and negative parts of an integrable function are integrable, thus we can say the same for $|f|$ and f.

Proposition 1.4. *Suppose $f, g \in L^1(E)$, then*

(i) $\|\alpha f\|_{L^1(\mathbb{R}^n)} = |a| \|f\|_{L^1(\mathbb{R}^n}$

(ii) $\|f + g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R}^n)} + \|g\|_{L^1(\mathbb{R}^n)}$

(iii) $\|f\|_{L^1(\mathbb{R}^n)} \equiv 0 \iff f \equiv 0$ *a.e*

(iv) $d(f, g) =: \|f - g\|_{L^1(\mathbb{R})}$ *defines a metric on* $L^1(\mathbb{R}^n)$

Exercise 1.5. Show that L^1 is a Banach Space (i.e it is complete in its metric).