Potential

A force is conservative $\longleftrightarrow$ a potential energy can be associated with it

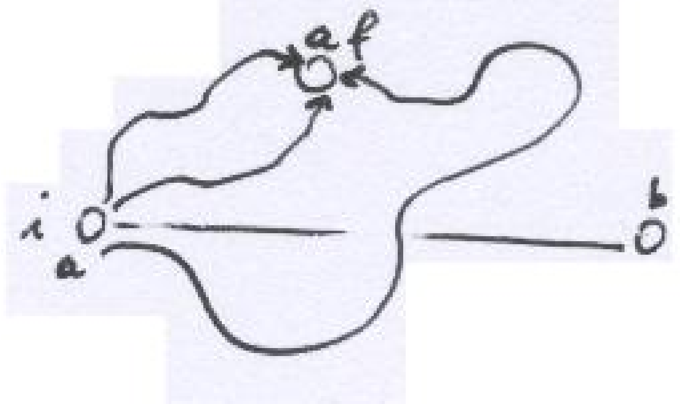
vector (3D) $\longleftrightarrow$ scalar

Electric Potential Energy

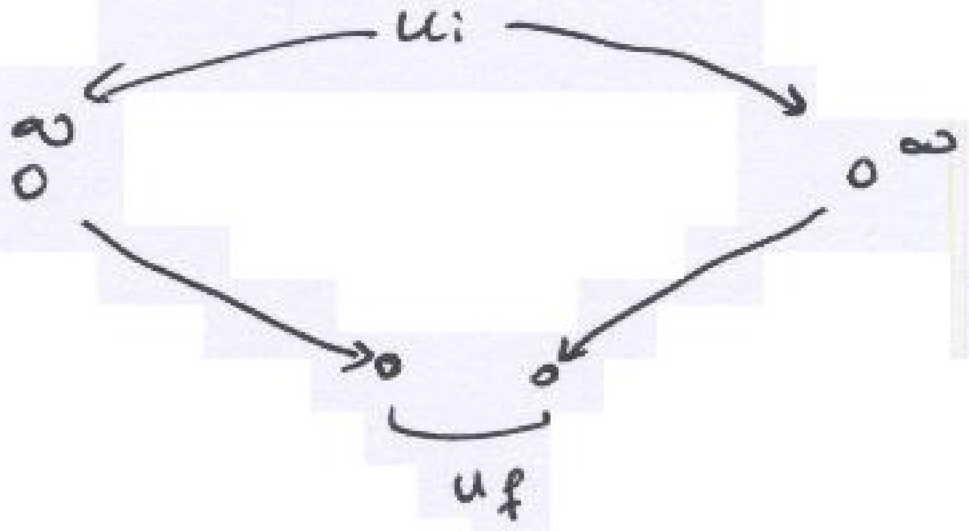
If a system of two or more charges changes configuration from i to f

$\rightarrow$ Resulting change ΔU in the pot. energy of the system is

$$\Delta U = U_f - U_i = -\underbrace{W}_{\hookrightarrow \text{work done (path independent)}}$$

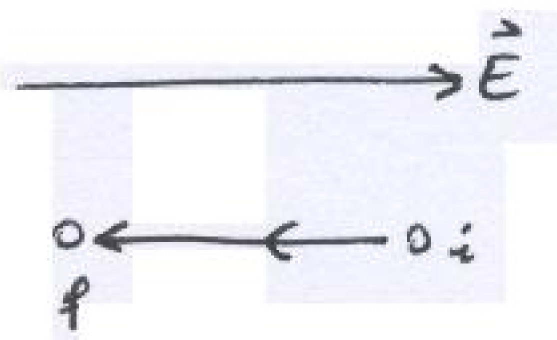


$U_i = 0$ when the particles are infinitely separated from each other



$U = -W_\infty$

Ex: A proton moves from i to f in a uniform Electric field



a.) Does the electric field do a positive or negative work? (negative)

b.) Does the electric potential Energy of the proton increase or decrease? (increases)

$W = \vec{F} \cdot \vec{d}$

$\vec{F} = q\vec{E}$

Electric Potential

Potential Energy depends on the magnitude of charge.
However, potential Energy per unit Charge has a unique value at any point in an electric Field.

Ex: $q_1 = 1.6 \times 10^{-19}$ C in an $\vec{E}$
where its potential Energy is 2.4×10^{-17} J

$$\frac{2.40 \times 10^{-17} \text{ J}}{1.60 \times 10^{-19} \text{ C}} = 150 \text{ J/C}$$

$$q_2 = 3.2 \times 10^{-19} \text{ C}, \quad U = 4.80 \times 10^{-17} \text{ J}$$

$$\rightarrow \quad \frac{U}{q} = 150 \text{ J/C}$$

Electric Potential: $V = \frac{U}{q}$ (scalar); $\frac{J}{C} = \text{Volt}$

$$\Delta V = V_f - V_i = \frac{U_f}{q} - \frac{U_i}{q} = \frac{\Delta U}{q}$$

$\Delta U = -W \rightarrow \Delta V = V_f - V_i = -\frac{W}{q}$ (potential Difference)

→ The potential Difference between two points is thus the negative of the work done by the electrostatic force to move a unit charge from one point to the other.

$U_i = 0$ at $\infty \longrightarrow V_\infty = 0 \Rightarrow V = -\frac{W_\infty}{q}$ (potential)

$$[1\text{ V}] = \frac{J}{C} \equiv \text{Volt}$$

$$[1 E 1] = \frac{N}{C} = \frac{N}{C} \underbrace{\left(\frac{VC}{J}\right)}_{=1} \underbrace{\left(\frac{J}{Nm}\right)}_{=1} = \frac{V}{m}$$

1 eV (electron volt) is the energy equal to the work required to move a single elementary charge $|e|$ through a ΔV of one volt.

$$1\,eV = e\,(1V) = (1.60\times10^{-19}\,C)(1\,J/C) = 1.60\times10^{-19}\,J$$

($\underline{H^+} = 13.6\,eV$
↳ ionization energy of hydrogen)

Work done by an applied Force

q, $i \longrightarrow f$ by applying a Force on it

W_{app}, while $\vec{E}$ does work W

$$\underbrace{\Delta K}_{\text{change in the kinetic Energy}} = K_f - K_i = W_{app} + W$$

Suppose v is 0 before and after the move

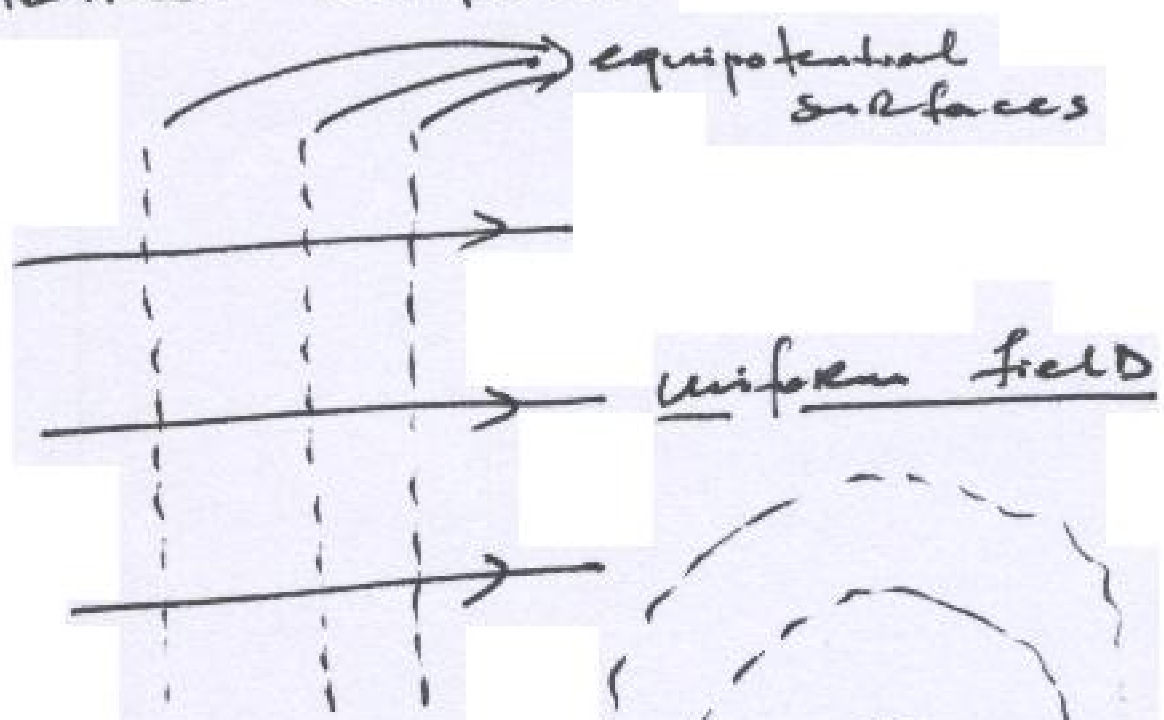
$$\rightarrow K_i = K_f = 0$$

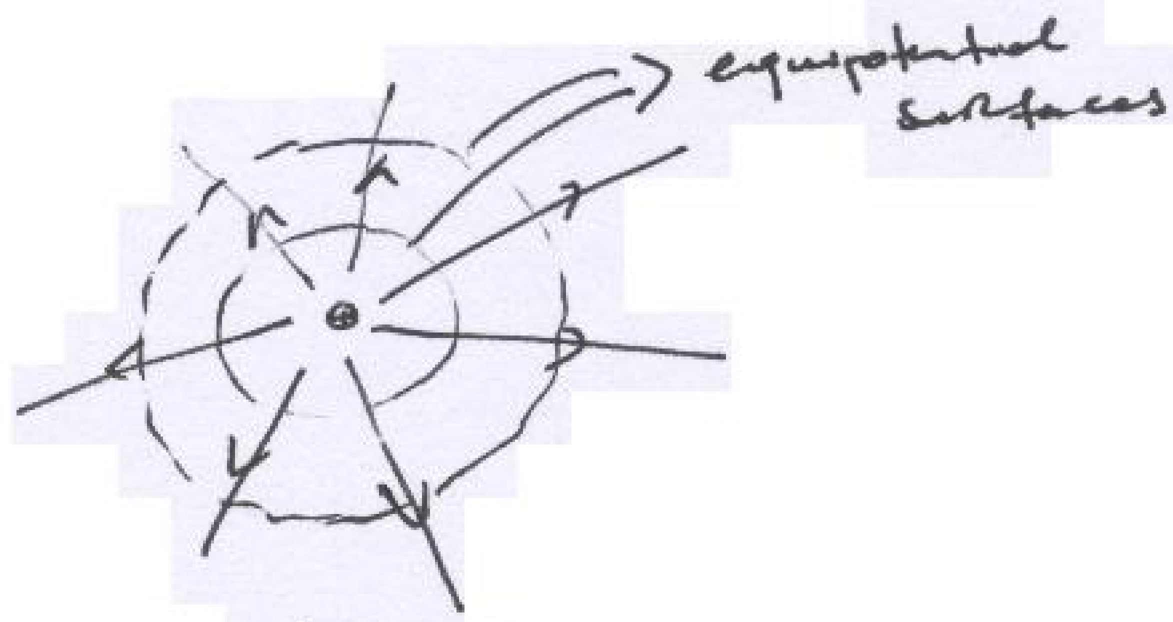
$$W_{app} = -W$$

$$\Delta U = U_f - U_i = W_{app}$$

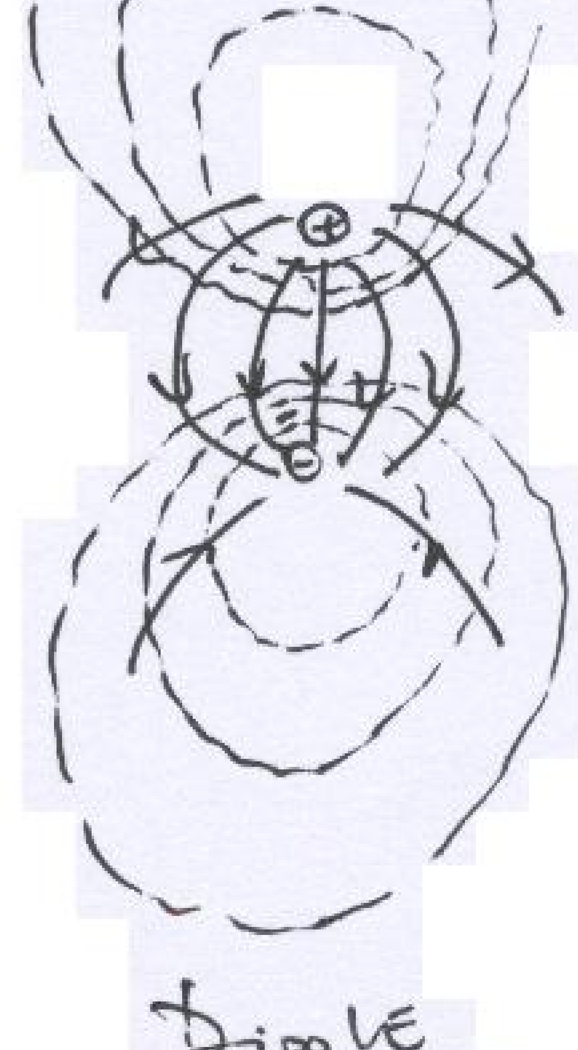
$$W_{app} = q\,\Delta V$$

Equipotential Surfaces





Charge

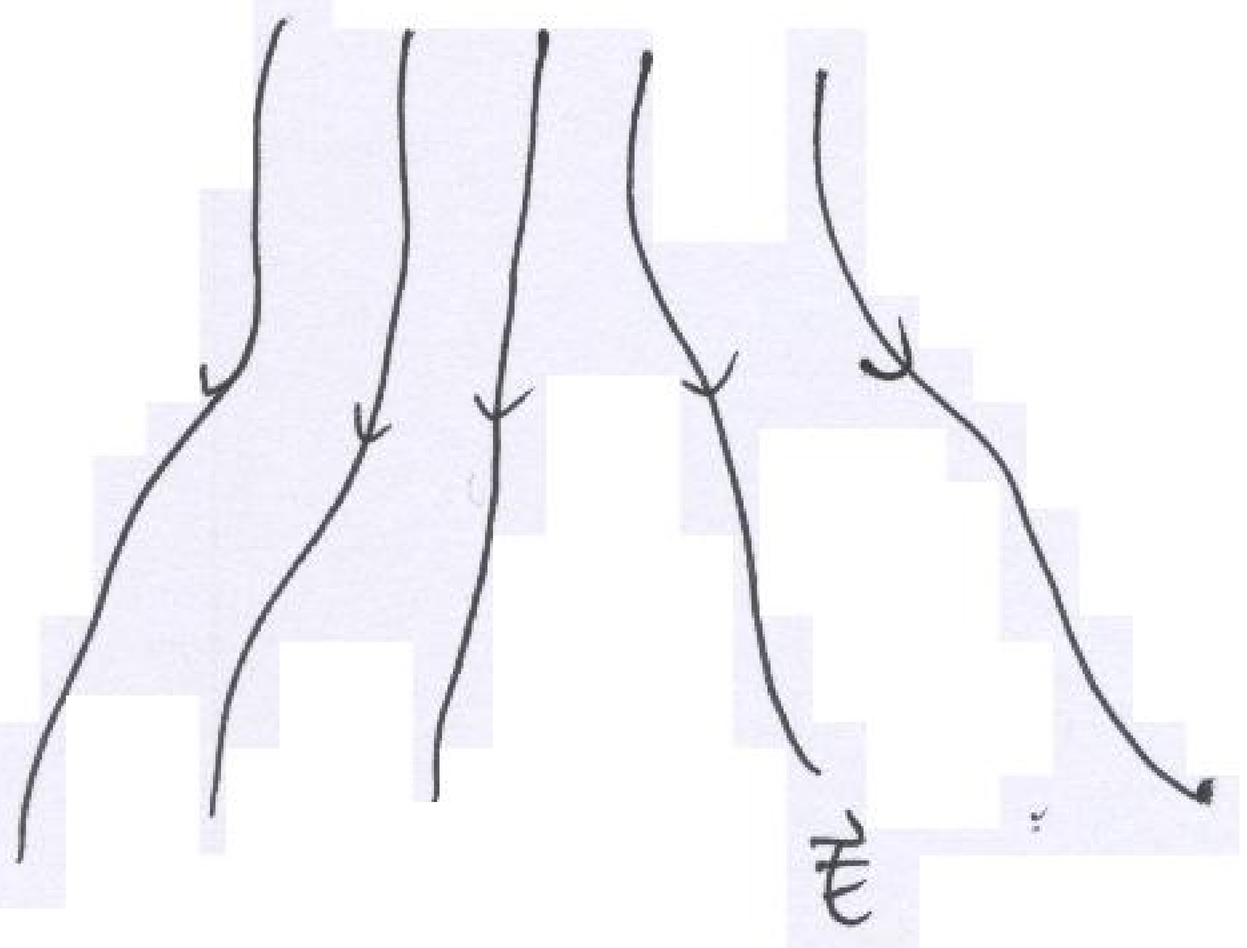


Dipole

Equipotential Surfaces are always perpendicular to the electric field lines

$\Rightarrow \vec{E}$ is always tangent

Calculating the Potential From the field



$q_0\vec{E}$

$dW = \vec{F}\cdot d\vec{s}$

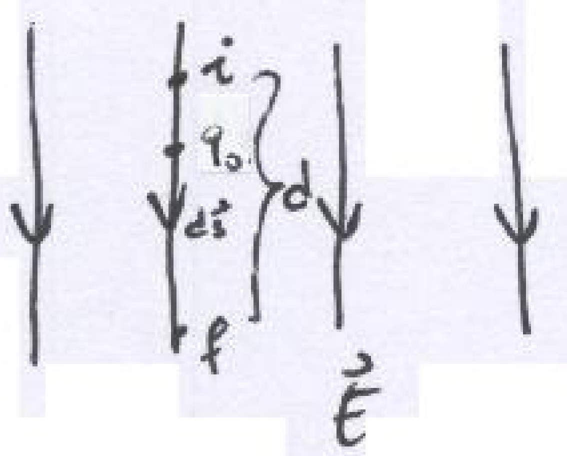
$= q_0\vec{E}\cdot d\vec{s}$

$W = q_0\int_i^f \vec{E}\cdot d\vec{s}$

$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}$

$V_i = 0 \rightarrow V = -\int_i^f \vec{E}\cdot d\vec{s}$

(relative to the zero potential at point i)

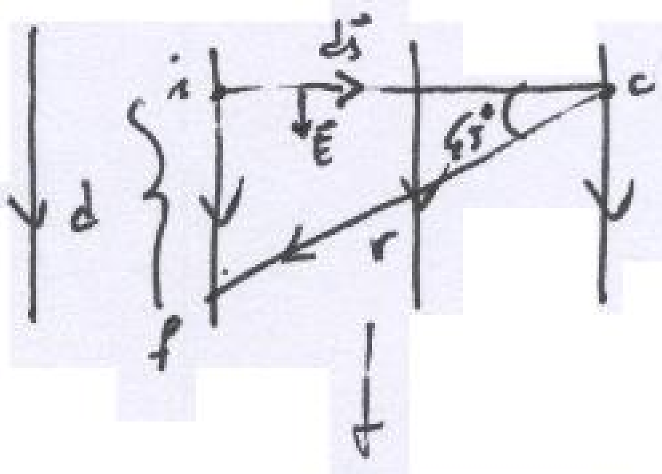


$\vec{E}\cdot d\vec{s} = E\,ds\,\cos\overset{0}{\theta} = E\,ds$

$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s} = -\int_i^f E\,ds = -E\int_i^f ds = -Ed$

$\downarrow$

$V_f < V_i$: the potential always decreases along a path that extends in the direction of $\vec{E}$ lines



$r\sin 45 = d$

$r\frac{\sqrt{2}}{2} = d \rightarrow r = \frac{2d}{\sqrt{2}}$

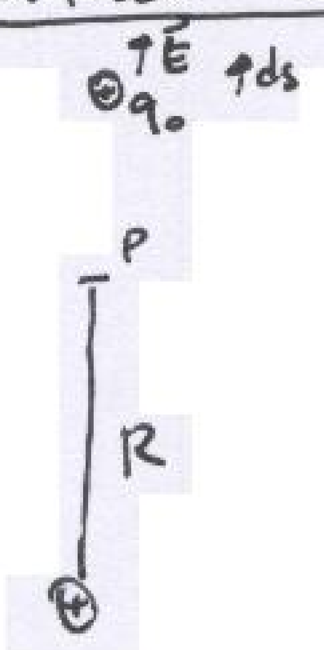
ic : $\vec{E}\cdot d\vec{s} = E\,ds\,\cos\overset{90}{\theta} = 0$

$\hookrightarrow$ same potential

cf : $V_f - V_c = -\int_c^f \vec{E}\cdot d\vec{s} = -\int_c^f E\cos 45^\circ ds$

$= -E\cos 45\int_c^f ds = -E\frac{\sqrt{2}}{2}\cdot\frac{2d}{\sqrt{2}} = \underline{\underline{-Ed}}$

Potential Due to a Point Charge



$$\vec{E} \cdot d\vec{s} = E \cos\theta \, ds$$

$$V_f - V_i = -\int_R^\infty E \, dr$$

$$V_f = 0 \ (\infty) \rightarrow V_i = V(R)$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

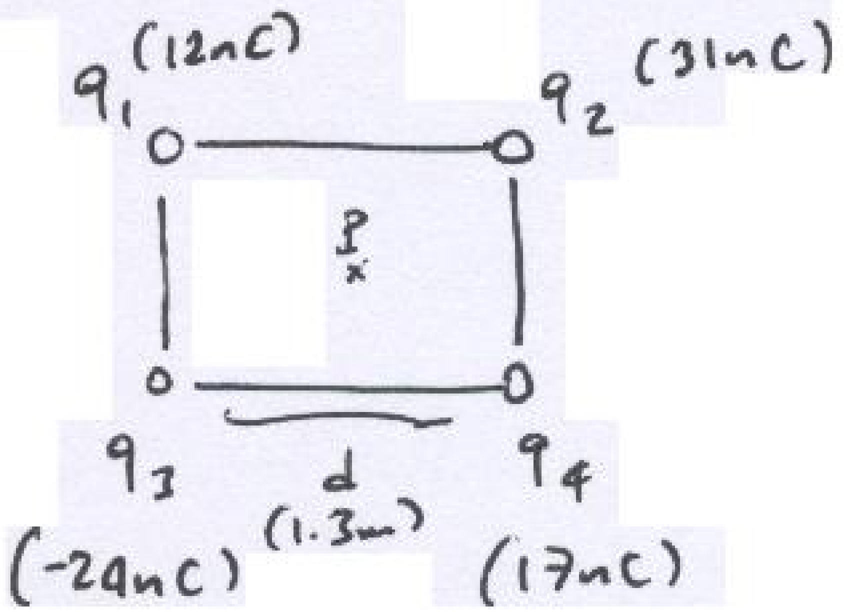
$$0 - V = -\frac{q}{4\pi\epsilon_0} \int_R^\infty \frac{1}{r^2} dr = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r}\right]_R^\infty = -\frac{1}{4\pi\epsilon_0} \frac{q}{R}$$

$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \rightarrow$$ the sign of V is the same as the sign of $\underline{\underline{q}}$

Potential Due to a Group of Point Charges

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} \quad \text{(n point charges)}$$

Ex: Electric potential at point P located at the center of the square of point charges:



$$V = \sum_{i=1}^{4} V_i = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1}{r} + \frac{q_2}{r} + \frac{q_3}{r} + \frac{q_4}{r} \right)$$

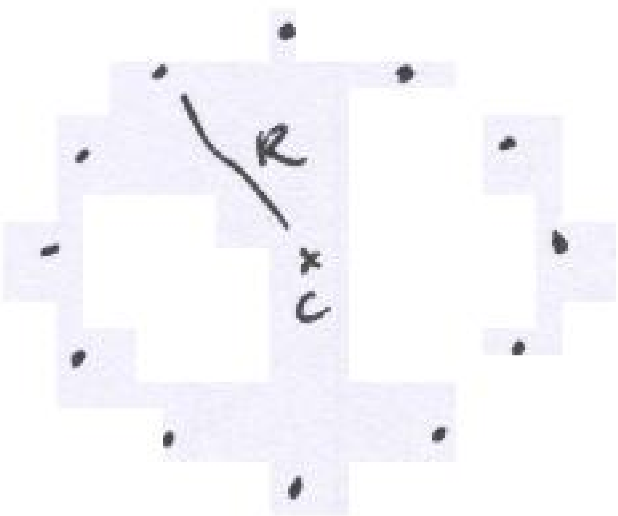
$$r = \frac{d}{\sqrt{2}} \simeq 0.919\text{m}$$

$$\sum q_i = (12 - 24 + 31 + 17) \times 10^{-9} \text{C} = 36 \times 10^{-9} \text{C}$$

$$\rightarrow V = \frac{(8.99 \times 10^9 \text{Nm}^2/\text{C}^2)(36 \times 10^{-9} \text{C})}{0.919 \text{m}}$$

$$V \simeq 350 \text{V}$$

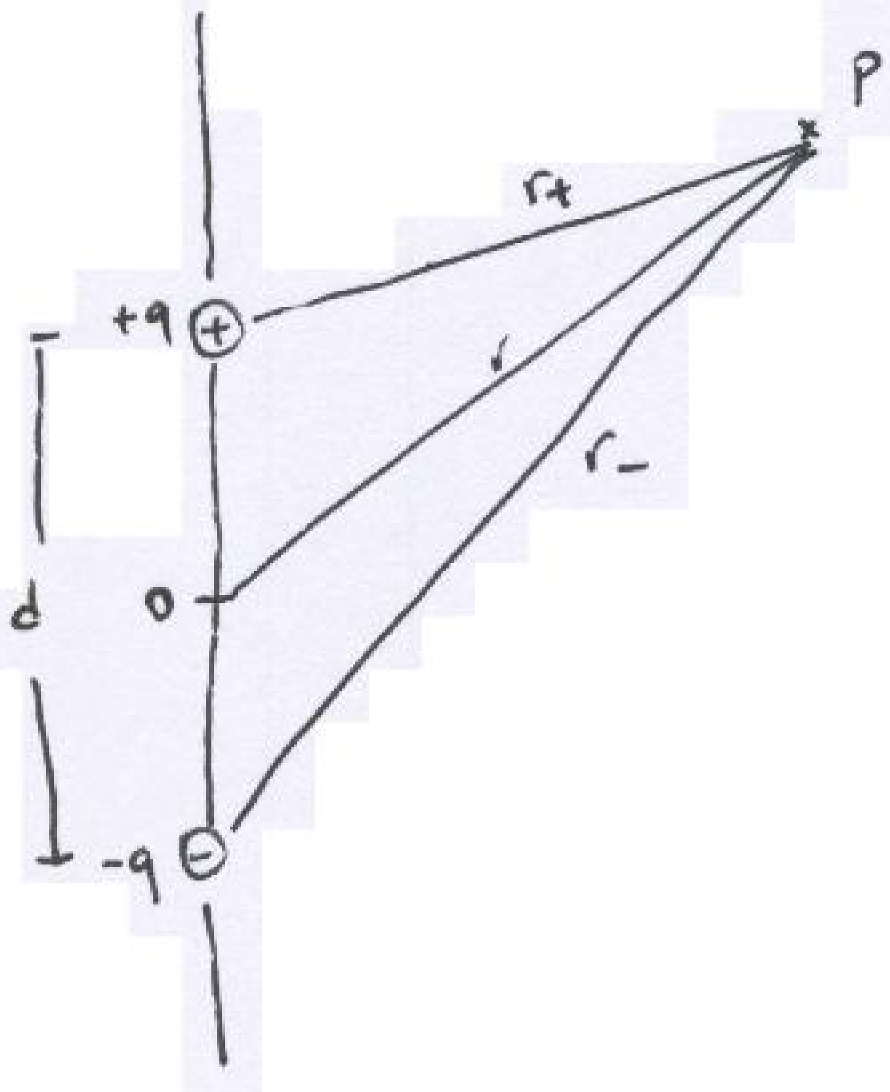
Ex: 12 e^- $(-|e^-|)$



$V = -12 \frac{1}{4\pi\epsilon_0} \frac{e}{R}$ [orientation of the electrons does not matter]

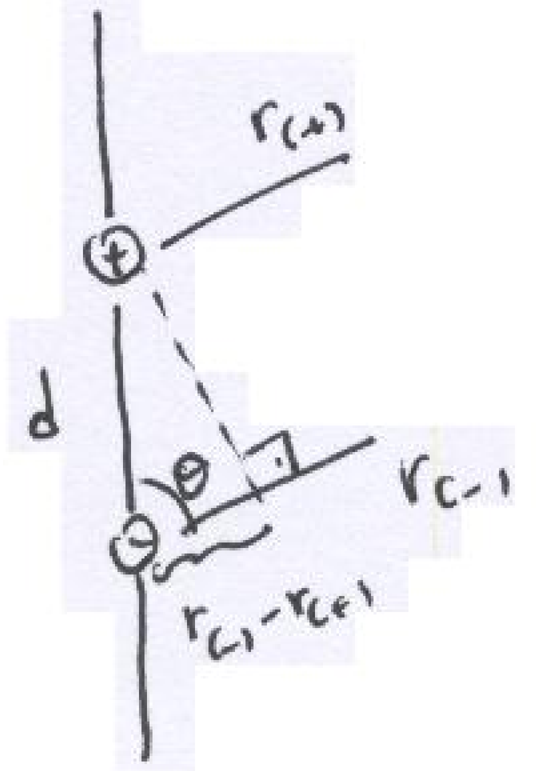
$\vec{E} = 0$ [orientation is important!]

Potential Due to an Electric Dipole



$$V = \sum_{i=1}^{2} V_i = V_{(+)} + V_{(-)} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_{(+)}} + \frac{-q}{r_{(-)}}\right) = \frac{q}{4\pi\epsilon_0}\,\frac{r_{(-)} - r_{(+)}}{r_{(-)}\, r_{(+)}}$$

$r >> d$:

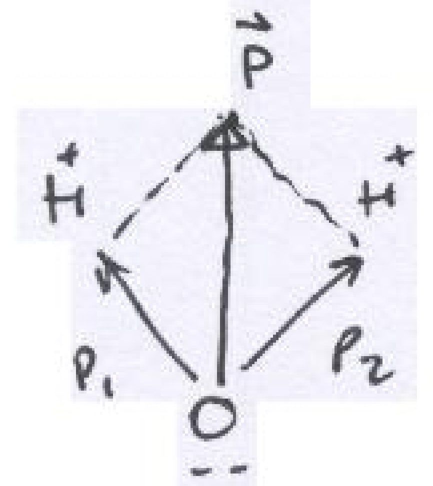


$$r_{(-)} - r_{(+)} \simeq d\cos\theta\ , \quad r_{(-)}\, r_{(+)} \approx r^2$$

→ p : dipole moment (magnitude)

$$V = \frac{q}{4\pi\epsilon_0}\,\frac{d\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\,\frac{p\cos\theta}{r^2}$$

Induced dipole moment

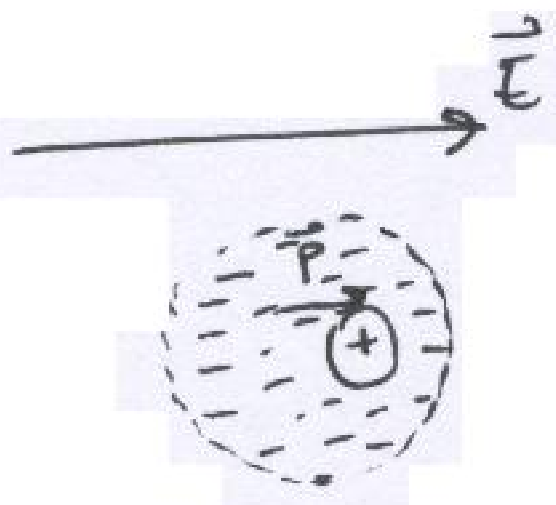


permanent electric dipole moment

non-polar molecules:

i)

ii)

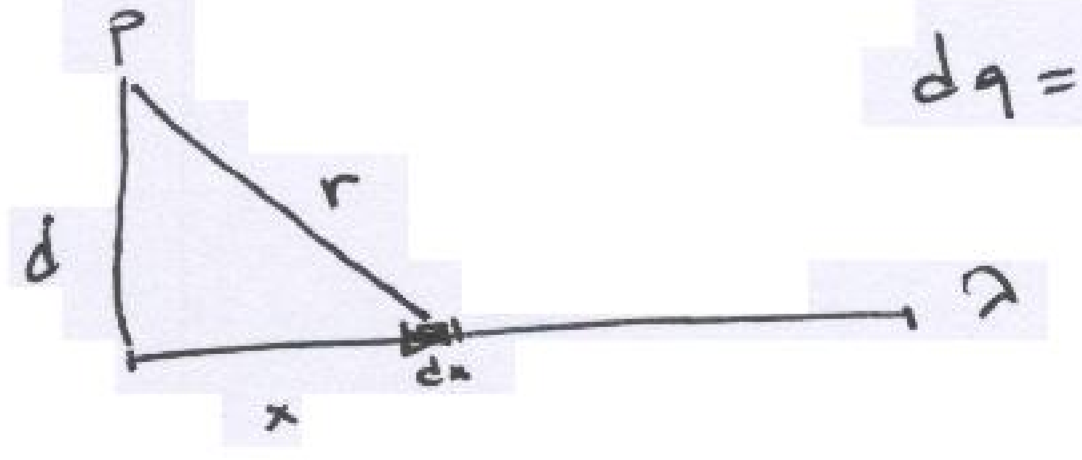


Potential Due to a Continuous Charge Distribution

dq , $V_\infty = 0$

$$dV = \frac{1}{4\pi\epsilon_0}\,\frac{dq}{r} \rightarrow V = \int dV = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$$

* line of Charge



$$dq = \lambda dx$$

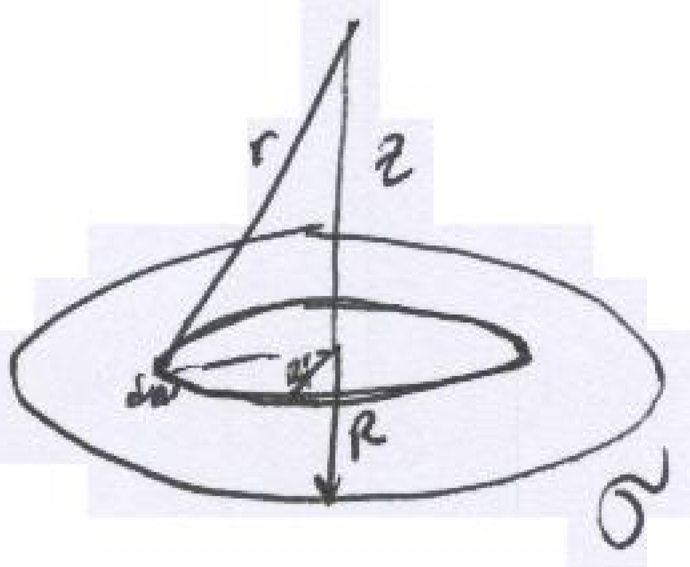
$$r = (x^2 + d^2)^{1/2}$$

$$dV = \frac{1}{4\pi\epsilon_0}\,\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\,\frac{\lambda\, dx}{(x^2+d^2)^{1/2}}$$

$$V = \int dV = \int_0^L \frac{1}{4\pi\epsilon_0}\,\frac{\lambda}{(x^2+d^2)^{1/2}}dx = \frac{\lambda}{4\pi\epsilon_0}\int_0^L\frac{dx}{(x^2+d^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(x + (x^2+d^2)^{1/2}\right)\right]_0^L = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L + (L^2+d^2)^{1/2}}{d}\right]$$

Charged Disk



$$dq = \sigma\,(2\pi R')(dR')$$

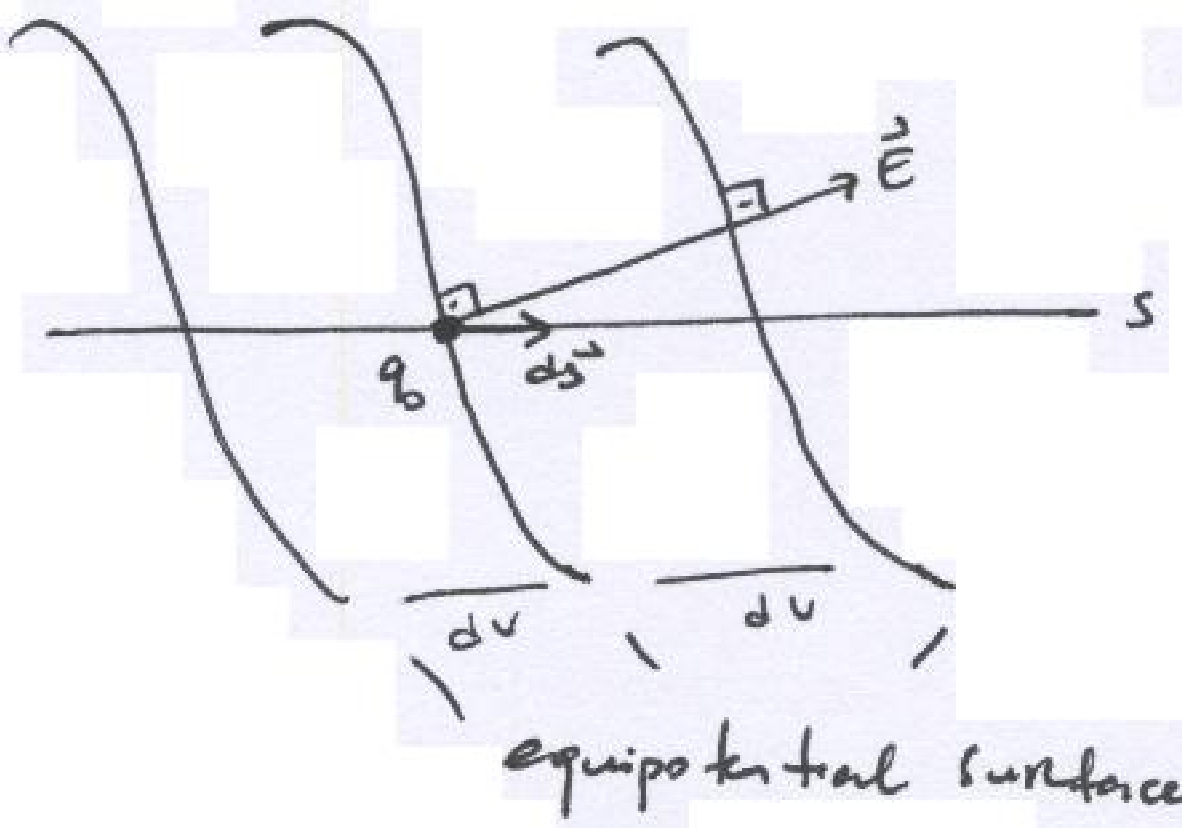
$$dV = \frac{1}{4\pi\epsilon_0}\,\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\,\frac{\sigma(2\pi R')(dR')}{\sqrt{z^2+R'^2}}$$

$$V = \int dV = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{R'\,dR'}{\sqrt{z^2+R'^2}} = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2}-z\right)$$

Calculating the Field From the Potential

$\vec{E} \to V$: $V = \int \vec{E}\cdot d\vec{s}$

$V \to \vec{E}$?



The work the electric field does on the test charge $\quad W = -q_0\,dV$

$$W = \underbrace{(q_0\vec{E})}_{\vec{F}}\cdot\underbrace{d\vec{s}}_{d\vec{x}} = q_0 E\cos\theta\,ds$$

$$\Rightarrow\ -q_0\,dV = q_0 E(\cos\theta)\,ds$$

$$\to\ \underline{E\cos\theta} = -\frac{dV}{ds}$$

E_s component $\to$ $E_s = -\dfrac{\partial V}{\partial s}$ $\quad (s = x, y, z)$

The component of $\vec{E}$ in any direction is the negative of the rate at which the electric potential changes with distance in that direction.

$$\Rightarrow E_x = -\frac{\partial V}{\partial x}\ ;\ E_y = -\frac{\partial V}{\partial y}\ ;\ E_z = -\frac{\partial V}{\partial z}$$

$$\vec{E} = E_x\,\hat{i} + E_y\,\hat{j} + E_z\,\hat{k}$$

Ex: The electric Potential at any point on the central axis of a uniformly charged disk:

$$V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - z\right)$$

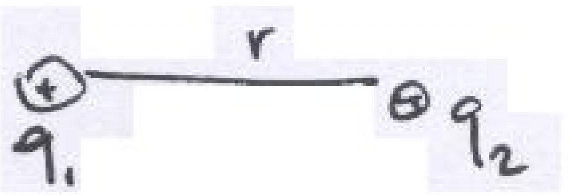
$$\longrightarrow \vec{E} = ? \longrightarrow \underline{E_z}$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\epsilon_0}\frac{d}{dz}\left(\sqrt{z^2+R^2} - z\right)$$

$$= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

Electric Potential Energy of a system of Point Charges

The electric potential energy of a system of a fixed point charges is equal to the work that must be done by an external agent to assemble the system, bringing each charge from an infinite distance.

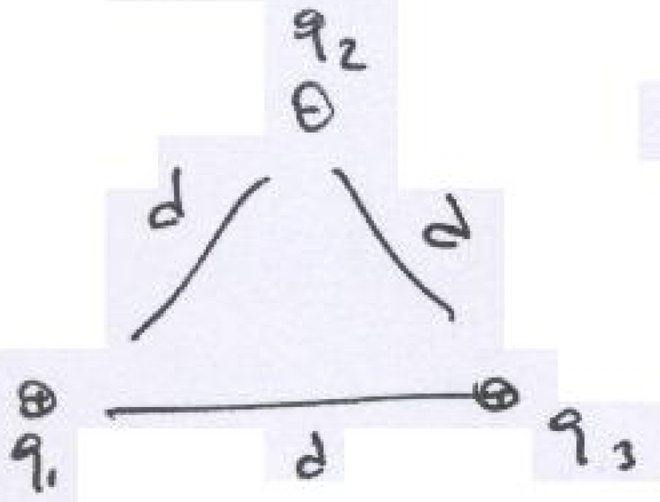


$$W = q_2 V \quad \left(\text{our work} \quad \overset{\text{Sign}}{(-) \longrightarrow (+)}\right)$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r}$$

$$U = W = q_2 V = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$$

Ex: Potential Energy of a system of 3 charged particles



$q_1 = q$, $q_2 = -4q$, $q_3 = 2q$ $\quad q = 150\,nC$

q_1 first (no work required)

$$q_2 : \; U_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{d}$$

$$q_3 : \; W_{13} + W_{23} = U_{13} + U_{23} = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{d}}_{U_{13}} + \underbrace{\frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{d}}_{U_{23}}$$

$U = U_{12} + U_{13} + U_{23} = \dots = -17\,mJ$ $\quad$ $U < 0 \rightarrow$ it is their tendency to stay like this (bound, stable)

Potential of a Charged Isolated Conductor

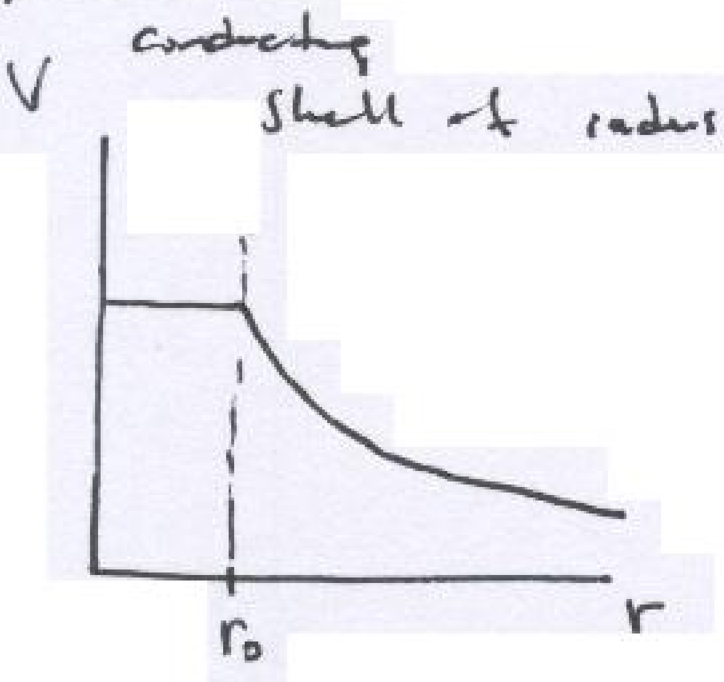
An excess charge placed on an isolated conductor will distribute itself on the surface of that conductor so that all points of the conductor – whether on the surface or inside – come to the same potential. This is true even if the conductor has an internal cavity and even if that cavity contains a net charge.

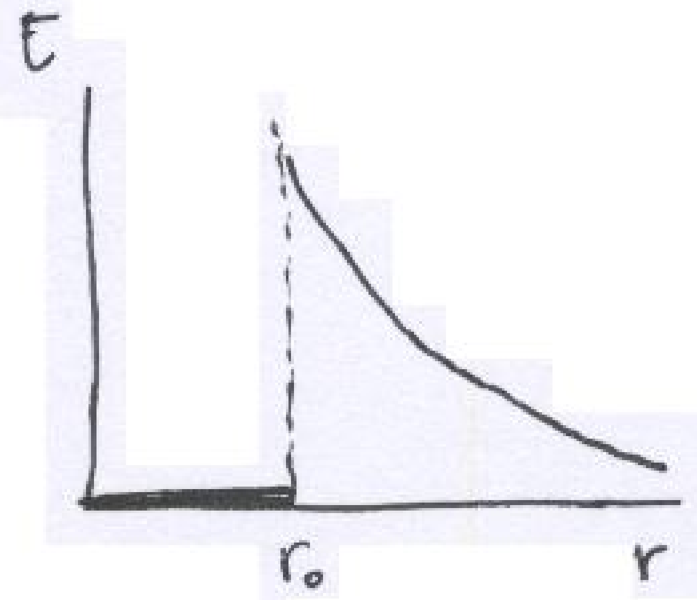
$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$$

$\vec{E} = 0$ for all points in the conductor

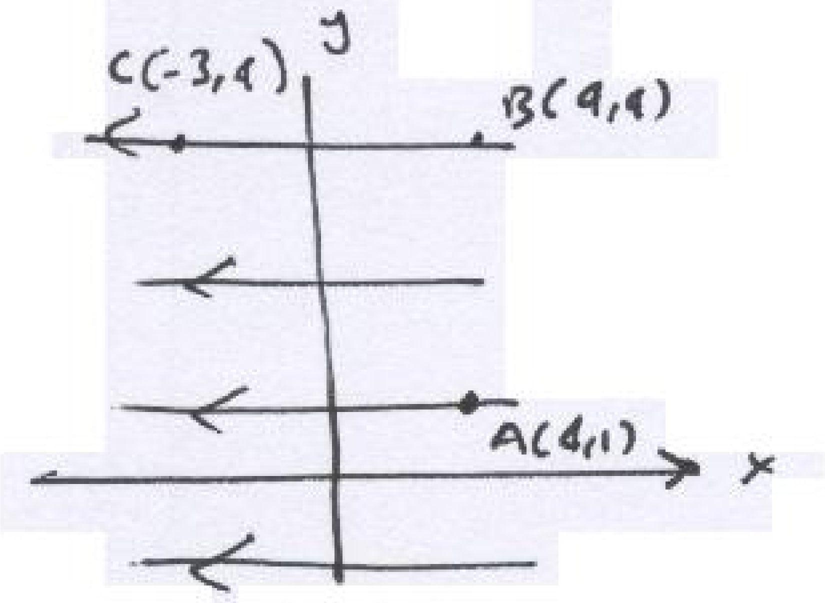
$V_f - V_i$ for all possible pairs of points i and f in the conductor.

Spherical conducting shell of radius r_o





Q A uniform electric field $\vec{E} = -300\,N/C\,\hat{i}$



V_{BA}

V_{CB} ?

V_{CA}

$$\Delta V = -\int \vec{E} \cdot d\vec{\ell}$$

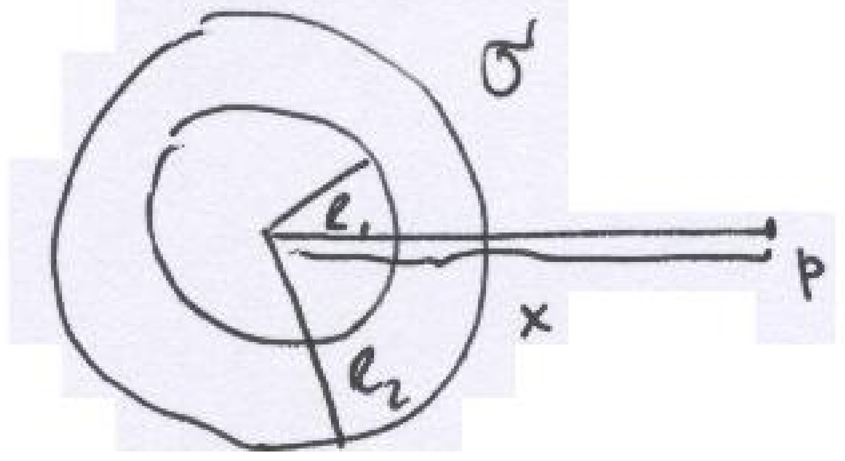
$$V_{BA} = -\int_A^B \vec{E} \cdot d\vec{\ell} = -\int_A^B (-300 N/c)\,\hat{i}\, dy\, \hat{j} = 0\,V$$

$$V_{CB} = -\int_B^C \vec{E} \cdot d\vec{\ell} = -\int_B^C (-300 N/c)\,\hat{i}\, dx\, \hat{i} = -2100\,V$$

$$V_{CA} = -\int_A^C \vec{E} \cdot d\vec{\ell} = -\int_A^C (-300 N/c)\,\hat{i}\,(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_{4m}^{-3m} (300\,N/c)\, dx = -2100\,V$$

Q.



$r\,dr$

$dq = \sigma\, 2\pi r\, dr$

$$dV = \frac{dq}{4\pi\epsilon_0}\,\frac{1}{\sqrt{x^2+r^2}} = \sigma\,\frac{2\pi r\, dr}{4\pi\epsilon_0}\,\frac{1}{\sqrt{x^2+r^2}}$$

$$= \frac{\sigma r\, dr}{2\epsilon_0\sqrt{x^2+r^2}}$$

$$V = \int_{R_1}^{R_2} \frac{\sigma r\, dr}{2\epsilon_0\sqrt{x^2+r^2}} = \frac{\sigma}{2\epsilon_0}\left(x^2+r^2\right)^{1/2}\Big|_{R_1}^{R_2}$$

$$\rightarrow V = \frac{\sigma}{2\epsilon_0}\left[\left(x^2+R_2^2\right)^{1/2} - \left(x^2+R_1^2\right)^{1/2}\right]$$

Q. $V = y^2 + 2xy - 4xyz \longrightarrow \vec{E} = ?$

$$\left.\begin{aligned} E_x &= -\frac{\partial V}{\partial x} = -(2y - 4yz) \\ E_y &= -\frac{\partial V}{\partial y} = -(2y + 2x - 4xz) \\ E_z &= -\frac{\partial V}{\partial z} = -(-4xy) \end{aligned}\right\} \vec{E} = E_x\,\hat{i} + E_y\,\hat{j} + E_z\,\hat{k}$$