# Quantum Subgroups

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# Quantum groups

$$X \xrightarrow{\quad \mathcal{O} \quad} \mathcal{O}(X)$$

(finite/affine alg/top) set $\quad = \{ f : X \longrightarrow \mathbb{C} \mid f$ alg/cont'l function $\}$

$$X \xrightarrow{f} Y \longrightarrow \mathcal{O}(Y) \xrightarrow{\alpha^*} \mathcal{O}(X)$$

$$\alpha^*(f) = f \circ \alpha$$

$$\operatorname{Spec} A = \operatorname{Alg}(A, \mathbb{C}) \xleftarrow{\quad \operatorname{Spec} \quad} A \text{ commut. alg}/\mathbb{C}$$

algebra maps

$$\operatorname{Spec}(\mathcal{O}(X)) \simeq X$$

$$\mathcal{O}(\operatorname{Spec}(A)) \simeq A$$

$$\text{AffAlgSets} \underset{\text{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \text{CommAlg}_{\mathbb{C}}$$

$$\mathbb{A}^n = \mathbb{C}^n \longleftrightarrow \mathbb{C}[x_1, \dots, x_n] = \mathcal{O}(\mathbb{C}^n)$$

$$M_n(\mathbb{C}) \longleftrightarrow \mathbb{C}[x_{11}, \dots, x_{nn}] = \mathcal{O}(M_n(\mathbb{C}))$$

$$SL_n(\mathbb{C}) \longleftrightarrow \mathbb{C}[x_{11}, \dots, x_{nn}] / (\det(x_{ij}) - 1) \text{''} \mathcal{O}(SL_n(\mathbb{C}))$$

Assume $G$ is a (finite/aff. alg) group

$$\left\{\begin{array}{l} G \times G \xrightarrow{m} G \\ \{*\} \xrightarrow{u} G \\ G \xrightarrow{(\,)^{-1}} G \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \mathcal{O}(G) \xrightarrow{m^*} \mathcal{O}(G \times G) \simeq \mathcal{O}(G) \otimes \mathcal{O}(G) \\ \mathcal{O}(G) \xrightarrow{u^*} \mathcal{O}\{*\} \simeq \mathbb{C} \\ \mathcal{O}(G) \xrightarrow{(-)^{-1*}} \mathcal{O}(G) \end{array}\right.$$

$\mathcal{O}(G)$ is a commutative Hopf alg $/\mathbb{C}$

$$\Delta = m^*, \quad \varepsilon = u^*, \quad S = (-)^{-1*}$$

**Thm (Cartier)**

The categories of affine alg groups and finitely generated commutative Hopf alg. without nilpotent elements are equivalent

$$\mathrm{AffAlgGr}_{\mathbb{C}} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathrm{CommHopf}_{\mathbb{C}, f.g}$$

Finite groups $\longleftrightarrow$ finite-dimensional Comm. Hopf. alg

$G$ group

$H \hookrightarrow G$ $\longleftrightarrow$ $\mathcal{O}(G) \twoheadrightarrow \mathcal{O}(H)$

subgroup

Hopf alg. quotient

# What is a quantum group?

A non-commutative & non-cocommutative Hopf algebra $\rightsquigarrow$ Deformations of $U(\mathfrak{g})$ or $\mathcal{O}(G)$

In our setting: $\mathcal{O}_q(G)$

deformation of $\mathcal{O}(G)$

$q$ = deformation multiparameter

$G$ simple (simply conn.) affine alg. group

# Example

$$\mathcal{O}(SL_2) = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}] \Big/ (\underbrace{x_{11}x_{22} - x_{11}x_{22}}_{\det X} - 1)$$

$q$ indeterminate

non-commutative

$$\mathcal{O}_q(SL_2) = \mathbb{C}\langle x_{11}, x_{12}, x_{21}, x_{22} \rangle \Big/ I$$

$I$ ideal of relations:

$$x_{11}x_{12} = q\, x_{12}x_{11} \qquad x_{12}x_{22} = q\, x_{22}x_{12}$$

$$x_{11}x_{21} = q\, x_{21}x_{11} \qquad x_{12}x_{21} = x_{21}x_{12}$$

$$x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21} \qquad \underbrace{x_{11}x_{22} - q\, x_{12}x_{21}}_{\det_q X} = 1$$

$$\mathcal{O}_1(SL_2) = \mathcal{O}(SL_2)$$

$\mathcal{O}_q(SL_2)$ is a Hopf algebra with

$$\Delta(x_{ij}) = \sum_{k=1}^{2} x_{ik} \otimes x_{kj}\ , \quad \varepsilon(x_{ij}) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{oth.} \end{cases}$$

$$S(x_{11}) = x_{22},\ S(x_{12}) = -q^{-1}x_{12},\ S(x_{21}) = -q x_{21},\ S(x_{22}) = x_{11}$$

For any (semi)simple simply connected affine alg. group $G$ we have a one-parameter quantum group $\mathcal{O}_q(G)$

There exists multiparameter versions $\mathcal{O}_{\mathbf{q}}(G)$ with $\mathbf{q} = (q_{ij})$ matrix of multiparameters $\rightsquigarrow$ (twist) deformation of $\mathcal{O}_q(G)$

# What is a quantum subgroup of $\mathcal{O}_q(G)$?

Following the analogy $H \hookrightarrow G$, a quantum subgroup of $\mathcal{O}_q(G)$ is a Hopf algebra quotient

$$\mathcal{O}_q(G) \twoheadrightarrow A$$

$A$ "should" correspond to a quantized coordinate algebra of an algebraic subgroup of $G$. $\rightsquigarrow$ a quantum subgroup

$H_q \hookrightarrow G_q$

**Problem** Determine all quantum subgroups of a given quantum group $G_q$ (all Hopf alg. quotients of $\mathcal{O}_q(G)$)

Quantum version of the classical problem of determining all (finite) subgroups of a simple affine algebraic group (still open)

# Some Results:

- Podleś '95: $SU_q(2)$ and $SU_q(3)$, $q \in (-1,1) - \{0\}$
- Müller '00: $GL_q(n)$ and $SL_q(n)$, $q$ odd root of 1
- Andruskiewitsch-G. '09: $\mathcal{O}_q(G)$, $G$ conn. simply conn aff. alg group $q$ odd root of 1 + ...
- G. '10: $GL_{\alpha,\beta}(n)$, $\alpha^{-1}\beta$ odd root of 1
- Bichon-Natale '11: $SU_{-1}(2)$ (partial)
- Bichon-Dubois Violette '13: $O_n^*$ compact q.subgroups
- Bichon-Yuncken '14: $SU_{-1}(3)$
- G-Gutiérrez '17: $\mathcal{O}_q^{\varphi}(G)$ twisted q. groups

# The case $q$ a root of unity

Let $q$ be a primitive $\ell$-th root of 1 ($\ell$ odd and $(\ell, 3) = 1$ if $G$ has type $G_2$)

The dual version of the quantum Frobenius map $U_q(\mathfrak{g}) \xrightarrow{F} U(\mathfrak{g})$ gives us an embedding

$$\mathcal{O}(G) \hookrightarrow \mathcal{O}_q(G)$$

where the image of $\mathcal{O}(G)$ is central

For example, for $G = SL_2(\mathbb{C})$, the image is generated by

$$x_{ij}^{\ell} \quad \forall\, 1 \le i,j \le 2$$

Moreover, we have a short exact seq

$$\mathcal{O}(G) \hookrightarrow \mathcal{O}_q(G) \twoheadrightarrow u_q(\mathfrak{g})^*$$

$\dim u_q(\mathfrak{g}) = \ell^{\dim \mathfrak{g}}$

with $u_q(\mathfrak{g})$ = small quantum group Frobenius-Lusztig kernel

$\mathcal{O}_q(G)$ has a classical part and a quantum finite-dim. part

Consequence: Any quantum group fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(G) \arrow[r, hook] \arrow[d] & \mathcal{O}_q(G) \arrow[r, two heads] \arrow[d, two heads] & u_q(\mathfrak{g})^* \arrow[d, two heads] \\
\mathcal{O}(\Gamma) \arrow[r, hook] & A \arrow[r, two heads] & H
\end{tikzcd}$$

of short exact seq. of Hopf algebras

- $\Gamma \hookrightarrow G$ algebraic subgroup
- $H^* \hookrightarrow u_q(\mathfrak{g})$ a Hopf subalg. (known)

Idea: Construct $A$ from $\Gamma$ and $H$

## By product of the classification:

- New examples of Hopf algebras with different properties
- Better understanding of the family of quantum groups

## Future work (in progress)

- Remove restrictions on the order of q
- Work with more general quantum groups
- Determine which families of Hopf alg are quantum subgroups