M E T U
Department of Mathematics

	Calculus with Analytic Geometry First Midterm Exam	
Code : MATH 119 Acad.Year : 2016 Semester : Fall Coord. : Muhiddin Uğuz Date : 19.11.2016 Time : 9.30 Duration : 110 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : 6 Questions on 4 Pages Total 100 Points	

Q1	Q2	Q3	Q4	Q5	Q6			
							SHOW YOUR WORK !	

Q.1 (15 pts) Show that the equation

$$\arctan x = 2 - x - x^3$$

has exactly ONE solution.

Consider the function $f(x) = \arctan x - 2 + x + x^3$

Note that $f(0) = -2 < 0$

$f(1) = \frac{\pi}{4} > 0$

Moreover f is continuous on $[0,1]$, and hence by I.V.T. (Intermediate Value Theorem), there exists $c \in (0,1)$ such that $f(c) = 0$. Hence given equation has at least one solution.

For the uniqueness of the solution;

Note that $f'(x) = \frac{1}{1+x^2} + 1 + 3x^2 \cancel{\geq} > 0 \quad \forall x.$

so $f(x)$ is a strictly increasing function and hence one-to-one.

Therefore the equation $f(x) = 0$ has at most one solution

Hence $f(x) = 0$ has exactly one solution.

OR/ since f is continuous on $[0,1]$ and differentiable on $(0,1)$, we can use Rolle's Thm:

if $f(x_0) = 0 = f(x_1)$ for some $x_0, x_1 \in (0,1)$ then $\exists\, r$ between x_0 & x_1 s.t. $f'(r) = 0$. But $f'(r) = \frac{1}{1+r^2} + 1 + 3r^2$ and hence $f'(r) \neq 0 \ \forall r$.

Q.2 ($4 \times 6 = 24$ pts) Without using L'Hospital's rule, evaluate the limits:

(a) $\lim_{x \to 0} \dfrac{\tan 5x}{\sin 3x}$ $= \lim_{x \to 0} \left[\dfrac{\sin 5x}{5x} \cdot \dfrac{3x}{\sin 3x} \cdot \dfrac{1}{\cos 5x} \cdot \dfrac{5}{3} \right]$

since each limit exists $\Rightarrow$ $= \lim_{u \to 0} \dfrac{\sin u}{u} \cdot \lim_{w \to 0} \dfrac{w}{\sin w} \cdot \lim_{x \to 0} \dfrac{1}{\cos 5x} \cdot \dfrac{5}{3}$

$= 1 \cdot 1 \cdot 1 \cdot \dfrac{5}{3} = \dfrac{5}{3}$

(b) $\lim_{x \to -\infty} (x - \sqrt{x^2 + 4x})$

Note that $x - \sqrt{x^2 + 4x} \le x \quad \forall\, x \le -4$ and $\lim_{x \to -\infty} x = -\infty$

so $\lim_{x \to -\infty} x - \sqrt{x^2+4x} = -\infty$

(c) $\lim_{x \to 0} \dfrac{\cos[(x+2)^2] - \cos 4}{x}$ $= f'(0)$ where $f(x) = [\cos(x+2)^2]$

and hence

(or, you may choose $g(x) = \cos(x^2)$ and find $g'(2)$)

$f'(x) = -2(x+2)\sin[(x+2)^2]$

$= -2(0+2)\sin((0+2)^2) = -4\sin(4)$

(d) $\lim_{x \to 0} x^2\, \mathrm{e}^{\sin(\frac{\pi}{x})}$

e^x is increasing

$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \implies e^{-1} \le e^{\sin(\frac{\pi}{x})} \le e$

$\implies x^2 e^{-1} \le x^2 e^{\sin(\frac{\pi}{x})} \le x^2 e$

since $\lim_{x \to 0} \dfrac{x^2}{e} = 0 = \lim_{x \to 0} x^2 e$, by Squeeze Thm, $\lim_{x \to 0} x^2 e^{\sin(\frac{\pi}{x})} = 0$

Q.3 (10 pts) Let $f(x) = \begin{cases} x^2 + \arcsin x & \text{for } \; x \ge 0 \\ x + \mathrm{e}^{x^2} & \text{for } \; x < 0 \end{cases}$. Find $f'(0)$ if it exists, or explain why it does NOT exist.

$f(0) = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + e^{x^2}) = 1 \ne 0 \implies f$ is not continuous and hence not differentiable at $x = 0$.

(OR $\lim_{t \to 0^+} \dfrac{f(t) - f(0)}{t} = \lim_{t \to 0^+} \dfrac{t^2 + \arcsin t}{t} = \cdots = 1$

$\lim_{t \to 0^-} \dfrac{f(t) - f(0)}{t} = \lim_{t \to 0^-} \dfrac{t + e^{t^2}}{t} = \cdots = -\infty$

$\therefore f$ is not diff'ble at $x = 0$)

Q.4 **(5 × 6 = 30 pts)**

(a) By definition, the limit $\lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{h}$ represents the derivative of a function $f(x)$ at the number $x = x_0$. Find $f(x)$ and x_0.

$f(x) = \sqrt[4]{x}$, $x_0 = 16$

[or $g(x) = \sqrt[4]{16+x}$ and $x_0 = 0$
or ⋮]

(b) Find $f'(x) = \frac{dy}{dx}$ if $y = f(x) = \frac{x^\pi + \cos x}{1+x^3}$.

$$f'(x) = \frac{(\pi x^{\pi-1} - \sin x)(1+x^3) - (x^\pi + \cos x)(3x^2)}{(1+x^3)^2}$$

(c) Find $f'(x) = \frac{dy}{dx}$ if $y = f(x) = \sec(\sec x)$.

Recall that $(\sec x)' = \sec x . \tan x$

$f'(x) = \sec(\sec x) . \tan(\sec x) . \sec x . \tan x$

(d) Find $f'(x) = \frac{dy}{dx}$ if $y = f(x) = 3^x + \log_x 3$.

$y = 3^x \rightarrow \ln y = x \ln 3$

$\rightarrow \frac{y'}{y} = \ln 3$

$\rightarrow y' = y \ln 3 = 3^x \ln 3$

$y = \log_x 3 = \frac{\ln 3}{\ln x}$

$\rightarrow y' = \frac{-\ln 3 . \frac{1}{x}}{(\ln x)^2}$

$f'(x) = 3^x \ln 3 + \ln 3 . \frac{-1}{x(\ln x)^2}$

$= \ln 3 \left[3^x - \frac{1}{x \ln^2 x}\right]$

(e) Find $f'(1) = \left.\frac{dy}{dx}\right|_{x=1}$ if $y = f(x) = \frac{x^{9x}(x-2)^3}{x^4 e^x}$.

We can use logarithmic differentiation:

$\ln y = 9x \ln x + 3 \ln(x-2) - 4 \ln x - x$

$\Rightarrow \frac{1}{y} . y' = 9(\ln x + 1) + \frac{3}{x-2} - \frac{4}{x} - 1$

put $x = 1$ ($\Rightarrow y = \frac{-1}{e}$) to get

$y'|_{x=1} = \frac{-1}{e}(9 - 3 - 4 - 1) = \frac{-1}{e}$

Q.5 (12 pts) Verify that the point $P_0(\pi, 0)$ is on the curve C defined implicitly by the equation $\sin(x+y) = xy$. Then find the two lines which are **normal** and **tangent** to C at P_0.

$$\left.\begin{array}{l} x=\pi \\ y=0 \end{array}\right\} \Rightarrow \quad \sin(\pi+0) \stackrel{?}{=} \pi \cdot 0$$

$$\sin\pi \stackrel{\checkmark}{=} 0 \quad \text{so } P_0 \text{ is on } C$$

To find the slope of tangent line m at P_0 (and hence slope of normal line $-\frac{1}{m}$ at P_0), we need to find $y'\big|_{P_0}$.

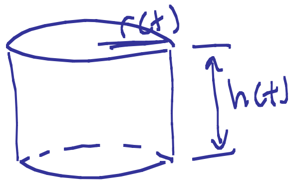
$$\sin(x+y) = xy \Rightarrow \cos(x+y)\cdot(1+y') = y + xy'$$

at P_0, we have $m = y'\big|_{P_0} = \dfrac{-1}{\pi+1}$ = slope of tangent line at P_0 to the curve C

Hence

- Equation of tangent line at P_0 is $y = \dfrac{-1}{\pi+1}(x-\pi)$
- " " normal " " " " $y = (\pi+1)(x-\pi)$

Q.6 (9 pts) The **radius** r of a right circular cylinder decreases at a rate of 0.3 cm/sec, and the **height** h increases at a rate of 0.2 cm/sec. Find the rate of change of the **volume** V of the cylinder when $r = 1$ cm and $V = 15\pi$ cm^3. Is V increasing or decreasing?



$$r'(t) = -0.3 \text{ cm/sec} \quad \forall t$$
$$h'(t) = +0.2 \text{ cm/sec} \quad \forall t$$

$$V(t) = \pi r^2(t) h(t) \quad \forall t$$

$$\Rightarrow V'(t) = \pi\left[2r(t)r'(t)h(t) + r^2(t)\cdot h'(t)\right] \quad \forall t$$

Let t_0 be the time when $r(t_0) = 1$ cm and $V(t) = 15\pi$ cm^3.

Then $h(t_0) = 15$ cm and hence

$$V'(t_0) = \pi\left[2\cdot 1\cdot(-0.3)\cdot 15 + 1^2\cdot(0{,}2)\right] = -8{,}8\pi \text{ cm}^3/\text{sec}$$

Thus when $r(t) = 1$ cm and $V(t) = 15\pi$ cm^3, Volume is decreasing at a rate of $8{,}8\pi$ cm^3/sec.