Solutions

① $\mathbb{R}_2$ is not a v.s. with $\oplus$ and $\odot$ because $1\odot(x,y) \neq (x,y)$ for any $(0,0)\neq(x,y)\in\mathbb{R}_2$.

② V is closed under $\oplus$ because if u, v are positive real numbers, then $u.v$ is also positive. $u\oplus v = u.v$

V is closed under $\odot$, because if $u>0$, then u^c is also positive.

$\oplus$ is commutative : $u\oplus v = u.v = v.u = v\oplus u$

$\oplus$ is associative : $u\oplus(v\oplus w) = u\oplus(v.w) = u.(v.w) = (u.v).w$
$= (u\oplus v).w = (u\oplus v)\oplus w$.

0_v element : $u\oplus v = u \Rightarrow u.v = u \Rightarrow v=1$. The zero element is $1\in V$.

$u\in V \Rightarrow -u = ?$: $u\oplus(-u) = 0_v = 1 \Rightarrow u.(-u) = 1 \overset{u>0}{\Rightarrow} -u := \frac{1}{u}\in V$.

Distributions : $c\odot(v\oplus w) = c\odot(v.w) = (vw)^c = v^c.w^c = (c\odot v)\oplus(c\odot w)$

$\left(\begin{matrix} c,d\in\mathbb{R} \\ u,v,w\in V\end{matrix}\right)$ $\quad (c+d)\odot u = u^{c+d} = u^c.u^d = \underbrace{(c\odot u)}_{\in V}.\underbrace{(d\odot u)}_{\in V} = (c\odot u)\oplus(d\odot u)$

Associativity : $(c.d)\odot u = u^{cd} = (u^d)^c = d\odot u^d = c\odot(d\odot u)$.

Identity : $1\odot u = u^1 = u$. Thus V is a vector space.

③ a) $W = \{(2t,-3t) \mid t\in\mathbb{R}\} < \mathbb{R}^2$: $W\neq\emptyset$ because $(0,0)\in W$.

$(2t,-3t) + (2s,-3s) = (2(t+s), -3(t+s)) \in W$.

$c\in\mathbb{R}$, $(2t,-3t)\in W \Rightarrow c.(2t,-3t) = (2(ct), -3(ct))\in W$.

b) $W = \{(2t+3, -4t) \mid t\in\mathbb{R}\}$ is not a subspace, because $(0,0)\notin W$. $(0,0) = (2t+3,-4t) \Rightarrow t=0,\ 0=3$ #.

c) $W = \{(a,b,c) \mid 3a+b-2c=0\} = \{(a, 2c-3a, c) \mid a,c\in\mathbb{R}\} < \mathbb{R}^3$.

$\underbrace{(a, 2c-3a, c)}_{\in W} + \underbrace{(a', 2c'-3a', c')}_{\in W} = \underbrace{(a+a', 2(c-c')-3(a-a'), c-c')}_{\in W}$

$r\in\mathbb{R}$, $r(a, 2c-3a, c) = (ra, 2rc-3ra, rc)\in W$.

d) $W = \{a_0 + 3a_0t + a_2t^2 \mid a_0, a_2\in\mathbb{R}\} < P_2$ (?) $\quad W\neq\emptyset$ $(a_0=a_2=0)$.

$(a_0+3a_0t+a_2t^2) + (b_0+3b_0t+b_2t^2) = a_0+b_0+3(a_0+b_0)t + (a_2+b_2)t^2\in W$.

$c\in\mathbb{R}$, $c.(a_0+3a_0t+a_2t^2) = (ca_0+3ca_0t+ca_2t^2\in W$.

e) $W = \left\{ \begin{bmatrix} a & b & -a \\ d & e & -b-d \end{bmatrix} \mid a,b,d,e \in \mathbb{R} \right\} < M_{23}$ (?)

$$\begin{bmatrix} a & b & -a \\ d & e & -b-d \end{bmatrix} + \begin{bmatrix} a' & b' & -a' \\ d' & e' & -b'-d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' & -(a+a') \\ d+d' & e+e' & -(b+b')-(d+d') \end{bmatrix} \in W.$$

$c \in \mathbb{R} \Rightarrow c[\quad] \in W.$

f) $W_1 = \{\text{constant functions}\}$: $f, g \in W_1 \Rightarrow f(x) = c_1 \in \mathbb{R}$, $g(x) = c_2 \in \mathbb{R}$

$\Rightarrow (f+g)(x) = f(x) + g(x) = c_1 + c_2$

$\Rightarrow f+g$ is constant.

$c \in \mathbb{R}, f \in W_1 \Rightarrow f(x) = c_1 \in \mathbb{R}.$

$(cf)(x) = c \cdot f(x) = cc_1 \Rightarrow cf \in W_1$

$\therefore W_1 < C[-\infty, \infty]$

$W_2 = \{\text{all functions } f \text{ s.t. } f(0) = 0\}$: $f(0) = 0, g(0) = 0 \Rightarrow (f+g)(0) = 0$

$(c \cdot f)(0) = c \underbrace{f(0)}_{0} = 0.$ $\Rightarrow f+g \in V$, $c \cdot f \in V$.

$W_3 = \{$ " " $f(0) = 3\}$: $f(0) = 3, g(0) = 3 \Rightarrow (f+g)(0) = 6 \neq 3$

$\Rightarrow f+g \notin W_3$. not a subspace.

$W_4 = \{\text{All differentiable functions}\}$: $f, g \in W_4 \Rightarrow f+g \in W_4$.

$c \in \mathbb{R}$, cf is also differentiable $\Rightarrow cf \in W_4$.

4 a) $\langle (4\ 2\ -6), (-2\ -1\ 3) \rangle = \{a(4,2,-6) + b(-2,-1,3) \mid a, b \in \mathbb{R}\}$

$= \{(4a-2b, 2a-b, -6a+3b) \mid a, b \in \mathbb{R}\}$. Then

$(x, y, z) = (4a-2b, 2a-b, -6a+3b) \Rightarrow$
$x = 4a - 2b$
$y = 2a - b$
$z = -6a + 3b.$

$$\left[\begin{array}{cc|c} 4 & -2 & x \\ 2 & -1 & y \\ -6 & 3 & z \end{array}\right] \xrightarrow[+3R_2+R_3]{-2R_2+R_1} \left[\begin{array}{cc|c} 0 & 0 & x-2y \\ 2 & -1 & y \\ 0 & 0 & z+3y \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -1 & y \\ 0 & 0 & x-2y \\ 0 & 0 & z+3y \end{array}\right]$$ we want the system consistent,

$\Rightarrow x - 2y = 0,\ z + 3y = 0 \Rightarrow (x, y, z) = (2t, t, -3t)$

$y = t \Rightarrow x = 2t,\ z = -3t$

$\therefore \langle (4\ 2\ -6), (-2\ -1\ 3) \rangle = \{t(2,1,-3) \mid t \in \mathbb{R}\} = \{$all lines passing through $(0,0,0)$ and $(2,1,-3)\}$.

b)

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b) $W = \langle (-1\ -3\ 2), (1\ 2\ -1), (1\ 1\ -1) \rangle$

$= \{ a(\quad) + b(\quad) + c(\quad) \mid a,b,c \in \mathbb{R} \}$

$= \{ (-a+b+c, -3a+2b+c, 2a-b-c) \mid \text{"} \}$

$(x,y,z) \in W \Rightarrow \begin{matrix} -a+b+c = x \\ -3a+2b+c = y \\ 2a-b-c = z \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 1 & x \\ -3 & 2 & 1 & y \\ 2 & -1 & -1 & z \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & x+z \\ 0 & 1 & 0 & x+y+2z \\ 0 & 0 & 1 & x-y-z \end{array}\right].$

$\Rightarrow a = x+z$, $b = x+y+2z$ $c = x-y-z$. Thus the system is always consistent and therefore $(x,y,z) \in W$. Since it is true for any vector (x,y,z), we have that $\mathbb{R}^3 \subseteq W$. Hence $\mathbb{R}^3 = W$.

Every vector (x,y,z) can be written as

$(x,y,z) = (x+z)(-1,-3,2) + (x+y+2z)(1,2,-1) + (x-y-z)(1,1,-1)$.

(5) a) $c_1\alpha_1 + c_2\alpha_2 + c_3(d\alpha_3) = 0 \Rightarrow c_1\alpha_1 + c_2\alpha_2 + (c_3 d)\alpha_3 = 0$, $\{\alpha_1,\alpha_2,\alpha_3\}$ lin. ind.

$\Rightarrow c_1 = c_2 = c_3 d = 0 \overset{d \neq 0}{\Rightarrow} c_1 = c_2 = c_3 = 0$.

b) $c_1(\alpha_1 + 2\alpha_3) + c_2(3\alpha_2 - \alpha_3) = 0 \Rightarrow c_1\alpha_1 + (3c_2)\alpha_2 + (2c_1 - c_2)\alpha_3 = 0$.

$\Rightarrow c_1 = 3c_2 = 2c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = 0$.

c) $c_1\alpha_1 + c_2(\alpha_2 + d\alpha_3) + c_3\alpha_3 = 0 \Rightarrow c_1\alpha_1 + c_2\alpha_2 + (c_2 d + c_3)\alpha_3 = 0 \Rightarrow$

$c_1 = c_2 = c_2 d + c_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$.

They are all lin. independent.

(6) $B = \{1-t, 3-t^2, t+4t^2\}$ span P_2: Let $p(t) = a_0 + a_1 t + a_2 t^2 \in P_2$.

$p(t) \in \langle B \rangle$ (?). $a_0 + a_1 t + a_2 t^2 = c_0(1-t) + c_1(3-t^2) + c_2(t+4t^2)$, $c_i = ?$

$= (c_0 + 3c_1) + (c_0 + c_2)t + (-c_1 + 4c_2)t^2$

$\Rightarrow \begin{matrix} c_0 + 3c_1 = a_0 \\ c_0 + c_2 = a_1 \\ -c_1 + 4c_2 = a_2 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & a_0 \\ 1 & 1 & 0 & a_1 \\ 0 & -1 & 4 & a_2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & a_0 \\ 0 & -2 & 0 & a_1 - a_0 \\ 0 & -1 & 4 & a_2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & a_0 \\ 0 & 0 & -8 & a_1 - a_0 - 2a_2 \\ 0 & -1 & 4 & a_2 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 3 & 0 & a_0 \\ 0 & 1 & -4 & -a_2 \\ 0 & 0 & +1 & (a_1 - a_0 - 2a_2)\cdot -\frac{1}{8} \end{array}\right] \to \begin{matrix} c_0 + 3c_1 = a_0 \\ c_1 - 4c_2 = -a_2 \\ c_2 = -\frac{1}{8}(a_1 - a_0 - 2a_2). \end{matrix}$ $\begin{matrix} c_1 = -a_2 + 4c_2 \\ c_0 = a_0 - 3c_1. \end{matrix}$ The system has a solution.

So $p(t) \in \langle B \rangle$.

B is lin. independent because if $c_0(1-t) + c_1(3-t^2) + c_2(t+4t^2) = 0$, then $a_0 = a_1 = a_2 = 0 \Rightarrow c_2 = 0 = c_1 = c_0$ by above.

$\left(\det \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix} = 1(4) - 3 \cdot 4 \neq 0. \right)$

(7) $\begin{bmatrix} a & c & a+2c \\ 3b & a-b & c+a \end{bmatrix} + \begin{bmatrix} a' & c' & a'+2c' \\ 3b' & a'-b' & c'+a' \end{bmatrix} = \begin{bmatrix} a+a' & c+c' & a+a'+2(c+c') \\ 3(b+b') & (a+a')-(b+b') & (c+c')+(a+a') \end{bmatrix}$

$\in H \qquad \in H \qquad \in H$

$r \in \mathbb{R},\ A \in H \Rightarrow r.A \in H.$

$$A = \begin{bmatrix} a & c & a+2c \\ 3b & a-b & c+a \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{X} + b\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 3 & -1 & 0 \end{bmatrix}}_{Y} + c\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{Z}$$

$A \in \langle X, Y, Z \rangle \Rightarrow H = \langle X, Y, Z \rangle.$

Is $\{X, Y, Z\}$ lin. independent? : $aX + bY + cZ = 0 \Rightarrow A = 0 \Rightarrow$
$a = c = 0 = 3b \Rightarrow a = b = c = 0$. So $\{X, Y, Z\}$ is lin. independent.
Hence it is a basis for H.

(8) a) $W = \langle (1\ 2\ 3) \rangle$, $(1\ 2\ 3)$ is a nonzero vector in $\mathbb{R}^3$ and therefore linearly independent. Hence $(1\ 2\ 3)$ is a basis for W.

b) $W = \langle (4,1,1), (-1,1,2) \rangle$. Since $(4,1,1)$ is not a multiple of $(-1,1,2)$, these vectors are linearly independent, and they span W. Therefore they are basis for W.

c) We will find the lin. independent vectors from these five vectors.

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -4 & 1 & -2 \\ 3 & -5 & -4 & 5 \\ 2 & -2 & -3 & 4 \end{bmatrix} \xrightarrow{-2,\,-3} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -4 & 1 & -2 \\ 0 & -8 & 2 & -4 \\ 0 & -4 & 1 & -2 \end{bmatrix} \xrightarrow[-2R_2 + R_3]{\frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -4 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & -1/4 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$u_1 \quad u_2 \quad u_3 \quad u_4$

$\Rightarrow u_1$ and u_2 are lin. independent. u_3 and u_4 must be a lin. combination of u_1 and u_2. Indeed,

$$\rightarrow \begin{bmatrix} 1 & 0 & \boxed{-7/4} & \boxed{5/2} \\ 0 & 1 & \boxed{-1/4} & \boxed{1/2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$u_3 = (-2\ \ 1\ \ -4\ \ -3) = -\frac{7}{4}u_1 - \frac{1}{4}u_2$
$= -\frac{7}{4}(1\ 0\ 3\ 2) - \frac{1}{4}(1\ -4\ -5\ -2)$
$= (-2, 1, -4, -3)$

$u_4 = \frac{5}{2}u_1 + \frac{1}{2}u_2$. Hence $W = \langle u_1, \dots, u_4 \rangle = \langle u_1, u_2 \rangle$ and $\{u_1, u_2\}$ is a basis for W.

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(16) (18) Eigenvalue and eigenvectors of $A=\begin{bmatrix}4&0&-6\\0&1&0\\3&0&-5\end{bmatrix}$.

Sol. $p(\lambda)=|\lambda I-A|=\begin{vmatrix}\lambda-4&0&6\\0&\lambda-1&0\\-3&0&\lambda+5\end{vmatrix}=(\lambda-1)\left((\lambda-4)(\lambda+5)+18\right)$

$=(\lambda-1)(\lambda^2+\lambda-2)$

$=(\lambda-1)^2(\lambda+2)$.

$\Rightarrow \lambda_1=1$, $\lambda_2=-2$.

are eigenvalues.

$\lambda_1=1$: $(\lambda_1 I-A)X=0 \Rightarrow X=?$ $I-A=\begin{bmatrix}-3&0&6\\0&0&0\\-3&0&6\end{bmatrix}$

$[I-A\,|\,0]\to\begin{bmatrix}1&0&-2\\0&0&0\\0&0&0\end{bmatrix} \Rightarrow X=\begin{bmatrix}2t\\s\\t\end{bmatrix}=s\begin{bmatrix}0\\1\\0\end{bmatrix}+t\begin{bmatrix}2\\0\\1\end{bmatrix}$

$\Rightarrow$ all eigenvectors of $\alpha_1=1$ is $\left\langle \begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\1\end{bmatrix}\right\rangle$ (eigenspace)

For exp, $\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\1\end{bmatrix}$ are eigenvectors of λ_1.

$\lambda_2=-2$: $-2I-A=\begin{bmatrix}-6&0&6\\0&-3&0\\-3&0&3\end{bmatrix} \Rightarrow [-2I-A\,|\,0]\sim\begin{bmatrix}1&0&-1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}$

$X=\begin{bmatrix}2t\\0\\t\end{bmatrix}$ All Eigenvectors are $\left\{t\begin{bmatrix}2\\0\\1\end{bmatrix}\,\middle|\,t\in\mathbb{R}\right\}$.

(19) Eigenvalue and eigenvectors of a) $A=\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&-1&1\end{bmatrix}$?

(16) b) $L:\mathbb{R}^2\to\mathbb{R}^2$, $L(u_1,u_2)=(3u_1-5u_2,2u_1-3u_2)$.

Sol (a) $p(\lambda)=|\lambda I-A|=(\lambda^2-1)[(\lambda-1)^2+1]$

Real roots are $1,-1$. $\left(\begin{matrix}\lambda^2-2\lambda+2\\ \Delta=4-4.2<0\end{matrix}\right)$

$\lambda_1=1 \Rightarrow (\lambda I-A)X=0 \Rightarrow X=t\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ $(I-A)X=\begin{bmatrix}2&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=0 \Rightarrow \begin{matrix}2x_1=0\\-x_4=0\\x_3=0\end{matrix}$

$\Rightarrow X=\begin{bmatrix}0\\x_2\\0\\0\end{bmatrix}=x_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$\lambda_2=-1 \Rightarrow (\lambda I-A)X=0 \Rightarrow X=t.\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

b) $A=[L]_S^S=\begin{bmatrix}3&-5\\2&-3\end{bmatrix}$ $S=\{e_1,e_2\}$.

$p(\lambda)=|\lambda I-A|=\lambda^2+1$; it has no real root.

So L has no (real) eigenvalue.

17) $A = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}$ skew ~~[illegible]~~ sym.

$H = \{A \mid A$ is skew symmetric$\} \Rightarrow \dim H = 3 \Rightarrow H \cong \mathbb{R}^3$.

18) $A = \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ -1 & 3 & 2 & 3 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 4 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$

row space A = row space B

$= \langle \underbrace{(1\ 0\ 4\ -3), (0\ 1\ 2\ 0)}_{\text{basis}} \rangle$. //.

rank $A = 2$.