Numerical Computing Homework 2

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1. Determine rigorously if each function has a unique fixed point on the given interval (follow Example 2.3).

(a) $g(x) = 1 - x^2/4$ on $[0, 1]$

(b) $g(x) = 2^{-x}$ on $[0, 1]$

(c) $g(x) = 1/x$ on $[0.5, 5.2]$

Solution:

(a) Clearly, $g \in C[0, 1]$. Also, because $g'(x) = -x/2 < 0$ on [0, 1], $g(x)$ is a decreasing function on [0, 1]; thus its range on [0, 1] is $[\frac{3}{4}, 1] \subseteq [0, 1]$. Thus condition(3) of the Theorem 2.2 is satisfied and g has a fixed point on [0, 1].

Finally, if $x \in (0, 1)$, then $|g'(x)| = |-x/2| = x/2 \leq 1/2 < 1$. Thus condition(4) of Theorem 2.2 is satisfied, and g has a unique fixed point in [0, 1].

(b) Clearly, $g \in C[0, 1]$. Also, because $g'(x) = -2^{-x}\ln 2 < 0$ on [0, 1], $g(x)$ is a decreasing function on [0, 1]; thus its range on [0, 1] is $[\frac{1}{2}, 1] \subseteq [0, 1]$. Thus condition(3) of the Theorem 2.2 is satisfied and g has a fixed point on [0, 1].

Finally, if $x \in (0, 1)$, then $|g'(x)| = |-2^{-x}\ln 2| = 2^{-x}\ln 2 \leq \ln 2 < 1$. Thus condition(4) of Theorem 2.2 is satisfied, and g has a unique fixed point in [0, 1].

(c) Clearly, $g \in C[0.5, 5.2]$. Also, because $g'(x) = -x^{-2} < 0$ on [0.5, 5.2], $g(x)$ is a decreasing function on [0.5, 5.2]; thus its range on [0.5, 5.2] is $[\frac{5}{26}, 2] \nsubseteq [0.5, 5.2]$. Thus condition(3) of the Theorem 2.2 is not satisfied and g doesn't have a fixed point on [0.5, 5.2].

Finally, g doesn't have a unique fixed point in [0.5, 5.2].

4. Let $g(x) = x^2 + x - 4$. Can fixed-point iteration be used to find the solution(s) to the equation $x = g(x)$? Why?

Solution:

Solve the equation $x = x^2 + x - 4$, we can conclude that the fixed points of $g(x)$ are $P_1 = 2$ and $P_2 = -2$. While $|g'(2)| = 5 > 1$ and $|g'(-2)| = 3 > 1$, fixed-point iteration will not converge to P_1 and P_2.

Thus fixed-point iteration cannot be used to find the solution(s) to the equation $x = g(x)$.

5. Let $g(x) = x \cos x$. Solve $x = g(x)$ and find all the fixed points of g (there are infinitely many). Can fixed-point iteration be used to find the solution(s) to the equation $x = g(x)$? Why?

Solution:

Solve the equation $x = x \cos x$, we can conclude that the fixed points of $g(x)$ are $P = 2k\pi, k \in \mathbb{Z}$. While $|g'(P)| = |\cos(2k\pi) - 2k\pi \sin(2k\pi)| = |1 - 0| = 1$

Thus Theorem 2.3 might not be used to find the solution(s) to the equation $x = g(x)$.

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4. Start with $[a_0, b_0]$, and use the false position method to compute c_0, c_1, c_2 and c_3.

$$e^x - 2 - x = 0, [a_0, b_0] = [-2.4, -1.6]$$

Solution:

We can use equation(22) to compute c_0, c_1, c_2, c_3.

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

The procedure of false position method is as follows.

k	Left endpoint,a_k	Midpoint,c_k	Right endpoint,b_k	Function value, $f(c_k)$
0	-2.40000000	-1.83007818	-1.60000000	-0.00952079
1	-2.40000000	-1.84092522	-1.83007818	-0.00040423
2	-2.40000000	-1.84138538	-1.84092522	-0.00001707
3	-2.40000000	-1.84140480	-1.84138538	-0.00000072

Thus,

$$c_0 = -1.83007818,$$
$$c_1 = -1.84092522,$$
$$c_2 = -1.84138538,$$
$$c_3 = -1.84140480.$$

9. What will happen if the bisection method is used with the function $f(x) = 1/(x-2)$ and

(a) the interval is $[3, 7]$?

(b) the interval is $[1, 7]$?

Solution:

(a) Since $f(3) \cdot f(7) > 0$, the initial condition is not satisfied.

(b) After infinite iterations, c will be infinitely close to 2, but $f(x)$ is undefined at 2.

11. Suppose that the bisection method is used to find a zero of $f(x)$ in the interval $[2, 7]$. How many times must this interval be bisected to guarantee that the approximation c_N has an accuracy of 5×10^{-9}?

Solution:

Use equation(15) to get the number N, where $a = 2, b = 7, \delta = 5 \times 10^{-9}$.

$$\begin{aligned} N &= \text{int}\left(\frac{\ln(b-a) - \ln(\delta)}{\ln(2)}\right) \\ &= \text{int}\left(\frac{\ln(7-2) - \ln(5 \times 10^{-9})}{\ln(2)}\right) \\ &= \text{int}(29.89735285) \\ &= 29 \end{aligned}$$

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4. Let $f(x) = x^3 - 3x - 2$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) Start with $p_0 = 2.1$ and find p_1, p_2, p_3 and p_4.

(c) Is the sequence converging quadratically or linearly?

Solution:

(a) According to Theorem 2.5,

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{x^3 - 3x - 2}{3x^2 - 3} \\ &= \frac{2x^3 + 2}{3x^2 - 3} \end{aligned}$$

Thus,

$$p_k = g(p_{k-1}) = \frac{2p_{k-1}^3 + 2}{3p_{k-1}^2 - 3}$$

(b) We could use the formula above to calculate:

$$p_1 = g(p_0) = \frac{2 \times 2.1^3 + 2}{3 \times 2.1^2 - 3} = 2.0060606061$$

$$\begin{aligned} p_2 = g(p_1) &= \frac{2 \times 2.0060606061^3 + 2}{3 \times 2.0060606061^2 - 3} = 2.0000243398 \\ p_3 = g(p_2) &= \frac{2 \times 2.0000243398^3 + 2}{3 \times 2.0000243398^2 - 3} = 2.0000000004 \\ p_4 = g(p_3) &= \frac{2 \times 2.0000000004^3 + 2}{3 \times 2.0000000004^2 - 3} = 2.0000000000 \end{aligned}$$

(c) Assume $R = 2$ and use equation(19) to examine the quadratical convergence, We find that $A \approx \frac{2}{3}$.

Thus the sequence converging quadratically.

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8. Use the secant method and formula (27) and compute the next two iterations p_2 and p_3. Let $f(x) = x^2 - 2x - 1$. Start with $p_0 = 2.6$ and $p_1 = 2.5$.

Solution:

$$\begin{aligned} p_2 &= p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} \\ &= 2.5 - \frac{(2.5^2 - 2 \times 2.5 - 1)(2.5 - 2.6)}{(2.5^2 - 2 \times 2.5 - 1)(2.6^2 - 2 \times 2.6 - 1)} \\ &= 2.41935484 \end{aligned}$$

$$
\begin{aligned}
p_3 &= p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} \\
&= 2.41935484 - \frac{(2.41935484^2 - 2 \times 2.41935484 - 1)(2.41935484 - 2.5)}{(2.41935484^2 - 2 \times 2.41935484 - 1)(2.5^2 - 2 \times 2.5 - 1)} \\
&= 2.41436464
\end{aligned}
$$