Numerical Computing Homework 4

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1(b). Let $f(x) = \sin(x)$ and apply Theorem 4.1. Show that if $|x| \leq 1$, then the approximation

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

has the error bound $|E_9(x)| < 1/10! \leq 2.75574 \times 10^{-7}$.

Solution:

Because $|x| \leq 1, |x_0| = |0| \leq 1$, from Theorem 4.1 we could obtain that

$$|E_9(x)| = |\frac{f^{10}(c)}{10!}(x-0)^{10}| = \frac{sin(c)}{10!}x^{10} \leq \frac{1}{10!} \leq 2.75574 \times 10^{-7}$$

for some value $c = c(x)$ that lies between 0 and 1.

2(b). Let $f(x) = \cos(x)$ and apply Theorem 4.1. Show that if $|x| \leq 1$, then the approximation

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

has the error bound $|E_8(x)| < 1/9! \leq 2.75574 \times 10^{-6}$.

Solution:

Because $|x| \leq 1, |x_0| = |0| \leq 1$, from Theorem 4.1 we could obtain that

$$|E_8(x)| = |\frac{f^{9}(c)}{9!}(x-0)^{9}| = \frac{sin(c)}{9!}x^{9} \leq \frac{1}{9!} \leq 2.75574 \times 10^{-6}$$

for some value $c = c(x)$ that lies between 0 and 1.

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2. Consider $P(x) = -0.04x^3 + 0.14x^2 - 0.16x + 2.08$, which passes through the four points $(0, 2.08), (1, 2.02), (2, 2.00)$, and $(4, 1.12)$.

(a) Find $P(3)$.

(b) Find $P'(3)$.

(c) Find the definite integral of $P(x)$ taken over $[0, 3]$.

(d) Find the extrapolated value $P(4.5)$.

(e) Show how to find the coefficients of $P(x)$.

Solution:

(a) We could use Horner's methods to find $P(3)$:

$$\begin{aligned}
b_3 &= a_3 = -0.04 \\
b_2 &= a_2 + b_3x = 0.14 + (-0.04) \times 3 = 0.02 \\
b_1 &= a_1 + b_2x = -0.16 + 0.02 \times 3 = -0.1 \\
b_0 &= a_0 + b_1x = 2.08 + (-0.1) \times 3 = 1.78
\end{aligned}$$

The interpolated value is $P(3) = 1.78$.

(b) Similarly, we could use Horner's methods to find $P'(3)$:

$$\begin{aligned}
d_2 &= 3a_3 = 3 \times (-0.04) = -0.12 \\
d_1 &= 2a_2 + d_2x = 2 \times 0.14 + (-0.12) \times 3 = -0.08 \\
d_0 &= a_1 + d_1x = -0.16 + (-0.08) \times 3 = -0.4
\end{aligned}$$

The numerical derivative is $P'(3) = -0.4$.

(c) Similarly, we could use Horner's methods to find $\int_0^3 P(x)\mathrm{d}x$:

$$\begin{aligned}
i_4 &= \frac{a_3}{4} = \frac{-0.04}{4} = -0.01 \\
i_3 &= \frac{a_2}{3} + i_4x = \frac{0.14}{3} + (-0.01) \times 3 = 0.0166667 \\
i_2 &= \frac{a_1}{2} + i_3x = \frac{-0.16}{2} + 0.0166667 \times 3 = -0.03 \\
i_1 &= a_0 + i_2x = 2.08 + (-0.03) \times 3 = 1.99 \\
i_0 &= i_1x = 1.99 \times 3 = 5.97
\end{aligned}$$

Hence $I(3) = 5.97$. Similarly, $I(0) = 0$.

Therefore, $\int_0^3 P(x)\mathrm{d}x = I(3) - I(0) = 5.97$.

(d) We could use Horner's methods to find $P(4.5)$:

$$
\begin{aligned}
b_3 &= a_3 = -0.04 \\
b_2 &= a_2 + b_3 x = 0.14 + (-0.04) \times 4.5 = -0.04 \\
b_1 &= a_1 + b_2 x = -0.16 + (-0.04) \times 4.5 = -0.34 \\
b_0 &= a_0 + b_1 x = 2.08 + (-0.34) \times 4.5 = 0.55
\end{aligned}
$$

The extrapolated value is $P(4.5) = 0.55$.

(e) The methods of Chapter 3 can be used to find the coefficients. Assume that $P(x) = A + Bx + Cx_2 + Dx_3$; then at each value $x = 0, 1, 2,$ and 4 we get a linear equation involving A, B, C, and D.

$$
\begin{array}{lccccccccc}
\text{At } x = 0: & A & + & 0 & + & 0 & + & 0 & = & 2.08 \\
\text{At } x = 1: & A & + & B & + & C & + & D & = & 2.02 \\
\text{At } x = 2: & A & + & 2B & + & 4C & + & 8D & = & 2.00 \\
\text{At } x = 4: & A & + & 4B & + & 16C & + & 64D & = & 1.12
\end{array}
$$

The solution is

$$
\begin{aligned}
A &= 2.08 \\
B &= -0.16 \\
C &= 0.14 \\
D &= -0.04
\end{aligned}
$$

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1. Find the Lagrange polynomials that approximation $f(x) = x^3$.

(a) Find the linear interpolation polynomial $P_1(x)$ using the nodes $x_0 = -1$ and $x_1 = 0$.

(b) Find the quadratic interpolation polynomial $P_2(x)$ using nodes $x_0 = -1, x_1 = 0, x_2 = 1$.

(c) Find the cubic interpolation polynomial $P_3(x)$ using the nodes $x_0 = -1, x_1 = 0, x_2 = 1$ and $x_3 = 2$.

(d) Find the linear interpolation polynomial $P_1(x)$ using the nodes $x_0 = 1$ and $x_1 = 2$.

(e) Find the quadratic interpolation polynomial $P_2(x)$ using the nodes $x_0 = 0, x_1 = 1$ and $x_2 = 2$.

Solution:

We know that a polynomial $P_N(x)$ of degree at most N that passes through the $N+1$ points $(x_0, y_0), (x_1, y_1), \cdots (x_N, y_N)$ and has the form

$$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}(x),$$

where $L_{N,k}$ is the Lagrange coefficient polynomial based on these nodes:

$$L_{N,k}(x) = \frac{\prod_{j=0, j\neq k}^{N}(x - x_j)}{\prod_{j=0, j\neq k}^{N}(x_k - x_j)}.$$

(a) Using $P_N(x)$ with the abscissas $x_0 = -1$ and $x_1 = 0$ and the ordinates $y_0 = (-1)^3 = -1$ and $y_1 = 0^3 = 0$ produces

$$\begin{aligned} P_1(x) &= -1\frac{x-0}{-1-0} + 0\frac{x-(-1)}{0-(-1)} \\ &= x \end{aligned}$$

(b) Using $x_0 = -1, x_1 = 0, x_2 = 1$ and $y_0 = (-1)^3 = -1, y_1 = 0^3 = 0, y_2 = 1^3 = 1$ produces

$$\begin{aligned} P_2(x) &= -1\frac{(x-0)(x-1)}{(-1-0)(-1-1)} + 0\frac{(x-(-1))(x-1)}{(0-(-1))(0-1)} \\ &\quad + 1\frac{(x-(-1))(x-0)}{(1-(-1))(1-0)} \\ &= -1\frac{x(x-1)}{2} + 0 + \frac{x(x+1)}{2} \\ &= x \end{aligned}$$

(c) Using $x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2$ and $y_0 = (-1)^3 = -1, y_1 = 0^3 = 0, y_2 = 1^3 = 1, y_3 = 2^3 = 8$ produces

$$\begin{aligned}
P_3(x) &= -1\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} \\
&\quad + 0\frac{(x-(-1))(x-1)(x-2)}{(0-(-1))(0-1)(0-2)} \\
&\quad + 1\frac{(x-(-1))(x-0)(x-2)}{(1-(-1))(1-0)(1-2)} \\
&\quad + 8\frac{(x-(-1))(x-0)(x-1)}{(2-(-1))(2-0)(2-1)} \\
&= -1\frac{x(x-1)(x-2)}{-6} + 0 \\
&\quad + \frac{x(x+1)(x-2)}{-2} \\
&\quad + 8\frac{x(x+1)(x-1)}{6} \\
&= x^3
\end{aligned}$$

(d) Same as (a), we could easily obtain $P_1(x)$:

$$\begin{aligned}
P_1(x) &= 1\frac{x-2}{1-2} + 8\frac{x-1}{2-1} \\
&= 7x - 6
\end{aligned}$$

(e) Same as (b), we could easily obtain $P_2(x)$:

$$\begin{aligned}
P_2(x) &= 0\frac{(x-1)(x-2)}{(0-1)(0-2)} + 1\frac{(x-0)(x-2)}{(1-0)(1-2)} \\
&\quad + 8\frac{(x-0)(x-1)}{(2-0)(2-1)} \\
&= 0 + 1\frac{x(x-2)}{-1} + \frac{x(x-1)}{2} \\
&= 3x^2 - 2x
\end{aligned}$$

2. Let $f(x) = x + 2/x$.

(a) Use quadratic Lagrange interpolation based on the nodes $x_0 = 1, x_1 = 2$, and $x_2 = 2.5$ to approximate $f(1.5)$ and $f(1.2)$.

(b) Use cubic Lagrange interpolation based on the nodes $x_0 = 0.5, x_1 = 1, x_2 = 2$, and $x_3 = 2.5$ to approximate $f(1.5)$ and $f(1.2)$.

Solution:

(a) Similar to Exercise 1, we could easily obtain $P_2(x)$:

$$
\begin{aligned}
P_2(x) = 3\frac{(x-2)(x-2.5)}{(1-2)(1-2.5)} + 3\frac{(x-1)(x-2.5)}{(2-1)(2-2.5)}\\
+ 3.3\frac{(x-1)(x-2)}{(2.5-1)(2.5-2)}\\
= 0 + 1\frac{x(x-2)}{-1} + \frac{x(x-1)}{2}\\
= 0.4x^2 - 1.2x + 3.8
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f(1.5) \approx P_2(1.5) = 2.9\\
f(1.2) \approx P_2(1.2) = 2.936
\end{aligned}
$$

(b) Similar to Exercise 1, we could easily obtain $P_3(x)$:

$$P_3(x) = -0.8x^3 + 4.8x^2 - 8.8x + 7.8$$

Thus,

$$
\begin{aligned}
f(1.5) \approx P_3(1.5) = 2.7\\
f(1.2) \approx P_3(1.2) = 2.7696
\end{aligned}
$$

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7. Given that $f(x) = 3\sin^2(\pi x/6), x = 1.5, 3.5$.

k	x_k	$f(x_k)$
0	0.0	0.00
1	1.0	0.75
2	2.0	2.25
3	3.0	3.00
4	4.0	2.25

(a) Compute the divide-difference table for the tabulated function.

(b) Write down the Newton polynomials $P_1(x), P_2(x), P_3(x)$, and $P_4(x)$.

(c) Evaluate the Newton polynomials in part (b) at the given values of x.

(d) Compare the values in part (c) with the actual function value $f(x)$.

Solution:

(a) The divide-difference table for the tabulated function is as follows:

x_k	$f[x_k]$	$f[,]$	$f[,,]$	$f[,,,]$	$f[,,,,]$
$x_0 = 0$	0.00	0	0	0	0
$x_1 = 1$	0.75	0.75	0	0	0
$x_2 = 2$	2.25	1.5	0.375	0	0
$x_3 = 3$	3.00	0.75	-0.375	-0.25	0
$x_4 = 4$	2.25	-0.75	-0.75	-0.125	0.03125

(b) The Newton polynomial is

$$P_N(x) = a_0 + a_1(x - x_0) + \cdots + a_N(x - x_0)(x - x_1)\cdots(x - x_N),$$

where $a_k = f_[x_0, x_1 \cdots x_k]$, for $k = 0, 1, \cdots N$.

Thus,

$$\begin{aligned}
&P_1(x) = 0 + 0.75(x - 0) \\
&P_2(x) = 0 + 0.75(x - 0) + 0.375(x - 0)(x - 1) \\
&P_3(x) = 0 + 0.75(x - 0) + 0.375(x - 0)(x - 1) + (-0.25)(x - 0)(x - 1)(x - 2) \\
&P_4(x) = 0 + 0.75(x - 0) + 0.375(x - 0)(x - 1) + (-0.25)(x - 0)(x - 1)(x - 2) \\
&\qquad + 0.03125(x - 0)(x - 1)(x - 2)(x - 3)
\end{aligned}$$

(c)

$$\begin{aligned}
P_1(1.5) &= 1.125 & P_1(3.5) &= 2.625 \\
P_2(1.5) &= 1.40625 & P_1(3.5) &= 5.90625 \\
P_3(1.5) &= 1.5 & P_1(3.5) &= 2.625 \\
P_4(1.5) &= 1.517578125 & P_1(3.5) &= 2.830078125
\end{aligned}$$

(d)

$$f(1.5) = 1.5 \qquad\qquad f(3.5) = 2.799038106$$

$|P_3(1.5) - f(1.5)| = 0$, which is the nearest value between $P_i(1.5)$, $i = 1, 2, 3, 4$.

$|P_4(3.5) - f(3.5)| = 0.03104$, which is the nearest value between $P_i(3.5)$, $i = 1, 2, 3, 4$.