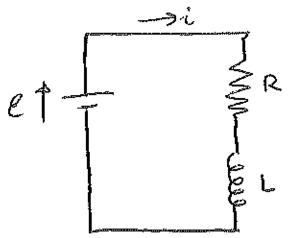
ENERGY STORED IN A MAGNETIC FIELD

When we pull two charged particles of opposite signs away from each other, we say that the resulting electric potential energy is stored in the electric field of the particles. We get it back from the field by letting the particles move close together again.

In the same way, we say energy is stored in a magnetic field, but now we deal with current instead of electric charges.



$$\mathcal{E} = L\frac{di}{dt} + iR$$

$$\mathcal{E}i = Li\frac{di}{dt} + i^2R$$

1) If a differential amount of charge dq passes through the battery of emf $\mathcal{E}$ in the dt, the battery does work on it in the amount $\mathcal{E}dq$. The rate at which the battery does work is $(\mathcal{E}dq)/dt$ or $\mathcal{E}i$. Thus, the left side of the equation represents the rate at which the emf device delivers energy to the rest of the circuit.

2) i^2R: the rate at which energy appears as thermal energy in the resistor.

3) Energy that is delivered to the circuit but does not appear as thermal energy must be stored somewhere in the magnetic field of the inductor.

$\rightarrow \dfrac{dU_B}{dt} = Li\dfrac{di}{dt}$: the rate at which magnetic potential energy is stored in the magnetic field.

$$dU_B = L\,i\,di$$

$$\int_0^{U_B} dU_B = \int_0^{i} L\,i\,di \Rightarrow U_B = \frac{1}{2}Li^2 \text{ : Magnetic Energy}$$

$\left(\leftrightarrow U_E = \dfrac{q^2}{2C}\right.$ Energy stored by a capacitor$\left.\right)$

Ex: A coil has an inductance 53 mH, $r = 0.35\,\Omega$

a.) If a 12V emf is applied across the coil, how much energy is stored in the magnetic field after the current has built up to its equilibrium value?

$$U_B = \frac{1}{2} L i^2$$

$$i_\infty = \frac{\mathcal{E}}{R} = \frac{12}{0.35} = 34.3\,A$$

$$U_{B\infty} = \frac{1}{2} L i_\infty^2 = \frac{1}{2}(53\times 10^{-3} H)(34.3A)^2$$

$$= 31\,J$$

b) After how many time constants will half of this equilibrium energy be stored in the magnetic field?

at what time $U_B = \frac{1}{2} U_{B\infty}$

$$\frac{1}{2} L i^2 = \left(\frac{1}{2}\right)\frac{1}{2} L i_\infty^2$$

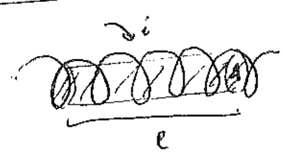
$$i = \frac{1}{\sqrt{2}} i_\infty$$

$$\frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right) = \frac{\mathcal{E}}{\sqrt{2}R} \rightarrow e^{-t/\tau_L} = 1 - \frac{1}{\sqrt{2}} = 0.293$$

$$\frac{t}{\tau_L} = -\ln 0.293 = 1.23$$

$$t = 1.2\tau_L \longrightarrow \tau_L = \frac{L}{R}$$

ENERGY DENSITY OF A MAGNETIC FIELD



Volume: $A\ell$

$$u_B = \frac{U_B}{A\ell}$$

$N\Phi_B = (n\ell)BA$

$$U_B = \frac{1}{2} L i^2$$

$B = \mu_0 i n$

$$L = \frac{N\Phi_B}{i} = \frac{(n\ell)(BA)}{i} = \frac{(n\ell)(\mu_0 i n)A}{i}$$

$$u_B = \frac{Li^2}{2A\ell} = \frac{L}{\ell}\frac{i^2}{2A}$$

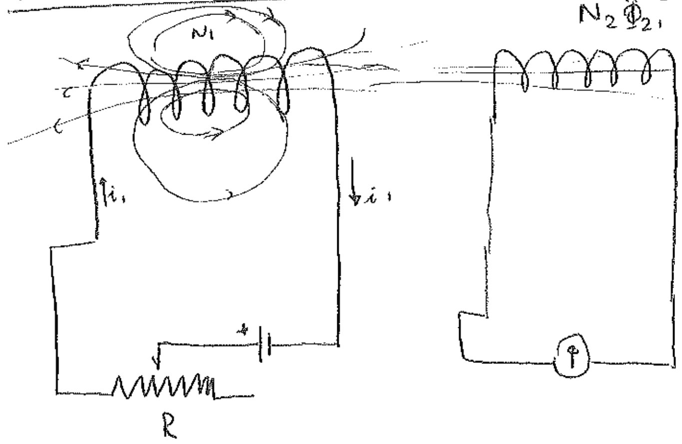
$\rightarrow L = \mu_0 n^2 \ell A$

$\Rightarrow \frac{L}{\ell} = \mu_0 n^2 A$

$$u_B = \frac{1}{2}\mu_0 n^2 i^2$$

$$B = \mu_0 i n \Rightarrow \underbrace{u_B = \frac{B^2}{2\mu_0}}_{\text{Magnetic Energy Density}} \qquad (\Leftrightarrow u_E = \frac{1}{2}\epsilon_0 E^2)$$

MUTUAL INDUCTION



$N_2 \Phi_{21}$ → the flux through coil 2 associated with the current in coil 1

Mutual induction M_{21} of coil 2 with respect to coil 1

$$M_{21} = \frac{N_2 \Phi_{21}}{i_1}$$

$$L = \frac{N\Phi}{i}$$

$$M_{21} i_1 = N_2 \Phi_{21}$$

$$M_{21} \frac{di_1}{dt} = \underbrace{N_2 \frac{d\Phi_{21}}{dt}}$$

→ Magnitude of the emf $\mathcal{E}_2$ appearing in coil 2 due to the changing current in coil 1

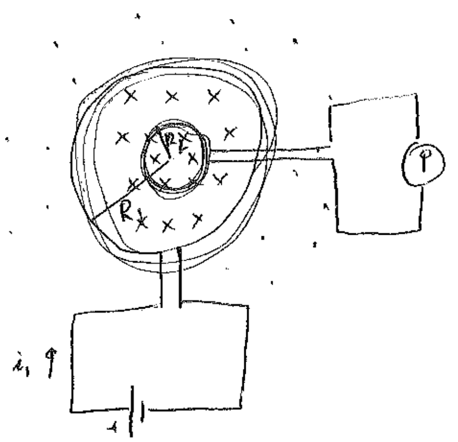
$$\rightarrow \mathcal{E}_2 = -M_{21} \frac{di_1}{dt} \qquad \left(\mathcal{E} = -L \frac{di}{dt} \right)$$

$$\mathcal{E}_1 = -M_{12} \frac{di_2}{dt}$$

$M_{12} = M_{21} = M$ (fact, derivation not shown)

$$\Rightarrow \mathcal{E}_2 = -M \frac{di_1}{dt}, \quad \mathcal{E}_1 = -M \frac{di_2}{dt}$$

Ex: R_1, N_1 & R_2, N_2



Derive an expression for the mutual inductance M ($R_1 >> R_2$)

$$M = \frac{N_2 \Phi_{21}}{i_1}$$

$$\Phi_{21} = B_1 A_2$$

$$N_2 \Phi_{21} = N_2 B_1 \underbrace{A_2}_{\pi R_2^2}$$

use : $B(z) = \dfrac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}}$ ($z=0$) same plane

$$B_1 = N_1 \frac{\mu_0 i_1}{2R_1}$$

$$N_2 \Phi_{21} = \frac{\pi \mu_0 N_1 N_2 R_2^2 i_1}{2R_1}$$

$$M = \frac{N_2 \Phi_{21}}{i_1} = \frac{\pi \mu_0 N_1 N_2 R_2^2}{2R_1}$$

ELECTROMAGNETIC OSCILLATIONS AND ALTERNATING CURRENT

Applied Physics: How energy produced in one location can be transferred to another location, so that it can be put to use.

In most parts of the world, electrical energy is transferred not as a direct current but as a sinusoidally oscillating current (alternating current – AC). The challenge is to design AC systems that transfer energy efficiently and to build appliances that make use of that energy.

LC Oscillations

(So far we have dealt with RC/RL, now: LC)

So far, the charge, current and potential Difference was growing and decaying exponentially, now we'll see that it's changing sinusoidally (with period T and angular frequency ω) $\Rightarrow$ electromagnetic oscillations

$$U_E = \frac{q^2}{2C}, \quad U_B = \frac{Li^2}{2}$$

q, i, v : instantenous value

Q, I, V : Amplitude (maximum value)

[REFER TO THE ILLUSTRATION & GRAPHS NEXT PAGE]

Electrical - Mechanical Analogy

LC $\leftrightarrow$ Block-Spring System $\rightarrow$ 2 Kinds of Energy: Potential Energy of the compressed spring; Kinetic Energy of the moving block

Block-Spring

Element	Energy
Spring	Pt. $\frac{1}{2}kx^2$
Block	Kin. $\frac{1}{2}mv^2$

$v = \frac{dx}{dt}$

$\omega = \sqrt{\frac{k}{m}}$

$q \leftrightarrow x$
$1/C \leftrightarrow k$
$i \leftrightarrow v$
$L \leftrightarrow m$

LC

Element	Energy
Capacitor	Electrical: $\frac{1}{2}\left(\frac{1}{C}\right)q^2$
Inductor	Magnetic: $\frac{1}{2}Li^2$

$i = \frac{dq}{dt}$

$\omega = \frac{1}{\sqrt{LC}}$

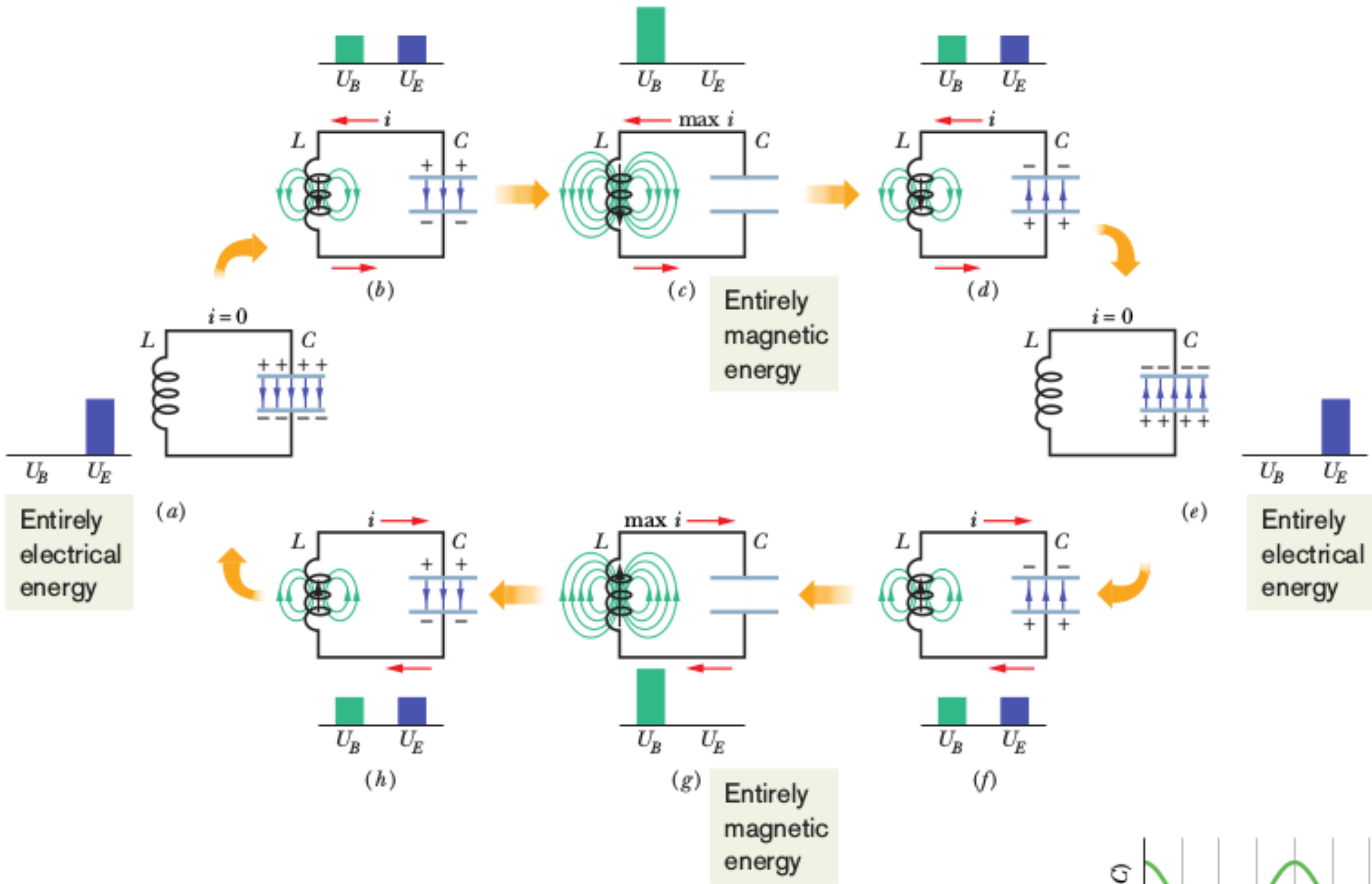


Fig. 31-1 Eight stages in a single cycle of oscillation of a resistanceless LC circuit. The bar graphs by each figure show the stored magnetic and electrical energies. The magnetic field lines of the inductor and the electric field lines of the capacitor are shown. (*a*) Capacitor with maximum charge, no current. (*b*) Capacitor discharging, current increasing. (*c*) Capacitor fully discharged, current maximum. (*d*) Capacitor charging but with polarity opposite that in (*a*), current decreasing. (*e*) Capacitor with maximum charge having polarity opposite that in (*a*), no current. (*f*) Capacitor discharging, current increasing with direction opposite that in (*b*). (*g*) Capacitor fully discharged, current maximum. (*h*) Capacitor charging, current decreasing.

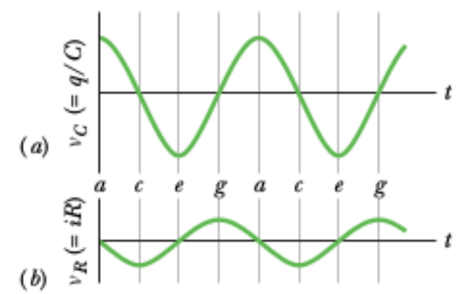


Fig. 31-2 (*a*) The potential difference across the capacitor of the circuit of Fig. 31-1 as a function of time. This quantity is proportional to the charge on the capacitor. (*b*) A potential proportional to the current in the circuit of Fig. 31-1. The letters refer to the correspondingly labeled oscillation stages in Fig. 31-1.

LC Oscillations, Quantitatively

Block - Spring

$$\underset{\text{total energy}}{U} = \underset{\text{kinetic}}{U_b} + \underset{\text{potential}}{U_s} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

no friction : Energy stays constant with respect to time $\rightarrow \frac{dU}{dt} = 0$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

$$v = \frac{dx}{dt}, \quad \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + kx = 0$$

$$x = X \cos(\omega t + \varphi)$$

X: amplitude, ω: angular frequency, φ: phase constant

$$\omega = \sqrt{\frac{k}{m}}$$

LC Oscillator

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

$\frac{Li^2}{2}$: magnetic energy, $\frac{q^2}{2C}$: electric energy

Circuit Resistance Zero: no thermal Energy dissipated

$\longrightarrow$ U is constant wrt time $\rightarrow \frac{dU}{dt} = 0$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0$$

$$i = \frac{dq}{dt}, \quad \frac{di}{dt} = \frac{d^2q}{dt^2}$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad \Rightarrow \quad q = Q\cos(\omega t + \varphi) \quad \text{(Charge)}$$

$$i = \frac{dq}{dt} = -\omega Q \sin(\omega t + \varphi)$$

Amplitude : $I = \omega Q$

$$i = -I \sin(\omega t + \varphi)$$

Angular Frequencies

Substitute $q = Q\cos(\omega t + \varphi)$ in $\dfrac{d^2q}{dt^2}$

$$\Rightarrow \frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t+\varphi)$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \Rightarrow -L\omega^2 Q\cos(\omega t+\varphi) + \frac{1}{C}Q\cos(\omega t+\varphi) = 0$$

$$\Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

φ : determined from initial conditions

Electrical and Magnetic Energy Oscillations

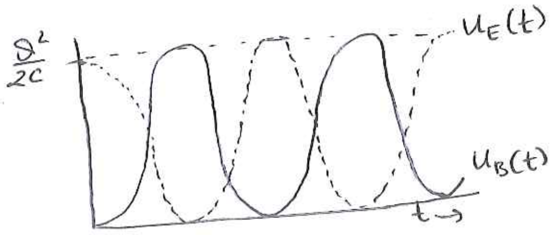
$$q = Q\cos(\omega t+\varphi) \Rightarrow U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t+\varphi)$$

$$i = -I\sin(\omega t+\varphi) \Rightarrow U_B = \frac{1}{2}Li^2 = \frac{1}{2}L\omega^2 Q^2 \sin^2(\omega t+\varphi)$$

$$\omega = \frac{1}{\sqrt{LC}} \Rightarrow U_B = \frac{Q^2}{2C}\sin^2(\omega t+\varphi)$$

$$U_{TOT} = U_E + U_B = \frac{Q^2}{2C}\Big[\overbrace{\cos^2(\omega t+\varphi) + \sin^2(\omega t+\varphi)}^{1}\Big] = \frac{Q^2}{2C}$$: Total Energy is conserved.

$\varphi = 0$:



1) Maximum values of U_E & U_B are both $Q^2/2C$
2) At any instant, the sum of U_E and U_B $= \dfrac{Q^2}{2C}$
3) When U_E is max, $U_B = 0$ (and vice versa)

Example: A 1.5 μF capacitor is charged to 57 V by a battery, which is then removed. At time t=0, a 12 mH coil is connected.

a.) What is the potential difference $V_L(t)$ across the inductor?

$V_L(t) = V_C(t)$

q max at $t=0 \rightarrow \varphi = 0$

$$q = Q\cos(\omega t) \Rightarrow \frac{q}{C} = \frac{Q}{C}\cos\omega t$$

$V_C = V_L$

b.) What is the maximum rate $\left(\dfrac{di}{dt}\right)_{max}$ at which the current i changes in the circuit?

$$i = -\omega Q\sin(\omega t)$$

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q\sin\omega t) = -\omega^2 Q\cos\omega t$$

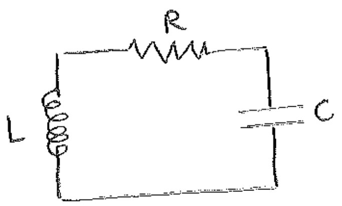
CV_c , $1/\sqrt{LC}$

$$\rightarrow \frac{di}{dt} = -\frac{1}{LC}CV_c\cos\omega t = -\frac{V_c}{L}\cos\omega t$$

Amplitude (max value) $= \ldots = 4750$ A/s

DAMPED OSCILLATIONS IN AN RLC CIRCUIT

RLC



With a Resistance R present, the total electromagnetic energy U of the circuit is no longer conserved. It decreases with time as the energy is transferred to thermal energy in the resistance (Like friction in the spring-block system).

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$ (No "R" term, because the Resistance does not store electromagnetic energy)

$$\frac{dU}{dt} = -i^2 R$$ (Rate of energy dissipation as Thermal energy in the resistor)

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2 R$$

$$i = \frac{dq}{dt} \rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$ (RLC Circuit)

$$q = Q e^{-Rt/2L} \cos(\omega' t + \varphi) \; ; \; \omega' = \sqrt{\omega^2 - (R/2L)^2}$$

we'll assume R is small, so we can replace ω' with ω ($R \ll L \rightarrow \omega' \sim \omega$)

→ Sinusoidal oscillation with an exponentially decaying amplitude: $Q e^{-Rt/2L}$

Example: A series RLC, $L = 12\,mH$, $C = 1.6\,\mu F$, $R = 1.5\,\Omega$

a.) At what time t will the amplitude of the charge oscillations in the circuit be 50% of its initial value?

$$Q e^{-Rt/2L} = 0.5\,Q$$

$$-\frac{Rt}{2L} = \ln 0.5 \Rightarrow t = -\frac{2L}{R}\ln\tfrac{1}{2} = \ldots = 11\,ms$$

b) How many oscillations are completed within This time?

$$T = \frac{2\pi}{\omega} \qquad \omega = \frac{1}{\sqrt{LC}} \qquad \frac{\Delta t}{t} = \frac{\Delta t}{2\pi\sqrt{LC}} = \ldots = 13$$

ALTERNATING CURRENT

The oscillations in an RLC circuit will not damp out if an external device supplies enough energy to make up for the energy dissipated as thermal energy in R. This is done via oscillating EMFs and currents ⟶ AC

(Standard) Reversing directions 120 times per second

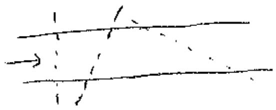
$f = 60Hz$

$v_d \simeq 4\times10^{-5} m/s$ if we reverse their direction every $1/120$ second, they can move only $\underbrace{3\times10^{-7} m}$ in half-cycle

↳ passes ~ 10 atoms

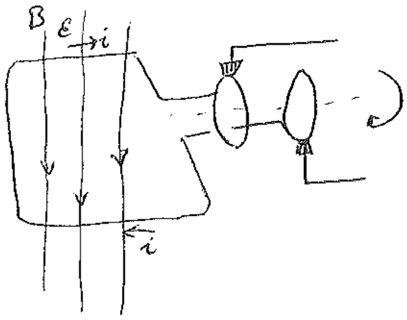
The signal to reverse the direction is propagated along the conductor at speeds ~ c.

However, the total distance traversed is not important since: the current is the charge passing through any plane cutting



Basic Advantage: As the current alternates, so does the magnetic field that surrounds the conductor. → By induction, we can step up/down the magnitude of an alternating potential difference at will using a transformer.

Also, it is more convenient to rotating machinery



$\mathcal{E} = \mathcal{E}_m \sin\omega_d t$

↑ amplitude

← the angular speed with which the loop rotates

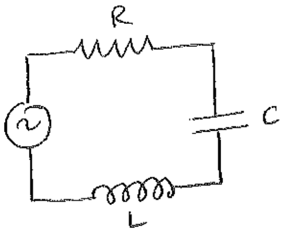
$i = I \sin(\omega_d t - \varphi)$

↳ driving angular frequency

FORCED OSCILLATIONS

$\omega = \frac{1}{\sqrt{LC}}$ (RLC also, with small R)

→ free oscillations, ω is circuit's national frequency



When an external alternating emf is connected to an RLC the oscillations of charge, potential difference and current are said to be driven oscillations / forced oscillations.

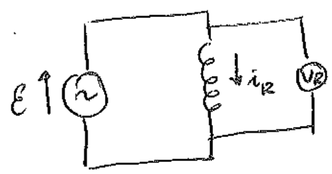
Whatever the natural frequency ω of a circuit may be, forced oscillations always occur at the driving angular frequency.

But its amplitude is closely related to $|\omega - \omega_d|$

if $\omega = \omega_d \Rightarrow I$ is maximum (resonance)

THREE SIMPLE CIRCUITS

i) A resistive Load:



$\mathcal{E} - V_R = 0$

$v_R = \mathcal{E}_m \sin(\omega_d t)$

$v_R = V_R \sin(\omega_d t)$

$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin(\omega_d t)$

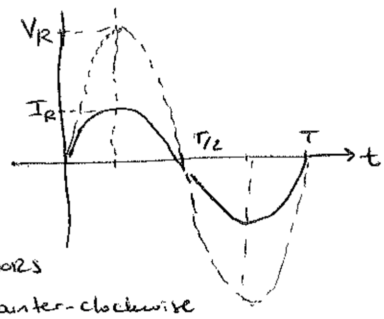
$i_R = I_R \sin(\omega_d t)$

$\varphi = 0°$

Amplitudes Relation: $V_R = I_R R$

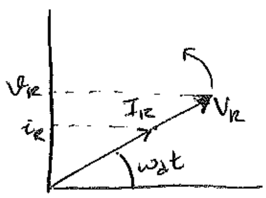
$v_R = V_R \sin(\omega_d t)$

$i_R = I_R \sin(\omega_d t)$



: no decay because the generator supplies energy to make up for the energy dissipated in R.

phasors



Angular speed: Both phasors rotate counter-clockwise with an angular speed equal to ω_d

Length: V_R, I_R

Rot. Angle: Phase at t

Example: $R = 200\,\Omega$, $\mathcal{E}_m = 36\,V$ $f_d = 60\,Hz$

a.) What is the potential difference $v_R(t)$?

$v_R(t) = \mathcal{E}(t)$ $\quad V_R = \mathcal{E}_m = 36\,V$

$$v_R(t) = \mathcal{E}_m \sin \omega_d t$$

$$\omega_d = 2\pi f_d = 2\pi\,(60\,Hz) = 120\,\pi/s$$

$$v_R = 36 \sin(120\pi t)$$

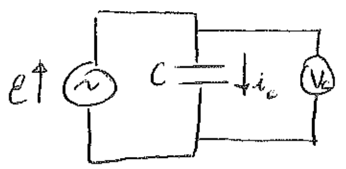
b.) What are the current $i_R(t)$ and I_R?

$$i_R = I_R \sin(\omega_d t - \varphi) = I_R \sin \omega_d t$$

$$I_R = \frac{V_R}{R} = \frac{36}{200} = 0.18A$$

$$\Longrightarrow \quad i_R = (0.18A) \sin(120\pi t)$$

ii) A Capacitive Load:



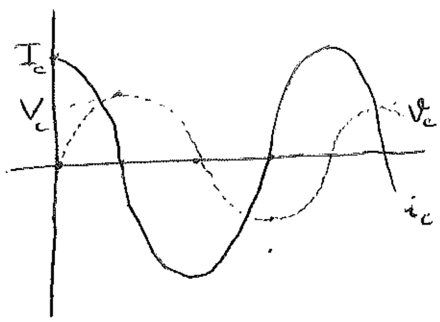
$$v_c = V_c \sin \omega_d t$$

$$q_c = C v_c = C V_c \sin \omega_d t$$

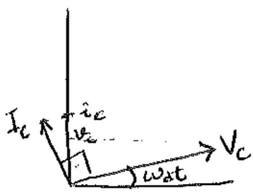
$$i_c = \frac{dq_c}{dt} = \omega_d C V_c \cos \omega_d t$$

$X_c \equiv \dfrac{1}{\omega_d C}$ "capacitive Reactance" of the capacitor

$([X] = \Omega)$



$$\cos \omega_d t = \sin(\omega_d t + 90°)$$



$$\Rightarrow \quad i_c = \left(\frac{V_c}{X_c}\right) \sin(\omega_d t + 90°)$$

"i_c leads v_c"

<u>Example:</u> $C = 15\mu F$ $\quad \mathcal{E}_m = 36V$ $\quad f_d = 60Hz$

v_c, i_c ?

$$V_c = \mathcal{E}_m = 36V$$

$$v_c(t) = \mathcal{E}(t) = \mathcal{E}_m \sin(\omega_d t)$$

$$\omega_d = 2\pi f_d = 120\pi \qquad v_c = (36V)\sin(120\pi t)$$

$$i_c = I_c \sin(\omega_d t - \varphi) = I_c \sin(\omega_d t + \pi/2)$$

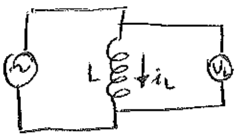
$$V_c = I_c X_c$$

$$X_c = \frac{1}{\omega_d C} = \frac{1}{2\pi(60Hz)(15\times10^{-6}F)} = 177\Omega$$

$$I_c = \frac{V_c}{X_c} = \frac{36}{177} = 0.203A$$

$$\longrightarrow \quad i_c = (0.203A)\sin(120\pi t + \pi/2)$$

iii) An inductive load:



$$v_L = V_L \sin\omega_d t$$

$$\left(\mathcal{E}_L = -L\frac{di}{dt}\right)$$

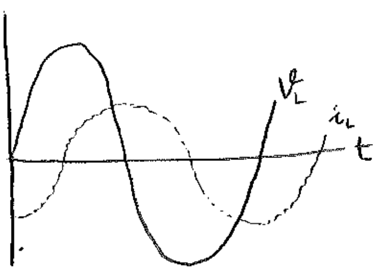
$$v_L = L\frac{di_L}{dt}$$

$$\longrightarrow \frac{di_L}{dt} = \frac{V_L}{L}\sin\omega_d t \Rightarrow di_L = \frac{V_L}{L}\sin(\omega_d t)\,dt$$

$$i_L = \int di_L = \frac{V_L}{L}\int \sin\omega_d t\,dt = -\left(\frac{V_L}{\omega_d L}\right)\cos\omega_d t$$

$$X_L = \omega_d L \quad \text{(inductive Reactance)}$$

$$[|X_L|] = \Omega$$

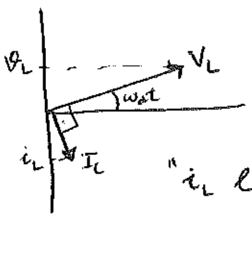


$$-\cos\omega_d t = \sin(\omega_d t - 90°)$$

$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90°)$$

$$i_L = I_L \sin(\omega_d t - \varphi)$$

$$V_L = I_L X_L$$



"i_L lags v_L"

Example: $L = 230\,mH$ $\quad \mathcal{E}_m = 36V$ $\quad f_d = 60Hz$

v_L, i_L ?

$v_L(t) = \mathcal{E}(t) \qquad V_L = \mathcal{E}_m = 36V$

$v_L(t) = \mathcal{E}_m \sin \omega_d t$

$\omega_d = 2\pi f_d = 120\pi \qquad v_L = (36V) \sin(120\pi t)$

$i_L = I_L \sin(\omega_d t - \varphi) = I_L \sin(\omega_d t - \pi/2)$

$V_L = I_L X_L$

$X_L = \omega_d L = 2\pi f_d L = 86.7\,\Omega$

$I_L = \frac{V_L}{X_L} = \frac{36}{86.7} = 0.415\,A$

$\underline{i_L = (0.415A) \sin(120\pi t - \pi/2)}$

SUMMARY

	Reactance	Current	Amplitude Relation
R	R	in phase with v_R	$V_R = I_R R$
C	$X_C = 1/\omega_d C$	Leads v_C by $90°$	$V_C = I_C X_C$
L	$X_L = \omega_d L$	Lags v_L by $90°$	$V_L = I_L X_L$