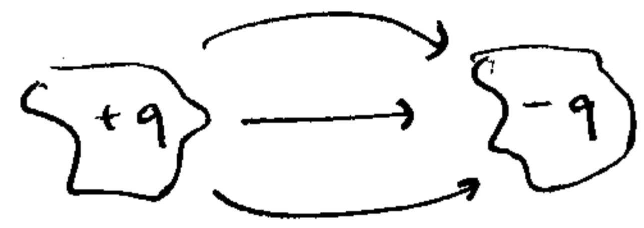
CAPACITANCE

Capacitor : A device in which electrical energy can be stored

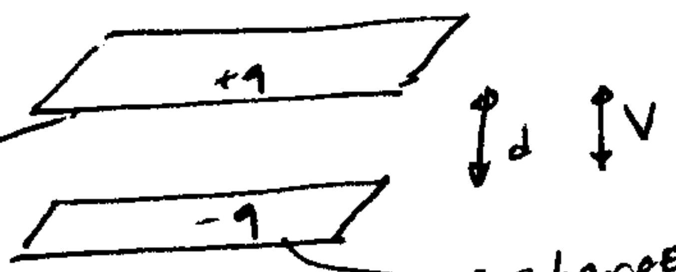
Battery ⟶ photoflash   transfer at a much greater rate
↘ capacitor ↗

How much charge can be stored by a capacitor? ⟶ "Capacitance"

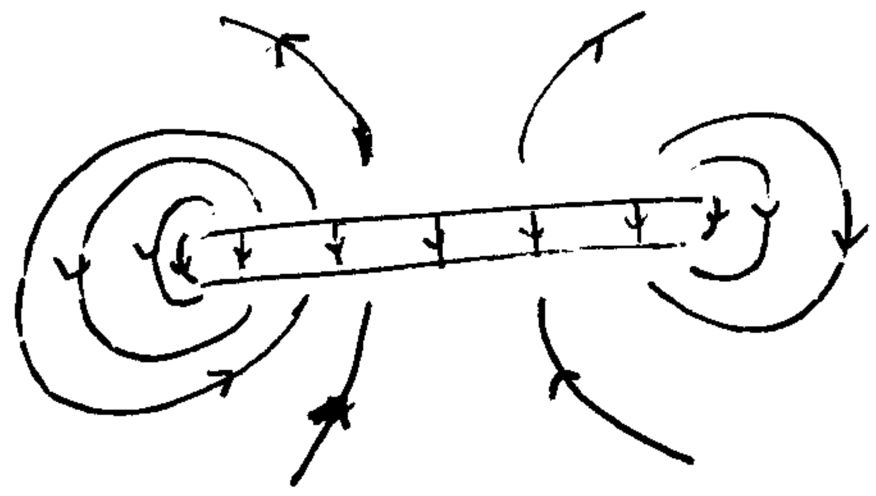


Two conductors, isolated from each other
same magnitude, opposite charges.

charges are at the bottom side of the top plate



⟶ charges are at the top side of the bottom plate



$\dashv\vdash$ charge of a capacitor being q although the net charge is ∅

plates are conductors ⟶ they are equipotential surfaces
⟶ all points have the same potential

$q \propto V \longrightarrow q = CV$
↳ proportionality constant

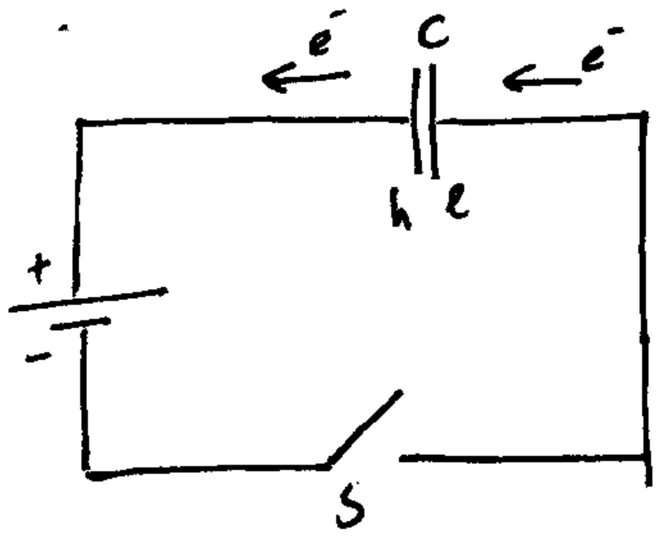
C : capacitance
depends <u>only</u> on the geometry

It is the measure of how much charge
must be put on plates to produce
a certain potential diff between them.

Greater Capacitance ⟶ more charge will be needed to maintain the voltage difference.
(& vice versa)

$[C] = \frac{C}{V} = F$ (Farad)

F is a very large unit, μF ($10^{-6} F$), pF ($10^{-12} F$)



h: positively charged

l: negatively charged

plates become charged until the potential difference equals to that of the Battery

then, h and + are at the same potential

$\longrightarrow$ no longer an electric field in the wire between them

$\Rightarrow$ capacitor is fully charged

Calculating the Capacitance

1) Assume charge q on the plates

q known $\rightarrow$ 2) Calculate $\vec{E}$ between plates via Gauss' Law

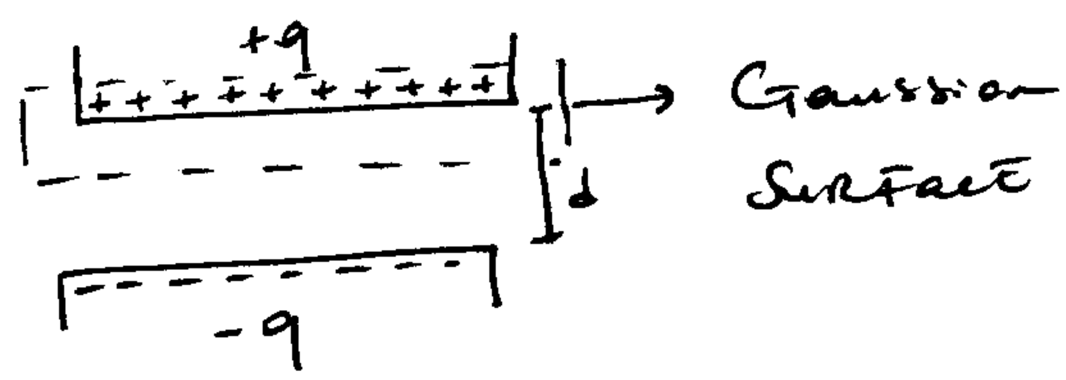
$\vec{E}$ known $\rightarrow$ 3) Calculate V

V known $\rightarrow$ 4) Calculate C

Calculating the Electric Field

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q$$

$$\longrightarrow q = \epsilon_0 E A \quad (\vec{E} \,//\, d\vec{A})$$



Calculating the Potential Difference

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$$

Convention: Start from the negative plate so that

$$\vec{E} \cdot d\vec{s} = -E\,ds$$

$$\longrightarrow V = \int_-^+ E\,ds$$

Parallel Plate Capacitor

Assumption: So large and so close → the fringing of $\vec{E}$ at the edges can be neglected

$$q = \epsilon_0 E A$$

$$V = \int_{-}^{+} E\,ds = E \int_0^d ds = E d$$

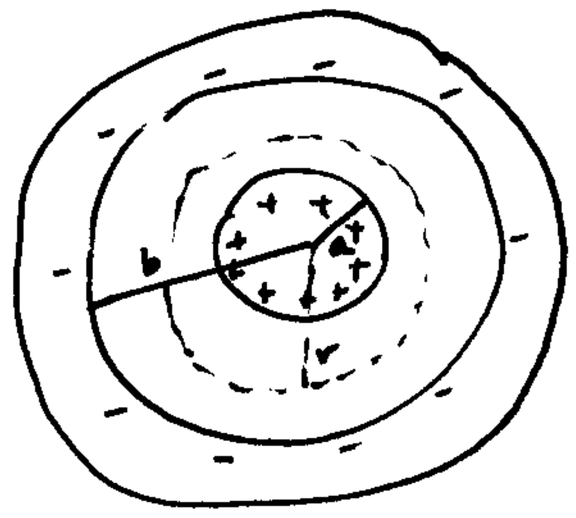
$$C = \frac{q}{V} \qquad C = \frac{\epsilon_0 A}{d}$$ geometrical factors

$A \uparrow$ $\quad C \uparrow$
$d \downarrow$

$$\epsilon_0 = 8.85 \times 10^{-12}\, F/m = 8.85\, pF/m$$ → more convenient unit

$$\epsilon_0 = 8.85 \times 10^{-12}\, C^2/Nm^2$$

A Cylendirical Capacitor



Length: L , $L \gg b$

$$q = \epsilon_0 E A = \epsilon_0 E 2\pi r L$$

$$\longrightarrow E = \frac{q}{2\pi\epsilon_0 L r}$$

$$V = \int_{-}^{+} E\,ds = -\frac{q}{2\pi\epsilon_0 L}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

$ds = -dr$

$$\Rightarrow C = 2\pi\epsilon_0 \frac{L}{\ln(b/a)}$$

A Spherical Capacitor

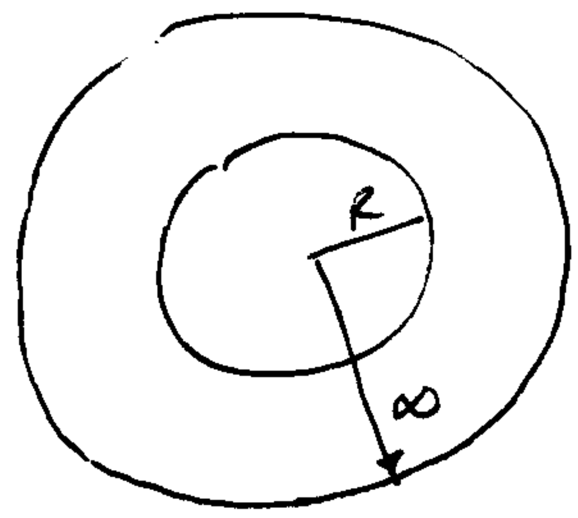
$$q = \epsilon_0 E A = \epsilon_0 E (4\pi r^2)$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad V = \int_{-}^{+} E\,ds = -\frac{q}{4\pi\epsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{b-a}{ab}$$

$$\Rightarrow C = 4\pi\epsilon_0 \frac{ab}{b-a}$$

An isolated sphere ($b \to \infty$)

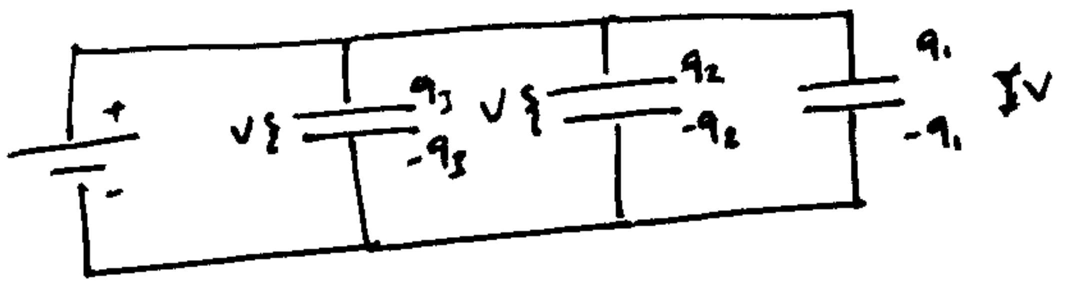


$$C = 4\pi\epsilon_0 \frac{a}{1 - a/b}$$

$$b \to \infty \Rightarrow C = 4\pi\epsilon_0 R$$

CAPACITORS IN PARALLEL AND IN SERIES

* in Parallel:



$\to$ V is same, q different

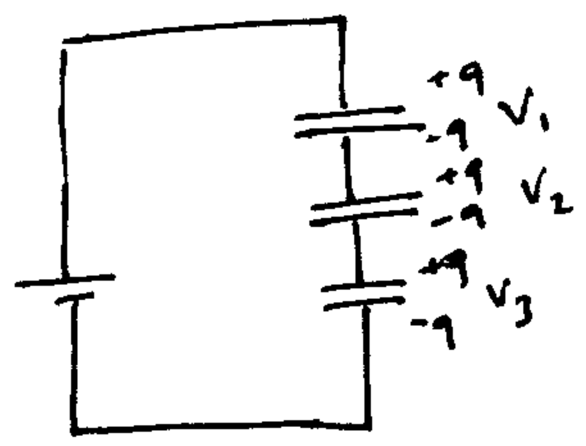
$$q_1 = C_1 V, \quad q_2 = C_2 V, \quad q_3 = C_3 V$$

$$q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V$$

$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3$$

$$\Rightarrow C_{eq} = \sum_{j=1}^{n} C_j \quad \text{(capacitors in parallel)}$$

* in Series



$\to$ q: same on all, V: different

$$V_1 = \frac{q}{C_1}, \quad V_2 = \frac{q}{C_2}, \quad V_3 = \frac{q}{C_3}$$

$$V = V_1 + V_2 + V_3 = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

$$C_{eq} = \frac{q}{V} = \frac{1}{1/C_1 + 1/C_2 + 1/C_3}$$

$$\to \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \Rightarrow \frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j}$$

(capacitors in series)

Energy Stored in an Electric FielD

q' is transferred from one plate to the other

$$V' \rightarrow q'/C$$

$$dq' \rightarrow dW = V' dq' = \frac{q'}{C} dq'$$

$$U = \frac{q^2}{2C} = \frac{1}{2} C V^2$$: independent of the geometry

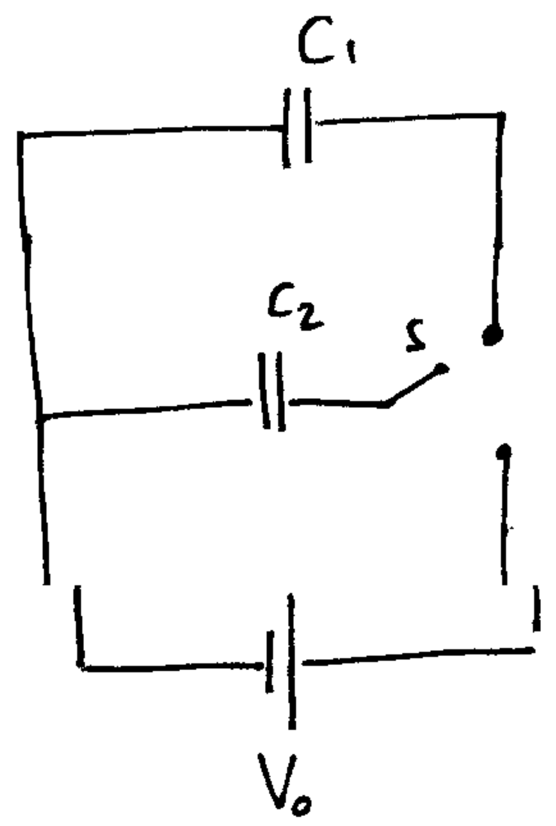
Energy Density

$$C = \frac{\epsilon_0 A}{d}$$

Pot. Energy per unit volume : $u = \frac{U}{Ad} = \frac{CV^2}{2Ad} = \frac{1}{2}\epsilon_0\left(\frac{V}{d}\right)^2 = \frac{1}{2}\epsilon_0 E^2$

$V = Ed$

Ex :



Switch is connected downward so that capacitor C_2 becomes fully charged by the Battery. Then switch is connected upwards.

→ ~~Determine~~ the Charge on each capacitor after switching

$$Q_2 = C_2 V_0$$

$$\rightarrow V_1 = V_2 = V$$

$$Q = Q_1' + Q_2' = Q_2$$

$$C_1 V + C_2 V = C_2 V_0 \rightarrow V = \frac{C_2 V_0}{(C_1 + C_2)}$$

$$Q_1' = C_1 V = \frac{C_1 C_2 V_0}{C_1 + C_2}$$

$$Q_2' = C_2 V = \frac{C_2^2 V_0}{C_1 + C_2}$$

CAPACITOR WITH A DIELECTRIC

What happens if you fill the space between the plates of a capacitor with a dielectric?

→ an insulator material that is polarizable.

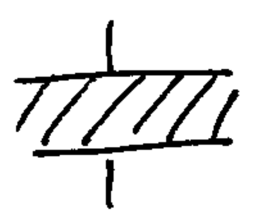
Michael Faraday, 1837

→ Capacitance increases by K : Dielectric constant

Material	K	Dielectric Strength (kV/m
Vacuum	1	
Air	1.00054	3
Paper	3.5	16
Silicon	12	
Germanium	16	
Water 20°C	80.1	
" 25°C	78.5	
TiO_2 ceramic	130	
$SrTiO_3$	310	8

if a V_{max} is exceeded, the dielectric material will break down and form a conducting path between the plates.

Dielectric Strength: Maximum value of the Electric Field that it can tolerate



→ V constant: charge increases by a factor K

→ Q constant: V decreases by a factor K

$$C = \epsilon_0 \mathcal{L} \rightarrow \text{some quantity in units of length } \left(\text{e.g. } \frac{A}{d}\right)$$

$$C = K \epsilon_0 \mathcal{L} = K\, C_{vacuum}$$

$$\epsilon_0 \longrightarrow K\epsilon_0$$

$$E = \frac{1}{4\pi K \epsilon_0} \frac{q}{r^2}$$: E due to point charge inside a dielectric

$$E = \frac{\sigma}{K\epsilon_0}$$: E just outside of an isolated conductor in a dielectric

$K > 1 \longrightarrow$ the Effect of a dielectric is to weaken the Electric field

Ex: $C = 13.5\,pF$, $V = 12.5\,V$ → plates are charged, battery disconnected and a porcelain slab is pushed between the plates ($K = 6.50$)

a) Potential Energy Before?

$$U_i = \frac{1}{2} C V^2 = \frac{q^2}{2C}$$

$$= \frac{1}{2} (13.5 \times 10^{-12} F)(12.5 V)^2$$

$$= 1055\,pJ$$

b) Potential Energy After?

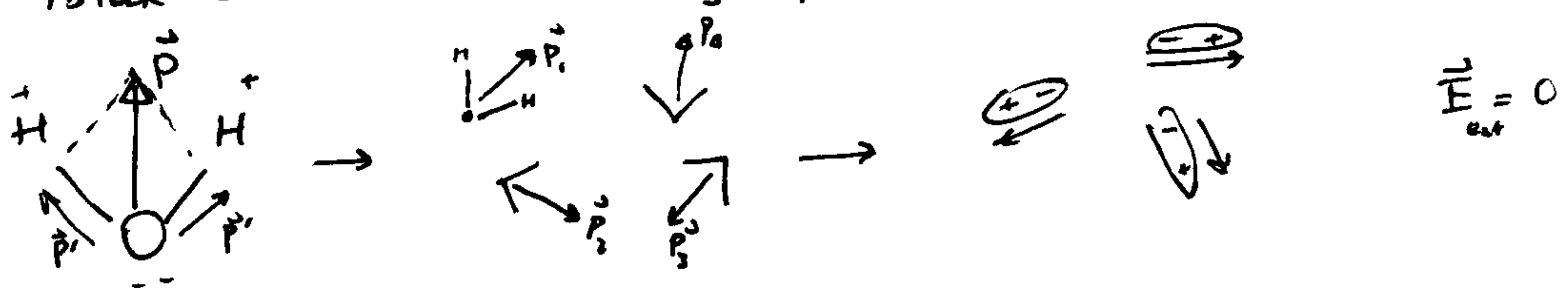
$$U_f = \frac{q^2}{2KC} = \frac{U_i}{K} = \frac{1055\,pJ}{6.5} = 162\,pJ$$

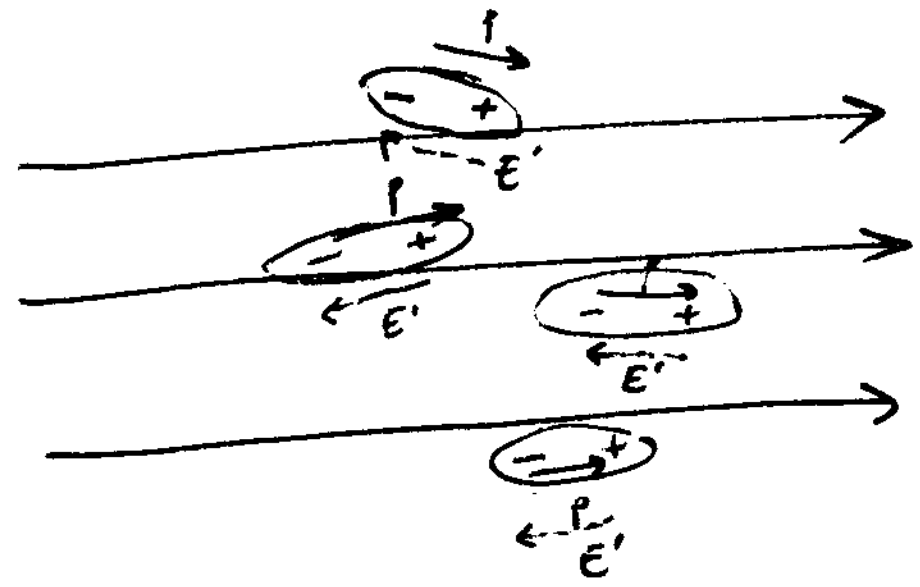
decreases
→ capacitor pulls the slab inside

$$W = U_i - U_f = 893\,pJ$$

Dielectrics : An atomic view

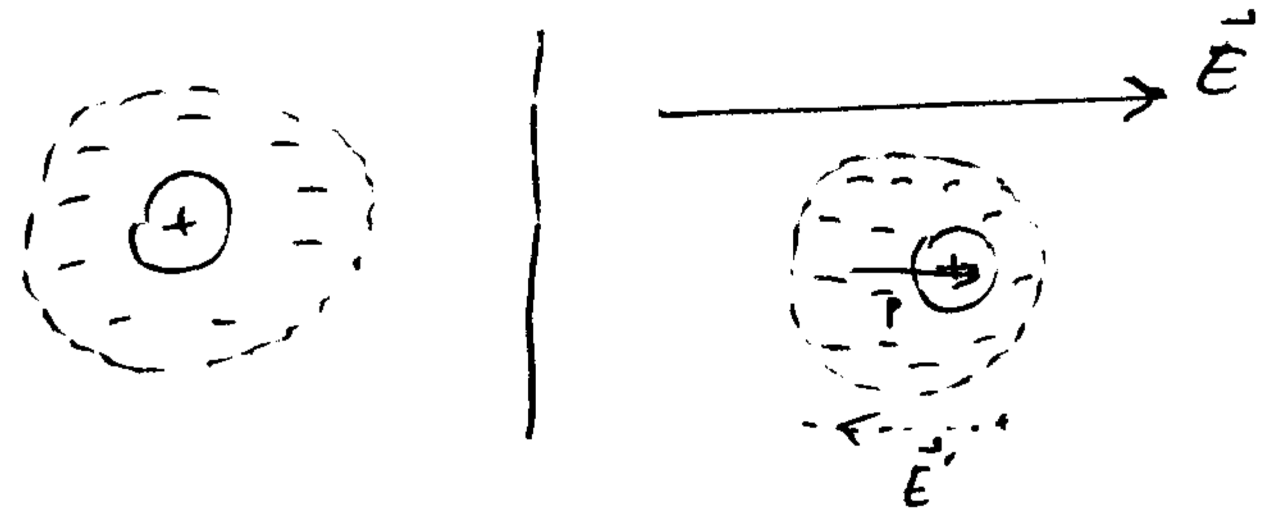
1) Polar dielectrics : Water, permanent electric dipole moments

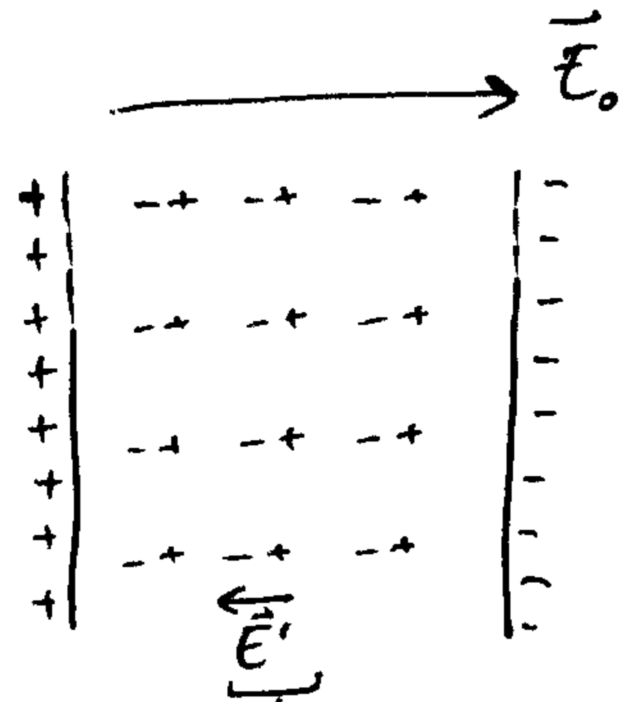




produce an electric field that is opposite of the applied field and smaller in magnitude

2) Non-polar dielectrics: Permanent or not, molecules acquire dipole moments by induction.



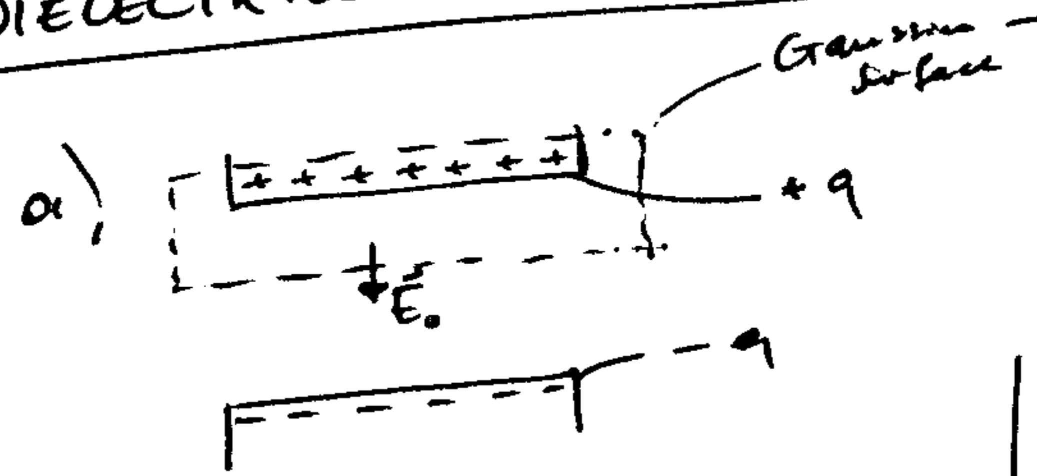


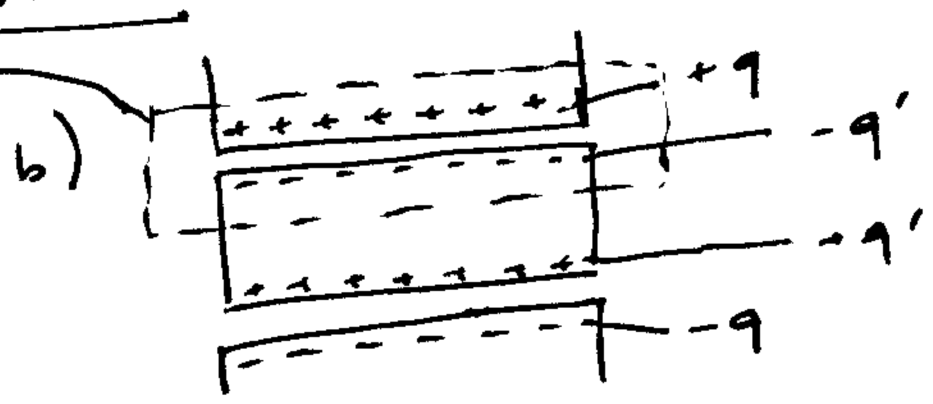
$$\vec{E} = \vec{E}_0 - \vec{E}'$$

Smaller in magnitude, in the opposite direction

Both ① and ② produce the same effect as they weaken any applied field within them.

DIELECTRICS AND GAUSS' LAW





a)

$$\epsilon_0 \oint \vec{E}\, d\vec{A} = \epsilon_0 E A = q$$

$$E_0 = \frac{q}{\epsilon_0 A}$$

b)

$$\epsilon_0 \oint \vec{E}\cdot d\vec{A} = \epsilon_0 E A = q - q' \rightarrow \frac{q-q'}{\epsilon_0 A} = E$$

$$E = \frac{q-q'}{\epsilon_0 A} \rightarrow \text{weakening}: \; E = \frac{E_0}{K} = \frac{q}{K\epsilon_0 A}$$

$$\Rightarrow \; q - q' = \frac{q}{K}$$

$$\Rightarrow \; \boxed{\epsilon_0 \oint \underbrace{K}_{\text{contains all the information about the charges in the dielectric}} \vec{E}\cdot d\vec{A} = \underbrace{q}_{\text{charges on the capacitor plates only!}}}$$

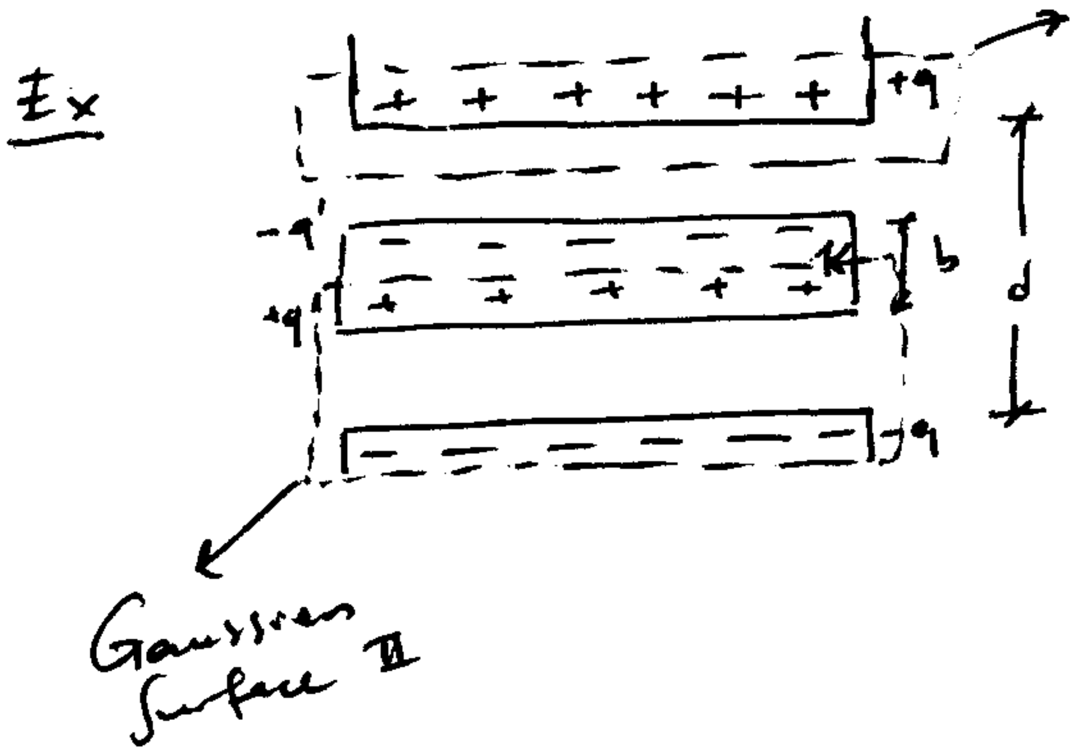
Comments on $\epsilon_0 \oint K\vec{E}\cdot d\vec{A} = q$ (Gauss' Law with Dielectrics)

i) $K\vec{E}$, $\epsilon_0 K \vec{E} \equiv \vec{D}$ $\qquad \oint \vec{D}\cdot d\vec{A} = q$

ii) The charge q enclosed by the Gaussian Surface is now taken to be the free charge only.

iii) $\epsilon_0 \longrightarrow K\epsilon_0$ (K is inside the integral because it may not be constant)

Ex



V_0, A, d

a) Capacitance before?

$$C_0 = \frac{\epsilon_0 A}{d}$$

b) Free charge on plates

$$q = C_0 V_0$$

c) E_0 in the gaps between the plates and slab?

Gauss. I $\qquad \vec{E}_0\, d\vec{A} = E_0\, dA$

$$\epsilon_0 K E_0 \int d\vec{A} = q$$

$$\epsilon_0 K E_0 A = q$$

$$E = \frac{q}{\epsilon_0 K A}$$

$$\underset{\text{(vacuum)}}{K=1} \longrightarrow E_0 = \frac{q}{\epsilon_0 A}$$

d) E_1 in the dielectric slab? $\qquad \vec{E}_1 \downarrow \; d\vec{A} \uparrow$

Gauss. II $\qquad$ $\vec{E}_1$ in the opposite dir. with $d\vec{A}$ $\qquad$ only the free charge is taken into account

$$\epsilon_0 \oint K\vec{E}_1\cdot d\vec{A} = -\epsilon_0 K E_1 A = -q$$

$$E_1 = \frac{q}{\epsilon_0 K A} = \frac{E_0}{K}$$

e) V between plates?

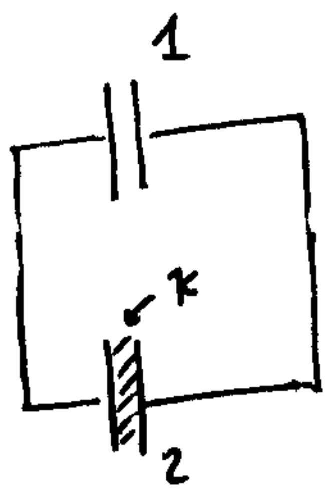
$$V = \int_{-}^{+} E\, ds = E_0(d-b) + E_1 b$$

f) Capacitance? $\qquad C = \frac{q}{V} > C_0$ $\quad$ from (b)

Ex Two identical capacitors in parallel

each Q_0 for V_0

The voltage is disconnected and K dielectric is inserted into one of them.



a) The new charge on each capacitor

$$C = \frac{Q_0}{V_0}$$

parallel → V' same $\quad V = \frac{Q_1}{C} = \frac{Q_2}{KC} \Rightarrow Q_2 = KQ_1$

Charge is conserved: $2Q_0 = Q_1 + Q_2$

$$(K+1)Q_1 = 2Q_0$$

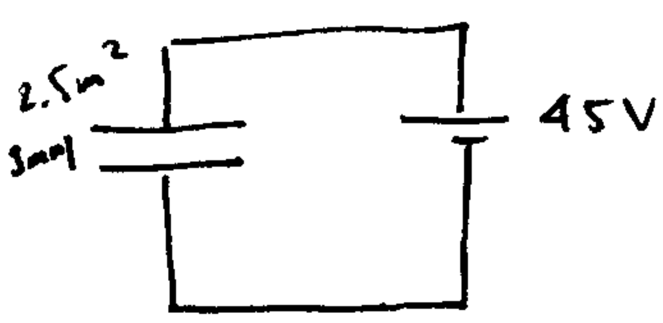
$$Q_1 = \frac{2Q_0}{K+1}$$

$$Q_2 = 2Q_0 - Q_1$$

b) New voltage?

$$V = \frac{Q_1}{C} = \frac{2Q_0}{K+1}\,\frac{V_0}{Q_0} = \frac{2}{K+1}V_0$$

Ex A parallel Capacitor $A = 2.5\,m^2$, $d = 3\,mm$, $V = 45\,V$



a.) C_0, Q_0, E_0, u_0?

$C_0 = \epsilon_0 \frac{A}{d}$ $(7.4\,nF)$

$Q_0 = C_0 V$ $(0.33\,\mu C)$

$E_0 = \frac{V}{d}$ $(1.5 \times 10^4\,V/m)$

$u_0 = \frac{1}{2} C_0 V^2$ $(7.5 \times 10^{-6}\,J)$

b) Capacitor still connected to the battery, a dielectric with $K = 3.6$ is inserted → C, Q, E, u?

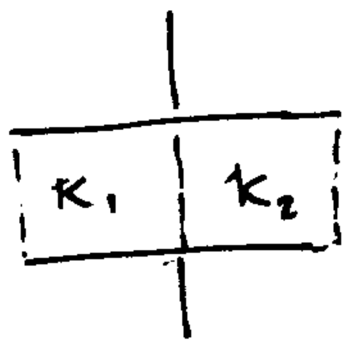
$C = K C_0$ $(27\,nF)$

$Q = CV$ $(1.2\,\mu C)$

$E = \frac{V}{d}$ $(1.5 \times 10^4\,V/m)$

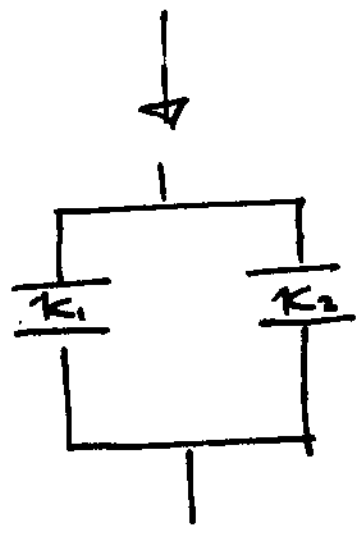
$u = \frac{1}{2} C V^2 = K u_0$ $(2.7 \times 10^{-5}\,J)$

Ex



$A = 5.56\,\text{cm}^2$ $\quad \kappa_1 = 7 \quad C = ?$

$d = 5.56\,\text{mm}$ $\quad \kappa_2 = 12$



$$C = C_1 + C_2 = \frac{\epsilon_0 (A/2)}{d}\kappa_1 + \frac{\epsilon_0 (A/2)}{d}\kappa_2 = \frac{\epsilon_0 A}{d}\left(\frac{\kappa_1 + \kappa_2}{2}\right)$$

$(8.41 \times 10^{-12}\,\text{F})$

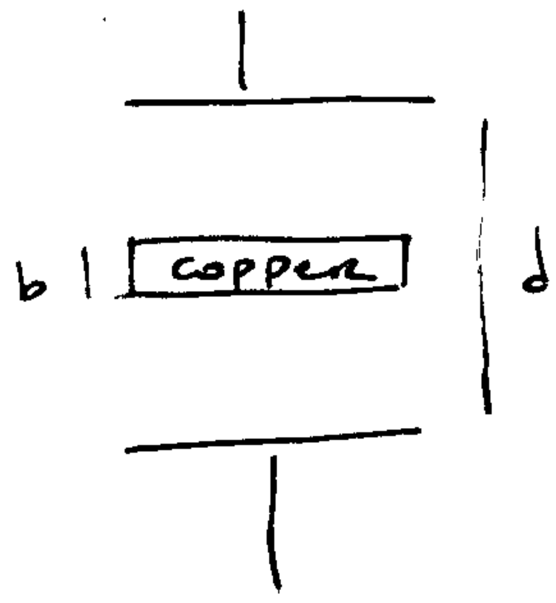
Ex Parallel plates, $A = 100\,\text{cm}^2$, $Q = 8.9 \times 10^{-7}$, $E = 1.4 \times 10^6\,\text{V/m}$

$\kappa = ?$

$$\kappa \epsilon_0 A = q \rightarrow \kappa = \frac{q}{\epsilon_0 E A} \quad (7.2)$$

$$q - q' = \frac{q}{\kappa} \rightarrow q' = q\left(1 - \frac{1}{\kappa}\right) \quad (7.7 \times 10^{-7}\,\text{C})$$

Ex

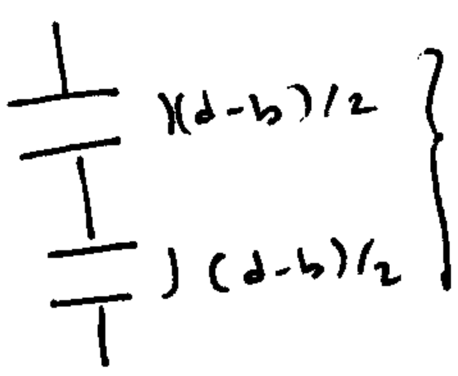


$A = 2.40\,\text{m}^2$

$d = 5\,\text{mm}$

$b = 2\,\text{mm}$

(Copper: conductor)



a) $C' = \dfrac{\epsilon_0 A}{d - b}$ $\quad (0.708\,\text{pF})$

(Series connection)

b) $q = 3.40\,\mu\text{C}$ is maintained:

$$\frac{U}{U'} = \frac{\frac{Q}{2C}}{\frac{Q}{2C'}} = \frac{C'}{C} = \frac{\epsilon_0 (A/(d-b))}{\epsilon_0 A/d} = \frac{d}{d-b} \quad (1.67)$$

c) $$W = \Delta U = U' - U = \frac{q^2}{2}\left(\frac{1}{C'} - \frac{1}{C}\right) = \frac{q^2}{2\epsilon_0 A}(d - b - d)$$

$$= -\frac{q^2 b}{2\epsilon_0 A} \quad (-5.44\,\text{J})$$

d) Is the slab sucked in or must be pushed?

$W < 0 \rightarrow$ slab is sucked in.

Ex Like the previous, but $V = 85\,\text{V}$ rather than the charge is kept constant.

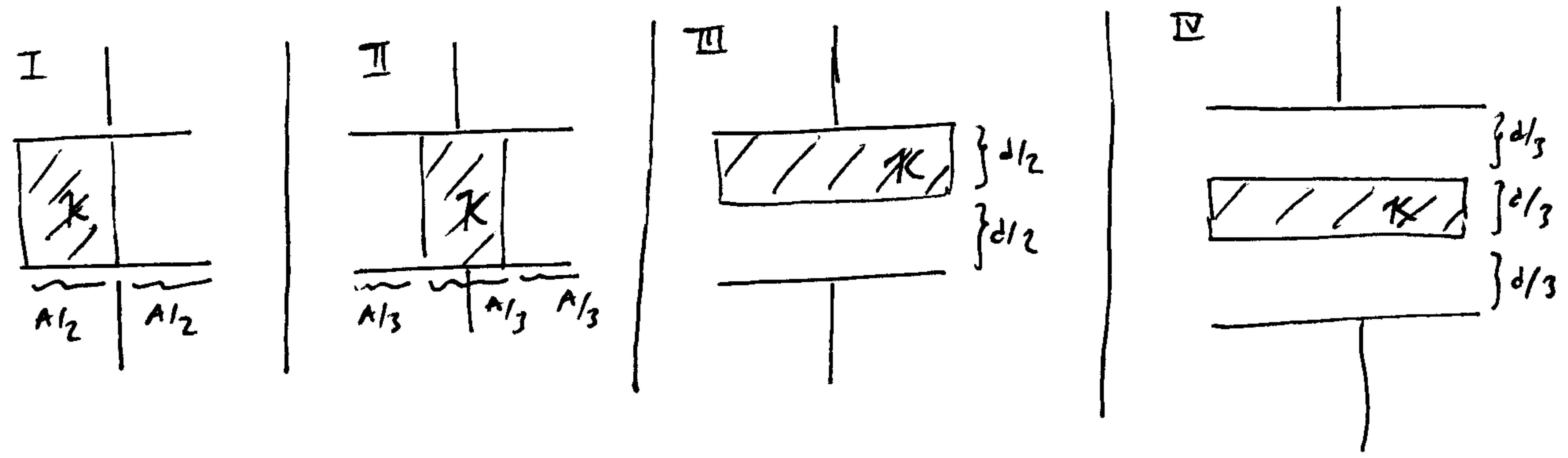
a) $C' = \dfrac{\epsilon_0 A}{d - b}$ $\quad (0.708\,\text{pF})$

b) $$\frac{U'}{U} = \frac{\frac{1}{2}CV^2}{\frac{1}{2}C'V^2} = \frac{C}{C'} = \frac{\epsilon_0 A/d}{\epsilon_0 A/(d-b)} = \frac{d-b}{d} \quad (0.6)$$

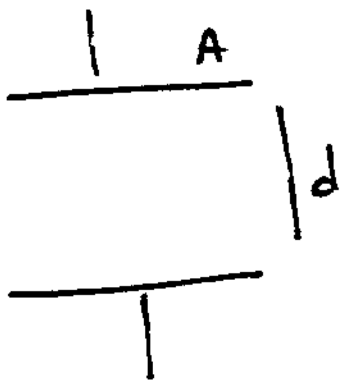
c) $$W = \Delta U = U' - U = \frac{1}{2}(C' - C)V^2 = \frac{\epsilon_0 A}{2}\left(\frac{1}{d-b} - \frac{1}{d}\right)V^2$$

$(1.02 \times 10^{-9}\,\text{J})$

Ex



0 :



$\to C_0 = \epsilon_0 \frac{A}{d}$

I : $C_{eq} = K C_p + C_p$

$= (K+1) C_p$

$C_p = \epsilon_0 \frac{A}{2d} = \frac{C_0}{2}$

$\Rightarrow \boxed{C_{eq} = (K+1) \frac{C_0}{2}}$

II : $C_{eq} = C_{3p} + K C_{3p} + C_{3p}$

$= (K+2) C_{3p}$

$C_{3p} = \epsilon_0 \frac{A}{3d}$

$\Rightarrow \boxed{C_{eq} = \frac{(K+2)}{3} C_0}$

III : $C_{eq} = \left(\frac{1}{K C_s} + \frac{1}{C_s} \right)^{-1}$

$= \left(\frac{K+1}{K C_s} \right)^{-1} \to \frac{K}{K+1} C_s$

$C_s = \epsilon_0 \frac{A}{\frac{d}{2}} = 2 C_0$

$\Rightarrow \boxed{C_{eq} = \frac{2K}{K+1} C_0}$

IV $C_{eq} = \left(\frac{1}{C_{3s}} + \frac{1}{K C_{3s}} +\right.$

$\frac{K}{2K+1} C_{3s}$

$C_{3s} = \epsilon_0 \frac{A}{\frac{d}{3}} = 3 C_0$

$\Rightarrow \boxed{C_{eq} = \frac{3K}{2K+1} C_0}$

K	$I \times C_0$	$II \times C_0$	$III \times C_0$	$IV \times C_0$
1	1	1	1	1
2	1.5	1.333	1.333	1.2
3	2	1.6667	1.5	1.28571
4	2.5	2	1.6	1.3333
5	3	2.3333	1.6666	1.36364

$\rightarrow \quad C_1 > C_2 > C_3 > C_4 > C_0$