Grassmannians and cluster structures

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Abstract

Cluster structures have been established on numerous algebraic varieties. These lectures focus on the Grassmannian variety and explain the cluster structures on it. The tools include dimer models on surfaces, associated algebras and the study of associated module categories.

Contents

1 Grassmannians

The main reference for this is Section 9 in R. Marsh's book on cluster algebras, [Mar13].

1.1 Exterior powers

Let $V := \mathbb{C}^n$, the tensor algebra is defined as $T(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V^{\otimes 3}) \oplus \dots$ and the **exterior algebra** is the quotient

$$\Lambda(V) = T(V)/J$$

where J is the ideal of $T(V)$ generated by $\{x \otimes x \mid x \in V\}$. We write the product in $\Lambda(V)$ as $(x, y) \mapsto x \wedge y$. The elements of $\Lambda(V)$ are called **alternating tensors**:

Lemma 1.1. *For all $x, y \in V$ we have $x \wedge y = -y \wedge x$.*

Exercise 1.2. Proof Lemma 1.1

The **k-th exterior power** $\Lambda^k(V)$ is the subspace of $\Lambda(V)$ spanned by the products $v_1 \wedge v_2 \wedge \cdots \wedge v_k$, $v_i \in V \forall i$,

$$\Lambda(V) = \bigoplus_{k=0}^{\infty} \Lambda^k(V)$$

(this is a finite sum since $\dim V < \infty$). Let $e_1, \dots, e_n$ be the natural basis of V, $(e_i)_j = \delta_{ij}$. Then the $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$ form a basis of $\Lambda^k(V)$ (note that $\dim \Lambda^k(V) = \binom{n}{k}$).

For $x \in \Lambda^k(V)$ we write $p_{i_1,\dots,i_k}(x)$ for the coefficient of x in terms of this basis: $x = \sum_{i_1 < \cdots < i_k} p_{i,1,\dots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$. In particular, the $p_{i_1,\dots,i_k}$ are linear maps $\Lambda^k(V) \to \mathbb{C}$.

An element $x \in \Lambda^k(V)$ is **decomposable** or **pure** if $x = v_1 \wedge \cdots \wedge v_k$ where $\{v_1, \dots, v_k\}$ is a linearly independent set of vectors in V.

Exercise 1.3. Check that the vectors $v_1, \dots, v_k$ are linearly dependent if and only if $v_1 \wedge \cdots \wedge v_k = 0$.

Let $v_1, \dots, v_k$ be linearly independent vectors, $k \leq n$. We can use them to form a $k \times n$ matrix $M = (M_{ij})_{ij}$ of rank k where $M_{ij} = (v_i)_j$, taking the v_i as rows.

We use the following notation for the minor of M of rows $a_1, \dots, a_r$ and columns $b_1, \dots, b_r$ (for $1 \leq r \leq k$):

$$\triangle^{a_1,\dots,a_r}_{b_1,\dots,b_r}(M)$$

Lemma 1.4. *Let $x = v_1 \wedge \cdots \wedge v_k \in \Lambda^k(V)$ be decomposable, M as above. Then we have*

$$p_{i_1,\dots,i_k}(x) = \triangle^{1,2,\dots,k}_{i_1,\dots,i_k}(M).$$

Exercise 1.5. Prove Lemma 1.4. Hint: use the expansion of the v_i in terms of $e_1, \dots, e_n$.

1.2 The Grassmannian

Let $1 < k < n$. The Grassmannian $\mathrm{Gr}(k, n)$ is the set of k-dimensional subspaces of $V = \mathbb{C}^n$. Take $U \in \mathrm{Gr}(k, n)$ and $\{v_1, \dots, v_k\}$ a basis of U. Consider

$$w := v_1 \wedge \cdots \wedge v_k \in \Lambda^k(V)$$

(Since the v_i's are linearly independent, $w \neq 0$, a decomposable alternating tensor).

Note that w does not depend on the choice of basis, up to multiplication by a non-zero scalar. If we associate to w all the coefficients $p_{i_1,\dots,i_k}(w)$, we get a well-defined element $(p_{i_1,\dots,i_k}(w))_{i_1 < \cdots < i_k}$ of the projective space $\mathbb{P}^N$ for $N = \binom{n}{k} - 1$. This gives us a map

$$\varphi : \mathrm{Gr}(k, n) \to \mathbb{P}^N$$

The $p_{i_1,\ldots,i_k}$ are called the **Plücker coordinates**. Note that in the definition of φ we have chosen the indices to be strictly increasing.

We want to describe the image of φ. For this, we extend the definition of the $p_{i_1,\ldots,i_k}$ to arbitrary (multi-) sets $\{i_1,\ldots,i_k\}$ with $i_j \in [n] = \{1,\ldots,n\}$ by setting $p_{i_1,\ldots,i_k} = 0$ if there are $r \neq s$ such that $i_r = i_s$ and by setting $p_{i_1,\ldots,i_k} = \operatorname{sgn}(\pi) p_{j_1,\ldots,j_k}$ in case the $i_1,\ldots,i_k$ are distinct, $\{i_1,\ldots,i_k\} = \{j_1,\ldots,j_k\}$ with $1 \leq j_1 < \cdots < j_k \leq n$ and π is the permutation with $\pi(i_k) = j_k$ for all k. With this notation, we can describe the relations, the image of $\operatorname{Gr}(k,n)$ under φ will satisfy.

The **Plücker relations for** $\operatorname{Gr}(k,n)$ are the relations

$$\sum_{r=0}^{k} (-1)^r p_{i_1,\ldots,i_{k-1},j_r} p_{j_0,\ldots,\hat{j_r},\ldots,j_k} \tag{1}$$

where the sum is taken over all tuples $(i_1,\ldots,i_{k-1})$, $(j_0,\ldots,j_k)$ satisfying $1 \leq i_1 < \cdots < i_{k-1} \leq n$ and $1 \leq j_0 < \cdots < j_k \leq n$.

Exercise 1.6. Write the Plücker relations for $\operatorname{Gr}(2,5)$.

Facts.

1. $x \in \Lambda^k(V)$ is decomposable $\Longleftrightarrow$ the Plücker relations on x are 0;
2. The image $\operatorname{im}\varphi \subseteq \mathbb{P}^N$ are the elements of $\mathbb{P}^N$ for which the Plücker relations are 0;
3. $\varphi : \operatorname{Gr}(k,n) \to \mathbb{P}^N$ is injective. It is called the **Plücker embedding**;
4. $\operatorname{im}\varphi$ is an irreducible projective variety, so $\operatorname{Gr}(k,n)$ is an irreducible projective variety.

For the proofs: (2) follows from (1) and from the definition of φ. For (1)-(3): [Jac96, §3.4] for (2),(3) [MS05, §14], for (4) W. Fulton, 1997, [Ful97, §8, §9.]

Example 1.7. For $\operatorname{Gr}(2,4)$, there is a single Plücker relation:

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$$

Remark 1.8. Let $Y \subseteq \mathbb{P}^r$ be a projective variety, $R := \mathbb{C}[x_0, x_1, \ldots, x_r]$ a graded ring, each x_i of degree 1. The **homogenous ideal** $J(Y)$ **of** Y is the ideal of R generated by the homogenous elements of R which vanish on Y. The **homogenous coordinate ring of** Y is $\mathbb{C}[Y] := R/J(Y)$. There is a natural projection

$$\operatorname{pr} : \mathbb{C}^{r+1} \setminus \{0\} \to \mathbb{P}^r, \quad (a_0,\ldots,a_r) \mapsto [a_0 : \ldots : a_r]$$

The **affine cone** $C(Y)$ **over** Y is the preimage of Y under pr:

$$C(Y) := \operatorname{pr}^{-1}(Y) \cup \{0\}$$

One can show that $C(Y)$ is an affine variety whose ideal is $J(Y)$ regarded as an ideal of R without the grading; the coordinate ring of $C(Y)$ coincides with the homogenous coordinate ring of Y, [Har77, Exercise 2.10]. From the facts above, we get that the affine cone of $\operatorname{Gr}(k,n)$ can be identified with the decomposable elements of $\Lambda^k(V)$ together with 0. And the coordinate ring of the affine cone of $\mathbb{C}[\operatorname{Gr}(k,n)]$ is the quotient of the polynomial ring in generators $x_{i_1,\ldots,i_k}$ with$1 \leq i_1 < \cdots \leq i_k \leq n$ by the ideal generated by the Plücker relations. For details: [Har77, §2]

1.3 Cluster algebra structure for $\operatorname{Gr}(2,n)$

For $k = 2$, the Plücker relations are

$$p_{i,j_0}p_{j_1,j_2} - p_{i,j_1}p_{j_0,j_2} + p_{i,j_2}p_{j_0.j_1}$$

where $1 \leq i \leq n$, $1 \leq j_0 < j_1 < j_2 \leq n$. We can rewrite these as

$$p_{ab}p_{cd} - p_{ac}p_{bd} + p_{ad}p_{bc} \quad \text{for all } a,b,c,d \text{ with } 1 \leq a < b < c < d \leq n \tag{2}$$

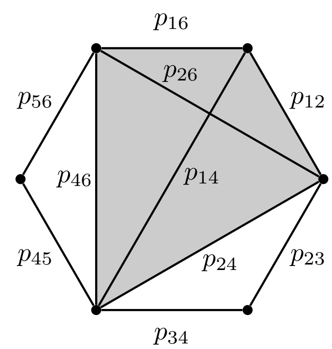
Exercise 1.9. Check the above.

Using this and the facts from above, we get the following result:

Lemma 1.10. *The homogenous coordinate ring of* $\mathrm{Gr}(2,n)$ *is the quotient of the polynomial ring in variables* p_{ab}*,* $1 \le a < b \le n$*, subject to the relations*

$$p_{ab}p_{cd} - p_{ac}p_{bd} + p_{ad}p_{bc} \quad \textit{for all } 1 \le a < b < c < d \le n$$

Remark 1.11. In this case, the Plücker coordinates can be parametrized by the diagonal and edges of a regular polygon P_n with vertices $1, 2, \dots, n$, say clockwise: the coordinate p_{ab}, $a < b$, corresponds to the diagonal or boundary edge connecting a and b. We can then interpret the relations (2) as "Ptolemy relations", e.g. $p_{14}p_{26} = p_{12}p_{46} + p_{16}p_{24}$ in the example $(n = 6)$:



1.4 The quiver of a triangulation of a surface

Let S be a connected, oriented surface with boundary. Let $M \neq \emptyset$ be a finite set of marked points in $\overline{S}$. The points of M are on the boundary or in the interior of S. Assume that M contains at least one marked point on each boundary component and that (S, M) is not one of the following:

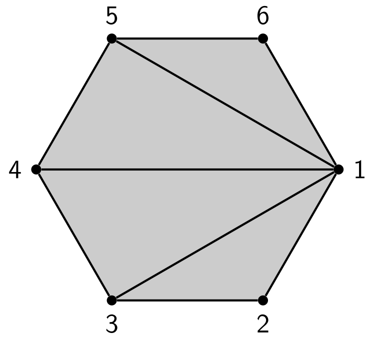
$$\begin{cases} \text{sphere with } 1, 2 \text{ or } 3 \text{ interior points} \\ \text{monogon with } 0 \text{ or } 1 \text{ interior points} \\ \text{digon or triangle with no interior points} \end{cases}$$

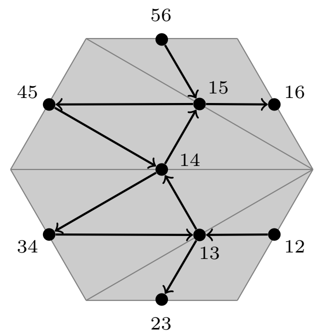
For details, we refer to [FST08].

We now restrict to the case where (S, M) has no punctures. Consider simple non-contractible arcs in (S, M), with endpoints in M (up to isotopy fixing endpoints). An **ideal triangulation** of (S, M) is a maximal collection of (isotopy classes of) such arcs which pairwise do not cross. Let T be an ideal triangulation of (S, M). Then we associate to T a quiver Q_T as follows.

The vertices of Q_T are the arcs of T and the boundary segments. We draw an arrow $i \to j$ if i and j are arcs of a common triangle of T and j is clockwise from i (around a common endpoint) and if i and j are not both boundary segments. In the example, the quiver of a triangulation of a hexagon is drawn.

If T is a triangulation of a polygon P_n, like below, there are n boundary edges $(i, i+1)$. By definition of Q_T, there are no arrows between these. We will later consider quiver with arrows between vertices on the boundary.





In these lectures, the surface is always a regular polygon P_n and $M = \{1, 2, \ldots, n\}$ the set of vertices of P_n. So for Q_T, the vertices are just the diagonals of the triangulation and the boundary edges of the polygon (sometimes called "frozen vertices").

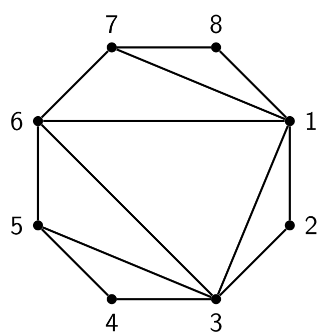
Theorem 1.12. *[FZ03, Proposition 12.6] Let $n \geq 5$. Let P_n be a convex n-gon. The homogenous coordinate ring $\mathbb{C}[\mathrm{Gr}(2,n)]$ of 2-planes in n-space is a cluster algebra:*
$\langle p_{ab} \mid 1 \leq a < b \leq n \rangle / \{$*Ptolemy relations*$\} \otimes \mathbb{C} = \mathbb{C}[\mathrm{Gr}(2,n)]$.

The cluster variables are the Plücker coordinates p_{ab}, where the (a,b) are the diagonals in P_n and the coefficients are the Plücker coordinates $p_{12}, p_{23}, \ldots, p_{n-1,n}, p_{1n}$ (corresponding to the boundary edges of P_n).

The seeds are in bijection with the triangulations of P_n. The quiver of the seed is Q_T. Cluster mutation corresponds to the quadrilateral flip in a triangulation (and to the Ptolemy relations (2)

By the above result, $\mathbb{C}[\mathrm{Gr}(2,n)]$ can be regarded as a cluster algebra of type A_{n-3} with coefficients (cf. [Mar13, Ex 8.2.3]). It is sometimes called **Ptolemy cluster algebra**.

Exercise 1.13. Find Q_T for the triangulation T given by the diagonals $(13), (35), (36), (16), (17)$ of an octagon.



1.5 Cluster algebra structure for $\mathrm{Gr}(k,n)$

From now on we assume $k \leq \frac{n}{2}$.

In order to find a cluster algebra structure for arbitrary k, we will use the notion of a "diagram in a disk with n marked points" instead of a triangulation of a polygon. We write S_n for the set of permutations of n. We write D_n for a disk with n marked points $\{1, 2, \ldots, n\}$ on the boundary (going clockwise). The following definition is due to Postnikov, [Pos06].

Definition 1.14. Let $\sigma \in S_n$ be a permutation. An **alternating strand diagram** or **Postnikov diagram of type** $\sigma \in S_n$ in the disk D_n with vertices $1, \ldots, n$, is a collection of n oriented strands (smooth curves) $\gamma_1, \ldots, \gamma_n$ (up to isotopy) in D_n with γ_i starting and i and ending at $\sigma(i)$ satisfying

(a) The γ_i have no self-intersections;

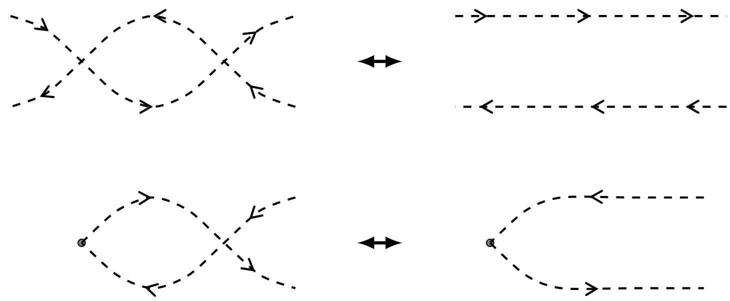


Figure 1: Untwisting and twisting moves in a Postnikov diagram

(b) There are finitely many intersections and they are transversal, of multiplicity 2;

(c) Crossings alternate (following any strand, the strands crossing it alternate between crossing from the left and crossing from the right);

(d) There are no "unoriented lenses": if two strands cross, they form an oriented disk.

For an illustration we refer to Figure 2. Postnikov diagrams can be simplified under two types of reductions as in Figure 1, also called twisting and untwisting moves. The moves obtained by reflecting the diagrams of Figure 1 in a horizontal line are also allowed. A Postnikov diagram is called **reduced** if it cannot be simplified with untwisting moves.

Remark 1.15. The conditions can be relaxed, i.e. if the surface has interior marked points or if it has several boundary components (like an annulus). Conditions (a) and (d) will no longer hold then. Strand diagrams appear as "webs" in [Gon17]. Strand diagrams for orbifolds are introduced in [BPV].

Any Postnikov diagram divides the surface into alternating and oriented regions. We label the alternating regions by i whenever γ_i is on the right of the region.

Let $\sigma_{k,n} \in S_n$ be the permutation $i \mapsto i + k$ (reducing modulo n).

Proposition 1.16. *[Pos06, Proposition 5]*

(a) Any $\sigma_{k,n}$-diagram in D_n has $k(n-k)+1$ alternating regions, $(k-1)(n-k-1)$ internal ones, n on the boundary.

(b) Each label is a k-subset of $[n]$.

(c) Every k-subset of $[n]$ appears as a label in a $\sigma_{k,n}$-diagram on D_n.

Note that if $\sigma = \sigma_{k,n}$, the n boundary alternating regions in (a) above have the labels $[1,k]$, $[2,k+1]$, $\dots, [n, n+k]$ (reducing modulo n), see Figure 2.

Example 1.17. Figure 2 shows an example of a Postnikov diagram of type $\sigma_{3,7}$ in D_7.

1.6 The quiver of a Postnikov diagram

Let D be a $\sigma_{k,n}$-diagram. Each label of D gives a Plücker coordinate, so we can associated to D a collection of Plücker coordinates, $D \mapsto \tilde{p}(D) = p(D) \cup C$ where C are the Plücker coordinates of the boundary alternating regions, i.e. the $p_{i,i+1,\dots,i+k-1}$ for $i = 1, \dots, n$. With this, we can associate a quiver $Q(D)$ to D.

This is similar to quiver of a triangulation (use remark on bijection 1.21). The boundary convention is however different as we will see.

Definition 1.18. Let D be a Postnikov diagram Then **the quiver** $Q(D)$ **of** D has as vertices the k-subsets of D. The **frozen vertices** are the k-subsets of the boundary alternating regions of D. The arrows of $Q(D)$ are given by the "flow": Whenever two k-subsets are separated by only two crossing

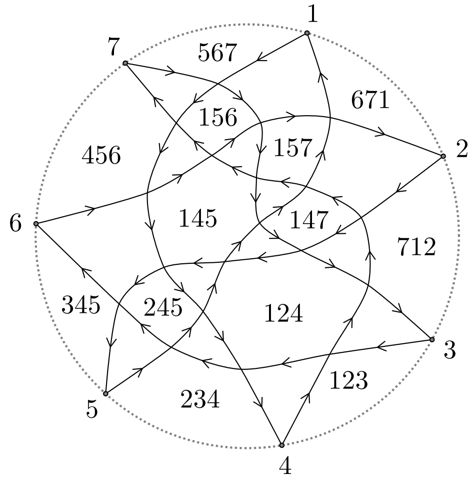


Figure 2: A $\sigma_{3,7}$ Postnikov diagram with its labels

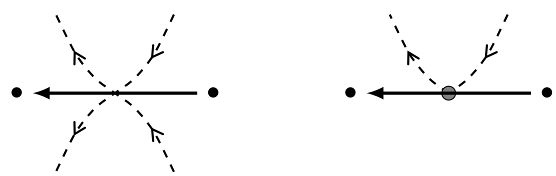


Figure 3: Orientation convention for the quiver $Q(D)$, on the right for the boundary.

strands, there is an arrow between them, following the orientation of the strands, see Figure 3. At the end, we remove all 2-cycles that may habe appeared through this.

Example 1.19. The quiver of the Postnikov diagram from Example 1.17 is in Figure 4.

Note that the quiver of a Postnikov diagram contains arrows between boundary vertices, unlike the quiver of a triangulation.

Definition 1.20. To any Postnikov diagram D of type $\sigma_{k,n}$, one can define a cluster algebra: The initial seed is given by the set $\{x_I\}_{I\in\tilde{p}(D)}$ and the quiver $Q(D)$. The cluster algebra $\mathcal{A}(D) := \mathcal{A}(\{x_I\}_{I\in\tilde{p}(D)}, Q(D))$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}((x_I)_I)$ generated by the x_I with $I \in C$ and by the x_I, $I \in p(D)$ and all elements obtained from the latter under arbitrary sequences of mutations.

Each $\sigma_{k,n}$-diagram gives rise to a seed in $\mathcal{A}(D)$. Scott proves [Sco06, Theorem 3] that there is an isomorphism $\varphi : \mathcal{A}(D) \xrightarrow{\sim} \mathbb{C}[Gr(k,n)]$ sending x_I to p_I for any k-subset $I \in \tilde{p}(D)$. In other words, $\mathbb{C}[Gr(k,n)]$ can be viewed as a cluster algebra where each Plücker coordinate is a cluster variable and where the $\sigma_{k,n}$-diagrams give some of its seeds.

Note that here we do not invert coefficients (see [Mar13, §9]). If we invert the coefficients, the cluster algebra is the coordinate ring of the Zariski-open subset of the Grassmannian defined by non-vanishing of the coefficients $p_{1,\dots,k}$, $p_{2,\dots,k+1}$, ... $p_{n,1,\dots,k-1}$.

Remark 1.21 (Scott map)**.** Let $k = 2$ and $n \geq 4$. Then there is a bijection between

$$\{\text{Triangulations of a convex } n\text{-gon}\} \quad \xleftrightarrow{1:1} \{\text{reduced } \sigma_{2,n}\text{-diagams}\}$$

arising from equipping each triangle in the triangulation with strand segments as shown in Figure 5, with a slight modification of Scott's definition from [Sco06, Section 3] in order for strands to end at the boundary vertices.

Exercise 1.22. Draw the $\sigma_{2,7}$-diagram D for the triangulation T from Exercise 1.13. Compare the two quivers Q_T and $Q(D)$.

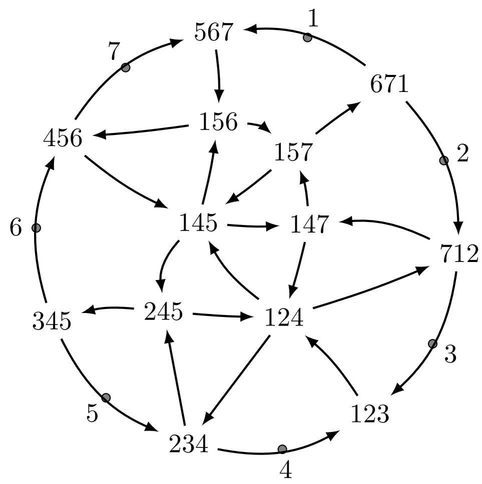


Figure 4: The quiver of the Postnikov diagram in Figure 2

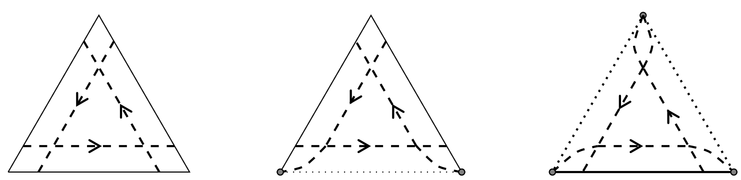


Figure 5: Modified version of Scott's construction. The dotted lines indicate boundary edges

Remark 1.23. A correspondence as the one given by Scott, see Remark 1.21 is not known for arbitrary k. The only other cases where a combinatorial construction is available are $\sigma_{k,n}$ diagrams for $n = 2m$ and $k = m + 1$; these arise from rhombic tilings, [Cos]).

By Remark 1.21, for $k = 2$, the $\sigma_{2,n}$-diagrams correspond to triangulations of polygons and thus in this case, the Postnikov diagrams give all the seeds (Theorem 1.12). If D is $\sigma_{3,n}$-diagram for $n \in \{6, 7, 8\}$ then the cluster algebra $\mathcal{A}(D)$ of any $\sigma_{3,n}$-diagram is of finite type, i.e. there are only finitely many cluster variables. However, not all seeds $\mathcal{A}(D)$ arise from $\sigma_{3,n}$-diagrams.

In all other cases (with $k \leq \frac{n}{2}$), the cluster algebra $\mathcal{A}(D)$ has infinitely many cluster variables.

2 Dimer models on surfaces and associated algebras

2.1 Dimer models with boundary

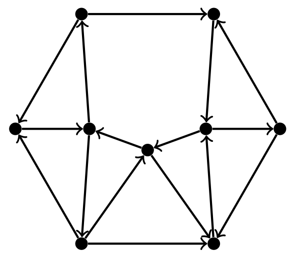
Dimer models with boundary have been introduced in [BKM16].

Definition 2.1. (1) A **dimer model (with boundary)** is a finite quiver Q that embeds into a surface S such that each connected component of $S \setminus Q$ is simply connected and bounded by an oriented cycle.

(2) The cycles bounding the connected components of $S \setminus Q$ are called the **unit cycles**. The arrows of Q are **internal** if they are contained in two faces and **boundary** if they are contained in one face. The vertices incident with boundary arrows are called **boundary vertices**.

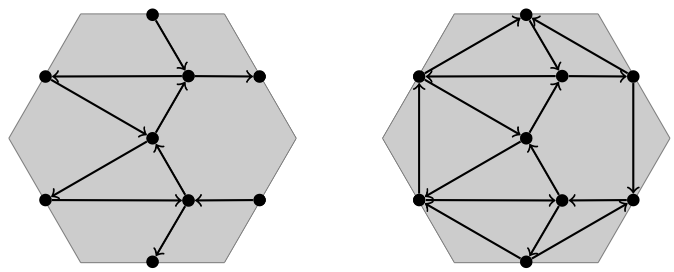
In the case without boundary, the dimer model have been studied by various authors, e.g. [Dav11], [Boc12], [Bro12]. The boundary convention has also been used independently in [BIRS11], [Fra12] and [DL16a].

Example 2.2. The following quiver is an example of a dimer model with boundary, on a disk with 6 boundary vertices.



Example 2.3. (a) The quiver of any Postnikov diagram is a dimer model in the disk.

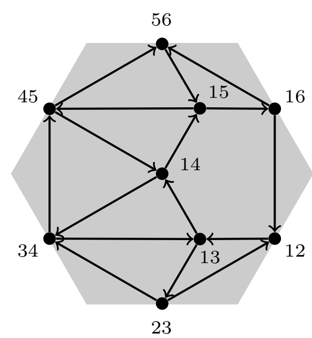
(b) Let T be a triangulation of a polygon P_n. If we complete the quiver Q_T by $n-2$ clockwise and 2 anti clockwise arrows between the boundary vertices in order to close all the cycles involving boundary vertices, we get a dimer model in P_n. We denote the resulting quiver by $Q(T)$.



Exercise 2.4. Find a triangulation T such that $Q(T)$ as defined in Example 2.3 (b) is the dimer of Example 2.2.

2.2 Algebras associated to dimer models

Most of the background for this section can be found in [BKM16] and [JKS16]. We associate associate algebras to dimer models. For this, we first describe the relations on paths we will use. If α is an internal arrow of a dimer model with boundary, then there are exactly two paths $\alpha^{\pm}$ back from its head to its tail. For example, in the dimer model from Example 2.3 (b), for the arrow $\alpha : 14 \to 15$, the two paths back are $p_\alpha^+ : (15 \to 45) \circ (45 \to 14)$ and p_α^- is $(15 \to 16) \circ (16 \to 12) \circ (12 \to 13) \circ (13 \to 14)$, composing arrows left to right:



Definition 2.5. Let Q be a dimer model with boundary. The **dimer algebra** A_Q **of** Q is quotient of the path algebra of $\mathbb{C}Q$ by the relations $p_\alpha^+ = p_\alpha^-$ for every internal arrow of Q.

We may also take the completed version of this path algebra, see Remark 2.14 below.

Another way to describe the relations for the definition of the dimer algebra is as follows. Let W be the (natural) potential associated to Q:

$$W := W(Q) := \sum_{p \text{ pos. unit cycle}} p - \sum_{p \text{ neg. unit cycle}} p.$$

Then the relations $p^+_\alpha = p^-_\alpha$ arise from taking all cyclic derivatives $\partial(W)$ of W with respect to internal arrows.

Observe that A_Q is an infinite dimensional algebra.

Exercise 2.6. Any two unit cycles at a vertex of a dimer model Q commute. Why is this?

Remark 2.7. Let $i \in Q_0$ be a vertex of a dimer model. Let U_i be a unit cycle at i. Then $t := \sum_{i \in Q_0} U_i$ is a central element of A_Q since any two unit cycles at i commute. So we get $\mathbb{C}[t] \subseteq Z(A_Q)$.

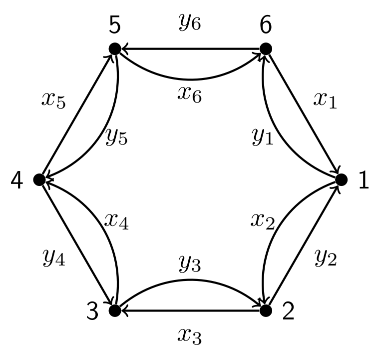
Let $e_1, e_2, \ldots, e_n$ be the idempotent elements of A_Q corresponding to the boundary vertices of Q and let $e := e_1 + \cdots + e_n$.

Definition 2.8. Let Q be a dimer model with boundary. The **boundary algebra** of Q is the idempotent subalgebra $B_Q := eA_Qe$.

B_Q has as basis all paths of Q between boundary vertices (up to the relations). Let $t := U_1 + \cdots + U_n$ be the sum of the unit cycles at the boundary vertices. Then $\mathbb{C}[t] \subseteq Z(B_Q)$.

Note that there is also a completed version of this, see Remark 2.14

Example 2.9. The boundary algebra from Example 2.2 is given by the following quiver:



One can show that

$$B_Q = \mathbb{C}[x_i, y_i \mid i = 1, \ldots, n] / \left\langle \{xy - yx, x^2 - y^4\} \right\rangle$$

and that $Z(B_Q) = \mathbb{C}[t]$.

Note that $\{xy - yx, x^2 - y^4\}$ is short for the twelve relations $x_iy_i - y_{i-1}x_{i-1}$, $x_ix_{i+1} - y_{i-1}y_{i-2}y_{i-3}y_{i-4}$ for $i = 1, \ldots, 6$ and where the indices are reduced modulo 6.

The boundary algebra B_Q is also an infinite dimensional algebra, in general its global dimension is infinite. A_Q and B_Q are not well understood apart from the case where the surface S is a disk and the dimer models arise from Postnikov diagrams of type $\sigma_{k,n}$ or from the cases where S is a surface and Q arise from a triangulation of S.

From now on we restrict to dimer models on disks. As we are interested in the cluster structure of the Grassmannian, we will also restrict the dimer models: we want them to correspond to $\sigma_{k,n}$-diagrams. Recall that any Postnikov diagram D on a disk determines a quiver $Q(D)$ and that this is a dimer model with boundary. There is also a way to go from dimer models to Postnikov diagrams ([Pos06, §14]): For any arrow in the dimer model, draw two segments of oriented curves, crossing on the arrow, pointing in the same direction as the arrow, then connect these; the strands correspond to zig-zag paths in the disk, cf. [Bro12, Section 5].

Exercise 2.10. Find the Postnikov diagram for the dimer model above. Determine its permutation.

Definition 2.11. If Q is a dimer model on a disk such that the associated Postnikov diagram is of type $\sigma_{k,n}$, we call Q a (k, n)-**dimer**.

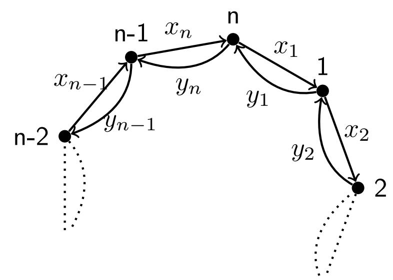


Figure 6: The quiver Γ_n

Remark 2.12.
- The $(2, n)$-dimers correspond to Q_T, T a triangulation of an n-gon.
- (k, n)-dimer exist for any (k, n). Examples of such arise from the rectangular $\sigma_{k,n}$-diagrams of Scott, [Sco06]

Theorem 2.13 ([BKM16])**.** *Let Q and Q' be two (k, n)-dimers. Let $e = e_1 + \cdots + e_n$ the sum of the boundary idempotents. Then*

$$eA_Qe \cong eA_{Q'}e \cong \mathbb{C}\,\Gamma_n / \left\langle \{xy - yx, x^k - y^{n-k}\} \right\rangle$$

where Γ_n is the quiver in Figure 6 and the relations are with indices as in Example 2.9.

Let $B := B_{k,n} := \mathbb{C}\,\Gamma_n / \left\langle \{xy - yx, x^k - y^{n-k}\} \right\rangle$. Then the boundary algebra of any (k, n)-dimer model is isomorphic to B.

Remark 2.14. The above theorem is about the completed versions of these algebras. For A_Q, the algebra $\widehat{A}_Q$ is the completion with respect to (U) for $U = \sum_{i \in Q_0} U_I$ and $\widehat{B}$ is the completion of B with respect to (t), $t = \sum_{i=1}^n x_i y_i$. Note that the completion $\widehat{B}$ of B with respect to (t) is the same as the completion with respect to the arrow ideal m, since $(m_{A_Q})^{N_1} \subseteq (U) \subseteq m_A$ and $(m_B)^{N_2} \subseteq (t) \subseteq m_B$ for some N_1, N_2, [JKS16, §3]. In particular, we have $e\widehat{A_Q}e \cong \widehat{B}$. See [BKM16, §11].

The centres of B and $\widehat{B}$ are polynomial rings, $Z(B) = \mathbb{C}[t]$ and $Z(\widehat{B}) = \mathbb{C}[|t|]$.

We will from now on work with the completed versions as we want the Krull-Schmidt Theorem to hold in the module categories. To simplify notation, we will write B and A_Q instead of $\widehat{B}$ and $\widehat{A_Q}$ to simplify notation.

We consider B-modules which are free over the centre and define

$$\begin{aligned} \mathcal{F}_{k,n} := \mathsf{CM}(B) \quad &:= \quad \{M \ B\text{-module} \mid M \text{ is free over } Z(B)\} \\ &= \quad \{M \mid \mathsf{Ext}^i_B(M, B) = 0 \text{ for all } i > 0\} \end{aligned}$$

Thee categories $\mathcal{F}_{k,n}$ have been introduced by Jensen-King-Su. In [JKS16], the authors prove that $\mathcal{F}_{k,n}$ provides an additive categorification of Scott's cluster algebra structure on the Grassmannian.

3 Grassmannian cluster categories

As before, let $k \leq \frac{n}{2}$. It is our goal to understand the categories $\mathcal{F}_{k,n}$ better, for example, to give a description of (some of) their indecomposable modules. The **rank** of a module $M \in \mathcal{F}_{k,n}$ is defined to be $\frac{1}{n}\mathsf{rk}_Z M$. Among the indecomposable modules, the rank 1 modules play a special role as they appear in filtrations of higher rank modules. Also, the rank 1 modules are well understood: As is shown in [JKS16, §5], there is a bijection between indecomposable rank 1 modules in $\mathcal{F}_{k,n}$ and k-subsets of $\{1, 2, \ldots, n\}$. As we know, these are in bijection with Plücker coordinates and hence with certain cluster variables, see Section 1.5. We write $\mathbb{M}_I$ for the indecomposable rank 1 module with k-subset I of $[n]$.

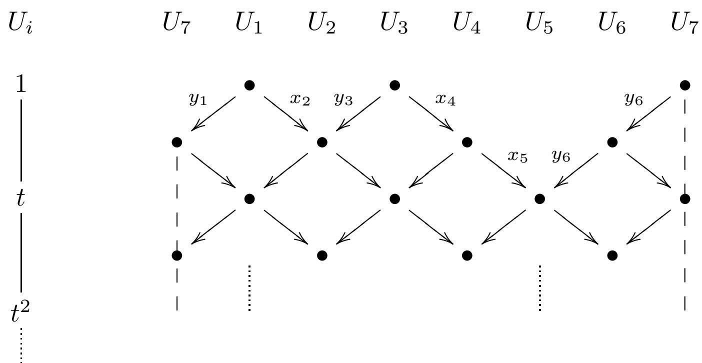
3.1 Rank 1 modules

We first describe how B acts on the rank 1 modules and then show how these modules can be understood as a lattice diagram with vertices for basis elements. Note that all modules in $\mathcal{F}_{k,n}$ are infinite-dimensional as they are free over the centre.

Any rank 1 module is given by n copies of the centre, $U_1, \ldots, U_n$, with $U_i := \mathbb{C}[|t|]$. Consider $\mathbb{M}_I$, I a k-subset of $[n]$. The actions of x_i and of y_i on the U_i are as follows:

$$x_i : U_{i-1} \to U_i \text{ acts as } \begin{cases} 1 & \text{if } i \in I \\ t & \text{if } i \notin I \end{cases} \qquad y_i : U_i \to U_{i-1} \text{ acts as } \begin{cases} t & \text{if } i \in I \\ 1 & \text{if } i \notin I \end{cases}$$

We can view this as a lattice diagram on a cylinder. For example, for $k = 3$ and $n = 7$, $I = \{2, 4, 5\}$. The arrows x_2, x_4, x_5 and the arrows y_1, y_3, y_6, y_7 all act as multiplication by 1, the other arrows as multiplication by t.



The **rim of a rank 1 module** $\mathbb{M}_I$ is formed by the top vertices in its lattice diagram and by the arrows connecting them. If the rim of $\mathbb{M}_I$ has two successive arrows y_i, x_{i+1} for some i, we say that the rim or $\mathbb{M}_I$ has a **peak (at i)**.

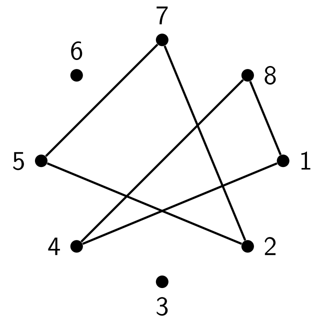
Remark 3.1. In $\mathcal{F}_{2,n}$ every indecomposable object is a rank 1 module, i.e. of the form $\mathbb{M}_{i,j}$ for some $1 \leq i \neq j \leq n$. The objects $\mathbb{M}_{i,i+1}$ are the indecomposable projective-injectives objects.

Exercise 3.2. Show that $\mathrm{Hom}_M(\mathbb{M}_I, \mathbb{M}_J) \cong \mathbb{C}[|t|]$ for all I, J.

Definition 3.3. Let I and J be k-subsets of $[n]$. We say that I and J **cross** if the complete graph $K_{I \setminus J}$ on $I \setminus J$ intersects $K_{J \setminus I}$.

Note that I and J do not cross if and only if the two k-subsets appear together in a $\sigma_{k,n}$-diagram. One can prove that the maximal non-crossing collections are exactly the ones arising from $\sigma_{k,n}$-diagrams: By [Sco06, Corollary 1], the k-subsets of any $\sigma_{k,n}$-diagram form a maximal non-crossing collection and by [OPS15, Thm. 7.1], every such collection arises in this way.

For an example of two crossing 4-subsets of [8] consider $I = \{1, 4, 6, 8\}$ and $J = \{2, 5, 6, 7\}$:



Crossing subsets are exactly the ones giving rise to non-trivial extensions:

Proposition 3.4. *[JKS16, Proposition 5.6]. $Ext^1_B(\mathbb{M}_I, \mathbb{M}_J) = 0$ of and only if I and J do not cross.*

One can use this to find has cluster-tilting objects in $\mathcal{F}_{k,n}$: let Q be a (k,n)-dimer, let $T := \oplus_{I \in Q_0} \mathbb{M}_I$. Then T is a maximal rigid object in $\mathcal{F}_{k,n}$ ([JKS16, BKM16]).

3.2 Dimer models as combinatorial approach to cluster categories

Theorem 2.13 above is a consequence of the following result:

Theorem 3.5. *[BKM16, Theorem 10.3] Let Q be a (k,n)-dimer and B as above, $e = e_1 + \cdots + e_n$ the sum of the idempotents for the boundary vertices of Q. Then*

$$A_Q \cong \mathit{End}_B(T)$$

and hence $eA_Qe \cong B^{op}$.

Sketch of proof. We define a map

$$\begin{array}{rcl} A_Q & \xrightarrow{g} & \mathsf{End}_B(T) = \mathsf{Hom}_B(\oplus_{I \in Q_0} \mathbb{M}_I, \oplus_{I \in Q_0} \mathbb{M}_I) \\ e & \mapsto & \mathsf{id}_{\mathbb{M}_I} \\ \alpha : I \to J & \mapsto & \varphi_{IJ} : \mathbb{M}_I \to \mathbb{M}_J(\text{“minimal codimension map”}) \end{array}$$

and extend to T (by 0's).

(a) One shows that g is an algebra homomorphism.

(b) For the surjectivity of g: to find $g^{-1}(\varphi_{IJ})$ we need a “minimal” path $p_{IJ} : I \to \cdots \to J$ in Q (unique in A_Q). It is difficult to see that this maps to φ_{IJ}. F0r this, we use weights on the arrows of Q, these are subsets of $[n]$, and show that the minimal path avoids at least one label of $[n]$ (using the proper ordering of strands around vertices).

(c) For the injectivity of g, we prove a Lemma stating that if $p : I \to J$ is a path in Q, then there exists $r \geq 0$ such that $p = u^r \circ p_{IJ}$ where u is any unit cycle at J. This yields injectivity: take $m \in A_Q$ with $g(m) = 0$. Without loss of generality, $m = e_I A_Q e_J$. By the lemma, $m = \sum_{r=0}^{l} \lambda_r u^r p_{IJ}$ for some coefficients λ_r and so $g(m) = \sum_{r=0}^{l} \lambda_r u^r \varphi_{IJ} = 0$. Since the $t^r \varphi_{IJ}$ are linearly independent elements of $\mathsf{Hom}_B(\mathbb{M}_I, \mathbb{M}_J)$, this implies $\lambda_r = 0$ for all r.

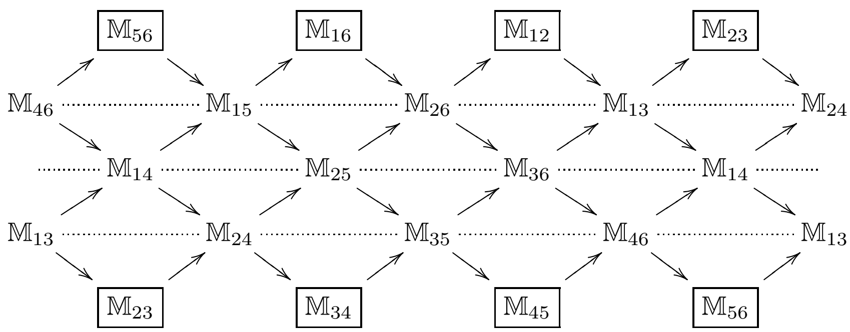
(d) For the statement $eA_Qe \cong B^{op}$, use the isomorphism in the theorem: $eA_Qe \cong g(e)\mathsf{End}_B T g(e) = \mathsf{End}_B P = B^{op}$

$\square$

3.3 Categories of finite type

The category $\mathcal{F}_{k,n}$ has of finitely many indecomposables if and only if $k = 2$ or $k = 3$ and $n \in \{6, 7, 8\}$. The Auslander-Reiten quiver of $\mathcal{F}_{k,n}$ has as vertices the isomorphism classes of indecomposable objects and it has an arrow for every irreducible map. The dotted lines in the Auslander-Reiten quiver are the Auslander-Reiten translate, sending an object to its neighbour on the left. The Auslander-Reiten quiver provides a good understanding of the category, since the $\mathcal{F}_{k,n}$ are Krull-Schmidt, every object can be uniquely written as a direct sum of indecomposables. In the case $k = 2$, the category $\mathcal{F}_{2,n}$ is a cluster category of type A_{n-3} (with projective-injective objects). Its Auslander-Reiten quiver sits on a Moebius strip. Recall that for $\mathcal{F}_{2,n}$, all the indecomposables are of the form $\mathbb{M}_{i,j}$ (Remark 3.1).

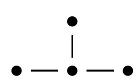
Example 3.6. The Auslander-Reiten quiver of $\mathcal{F}_{2,6}$ has the following form:



There are $\binom{6}{2}$ indecomposable modules, indexed by the 2-subsets of 6. The projective-injective indecomposables are the $\mathbb{M}_{i,i+1}$ (reducing modulo 6), drawn in a box in the quiver.

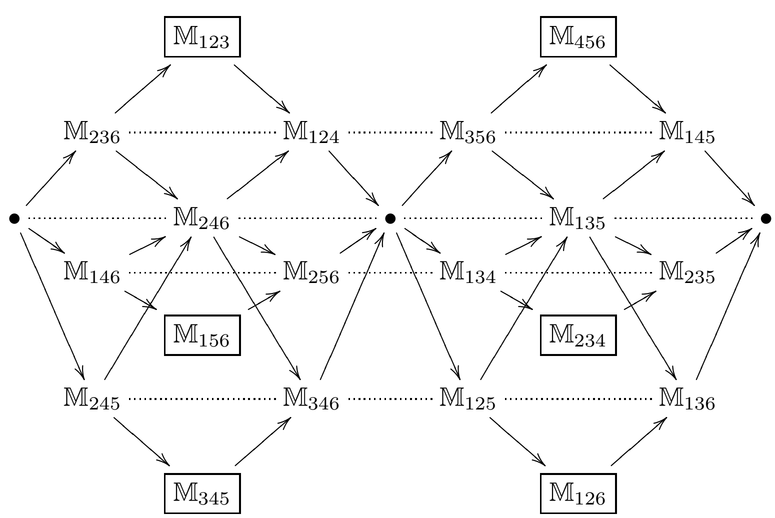
The Auslander-Reiten quivers of the categories $\mathcal{F}_{3,n}$ have been described in [JKS16, Fig 10,11,12]. We recall them here. Some of the indecomposable objects are rank 1 modules, but there are also rank 2 modules ($n = 6, 7, 8$) and rank 3 modules ($n = 8$).

Example 3.7. Auslander-Reiten quiver of $\mathcal{F}_{3,6}$. It is formed by 4 slices of shape D_4



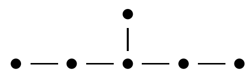
Dynkin diagram of type D_4

and 6 additional vertices, corresponding to the projective-injective modules $\mathbb{M}_{i,i+1,i+2}$ (reducing indices modulo 6).



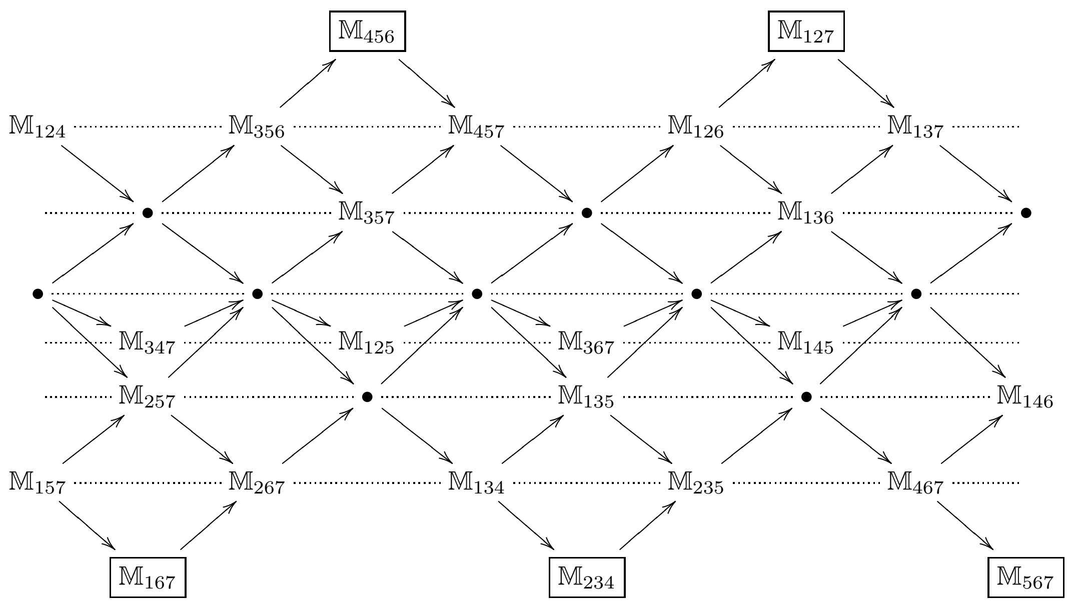
There are $\binom{6}{3} = 20$ rank 1 indecomposables and two rank 2 indecomposables, the latter indicated by a vertex $\bullet$ (the left most and the right most are identified). The projective-injectives are drawn in boxes.

Example 3.8. The Auslander-Reiten quiver of $\mathcal{F}_{3,7}$ is is formed by 7 slices of shape E_6



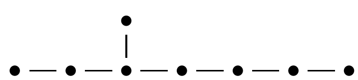
Dynkin diagram of type E_6

and 7 additional vertices, corresponding to the projective-injective modules $\mathbb{M}_{i,i+1,i+2}$ (reducing modulo 7). This figure shows part of it. To complete it, one can continue along the dotted lines, using the fact that $\tau^{-2}(\mathbb{M}_{i,j,k}) = \mathbb{M}_{i+3,j+3,k+3}$.



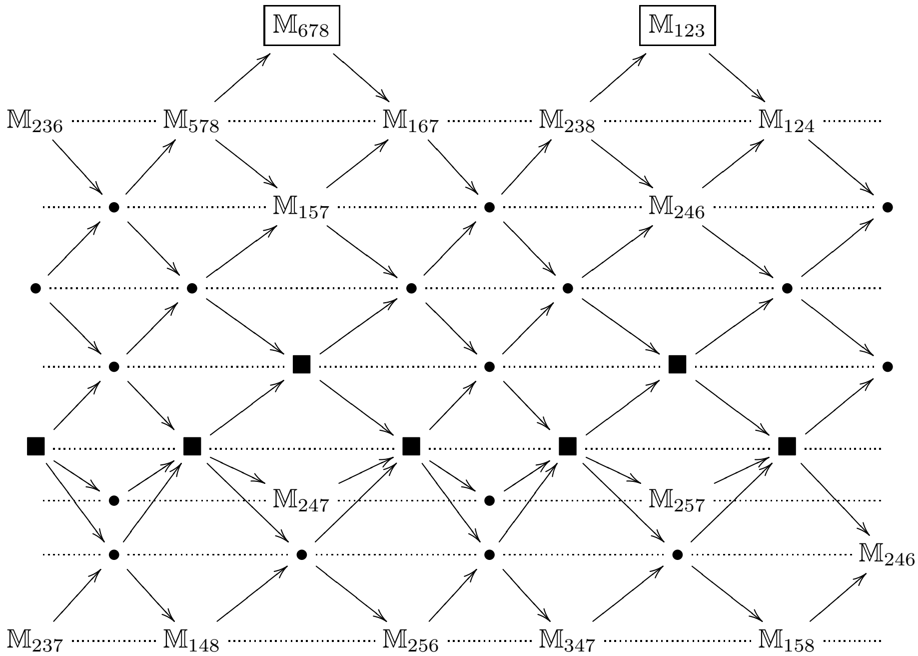
There are $\binom{7}{3} = 35$ rank 1 indecomposables and 14 rank 2 indecomposables, indicated by $\bullet$.

Example 3.9. The Auslander-Reiten quiver of $\mathcal{F}_{3,8}$ is formed by 16 slices of shape E_8



Dynkin diagram of type E_8

and 8 additional vertices, corresponding to the projective-injective modules $\mathbb{M}_{i,i+1,i+2}$ (reducing modulo 8), drawn in boxes. Apart from the rank 1 modules, there are rank 2 modules drawn as $\bullet$ and rank 3 modules drawn as $\blacksquare$. The rest of the shape of the Auslander-Reiten quiver can be obtained using the fact that for rank 1 modules, $\tau^{-2}(\mathbb{M}_{i,j,k}) = \mathbb{M}_{i+3,j+3,k+3}$.



There are $\binom{8}{3} = 56$ rank 1 indecomposables, 56 rank 2 indecomposables and 24 rank 3 indecomposables.

3.4 Structure of $\mathcal{F}_{k,n}$ in infinite types

The aim of the remainder of these notes is to provide more information about the categories $\mathcal{F}_{k,n}$ which have infinitely many indecomposables. We will describe part of the Auslander-Reiten quiver in the general situation. The main tool in this section is a result determining Auslander-Reiten sequences.

Remark 3.10. (i) If $M \in \mathcal{F}_{k,n}$ is indecomposable rigid, then it has a filtration $M \cong \begin{matrix} \mathbb{M}_{I_1} \\ \vdots \\ \mathbb{M}_{I_d} \end{matrix}$ by rank 1 modules $\mathbb{M}_{I_1}, \dots, \mathbb{M}_{I_d}$ and this filtration is unique. The rank of M is d.

(ii) Note that the rank is additive on Auslander-Reiten sequences.

There are certain canonical Auslander-Reiten sequences which involve rank 1 modules. If $\mathbb{M}_I$ is an indecomposable whose rim has s peaks (see Section 3.1), then $\Omega(\mathbb{M}_I)$ is a rank $(s-1)$ module ([BB17]), where Ω is the syzygy functor. In particular, if $I = \{i, j, j+1, \dots, j+k-2\}$ with $i+1 \neq j$ and $i-1 \neq j+k-2$, the rim of $\mathbb{M}_I$ has two peaks and $\Omega(\mathbb{M}_I) = \mathbb{M}_J$ is also a rank 1 module, with $J = \{i+1, i+2, \dots, i+k-1, j+k-1\}$, cf. [BB17, §2].

Theorem 3.11 ([BBGE19]). *Let $3 \leq k \leq \frac{n}{2}$ and $I = \{i, j, j+1, \dots, j+i-2\}$ and J such that $\Omega(\mathbb{M}_I) = \mathbb{M}_J$. Then there exists an Auslander-Reiten sequence*

$$\mathbb{M}_I \hookrightarrow \frac{\mathbb{M}_X}{\mathbb{M}_Y} \twoheadrightarrow \mathbb{M}_J$$

with rigid middle term $\frac{\mathbb{M}_X}{\mathbb{M}_Y}$ for $X = \{i+1, j, \dots, j+k-3, j+k-1\}$ and $Y = I \cup J \setminus X$.

The middle term is indecomposable if and only if $j \neq i+2$ and if $j = i+2$, $\frac{\mathbb{M}_X}{\mathbb{M}_Y} = P_i \oplus \mathbb{M}_{\{i,i+2,\dots,i+k-1,i+k+1\}}$ for $P_i = \mathbb{M}_{\{i+1,i+2,\dots,i+k\}}$.

Remark 3.12.
- To proves the rigidness of the middle term, one shows that $\dim\mathrm{Ext}^1(\mathbb{M}_I, \mathbb{M}_J) = 1$ (using the description of extensions between rank 1 modules from [BB17]).

- In case $k = 3$, the theorem above covers all Auslander-Reiten sequences where both end terms are rank 1 modules.

Remark 3.13. For $(k, n) \in \{(3, 9), (4, 8)\}$ the category $\mathcal{F}_{k,n}$ is of infinite type but tame (i.e. these are the first infinite cases to study). These are tubular categories: their Auslander-Reiten quiver is formed by tubes. Using Theorem 3.11 and computing syzygies for rank 2 modules, we are able to determine the tubes with low rank rigid modules ([BBGE19]). The tubes for $\mathcal{F}_{3,9}$ are of rank $2, 3, 6$ and $\mathcal{F}_{4,8}$ of rank $2, 4, 4$.

3.5 Periodicity in $\mathcal{F}_{k,n}$

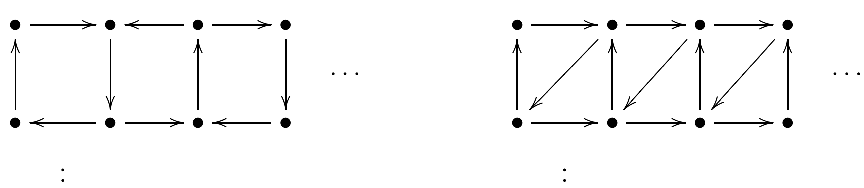
For any k-subset I of $[n]$, $I = \{i_1, i_2, \ldots, i_k\}$, we define $I+k$ to be the k-subset $\{i_1+k, i_2+k, \ldots, i_k+k\}$. Let τ be the Auslander-Reiten translate in $\mathcal{F}_{k,n}$.

Example 3.14. Let $\mathbb{M}_I$ be an indecomposable in $\mathcal{F}_{k,n}$. Then $\Omega^2(\mathbb{M}_I) = \mathbb{M}_{I+k}$.

Proposition 3.15. *Every indecomposable in* $\underline{\mathcal{F}_{k,n}}$ *is τ-periodic with period a factor of* $\frac{2}{k}\operatorname{lcm}(k, n)$.

The proof of this proposition uses $\mathcal{F}_{k,n} \simeq \mathsf{CM}^{\mathbb{Z}_n}(R_{k,n})$ (graded Morita equivalence) for the ring $R_{k,n} := \mathbb{C}[x, y]/(x^k - y^{n-k})$ where x has degree 1 and y has degree -1, cf. [DL16b, Theorem 3.22]. Alternatively, one can use [Kel13, Theorem 8.3] to find $\tau^n(M) = M$ for any indecomposable M in $\underline{Fk}$.

Remark 3.16. The quivers of the cluster-tilting objects that are given by Scott's quadrilateral arrangements have the following shapes. If we write $T = T' \oplus P_1 \oplus P_2 \oplus \cdots \oplus P_n$ where the P_i are projective-injective, then the quiver of the endomorphism algebra of T', for T' arising from the quadrilateral arrangement is as on the left, with $k-1$ rows and $n-k-1$ columns. It can be mutated to the quiver on the right:



These quivers are also appear in Keller's work [Kel13] on the periodicity conjecture. In terms of these quivers, $\mathcal{F}_{k,n}$ has finite representation type if and only if it has a cluster-tilting object $T = T' \oplus (\oplus_i P_i)$ such that the quiver of the endomorphism algebra of T' is a linear orientation of A_n or it is formed by 2, 4 or 6 oriented triangles forming one rectangle with side lengths 2 and 2,3 or 4.

3.6 Root systems associated to $\mathcal{F}_{k,n}$

Recall that $k \leq \frac{n}{2}$. The main references for this section are [JKS16, §8] and [BBGE19, §2].

Consider the graph

$$\begin{tikzcd} & & n \arrow[d, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dotted, dash] & k \arrow[r, dash] & (k+1) \arrow[r, dotted, dash] & n-1 \end{tikzcd}$$

with nodes $1, 2, \ldots, n-1$ on the horizontal line and node n branching off from node k. To this graph, a root system $\Phi_{k,n}$ is associated, each node corresponds to a simple root and edges indicate simple roots which can be added. The root system of $J_{k,n}$ has simple roots $\alpha_i := -e_i + e_{i+1}$ for $i = 1, \ldots, n-1$ and $\beta = e_1 + \cdots + e_k$ for $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{C}^n$. Note that if $k = 2$, $J_{2,n}$ is a Dynkin diagram of type D_n. If $k = 3$ and $n \in \{6, 7, 8\}$, the graph $J_{3,n}$ is a Dynkin diagram of type E_6, E_7 or E_8, respectively.

Let $\mathbb{Z}^n(k) := \{\underline{x} \in \mathbb{Z}^n \mid k \text{ divides } \sum_i x_i\}$. The root system $\Phi_{k,n}$ of $J_{k,n}$ can be identified with $\mathbb{Z}^n(k)$ via $\alpha_i \leftrightarrow -e_i + e_{i+1}$, for $i \leq n-1$ and $\beta \leftrightarrow e_1 + \cdots + e_k$.

Define $q : \mathbb{Z}^n(k) \to \mathbb{Z}$ to be $q(\underline{x}) = \sum_{i=1}^n x_i^2 + \frac{2-k}{k^2}(\sum_{i=1}^n x_i)^2$.

Then the roots for $J_{k,n}$ correspond to the vectors $\underline{a}$ of $\mathbb{Z}^n(k)$ with $q(\underline{a}) \leq 2$. The vectors with $q(\underline{a}) = 2$ are the **real** roots, the vectors with $q(\underline{a}) < 2$ are **imaginary roots**. The **degree of a root** γ is its coefficent at β: If $\gamma = \sum_{i=1}^{n-1} m_i\alpha_i + m\beta$, $m_i, m \in \mathbb{Z}$, then $\deg\gamma = m$.

In Section 8, [JKS16] define a map ind $\mathcal{F}_{k,n} \xrightarrow{\varphi} \Phi_{k,n}$. The map φ associates to each indecomposable a positive root of $\Phi_{k,n}$.

Let M be an indecomposable rank d module. Assume that M has a filtration by rank 1 modules, $M = \dfrac{\mathbb{M}_{I_1}}{\dfrac{\vdots}{\mathbb{M}_{I_d}}}$ for k-subsets $I_1, \ldots, I_d$. For $i = 1, \ldots, n$ let a_i be the multiplicity of i in $I_1 \cup \cdots \cup I_d$. We associate to M the vector $\underline{a}(M) = (a_1, \ldots, a_n)$. Then $\varphi(M)$ is defined to be the root corresponding to $(a_1, \ldots, a_n) \in \mathbb{Z}^n(k)$.

Example 3.17. Let $n = 6$, $k = 3$. So $J_{3,6}$ is as follows:

$$\begin{array}{ccccccccc} & & & & \beta & & & & \\ & & & & | & & & & \\ \alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 \end{array}$$

Let $I = \{1, 3, 5\}$ and $J = \{2, 4, 6\}$.

1. $M := \mathbb{M}_I$: $\underline{a} = (1, 0, 1, 0, 1, 0)$ and $\varphi(M) = \beta + \alpha_2 + \alpha_3 + \alpha_4$.
2. Let $M = \mathbb{M}_I \,/\, \mathbb{M}_J$. Then $\underline{a} = (1, 1, 1, 1, 1, 1)$ which corresponds to $2\beta+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5$, the highest root for E_6.

Exercise 3.18. Compute $\varphi(M)$ for $M = \mathbb{M}_J$ and for $M = \mathbb{M}_J \,/\, \mathbb{M}_I$ from Exercise 3.17.

Remark 3.19. Let M be indecomposable with filtration $M = \mathbb{M}_{I_1} \,/\, \mathbb{M}_{I_2} \,/\, \ldots \,/\, \mathbb{M}_{I_d}$. Then one observes that $\varphi(M)$ is a root of degree d.

Question 1. What is the connection between indecomposable rank r-modules in $\mathcal{F}_{k,n}$ and roots for $J_{k,n}$? What is the connection between rigid indecomposables in $\mathcal{F}_{k,n}$ and real roots for $J_{i,n}$

For $r = 1$, there is a bijection

$$\{\text{indecomposable rank 1-modules}\}/_\sim \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{real roots for } J_{k,n} \text{ of degree } 1\}$$

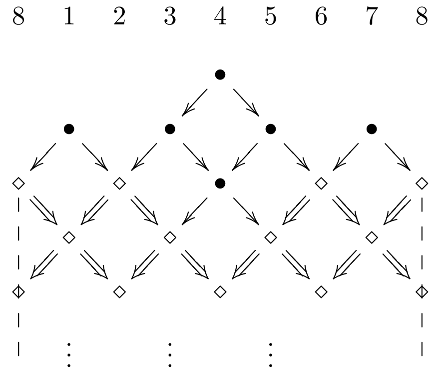
In finite types, one finds

$$\{\text{indecomposable rank } r\text{-modules}\}/_\sim \quad \overset{r:1}{\longleftrightarrow} \quad \{\text{real roots for } J_{k,n} \text{ of degree } r\}$$

as in these types, the higher rank modules "cycle": Let $k = 3$ and $n \in \{6, 7, 8\}$, then the indecomposables are all of rank ≤ 3. And $M = \mathbb{M}_I \,/\, \mathbb{M}_J$ is an indecomposable rank 2 module if and only if $\mathbb{M}_J \,/\, \mathbb{M}_I$ is indecomposable. Furthermore, $\varphi(\mathbb{M}_I \,/\, \mathbb{M}_J) = \varphi(\mathbb{M}_J \,/\, \mathbb{M}_I)$, cf. Exercise 3.18 for $n = 6$. For the rank 3 modules, $n = 8$, we have $\mathbb{M}_I \,/\, \mathbb{M}_J \,/\, \mathbb{M}_L$ is indecomposable if and only if $\mathbb{M}_J \,/\, \mathbb{M}_L \,/\, \mathbb{M}_I$ is indecomposable if and only if $\mathbb{M}_L \,/\, \mathbb{M}_I \,/\, \mathbb{M}_J$ is indecomposable, all with the same root in $\Phi_{3,8}$.

The first cases that are not fully understood are the tame cases $\mathcal{F}_{3,9}$ and $\mathcal{F}_{4,8}$, cf. Remark 3.13. It is known that in these cases every real root of degree 2 yields exactly two rigid indecomposable rank 2 modules. But there are indecomposable rank 2 modules in $\mathcal{F}_{4,8}$ whose root is an imaginary degree 2 root. See Example 3.20 and Exercise 3.21.

Example 3.20. Let $n = 8$, $k = 4$, $I = \{2, 5, 6, 8\}$ and $J = \{1, 3, 4, 7\}$. The lattice diagram of the indecomposable $\mathbb{M}_I \,/\, \mathbb{M}_J$ is here - vertices indicated with $\bullet$ correspond to one-dimensional vector spaces and vertices indicated with $\diamond$ correspond to two-dimensional vector spaces and arrows $=>$ to maps between two-dimensional vector spaces. The module $\mathbb{M}_J$ is seen as the submodule on the boxed vertices. One can check that the associated root is imaginary.



Note that $\mathbb{M}_J \,/\, \mathbb{M}_I$ is not indecomposable.

Exercise 3.21. Find $\varphi(\mathbb{M}_I \,/\, \mathbb{M}_J)$ for I, J as in Example 3.20. Compute $q(\underline{a}(\mathbb{M}_I \,/\, \mathbb{M}_J)$.

3.7 Rank 2 modules

Let I and J be two k-subsets. Assume that I and J are strictly 3-interlacing, i.e. that $|I \backslash J| = |J \backslash I| = 3$ and that the non-common elements of I and J interlace. We want to define a rank 2 module $\mathbb{M}(I, J)$ in similar way as rank 1 modules are defined. Let $V_i := \mathbb{C}[|t|] \oplus \mathbb{C}[|t|]$, $i = 1, \dots, n$. We will need to say how x_i, y_i act. For this, define matrices

$$A_1 := \begin{pmatrix} t & -2 \\ 0 & 1 \end{pmatrix} \quad B_1 := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \quad C_1 := \begin{pmatrix} t & -1 \\ 0 & 1 \end{pmatrix} \quad D_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$A_2 := \begin{pmatrix} 1 & 2 \\ 0 & t \end{pmatrix} \quad B_2 := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \quad C_2 := \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix} \quad D_2 := \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Note that these are all matrix factorisations of $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$: $A_1A_2 = B_1B_2 = C_1C_2 = D_1D_2 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$.

Example 3.22. Let $k = 3$, $n = 6$, $I = \{1, 3, 5\}$ and $J = \{2, 4, 6\}$. We define $\mathbb{M}(I, J)$ as follows: The vertices of Γ_n have the vector spaces V_i. The maps x_i, y_i are:

$$x_i : V_{i-1} \to V_i \text{ acts as } \begin{cases} A_1 & \text{if } i = 1 \\ B_2 & \text{if } i = 2 \\ B_1 & \text{if } i = 3 \\ C_2 & \text{if } i = 4 \\ C_1 & \text{if } i = 5 \\ A_2 & \text{if } i = 6 \end{cases} \qquad y_i : V_i \to V_{i-1} \text{ acts as } \begin{cases} A_2 & \text{if } i = 1 \\ B_1 & \text{if } i = 2 \\ B_2 & \text{if } i = 3 \\ C_1 & \text{if } i = 4 \\ C_2 & \text{if } i = 5 \\ A_1 & \text{if } i = 6 \end{cases}$$

We can define a rank 2 module in $\mathcal{F}_{k,n}$ more generally. Let I and J be strictly 3-interlacing, write $I \setminus J$ as $\{a_1, a_2, a_3\}$ and $J \setminus I = \{b_1, b_2, b_3\}$ so that $1 \le a_1 < b_1 < a_2 < b_2 < a_3 < b_3 \le n$. For the arrows incident with the a_is and the b_is, we use the construction from Example 3.22. For all other arrows, we use the maps D_1, D_2.

Definition 3.23. Let I, J be strictly 3-interlacing k-subsets of $[n]$. At the vertices of Γ_n, $\mathbb{M}(I, J)$ has the $V_1, \dots, V_n$. We define the maps x_i, y_i as follows:

$$x_i : V_{i-1} \to V_i \text{ acts as } \begin{cases} A_1 & \text{if } i = a_1 \\ B_2 & \text{if } i = b_1 \\ B_1 & \text{if } i = a_2 \\ C_2 & \text{if } i = b_2 \\ C_1 & \text{if } i = a_3 \\ A_2 & \text{if } i = b_3 \\ D_1 & \text{if } i \in I \cap J \\ D_2 & \text{if } i \in I^c \cap J^c \end{cases} \qquad y_i : V_i \to V_{i-1} \text{ acts as } \begin{cases} A_2 & \text{if } i = a_1 \\ B_1 & \text{if } i = b_1 \\ B_2 & \text{if } i = a_2 \\ C_1 & \text{if } i = b_2 \\ C_2 & \text{if } i = a_3 \\ A_1 & \text{if } i = b_3 \\ D_2 & \text{if } i \in I \cap J \\ D_1 & \text{if } i \in I^c \cap J^c \end{cases}$$

Exercise 3.24. Check that $\mathbb{M}(I,J) \in \mathcal{F}_{k,n}$. For this, check that $xy = yx$ and $x^k = y^{n-k}$ at all vertices (hence is a B-module) and that $\mathbb{M}(I,J)$ is free over the centre.

Question 2. When is $\mathbb{M}(I,J)$ indecomposable?

It is known that if a rank two module with filtration $\mathbb{M}_I \,/\, \mathbb{M}_J$ is indecomposable then I and J have to be strictly 3-interlacing ([BBGE19]). We claim that the converse is also true.

Proposition 3.25. *Let I, J be strictly 3-interlacing. Then $\mathbb{M}(I,J)$ is indecomposable.*

Idea of proof. Recall that a module is indecomposable if and only if its endomorphism ring is local and this is the case if and only if any idempotent endomorphism of the module is 0 or the identity.

(1) First consider $n = 6$. Any endomorphism of $\mathbb{M}(I,J)$ is of the form $\varphi = (\varphi_i)_{1\le i\le 6}$ with $\varphi_i : V_i \to V_i$. One computes that φ is as follows:

$$\left(\varphi_1, \begin{pmatrix} a & b \\ ct & d \end{pmatrix}, \varphi_1, \begin{pmatrix} a+c & b+(d-a-c)t^{-1} \\ ct & d-c \end{pmatrix}, \varphi_1, \begin{pmatrix} a+2c & b+2(d-a-2c)t^{-1} \\ ct & d-2c \end{pmatrix}\right)$$

for $\varphi_1 = \begin{pmatrix} a & bt \\ c & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{C}[[t]]$ and where $t \mid c$ and $t \mid (a-d)$. `To see this, use the relations` $x_i\varphi_{i+1} = \varphi_i x_i$ `for` $i = 1, \dots, 6$.

(2) Now one shows that if φ is an idempotent endomorphism of $\mathbb{M}(I,J)$, then $b = c = 0$ and $a = d = 0$ or $a = d = 1$. So any idempotent endomorphism of $\mathbb{M}(I,J)$ is the identity or 0. Hence $\mathbb{M}(I,J)$ is indecomposable in case $n = 6$.

In more detail: we consider $\varphi_1^2 = \varphi_1$:

$$\varphi_1^2 = \begin{pmatrix} a^2+bct & (a+d)b \\ (a+d)ct & d^2+bct \end{pmatrix} = \begin{pmatrix} a & b \\ ct & d \end{pmatrix}$$

The equations $a^2 + bct = a$ and $d^2 + bct = d$ on the diagonal entries give $a - a^2 = d - d^2$, i.e. $a - d = a^2 - d^2 = (a-d)(a+d)$ and hence $a = d$ or $a + d = 1$. The equations also show that t *divides* $a(1-a)$ *and that* t *divides* $d(1-d)$.

Assume first $a = d$. If $b \neq 0$, we get $a = \frac{1}{2}$ but then $bct = a - a^2 = -\frac{1}{4}$, impossible. Analogously for $c \neq 0$. Thus $b = c = 0$ and $a = d = 0$ or $a = d = 1$, the two trivial cases.

So assume $a \neq d$, so $d = 1 - a$. Combining $t \mid a(1-a)$ with the fact that t divides $a - d = 2a - 1$ implies to $t \mid 1$, a contradiction.

(3) In the general case, let $\varphi = (\varphi_i)_{1\le\le n}$. One checks that for for all $i \in I \cap J \cup I^c \cap J^c$, $\varphi_i = \varphi_{i+1}$. Hence this reduces to the case $n = 6$. $\square$

Question 3. How does a construction of indecomposable rank 3 modules look like?

3.8 Friezes from $\mathcal{F}_{k,n}$

The main reference for this Section is [BFG+18]. Let $\mathcal{F}_{k,n}$ be of finite type.

Definition 3.26. A **mesh frieze $M_{k,n}$ for $\mathcal{F}_{k,n}$** is a collection of positive integers, one for each indecomposable of $\mathcal{F}_{k,n}$ (up to isomorphism) such that $M_{k,n}(P) = 1$ for every indecomposable projective P and such that all mesh relations evaluate to 1. In order words: whenever we have an Auslander-Reiten sequence $A \to \oplus_i B_i \to C$ with B_i indecomposable, $M_{k,n}(A) \cdot M_{k,n}(C) = \prod_i M_{k,n}(B_i) + 1$.

Remark 3.27. A mesh frieze $M_{2,n}$ is a Conway-Coxeter frieze, also called SL_2-frieze. Such friezes are in bijection with triangulations of polygons ([CC73a, CC73b]) and thus arise from specialising a cluster-tilting object in a cluster category $\mathcal{F}_{2,n}$ of type A_{n-3} to 1.

Remark 3.28. The friezes of Dynkin types of [ARS10] correspond to our mesh friezes for $\underline{\mathcal{F}_{k,n}}$ in types A, D_4, E_6, E_8.

Definition 3.29. An $\mathbf{SL}_3$**-frieze** is an array starting and ending with $k-1$ rows of 0s and with finitely many rows of positive integers in between, arranged as below, and such that each 3×3 matrix has determinant 1. The **width** of an SL_3-frieze is the number of rows of positive integers between the two rows of 1s.

$$
\begin{array}{cccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 \\
1 & & 1 & & 1 & & 1 & & 1 & \\
 & * & & * & & * & & * & & * \\
* & & * & & * & & * & & * & \\
 & & \vdots & & & & \vdots & & & \\
 & * & & * & & * & & * & & * \\
1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

Proposition 3.30. *Let $n\in\{6,7,8\}$. There is a bijection*

$$\{\textit{mesh friezes for } \mathcal{F}_{3,n}\} \xleftrightarrow{1:1} \{\textit{SL}_3\textit{-friezes of width } n-4\}.$$

For $n=6$, this is in [MG15], for $n=7,8$ in [BFG$^+$18].

By the above result, in order to study mesh friezes for $\mathcal{F}_{3,n}$ it is equivalent to study SL_3-friezes of width $n-4$. We will use the two notions interchangeably.

Most of the mesh friezes for $k=3$ arise from specialising a cluster-tilting object in $\mathcal{F}_{3,n}$ to 1, but there are also other mesh friezes $M_{3,n}$, see Remark 3.31. If a mesh frieze arises from specialising a cluster-tilting object to 1, it is called **unitary**. Otherwise, the mesh frieze is non-unitary.

Remark 3.31. The number of mesh friezes for $\mathcal{F}_{3,n}$ are not known for $n=8$:

n	6	7	8
unitary	50	833	25080
non-unitary	1	35	1872 ?
all	51	868	26952 ?

The non-unitary SL_3 frieze of width 2 arises from specialising the non projective-injective summands of a cluster-tilting object to 2's. Cuntz-Plamondon prove in an appendix to [BFG$^+$18] that the number of non-unitary SL_3-friezes of width 3 is 35. These arise from specialising the non projective-injective summands of a cluster-tilting object to four 2's and two 1's. See Example 3.33.

Remark 3.32. We can use Iyama-Yoshino reduction to show that all known non-unitary mesh friezes arise from non-unitary friezes of type D_4 (i.e. from the non-unitary mesh frieze for $\mathcal{F}_{3,6}$ or from type D_6 (these are in [FP16]) or from specialising the non projective-injective summands of a cluster-tilting object of $\mathcal{F}_{3,8}$ to 3's. See Example 3.33.

Example 3.33. The mesh frieze for $\mathcal{F}_{3,6}$ arises from (two ways of) specialising a cluster-tilting object to 2's. The quiver of the endomorphism algebras (of the non projective-injective summands) of these two ways are belows. For $\mathcal{F}_{3,8}$ we obtain 4 non-unitary mesh friezes by specialising (the non projective-injective summands of) a cluster-tilting object to 3's. The quiver of the endomorphism algebra is on the right. In these mesh friezes, **all** entries are >1.

$$
\begin{tikzcd}
2 \arrow[r] & 2 \arrow[d] \\
2 \arrow[u] & 2 \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[r] & 2 \arrow[dl] \\
2 \arrow[u] \arrow[r] & 2 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
3 \arrow[d] & 3 \arrow[l] \arrow[r] & 3 \arrow[d] & 3 \arrow[l] \\
3 \arrow[r] & 3 \arrow[u] & 3 \arrow[l] \arrow[r] & 3 \arrow[u]
\end{tikzcd}
$$

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