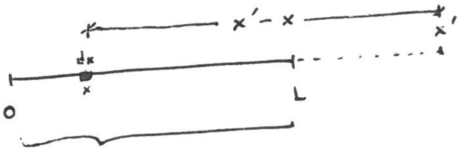
* Linear Charge Density



↳ charge density: $\lambda(x) = \lambda_0 \frac{x}{L}$

1-Dimensional: $\hat{E} = \hat{i}$ (direction along x-axis)

$$\vec{E} = \int_0^L k \frac{\overbrace{\lambda}^{\lambda_0 \frac{x}{L}} dx}{\underbrace{r^2}_{(x'-x)^2}} \underbrace{\hat{r}}_{\hat{i}} = \int_0^L k \frac{\hat{i}}{(x'-x)^2} \lambda_0 \frac{x}{L} dx$$

$\int \frac{x}{(x-a)^2} dx = \frac{a}{a-x} + \ln(a-x)$ → $$\vec{E} = k \frac{\lambda_0 \hat{i}}{L} \left[\ln(x'-x) + \frac{x'}{x'-x} \right]_0^L$$

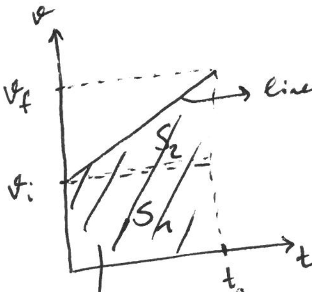
$$\vec{E} = k \frac{\lambda_0 \hat{i}}{L} \left[\ln\left(\frac{x'-L}{x'}\right) + \frac{L}{x'-L} \right]$$

* A charged particle in a uniform Electric field

An electron with an initial ~~spee~~ velocity of 3.99×10^4 m/s in the $\hat{i}$ direction enters into a uniform Electric field of 2.76×10^2 N/C $\hat{i}$. Find the distance it travels before it comes to a stop(*).

↳ After decelarating and $v = 0$, it will start accellarating in the opposite direction but that's not our concern here.

Classical Mechanics



→ linear ↔ const. a:

→ $v_f = v_i + a t_0$

→ $t_0 = \frac{v_f - v_i}{a}$

↳ total distance covered = the area under curve

$$\rightarrow d = \underbrace{v_i t_0}_{S_1} + \underbrace{\frac{(v_f - v_i)}{2} t_0}_{S_2} = \frac{v_f + v_i}{2} t_0$$

$$\Rightarrow d = \frac{(v_f + v_i)}{2} \frac{(v_f - v_i)}{a}$$

$$\Rightarrow d = \frac{v_f^2 - v_i^2}{2a}$$

$$\vec{F} = m\vec{a} \rightarrow \vec{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}$$

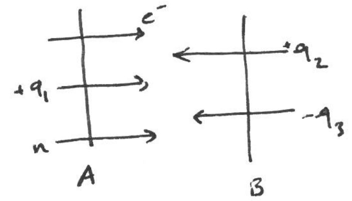
$$\rightarrow d = \frac{v_f^2 - v_i^2}{2a} = \frac{v_f^2 - v_i^2}{2} \left(\frac{1}{\frac{qE}{m}} \right)$$

$$= \frac{v_f^2 - v_i^2}{2} \frac{m}{qE} = \frac{[0 - (3.99 \times 10^4 m/s)^2](9.11 \times 10^{-31} kg)}{2(-1.6 \times 10^{-19} C)(2.76 \times 10^2 N/C)}$$

$$d = 1.64 \times 10^{-5} m$$

* Charged and Uncharged Particles in Electric Field

($\vec{E}$ is uniform)



If e^- is slowed without deflecting from its path,

a.) What is the direction of the field?

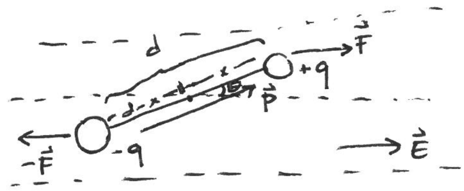
→ A negative charge is slowed down, so the field must be from A to B. In addition, the electron is not deflected, hence it must be in the same direction as the e^-'s movement.

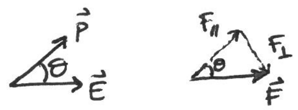
b) Consider the 4 particles with charges $+q_1, +q_2, -q_3$ and 0 (n). According to the figure, how would their speed is effected by the Electric field?

Particle	Speed
$+q_1$	Increases
$+q_2$	Decreases
$-q_3$	Increases
n	Does NOT change

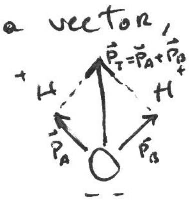
* A Dipole in an Electric Field

$\vec{P}$, ⊖ $\xrightarrow{\vec{P}}$ ⊕





Dipole Moment is a vector, so it can be summed as a vector, e.g.,



$F_{//}$ (parallel component to $\vec{p}$ is cancelled out by the opposite end's $F_{//}$)

$F_\perp$ (Contributing component to the Torque!)

$$\tau = F \cdot x \cdot \sin\theta + F(d-x)\sin\theta = \underline{\underline{Fd\sin\theta}}$$

$$\left.\begin{array}{l} F = qE \\ p = qd \rightarrow d = \frac{p}{q} \end{array}\right\} \longrightarrow \tau = pE\sin\theta$$

($\vec{\tau} = \vec{p} \times \vec{E}$: Torque on a dipole due to $\vec{E}$)

$$\vec{p} \,//\, \vec{E} \rightarrow \vec{\tau} = \vec{p} \times \vec{E} = 0$$

Potential Energy:

$$U = -W = -\int_{\theta_0}^{\theta} \tau \, d\theta = -\int_{\theta_0}^{\theta} pE\sin\theta \, d\theta = -pE\cos\theta \rightarrow U = -\vec{p} \cdot \vec{E}$$

$$\left.\begin{array}{l} \theta = 0 : \text{least } (U = -pE) \\ \theta = 180 : \text{Max } (U = pE) \end{array}\right\} \text{the differences } \Delta U \text{ is what matters. } (\Delta U_{max} = 2pE)$$

* Rotation of a Dipole due to the torque by a uniform Field

Dipole Data

$q = 6.32\,\mu C$

$d = 3.45$ cm

$\hat{p} = -\hat{\imath}$

Moment of Inertia : $I = 1.31 \times 10^{-4}\ kg\,m^2$ for rotations around the $\hat{k}$ direction.

Electric Field

$\vec{E} = 3.11 \times 10\ N/C\ (2\hat{\imath} + \hat{\jmath})$

a) Calculate the Torque on the Dipole

$$p = qd = (6.32 \times 10^{-6} C)(3.45 \times 10^{-2} m) = 2.18 \times 10^{-7} Cm$$

$$\vec{p} = -2.18 \times 10^{-7}\ Cm\ \hat{\imath}$$

$$\vec{\tau} = \vec{p} \times \vec{E} = (-2.18 \times 10^{-7} Cm)\,\hat{\imath} \times (3.11 \times 10 N/C)(2\hat{\imath} + \hat{\jmath})$$

$$= -6.78 \times 10^{-6}\ Nm\ \hat{k}$$

b) Calculate the rotational speed of the dipole when it is pointing in the $\hat{\imath}$ direction

Kinetic Energy ↘ ↙ Potential Energy

$$\Delta K + \Delta U = 0$$

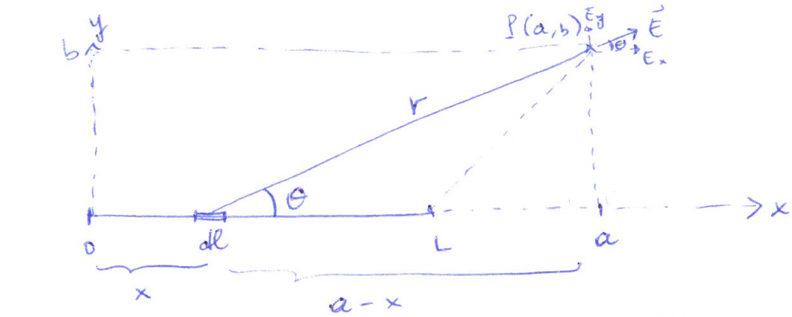
initially at rest → 0

$$\underbrace{\frac{1}{2} I (\omega_f^2 - \omega_i^2)}_{\Delta K} + \underbrace{\left[(-\vec{P}_f \vec{E}) - (-\vec{P}_i \vec{E})\right]}_{\Delta U} = 0$$

$$\rightarrow \omega_f = \sqrt{\frac{2(\vec{P}_f - \vec{P}_i)\vec{E}}{I}} = \sqrt{\frac{2(2.18 \times 10^{-7} Cm)(\hat{\imath} - (-\hat{\imath}))(3.11 \times 10\ N/C)}{1.31 \times 10^{-4}\ kg\,m^2}}$$

$$\Rightarrow \omega_f = 0.207\ s^{-1}$$

Electric Field due to a Linear Charge Density (General Case)



$r = \sqrt{(a-x)^2 + b^2}$

$$\lambda = \lambda_0 \frac{x}{L}$$

$$\sin\theta = \frac{b}{\sqrt{(a-x)^2+b^2}}, \quad \cos\theta = \frac{a-x}{\sqrt{(a-x)^2+b^2}}$$

$$\vec{E}_x = \vec{E}\cdot\cos\theta = E\,\frac{(a-x)}{\sqrt{(a-x)^2+b^2}}\,\hat{i}$$

$$\vec{E}_y = \vec{E}\cdot\sin\theta = E\,\frac{b}{\sqrt{(a-x)^2+b^2}}\,\hat{j}$$

$$\rightarrow \vec{E} = k\lambda_0 \int_0^L \frac{x}{L}\,\frac{dx}{(a-x)^2+b^2}\left[\frac{a-x}{\sqrt{(a-x)^2+b^2}}\,\hat{i} + \frac{b}{\sqrt{(a-x)^2+b^2}}\,\hat{j}\right]$$

Integral Table:

$$\int \frac{x(a-x)\,dx}{[(a-x)^2+b^2]^{3/2}} = \frac{x}{\sqrt{(a-x)^2+b^2}} - \ln\left(\sqrt{(a-x)^2+b^2} - a + x\right)$$

$$\int \frac{x}{(x-a)^2}\,dx = \frac{a}{a-x} + \ln(a-x)$$

$$\int \frac{x\,dx}{[(a-x)^2+b^2]^{3/2}} = \frac{a(x-a)-b^2}{b^2\sqrt{(a-x)^2+b^2}}$$

INTEGRAL TABLES

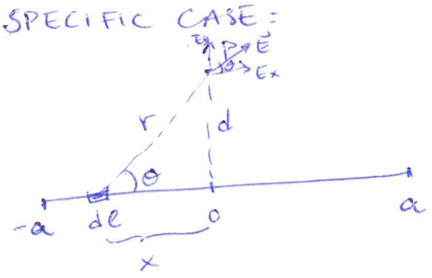
$$\int \frac{1}{(x-a)^2}\,dx = \frac{1}{a-x}$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\Rightarrow \vec{E} = \frac{k\lambda_0}{L}\left\{\underbrace{\left[\frac{L}{\sqrt{(a-L)^2+b^2}} - \ln\left(\sqrt{(a-L)^2+b^2} - a + L\right) + \ln\left(\sqrt{a^2+b^2} - a\right)\right]}_{E_x}\hat{i}\right.$$

$$\left. + \underbrace{\left[\frac{a(L-a)-b^2}{b^2\sqrt{(a-L)^2+b^2}} + \frac{a^2+b^2}{b^2\sqrt{a^2+b^2}}\right]}_{E_y}\hat{j}\right\}$$

SPECIFIC CASE:



$\lambda =$ const.

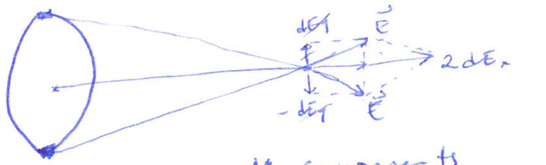
$$\vec{E} = k\lambda_0 d \int_{-a}^{a} \frac{dx}{x^2+d^2}\,\sin\theta\,\hat{j}$$

$$\sin\theta = \frac{d}{r} = \frac{d}{\sqrt{x^2+d^2}}$$

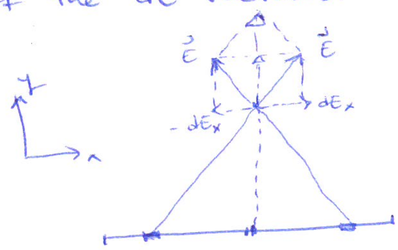
$$E = k\lambda d \int \frac{dx}{[x^2+d^2]^{3/2}} = k\lambda d\,\frac{1}{d^2}\left[\frac{x}{\sqrt{d^2+x^2}}\right]_{-a}^{a} = \frac{k\lambda}{d}\,\frac{2a}{\sqrt{a^2+d^2}}$$

Gauss' Law

When we were considering the Electric field with respect to a charged line or a ring, we used symmetry to cancel out the perpendicular components of the $d\vec{E}$ vectors.

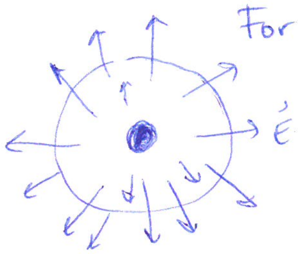


y-components cancel each other



x-components cancel each other

We can even further simplify things thanks to Carl Friedrich Gauss (1777-1855). Imagine a closed surface enclosing the charge distribution (Gaussian surface). If we can manage to imagine a surface such that the Electric field is one that mimics the symmetry of the charge density, we are saved! 8)



For example: If the charge is distributed uniformly over a sphere, we enclose the sphere with a spherical Gaussian surface.

The inverse argument is also correct:

If we know the electric field on a Gaussian surface, we can find the net charge enclosed by the surface. We can deduce the sign of the charge, but we need to know how much electric field is intercepted by the Gaussian surface to calculate how much charge is inside. This is related to some field related quantity called the flux (Φ).

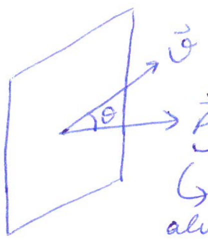
Flux (Φ)

Suppose that water is flowing in a River with uniform velocity $\vec{v}$ and you have a square loop (net) of area A and you want to catch fishes (moving along the water uniformly, as well).

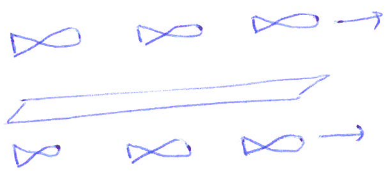
Let Φ represent the volume (fish) flow rate ($\frac{\text{volume}}{\text{(fish)}}$ per unit Time) at which water flows through the Loop. This rate depends on the angle between $\vec{v}$ and the plane of the loop. Also faster current or bigger plane means more volume (fish). If $\vec{v}$ is perpendicular ($\perp$) to the plane, the rate Φ is equal to vA $\longrightarrow$ if the loop is bigger, more fish will be caught inside the loop.

$\downarrow$ the faster the fish move, more will be inside the loop for the unit time

On the other hand, if $\vec{v}$ is parallel (//) to the plane of the Loop, no fish gets caught inside, no matter how fast they move or how big your loop is!



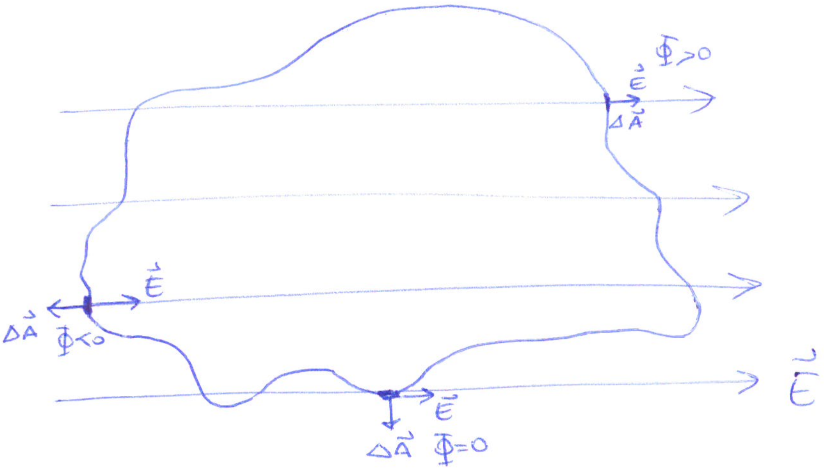
↳ $\vec{A}$ is always directed outwards!



$$\Phi = v \cdot A \cdot \cos\theta = \vec{v} \cdot \vec{A}$$

↳ a scalar quantity, no direction!

(number of fish caught)



$$\Phi_{Total} = \sum \vec{E} \cdot \Delta\vec{A}$$

Limit $\Delta\vec{A} \rightarrow d\vec{A}$

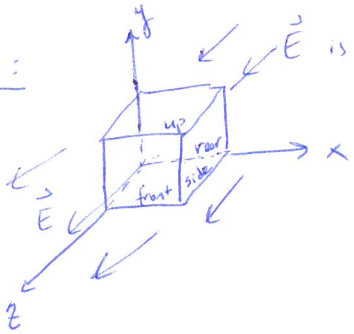
$$\Rightarrow \Phi = \oint \vec{E} \cdot d\vec{A}$$

the $\oint$ means that the Integral is through all surface!

Electric flux through a Gaussian Surface

$\Longrightarrow$ The Electric flux Φ through a Gaussian surface is proportional to the net number of electric field lines passing through that surface.

Example:



$\vec{E}$ is in the $\hat{k}$ direction, uniform.

Gaussian Surface is a cube with each side area = **A**

Φ front face: $\left.\begin{matrix}\vec{E} = E\hat{k} \\ \vec{A} = A\hat{k}\end{matrix}\right\}$ $\Phi_{front} = EA$ $\quad (\hat{k} \cdot \hat{k} = 1)$

$\Phi_{rear\ face}$: $\left.\begin{matrix}\vec{E} = E\hat{k} \\ \vec{A} = -A\hat{k}\end{matrix}\right\}$ $\Phi_{rear} = -EA$ $\quad (\hat{k} \cdot -\hat{k} = -1)$

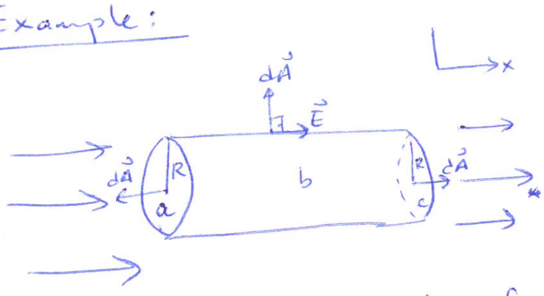
$\Phi_{side\ face}$: $\left.\begin{matrix}\vec{E} = E\hat{k} \\ \vec{A} = A\hat{i} \\ \text{-or-} \\ -A\hat{i}\end{matrix}\right\}$ $\Phi_{side} = 0$ $\quad (\hat{k} \cdot (\pm)\hat{i} = 0)$

$\Phi_{up/down}$: $\left.\begin{matrix}\vec{E} = E\hat{k} \\ \vec{A} = A\hat{j} \\ \text{-or-} \\ -A\hat{j}\end{matrix}\right\}$ $\Phi_{up/down} = 0$ $\quad (\hat{k} \cdot (\pm)\hat{j} = 0)$

$$\Phi_{Total} = \underbrace{EA}_{front} + \underbrace{(-EA)}_{rear} + \underbrace{0}_{left} + \underbrace{0}_{right} + \underbrace{0}_{up} + \underbrace{0}_{down}$$

$$\boxed{\Phi_{Total} = 0}$$

Example:



$\vec{E}$ is uniform, $\vec{E} = E.\hat{i}$

$$\Phi = \oint \vec{E}\cdot d\vec{A} = \int_a \vec{E}\, d\vec{A} + \int_b \vec{E}\cdot d\vec{A} + \int_c \vec{E}\cdot d\vec{A}$$

$$\int_a \vec{E}\cdot d\vec{A} = \int E\,(\cos 180^\circ)\, dA = -E\int dA = -EA \quad (A = \pi R^2)$$

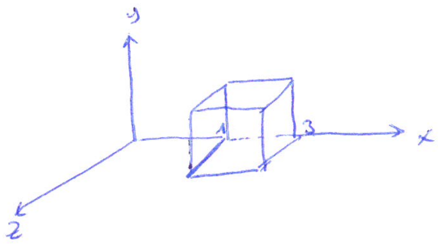
$$\int_c \vec{E}\cdot d\vec{A} = \int E\,(\cos 0^\circ)\, dA = E\int dA = EA$$

$$\int_b \vec{E}\cdot d\vec{A} = \int E\,(\cos 90^\circ)\, dA = 0$$

$$\Phi = -EA + 0 + EA = 0$$

Example: Non-uniform $\vec{E} = 3x\hat{i} + 4\hat{j}$

Gaussian surface: Cube with side length $2m \rightarrow \underline{A = 4m^2}$
Side-Area



Right Side:

$d\vec{A}_r = dA\,\hat{i}$

$$\Phi_R = \int \vec{E}\cdot d\vec{A} = \int (3x\hat{i} + 4\hat{j})(dA\,\hat{i})$$
$$= \int (3x\,dA + 0) = 3\int x\,dA$$
$x = 3 = \text{const}$
$$\rightarrow \Phi_R = 3\int 3\,dA = 9\int dA = 9.4 = 36\,Nm^2/C$$

Left Side: $(d\vec{A}_L = -dA\,\hat{i})$

$$\Phi_L = -3\int x\,dA = -3\int dA = -3.4 = -12\,Nm^2/C$$

Top Side: $(d\vec{A}_T = dA\,\hat{j})$

$$\Phi_T = \int (3x\hat{i} + 4\hat{j})(dA\,\hat{j}) = \int (0 + 4\,dA) = 4\int dA = 4.4 = 16\,Nm^2/C$$

Back and front faces: $(d\vec{A}_B = -dA\,\hat{k},\ d\vec{A}_F = dA\,\hat{k})$ ... $= 0$ $\quad (\hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0)$

$$\Phi_{TOTAL} = \underset{\text{Right}}{36} \underset{\text{Left}}{-12} + \underset{\text{Top}}{16} \underset{\text{Bottom}}{-16} = \underline{24\,Nm^2/C}$$

GAUSS' LAW

Relates the net flux Φ of an Electric field through a closed surface to the <u>net</u> charge q_{enc} that is <u>enclosed</u> by that surface.

$$\epsilon_0 \Phi = q_{enc}$$

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc} \longrightarrow \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

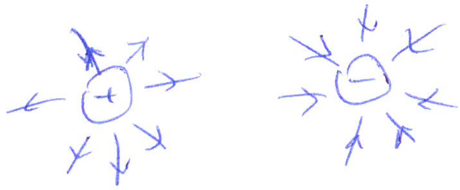
$\epsilon_0 \to$ vacuum/air

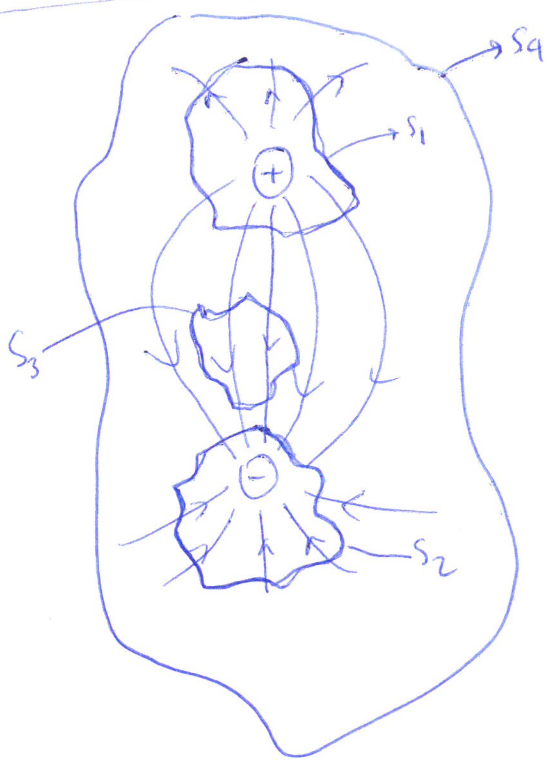
($8.85 \times 10^{-12}\ C^2/Nm^2$)

if $q_{enc} > 0$ Net flux: outward

if $q_{enc} < 0$ Net flux: inward.

Nothing surprising about the signs:





S_1: Electric field is outward for all points on the surface $\longrightarrow \Phi > 0$

$\longrightarrow q_{enc} > 0$

S_2: inward $\longrightarrow \Phi < 0$

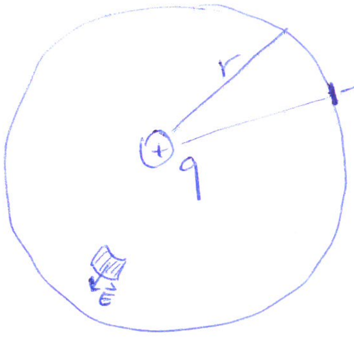
$\longrightarrow q_{enc} < 0$

S_3: No charge inside, no net electric field line $\Big\} \longrightarrow$ $q_{enc} = 0$, net flux $= 0$ (every line entering it, leaves it)

S_4: No net charge $\longrightarrow q_{enc} = 0$

There are as many field lines leaving surface as entering it

Gauss' Law & Coulomb's Law



$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = \epsilon_0 \oint E\, dA = q_{enc} = q$

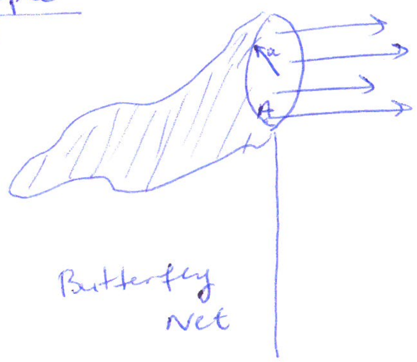
$(8.85 \times 10^{-12}\, C^2/Nm^2)$

$\epsilon_0 E \oint dA = q$

$\epsilon_0 E (4\pi r^2) = q$

$$\boxed{E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}}$$

Example
(p4)



$E = 3\, N/C$ the flux for the net?

$a = 11 \times 10^{-2}\, m$

No ~~net~~ charge inside $\rightarrow \Phi_{TOT} = 0$

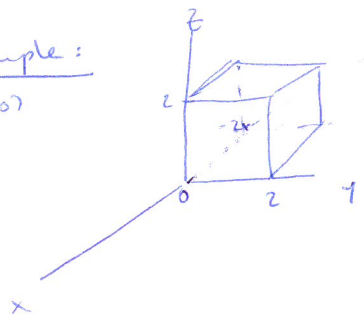
Loop: $\Phi_A = \int \vec{E}\, d\vec{A} = 3 . \pi . a^2 . E$

$= 1.1 \times 10^{-4}\, Nm^2/C$

$\underbrace{\Phi_{net} = -\Phi_A}$ $= -1.1 \times 10^{-4}\, Nm^2/C$

because $\Phi_{Total} = \Phi_A + \Phi_{net} = 0$

Example:
(p10)



$\vec{E} = (3x+4)\hat{i} + 6\hat{j} + 7\hat{k}\ (N/C)$

$E_{non\text{-}constant} = 3x\,\hat{i}$

$A = 4$

Front Face: $x=0$: $\left.\begin{matrix} A = 4\hat{i} \\ E = 0 \end{matrix}\right\}$ $\Phi_F = EA = 0\ Nm^2/C$

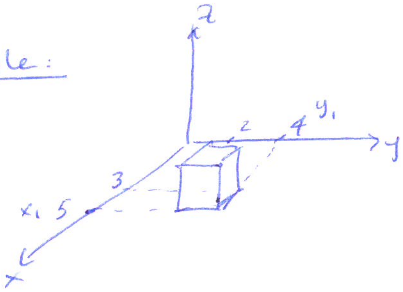
Rear Face $x=2$: $\left.\begin{matrix} A = -4\hat{i} \\ E = -6\hat{i} \end{matrix}\right\}$ $\Phi_R = EA = 24\ Nm^2/C$

$\Phi_{TOT} = 24\, Nm/C^2$

$q_{enc} = \epsilon_0 \Phi_{TOT} = (8.85 \times 10^{-12}\, C^2/Nm^2)(24\, Nm^2/C) = \underline{2.13 \times 10^{-10}\, C}$

4-7

Example:
(p11)



$a = 2\,m$
$x_1 = 5\,m$
$y_1 = 4\,m$

$\vec{E} = -3\hat{i} - 4y^2\hat{j} + 3\hat{k}$ N/C

$\rightarrow \vec{E}_{non-const} = (-4y^2)\hat{j}$, $A = 4\,m^2$

$\Phi_{Right} = -4\overset{y=4}{(4)^2} \cdot \overset{dA}{4} = -256\ Nm^2/C$

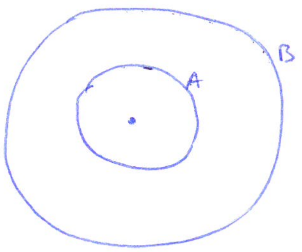
$\Phi_{Left} = -4\overset{y=2}{(2)^2}\underset{dA}{(-4)} = 64\ Nm^2/C$

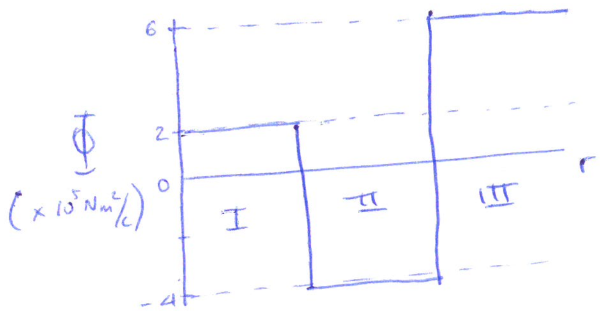
$\Phi_{Total} = -192\ Nm^2/C \rightarrow q_{enc} = \epsilon_0 \Phi$

$= (8.85\times10^{-12}\ C^2/Nm^2)(-192\ Nm^2/C)$

$q_{enc} = -1.70\times10^{-9}\,C$

Example: A charged particle is suspended at the center of two
(p14) concentric spherical shells (non-conducting).





a) What is the charge of the central particle?

$\Phi_I = 2\times10^5 = \dfrac{q}{\epsilon_0} \rightarrow q = \epsilon_0 \Phi = (8.85\times10^{-12})(2\times10^5)$

$= 1.77\times10^{-6}\,C$

b) Charges of shells A and B?

$\Phi_{II} = -4\times10^5\ Nm^2/C \rightarrow q_{enc} = -3.54\times10^{-6}\,C$

$q_A = -3.54\times10^{-6} - 1.77\times10^{-6}$

$= -5.33\times10^{-6}\,C$

$\Phi_{III} = 6\times10^5\ Nm^2/C \rightarrow q_{total\ enclosed} = 5.31\times10^{-6}\,C$

$q_B = q_{TOT} - q_A - q_{particle} = 8.84\times10^{-6}\,C$