Numerical Computing Homework 3

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3. Solve the upper-triangular system and find the value of the determinant of the coefficient matrix.

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 2x_4 - x_5 &= 4 \\ -2x_2 + 6x_3 + 2x_4 + 7x_5 &= 0 \\ x_3 - x_4 - 2x_5 &= 3 \\ -2x_4 - x_5 &= 10 \\ 3x_5 &= 6 \end{aligned}$$

Solution:

We find that all the diagonal elements are non-zero. So we could solve the upper-triangular system by the method of back substitution.

The coefficient matrix $\boldsymbol{A}$ and the matrix $\boldsymbol{B}$ are:

$$\boldsymbol{A} = \begin{bmatrix} 4 & -1 & 2 & 2 & -1 \\ 0 & -2 & 6 & 2 & 7 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 10 \\ 6 \end{bmatrix}.$$

From Theorem 3.5's equation(6), we could calculate x_1, x_2, x_3, x_4, x_5:

$$x_k = \frac{b_k - \sum_{j=k+1}^{N} a_{kj}x_j}{a_{kk}}, \text{for } k = N-1, N-2, \ldots, 1.$$

Solving for x_5 in the last equation yields

$$x_5 = \frac{6}{3} = 2$$

Then we could obtain:

$$\begin{aligned} x_4 &= \frac{10 - (-1 \times 2)}{-2} = -6 \\ x_3 &= \frac{3 - (-1 \times (-6) + (-2) \times 2)}{1} = 1 \\ x_2 &= \frac{0 - (6 \times 1 + 2 \times (-6) + 7 \times 2)}{-2} = 4 \\ x_1 &= \frac{4 - ((-1) \times 4 + 2 \times 1 + 2 \times (-6) + (-1) \times 2)}{4} = 5 \end{aligned}$$

And we could calculate $\det(\boldsymbol{A}) = 4 \times (-2) \times 1 \times (-2) \times 3 = 48$

4(a). Consider the two upper-triangular matrices.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}.$$

Show that their product $\boldsymbol{C} = \boldsymbol{AB}$ is also upper triangular.

Solution:

Using Definition 3.1's equation(7), we could obtain $\boldsymbol{C}$:

$$\boldsymbol{C} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} + b_{13}b_{33} \\ 0 & a_{22}b_{22} & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}$$

We could see that $\boldsymbol{C}$ is upper triangular.

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1. Show that $\boldsymbol{AX} = \boldsymbol{B}$ is equivalent to the upper-triangular system $\boldsymbol{UX} = \boldsymbol{Y}$ and find the solution.

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= -4 \\ x_1 + 5x_2 + 3x_3 &= 10 \\ x_1 + 3x_2 + 2x_3 &= 5 \end{aligned} \qquad \begin{aligned} 2x_1 + 4x_2 - 6_x3 &= -4 \\ 3x_2 + 6x_3 &= 12 \\ 3x_3 &= 3 \end{aligned}$$

Solution:

The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$$

The first row is used to eliminate elements in the first column below the diagonal.The result after elimination is

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array}\right]$$

The second row is used to eliminate elements in the second column that lie below the diagonal.The result after elimination is

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

Thus, $\boldsymbol{AX} = \boldsymbol{B}$ is equivalent to the upper-triangular system $\boldsymbol{UX} = \boldsymbol{Y}$.

The back-substitution algorithm can be used to solve the system, and we get

$$\boldsymbol{X} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

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9. Show that $\boldsymbol{AX} = \boldsymbol{B}$ is equivalent to the upper-triangular system $\boldsymbol{UX} = \boldsymbol{Y}$ and find the solution.

$$\begin{aligned} 2x_1 + 4x_2 - 4x_3 + 0x_4 &= 12 \\ x_1 + 5x_2 - 5x_3 - 3x_4 &= 18 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 8 \\ x_1 + 4x_2 - 2x_3 + 2x_4 &= 8 \end{aligned} \qquad \begin{aligned} 2x_1 + 4x_2 - 4x_3 + 0x_4 &= 12 \\ 3x_2 - 3x_3 - 3x_4 &= 12 \\ 4x_3 + 2x_4 &= 0 \\ 3x_4 &= -6 \end{aligned}$$

Solution:

The augmented matrix is

$$\left[\begin{array}{cccc|c} 2 & 4 & -4 & 0 & 12 \\ 1 & 5 & -5 & -3 & 18 \\ 2 & 3 & 1 & 3 & 8 \\ 1 & 4 & -2 & 2 & 8 \end{array}\right]$$

The first row is used to eliminate elements in the first column below the diagonal.The result after elimination is

$$\left[\begin{array}{rrrr|r} 2 & 4 & -4 & 0 & 12 \\ 0 & 3 & -3 & -3 & 12 \\ 0 & -1 & 5 & 3 & -4 \\ 0 & 2 & -4 & 2 & 2 \end{array}\right]$$

The second row is used to eliminate elements in the second column that lie below the diagonal.The result after elimination is

$$\left[\begin{array}{rrrr|r} 2 & 4 & -4 & 0 & 12 \\ 0 & 3 & -3 & -3 & 12 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & -6 & 0 & -6 \end{array}\right]$$

The third row is used to eliminate elements in the third column that lie below the diagonal.The result after elimination is

$$\left[\begin{array}{rrrr|r} 2 & 4 & -4 & 0 & 12 \\ 0 & 3 & -3 & -3 & 12 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right]$$

Thus, $\boldsymbol{AX} = \boldsymbol{B}$ is equivalent to the upper-triangular system $\boldsymbol{UX} = \boldsymbol{Y}$.

The back-substitution algorithm can be used to solve the system, and we get

$$\boldsymbol{X} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$$

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1(a). Solve $\boldsymbol{LY} = \boldsymbol{B}, \boldsymbol{UX} = \boldsymbol{Y}$, and verify that $\boldsymbol{B} = \boldsymbol{AX}$ for $\boldsymbol{B} = [-4\ 10\ 5]'$, where $\boldsymbol{A} = \boldsymbol{LU}$ is

$$\begin{bmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

Use the forward-substitution method to solve $\boldsymbol{LY} = \boldsymbol{B}$:

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -4 \\ 1/2 & 1 & 0 & 10 \\ 1/2 & 1/3 & 1 & 5 \end{array}\right]$$

And we obtain

$$\boldsymbol{Y} = \begin{bmatrix} -4 \\ 12 \\ 3 \end{bmatrix}$$

Next write the augmented matrix $\boldsymbol{UX} = \boldsymbol{Y}$:

$$\left[\begin{array}{rrr|r} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

And we obtain

$$\boldsymbol{X} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Thus,

$$\boldsymbol{AX} = \begin{bmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ 5 \end{bmatrix} = \boldsymbol{B}$$

4. Find the triangular factorization $\boldsymbol{A} = \boldsymbol{LU}$ for the matrices.

(a)

$$\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -2 & 7 \\ 4 & 2 & 1 \\ 2 & 5 & -2 \end{bmatrix}$$

Solution:

(a)

$$\begin{aligned} \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & -2 \\ 1 & -2 & 7 \end{bmatrix} &= \begin{bmatrix} 4 & 2 & 1 \\ 1/2 & 4 & -5/2 \\ 1/4 & -5/8 & 83/16 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/4 & -5/8 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 4 & -5/2 \\ 0 & 0 & 83/16 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{bmatrix} 1 & -2 & 7 \\ 4 & 2 & 1 \\ 2 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 \\ 4 & 10 & -27 \\ 2 & 9/10 & 83/10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 9/10 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 7 \\ 0 & 10 & -27 \\ 0 & 0 & 83/10 \end{bmatrix}$$

6. Find the triangular factorization $\boldsymbol{A} = \boldsymbol{LU}$ for the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -3 & 5 & -8 \\ 5 & 1 & -4 & -10 \\ -3 & -1 & -7/4 & -15/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -1 & -7/4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -3 & 5 & -8 \\ 0 & 0 & -4 & -10 \\ 0 & 0 & 0 & -15/2 \end{bmatrix}$$

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1.

$$\begin{aligned} 4x - y &= 15 \\ x + 5y &= 9 \end{aligned}$$

(a) Start with $\boldsymbol{P}_0 = \mathbf{0}$ and use Jacobi iteration to find $\boldsymbol{P}_k$ for $k = 1, 2, 3$. Will Jacobi iteration converge to the solution?

(b) Start with $\boldsymbol{P}_0 = \mathbf{0}$ and use Gauss-Seidel iteration to find $\boldsymbol{P}_k$ for $k = 1, 2, 3$. Will Gauss-Seidel iteration converge to the solution?

Solution:

(a) These equations can be written in the form

$$\begin{aligned} x &= \frac{15 + y}{4} \\ y &= \frac{9 - x}{5} \end{aligned}$$

This suggests the following Jacobi iterative process:

$$\begin{aligned} x_{k+1} &= \frac{15 + y_k}{4} \\ y_{k+1} &= \frac{9 - x_k}{5} \end{aligned}$$

Substitute $x_0 = 0, y_0 = 0$ into the right-hand side of each equation to obtain the new values

$$\begin{aligned} x_1 &= \frac{15 + y_0}{4} = \frac{15}{4} = 3.75 \\ y_1 &= \frac{9 - x_0}{5} = \frac{9}{5} = 1.8 \end{aligned}$$

Similarly, we could obtain

$$\begin{aligned} x_2 &= \frac{15 + y_1}{4} = \frac{15 + 1.8}{4} = 4.2 \\ y_2 &= \frac{9 - x_1}{5} = \frac{9 - 3.75}{5} = 1.05 \end{aligned}$$

and

$$\begin{aligned} x_3 &= \frac{15 + y_2}{4} = \frac{15 + 1.05}{4} = 4.0125 \\ y_3 &= \frac{9 - x_2}{5} = \frac{9 - 4.2}{5} = 0.96 \end{aligned}$$

Thus, $\boldsymbol{P}_1 = (3.75, 1.8), \boldsymbol{P}_2 = (4.2, 1.05), \boldsymbol{P}_3 = (4.0125, 0.96)$

The coefficient matrix of the linear system is strictly diagonally dominant because

$$\begin{array}{ll} \text{In row 1:} & |4| > |-1| \\ \text{In row 2:} & |5| > |1| \end{array}$$

According to **Theorem 3.15**, Jacobi iteration will converge to the solution, which is (4, 1).

(b) These equations can be written in the form

$$\begin{aligned} x &= \frac{15 + y}{4} \\ y &= \frac{9 - x}{5} \end{aligned}$$

This suggests the following Gauss-Seidel iterative process:

$$\begin{aligned} x_{k+1} &= \frac{15 + y_k}{4} \\ y_{k+1} &= \frac{9 - x_{k+1}}{5} \end{aligned}$$

Substitute $x_0 = 0, y_0 = 0$ into the right-hand side of each equation to obtain the new values

$$\begin{aligned} x_1 &= \frac{15 + y_0}{4} = \frac{15}{4} = 3.75 \\ y_1 &= \frac{9 - x_1}{5} = \frac{9 - 3.75}{5} = 1.05 \end{aligned}$$

Similarly, we could obtain

$$\begin{aligned} x_2 &= \frac{15 + y_1}{4} = \frac{15 + 1.05}{4} = 4.0125 \\ y_2 &= \frac{9 - x_2}{5} = \frac{9 - 4.0125}{5} = 0.9975 \end{aligned}$$

and

$$\begin{aligned} x_3 &= \frac{15 + y_2}{4} = \frac{15 + 0.9975}{4} = 3.999375 \\ y_3 &= \frac{9 - x_3}{5} = \frac{9 - 3.999375}{5} = 1.000125 \end{aligned}$$

Thus, $\boldsymbol{P}_1 = (3.75, 1.05), \boldsymbol{P}_2 = (4.0125, 0.9975), \boldsymbol{P}_3 = (3.999375, 1.000125)$

The coefficient matrix of the linear system is strictly diagonally dominant because

$$\begin{array}{ll} \text{In row 1:} & |4| > |-1| \\ \text{In row 2:} & |5| > |1| \end{array}$$

According to **Theorem 3.15**, Gauss-Seidel iteration will converge to the solution, which is (4, 1).

3.

$$\begin{aligned} -x + 3y &= 1 \\ 6x - 2y &= 2 \end{aligned}$$

(a) Start with $\boldsymbol{P}_0 = \mathbf{0}$ and use Jacobi iteration to find $\boldsymbol{P}_k$ for $k = 1, 2, 3$. Will Jacobi iteration converge to the solution?

(b) Start with $\boldsymbol{P}_0 = \mathbf{0}$ and use Gauss-Seidel iteration to find $\boldsymbol{P}_k$ for $k = 1, 2, 3$. Will Gauss-Seidel iteration converge to the solution?

Solution:

(a) Similar to **Exercise 1(a)**, we could obtain

$$\begin{aligned} x_{k+1} &= 3y_k - 1 \\ y_{k+1} &= \frac{6x_k - 2}{2} \end{aligned}$$

Thus, we can similarly get

$$\begin{aligned} \boldsymbol{P}_1 &= (-1, -1), \\ \boldsymbol{P}_2 &= (-4, -4), \\ \boldsymbol{P}_3 &= (-13, -13). \end{aligned}$$

The coefficient matrix of the linear system is not strictly diagonally dominant because

$$\begin{array}{ll} \text{In row 1:} & |-1| < |3| \\ \text{In row 2:} & |-2| < |6| \end{array}$$

According to **Theorem 3.15**, Jacobi iteration will not converge to the solution.

(b) Similar to **Exercise 1(b)**, we could obtain

$$\begin{aligned} x_{k+1} &= 3y_k - 1 \\ y_{k+1} &= \frac{6x_{k+1} - 2}{2} \end{aligned}$$

Thus, we can similarly get

$$\begin{aligned} \boldsymbol{P}_1 &= (-1, -4), \\ \boldsymbol{P}_2 &= (-13, -40), \\ \boldsymbol{P}_3 &= (-121, -364). \end{aligned}$$

The coefficient matrix of the linear system is not strictly diagonally dominant because

$$\begin{array}{ll} \text{In row 1:} & |-1| < |3| \\ \text{In row 2:} & |-2| < |6| \end{array}$$

According to **Theorem 3.15**, Gauss-Seidel iteration will not converge to the solution.

5.

$$\begin{aligned} 5x - y + z &= 10 \\ 2x + 8y - z &= 11 \\ -x + y + 4z &= 3 \end{aligned}$$

(a) Start with $\boldsymbol{P}_0 = \mathbf{0}$ and use Jacobi iteration to find $\boldsymbol{P}_k$ for $k = 1, 2, 3$. Will Jacobi iteration converge to the solution?

(b) Start with $\boldsymbol{P}_0 = \mathbf{0}$ and use Gauss-Seidel iteration to find $\boldsymbol{P}_k$ for $k = 1, 2, 3$. Will Gauss-Seidel iteration converge to the solution?

Solution:

(a) Similar to **Exercise 1(a)**, we could obtain

$$\begin{aligned} x_{k+1} &= \frac{10 + y_k - z_k}{5} \\ y_{k+1} &= \frac{11 - 2x_k + z_k}{8} \\ z_{k+1} &= \frac{3 + x_k - y_k}{4} \end{aligned}$$

Thus, we can similarly get

$$\begin{aligned} \boldsymbol{P}_1 &= (2, 1.375, 0.75), \\ \boldsymbol{P}_2 &= (2.125, 0.96875, 0.90625), \\ \boldsymbol{P}_3 &= (2.0125, 0.95703125, 1.0390625). \end{aligned}$$

The coefficient matrix of the linear system is strictly diagonally dominant because

$$\begin{array}{ll} \text{In row 1:} & |5| > |1| + |1| \\ \text{In row 2:} & |8| > |2| + |-1| \\ \text{In row 3:} & |4| > |-1| + |1| \end{array}$$

According to **Theorem 3.15**, Jacobi iteration will converge to the solution, which is (2, 1, 1).

(b) Similar to **Exercise 1(b)**, we could obtain

$$\begin{aligned} x_{k+1} &= \frac{10 + y_k - z_k}{5} \\ y_{k+1} &= \frac{11 - 2x_{k+1} + z_{k+1}}{8} \\ z_{k+1} &= \frac{3 + x_{k+1} - y_{k+1}}{4} \end{aligned}$$

Thus, we can similarly get

$$
\begin{aligned}
\boldsymbol{P}_1 &= (2, 0.875, 1.03125),\\
\boldsymbol{P}_2 &= (1.96875, 1.01171875, 0.9892578125),\\
\boldsymbol{P}_3 &= (2.0044921875, 0.9975341796875, 1.001739501953125).
\end{aligned}
$$

The coefficient matrix of the linear system is strictly diagonally dominant because

$$
\begin{array}{lc}
\text{In row 1:} & |5| > |1| + |1| \\
\text{In row 2:} & |8| > |2| + |-1| \\
\text{In row 3:} & |4| > |-1| + |1|
\end{array}
$$

According to **Theorem 3.15**, Gauss-Seidel iteration will converge to the solution, which is (2, 1, 1).