SPLIT APPROXIMATIONS IN ATMOSPHERIC GENERAL CIRCULATION MODELS

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$$\frac{\partial \psi}{\partial t} = -U\frac{\partial \psi}{\partial x} - V\frac{\partial \psi}{\partial y}$$

$$\psi^{n+1} = \psi^n - \Delta t\frac{U}{2}\delta_{2x}\left(\psi^{n+1} + \psi^n\right) - \Delta t\frac{V}{2}\delta_{2y}\left(\psi^{n+1} + \psi^n\right)$$

$$\psi^* = \psi^n - \Delta t\frac{U}{2}\delta_{2x}\left(\psi^* + \psi^n\right)$$

$$\psi^{n+1} = \psi^* - \Delta t\frac{V}{2}\delta_{2y}\left(\psi^{n+1} + \psi^*\right)$$

$$\delta_{2x}\left(\psi\right) = \frac{\psi(x+\Delta x) - \psi(x-\Delta x)}{2} \quad , \quad \delta_{2y}\left(\psi\right) = \frac{\psi(y+\Delta y) - \psi(y-\Delta y)}{2}$$

STRANG SPLITTING

$$\psi^* = \psi^n - \frac{\Delta t}{2}\frac{U}{2}\delta_{2x}\left(\psi^* + \psi^n\right)$$

$$\psi^{**} = \psi^* - \Delta t\frac{V}{2}\delta_{2y}\left(\psi^{**} + \psi^*\right)$$

$$\psi^{n+1} = \psi^{**} - \frac{\Delta t}{2}\frac{U}{2}\delta_{2x}\left(\psi^{n+1} + \psi^{**}\right)$$

Strang, 1968, SIAM Journal on Numerical Analysis, 5, 506-517

$$\frac{\partial T}{\partial t} = -\mathbf{V} \cdot \nabla T - \dot{\eta}\frac{\partial T}{\partial \eta} + \kappa T \frac{\omega}{p} + F_{T_H} + Q(T, q)$$

$$\frac{\partial q}{\partial t} = -\mathbf{V} \cdot \nabla q - \dot{\eta}\,\frac{\partial q}{\partial \eta} + S(T, q)$$

Q and S consist of:

- **Cloud**
- **Radiation**
- **Surface Fluxes**
- **PBL**
- **Convection**
- **Large-scale condensation**

$$\frac{\partial \psi}{\partial t} = D(\psi) + P(\psi)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{n+1}, \psi^n) + \Delta t P(\psi^{n+1}, \psi^n)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{n+1}, \psi^n) + \Delta t P(\psi^{n+1}, \psi^n)$$

PROCESS SPLIT

$$\psi^* = \psi^n + \Delta t P(\psi^*, \psi^n)$$

$$\psi^{**} = \psi^n + \Delta t D(\psi^{**}, \psi^n)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{**}, \psi^n) + \Delta t P(\psi^*, \psi^n)$$

TIME SPLIT

$$\psi^* = \psi^n + \Delta t P(\psi^*, \psi^n)$$

$$\psi^{n+1} = \psi^* + \Delta t D(\psi^{n+1}, \psi^*)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{n+1}, \psi^n) + \Delta t P(\psi^{n+1}, \psi^n)$$

PROCESS SPLIT

$$\psi^* = \psi^n + \Delta t P(\psi^*, \psi^n)$$

$$\psi^{**} = \psi^n + \Delta t D(\psi^{**}, \psi^n)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{**}, \psi^n) + \Delta t P(\psi^*, \psi^n)$$

CAM3

$$\psi^* = \psi^n + \Delta t P(\psi^*, \psi^n)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{n+1}, \psi^n) + \Delta t P(\psi^*, \psi^n)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{n+1}, \psi^n) + \Delta t P(\psi^{n+1}, \psi^n)$$

TIME SPLIT

$$\psi^* = \psi^n + \Delta t P(\psi^*, \psi^n)$$

$$\psi^{n+1} = \psi^* + \Delta t D(\psi^{n+1}, \psi^*)$$

SPORTISSE SPLITTING

$$\psi^* = \psi^n + \Delta t D(\psi^*, \psi^n)$$

$$\psi^{n+1} = \psi^* + \Delta t P(\psi^{n+1}, \psi^*)$$

Sportisse, 2000: J. Comput. Phys., 161, 140-168

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{n+1}, \psi^n) + \Delta t P(\psi^{n+1}, \psi^n)$$

PROCESS SPLIT

$$\psi^* = \psi^n + \Delta t P(\psi^*, \psi^n)$$

$$\psi^{n+1} = \psi^n + \Delta t D(\psi^{n+1}, \psi^n) + \Delta t P(\psi^*, \psi^n)$$

TIME SPLIT

$$\psi^* = \psi^n + \Delta t D(\psi^*, \psi^n)$$

$$\psi^{n+1} = \psi^* + \Delta t P(\psi^{n+1}, \psi^*)$$

Comparison of Time- and Process-Split coupling of dynamical core and parameterizaton suite in CCM3

Differences between simulations relatively small
- Effect of different time truncation errors have less effect than other arbitrary aspects of model design

Does not imply time truncation errors are insignificant

There are regions where differences are statistically significant
- Largest difference Antarctica
  - Summer – different sign in sensible heat flux
  - Winter – grid-scale structure in clouds
    - dynamics cannot respond

Williamson, 2002, Mon. Wea. Rev., 130, 2024-2041

Numerical method and time step size for parameterization suite?

Numerical schemes to solve the parameterization component have received relatively little attention

Parameterizations often thought to be too inaccurate to justify sophisticated and expensive numerical methods

Conservation and stability are dominant concerns

Truncation errors often concealed by problems in parameterizations

Beljaars, 1991, Proc. ECMWF Seminar on Numerical Methods in Atm. Models, Vol.2, 1-42

Time step size for parameterization suite?

Parameterization suite more expensive than dynamical core
Short time steps avoided

CCM0 through CAM3 –
same as used by semi-implicit Eulerian spectral transform core

But longer semi-Lagrangian time steps a problem

Caya et al.: serious errors with long semi-Lagrangian time steps
Murthy and Nanjundiah: variants to avoid certain splitting errors
Dubal et al: erroneous solutions with long semi-Lagrangian time steps

Caya et al., 1998, Mon. Wea. Rev., 126, 1707-2007
Murthy and Nanjundiah, 2000, Mon. Wea Rev., 128, 3921-3926
Dubal et al., 2004, Mon. Wea. Rev., 132, 989-1002

Studies of coupling parameterizations to dynamical core with simplified canonical model problems

Staniforth et al., 2002a, Mon. Wea. Rev., 130, 3129-3135
Staniforth et al., 2002b, Quart. J. Roy. Meteor. Soc., 128, 2779-2800
Dubal et al., 2005, Mon. Wea. Rev., 133, 989-1002
Dubal et al,. 2006, Quart. J. Roy. Meteor. Soc., 132, 27-42

Model problems are considerable simplifications
limit generality of conclusions
provide some insight
But bridge to full models still needs to be made

Time step size for parameterization suite?

CAM4 and CAM5 use explicit Finite Volume dynamical core

Time step smaller than semi-implicit CAM3

Higher resolution CAM3 also requires smaller time steps

Parameterizations become too expensive

and might misbehave

Sub stepping of dynamics allows longer parameterization time step

Process split with sub-stepped dynamics

$$\psi^* = \psi^n + \Delta t P(\psi^*, \psi^n)$$

$$P(\psi^*, \psi^n) = \frac{\psi^* - \psi^n}{\Delta t}$$

$$\psi^{n+1/m} = \psi^n + \frac{\Delta t}{m} D(\psi^{n+1/m}, \psi^n) + \frac{\Delta t}{m} P(\psi^*, \psi^n)$$

$$\psi^{n+2/m} = \psi^{n+1/m} + \frac{\Delta t}{m} D(\psi^{n+2/m}, \psi^{n+1/m}) + \frac{\Delta t}{m} P(\psi^*, \psi^n)$$

$$\vdots$$

$$\psi^{n+1} = \psi^{n+(m-1)/m} + \frac{\Delta t}{m} D(\psi^{n+1}, \psi^{n+(m-1)/m}) + \frac{\Delta t}{m} P(\psi^*, \psi^n)$$

Time split with sub-stepped dynamics

$$\psi^{*(1/m)} = \psi^n + \frac{\Delta t}{m} D(\psi^{*(1/m)}, \psi^n)$$

$$\psi^{*(2/m)} = \psi^{*(1/m)} + \frac{\Delta t}{m} D(\psi^{*(2/m)}, \psi^{*(1/m)})$$

$$\vdots$$

$$\psi^{*} = \psi^{*(m-1)/m} + \frac{\Delta t}{m} D(\psi^{*}, \psi^{*(m-1)/m})$$

$$\psi^{n+1} = \psi^{*} + \Delta t P(\psi^{n+1}, \psi^{*})$$

Dynamics - Parameterization coupling in CAM3 through CAM5

Process split for

semi-implicit Eulerian spectral transform dynamical core

Time split for

explicit Finite Volume dynamical core

Parameterization time step similar to that used for earlier versions

Coupling within the parameterization suite

- (P) PARAMETERIZATION SUITE
  - (M) Moist processes
    - Deep convection
    - Shallow convection
    - Grid-scale precipitation
  - (R) Radiation
    - Clouds
    - Radiation
  - (S) Surface exchange
  - (T) Turbulent mixing (PBL)

$$\mathbf{P} = \{\mathbf{M}, \mathbf{R}, \mathbf{S}, \mathbf{T}\}$$

PROCESS SPLIT

$$\psi^{n+1} = \psi^n + \Delta t \mathbf{P}\left(\psi^n\right)$$

$$\mathbf{P}\left(\psi^n\right) = \mathbf{T}\left(\psi^n\right) + \mathbf{S}\left(\psi^n\right) + \mathbf{R}\left(\psi^n\right) + \mathbf{M}\left(\psi^n\right)$$

TIME SPLIT

$$\psi^{n+1} = \mathcal{P}\left(\psi^n\right)$$

$$\mathcal{P}\left(\psi^n\right) = \mathcal{T}\left(\mathcal{S}\left(\mathcal{R}\left(\mathcal{M}\left(\psi^n\right)\right)\right)\right)$$

Total parameterization time evolution is much slower than that due to a single process

Large single process tendencies are compensated by other processes

Implicit parameterization can bring the profile into equilibrium

Without taking into account other processes can give incorrect equilibrium

With large time step must balance processes within the time step

Beljaars, 1991, Proc. ECMWF Seminar on Numerical Methods in Atm. Models, Vol.2, 1-42

Order by time scale

slow processes first (possibly explicit)

followed by fast processes (probably implicit)

acting on field incremented by slow processes

Iteration of fast processes might be necessary

to ensure equilibrium achieved

Time splitting desirable

Beljaars, 1991, Proc. ECMWF Seminar on Numerical Methods in Atm. Models, Vol.2, 1-42

Implicit methods well-suited

- for coupling between different processes
- but not practical

Predictor-corrector scheme yields

- some advantages of fully-implicit scheme

Significant impact on large-scale performance in ECMWF model

- comparisons at equal cost suggest competitive

But different formulation might be required for best performance

- parameterizations need to vary smoothly with input data

Cullen and Salmond, 2003, Quart. J. Roy. Metero. Soc., 129, 1217-1236

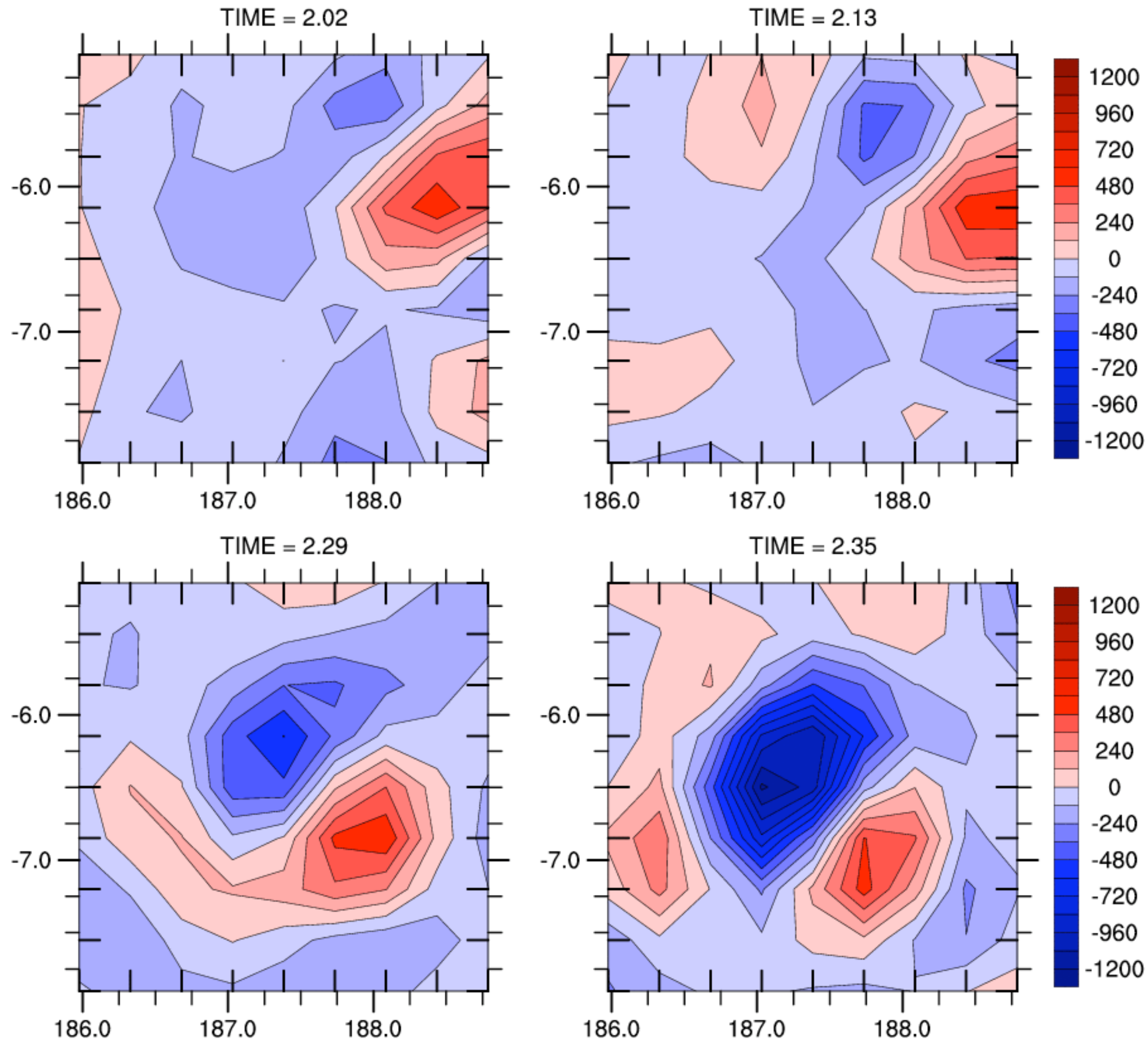
Time step should not be too large

Parameterization suite should converge
to a reasonable partition of processes
as time step goes to zero

Model should produce an atmospheric-like state
at end of time step

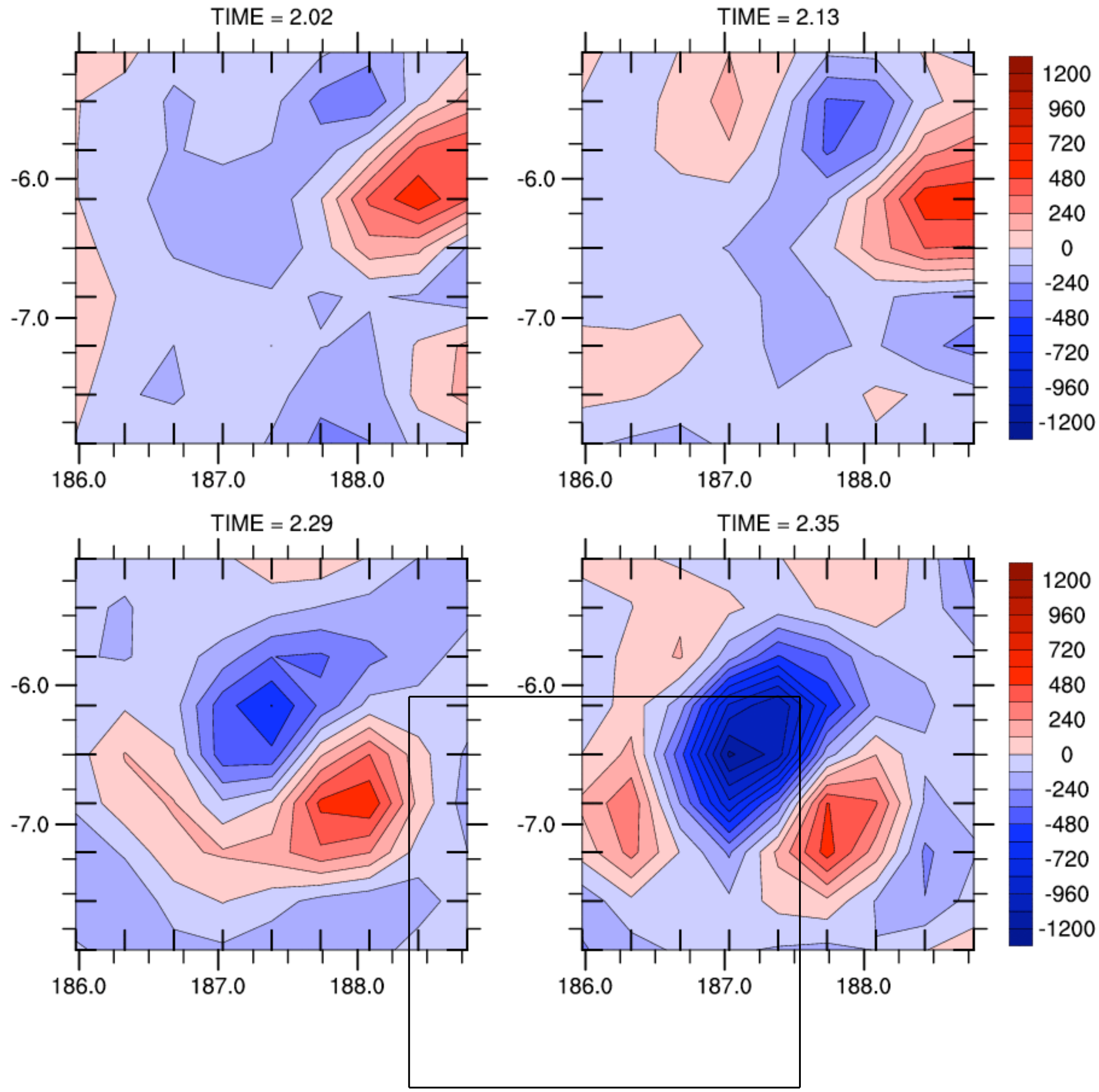
Following is an example of problem
when time step too small

OMEGA AT 600 mb (mb / day) EARLY PERIOD

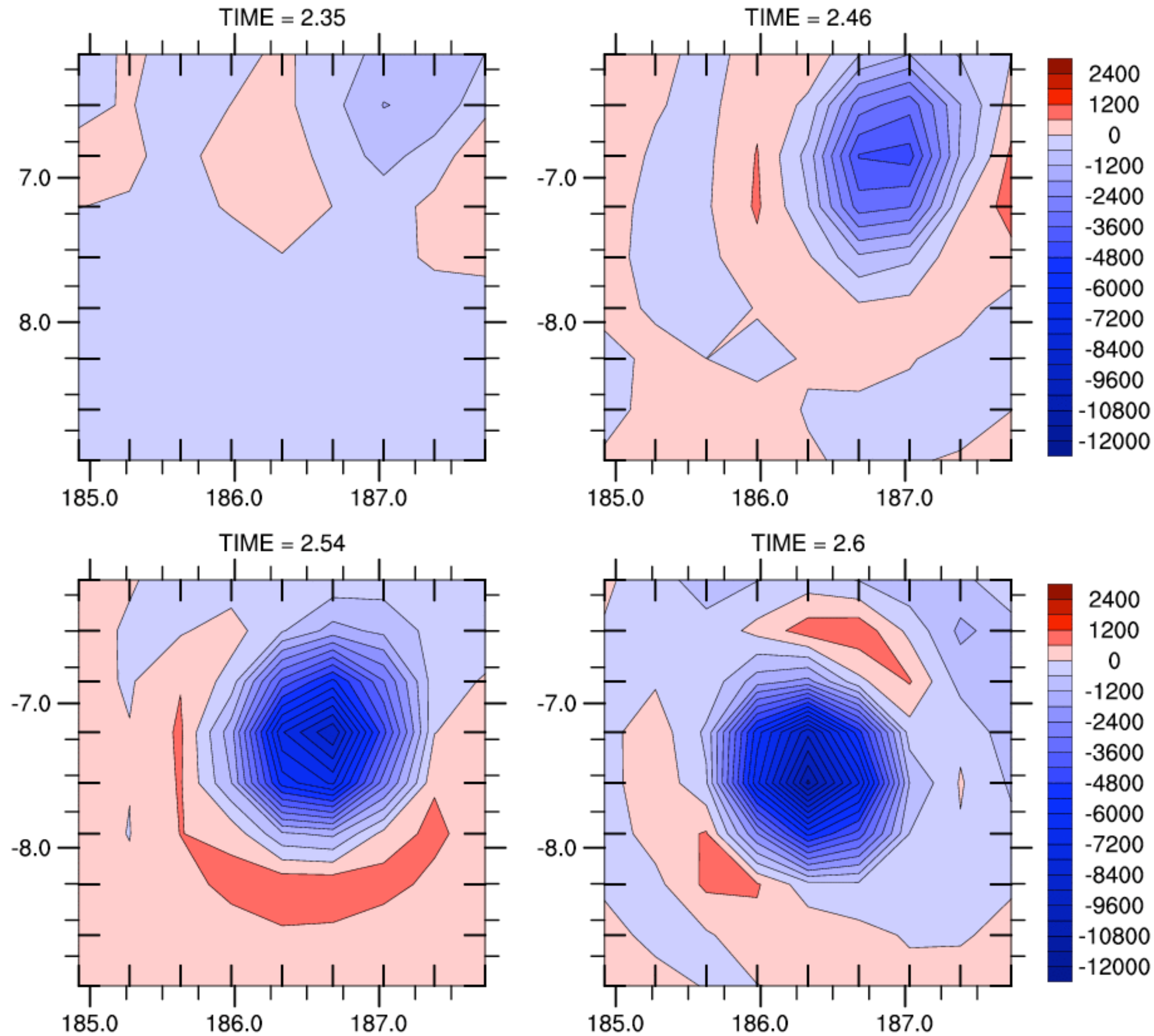


Williamson, 2012, Quart. J. Roy. Meteror. Soc., in press

OMEGA AT 600 mb (mb / day) EARLY PERIOD



OMEGA AT 600 mb (mb / day) LATE PERIOD



$$\frac{dq}{dt} = \mathrm{D} + \mathrm{P}$$

$$\mathrm{D}: \quad \frac{dq}{dt} = \alpha$$

$$q^{t+\Delta t} = q^t + \alpha \Delta t$$

$$\mathrm{P}: \quad \frac{d(q-q_s)}{dt} = \begin{cases} -(q-q_s)/\tau & \text{if } q > q_s \\ 0 & \text{if } q \leq q_s \end{cases}$$

$$\left(q^{t+\Delta t} - q_s\right) = \left(q^t - q_s\right) e^{-\Delta t/\tau}$$

Let $t = n\Delta t$

$$q^* = q^{n\Delta t} + \alpha \Delta t$$

$$\left(q^{(n+1)\Delta t} - q_s\right) = \left(q^* - q_s\right) e^{-\Delta t/\tau}$$

$$\left(q^{(n+1)\Delta t} - q_s\right) = \left[\left(q^{n\Delta t} - q_s\right) + \alpha \Delta t\right] e^{-\Delta t/\tau}$$

ASSUME $\quad q^0 = q_s, \quad$ THEN

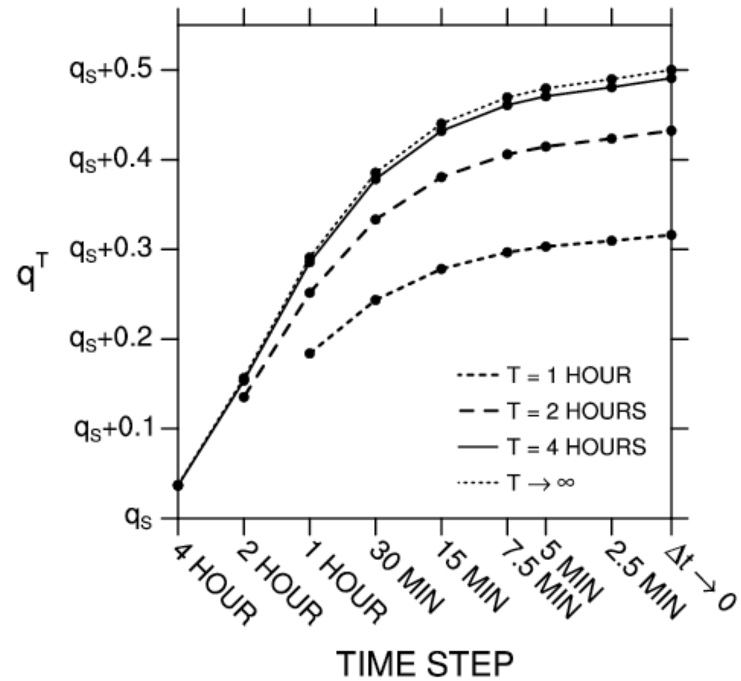
$$\left(q^{n\Delta t} - q_s\right) = \alpha \Delta t \left[\frac{e^{-(n+1)\Delta t/\tau} - e^{-\Delta t/\tau}}{e^{-\Delta t/\tau} - 1}\right]$$

FOR FIXED TIME $\quad T = n\Delta t \quad$ AS $\quad \Delta t \to 0$

$$\left(q^{n\Delta t} - q_s\right) \to \tau\alpha \left(1 - e^{-T/\tau}\right)$$

$$\tau = 1 \text{ hour}$$

$$\alpha = \frac{1}{2} \text{ hour}^{-1}$$



$$\frac{dq}{dt} = \mathrm{D} + \mathrm{P} + \mathrm{Q}$$

$$\mathrm{D}: \quad q^{t+\Delta t} = q^t + \alpha \Delta t$$

$$\mathrm{P}: \quad \left(q^{t+\Delta t} - q_s\right) = \left(q^t - q_s\right) e^{-\Delta t/\tau}$$

$$\mathrm{Q}: \quad q^{t+\Delta t} = \begin{cases} q_s & \text{if } q^t > q_s \\ q^t & \text{if } q^t \le q_s \end{cases}$$

Let $t = n\Delta t$

$$q^* = q^{n\Delta t} + \alpha \Delta t$$

$$(q^{**} - q_s) = (q^* - q_s)\, e^{-\Delta t/\tau}$$

$$q^{(n+1)\Delta t} = q_s$$

$$\mathrm{D}: \quad q^* - q_s = \alpha \Delta t$$

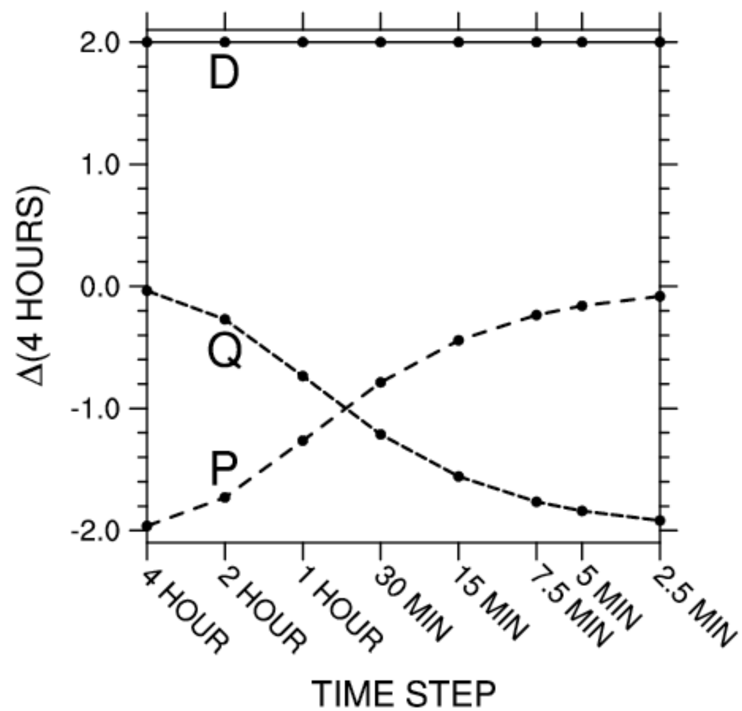
$$\mathrm{P}: \quad q^{**} - q^* = \alpha \Delta t \left(e^{-\Delta t/\tau} - 1\right)$$

$$\mathrm{Q}: \quad q^{(n+1)\Delta t} - q^{**} = -\alpha \Delta t \; e^{-\Delta t/\tau}$$

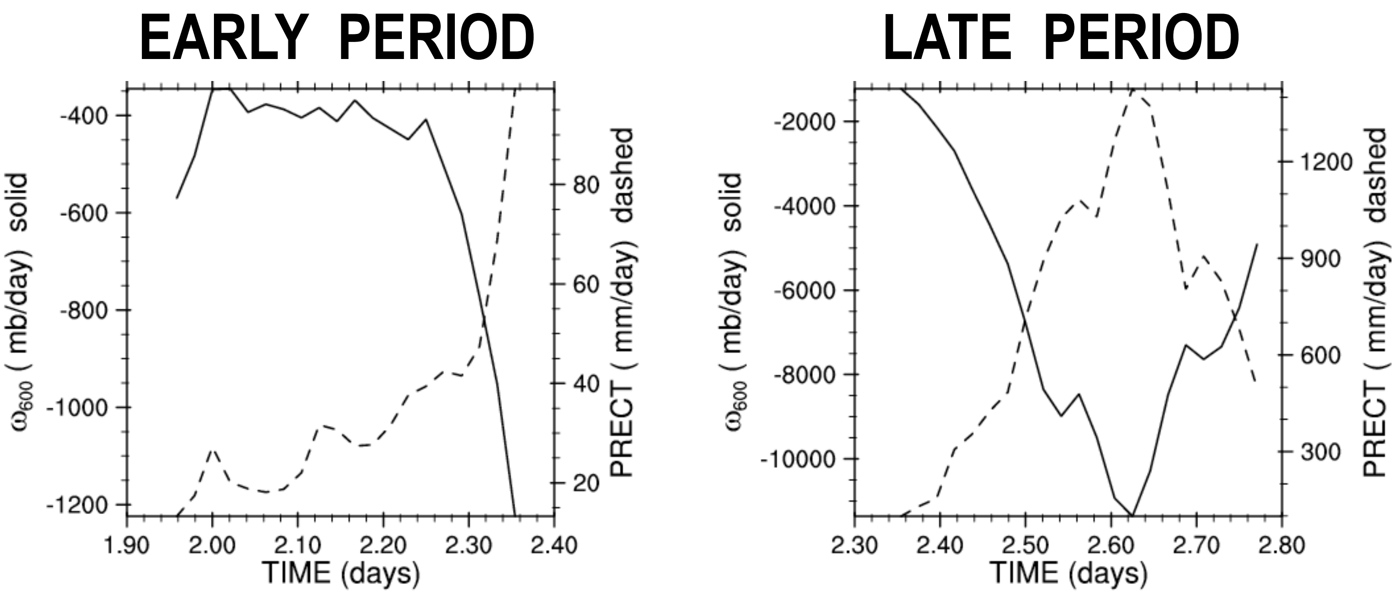
$$T = 4 \text{ hours}$$

$$\tau = 1 \text{ hour}$$

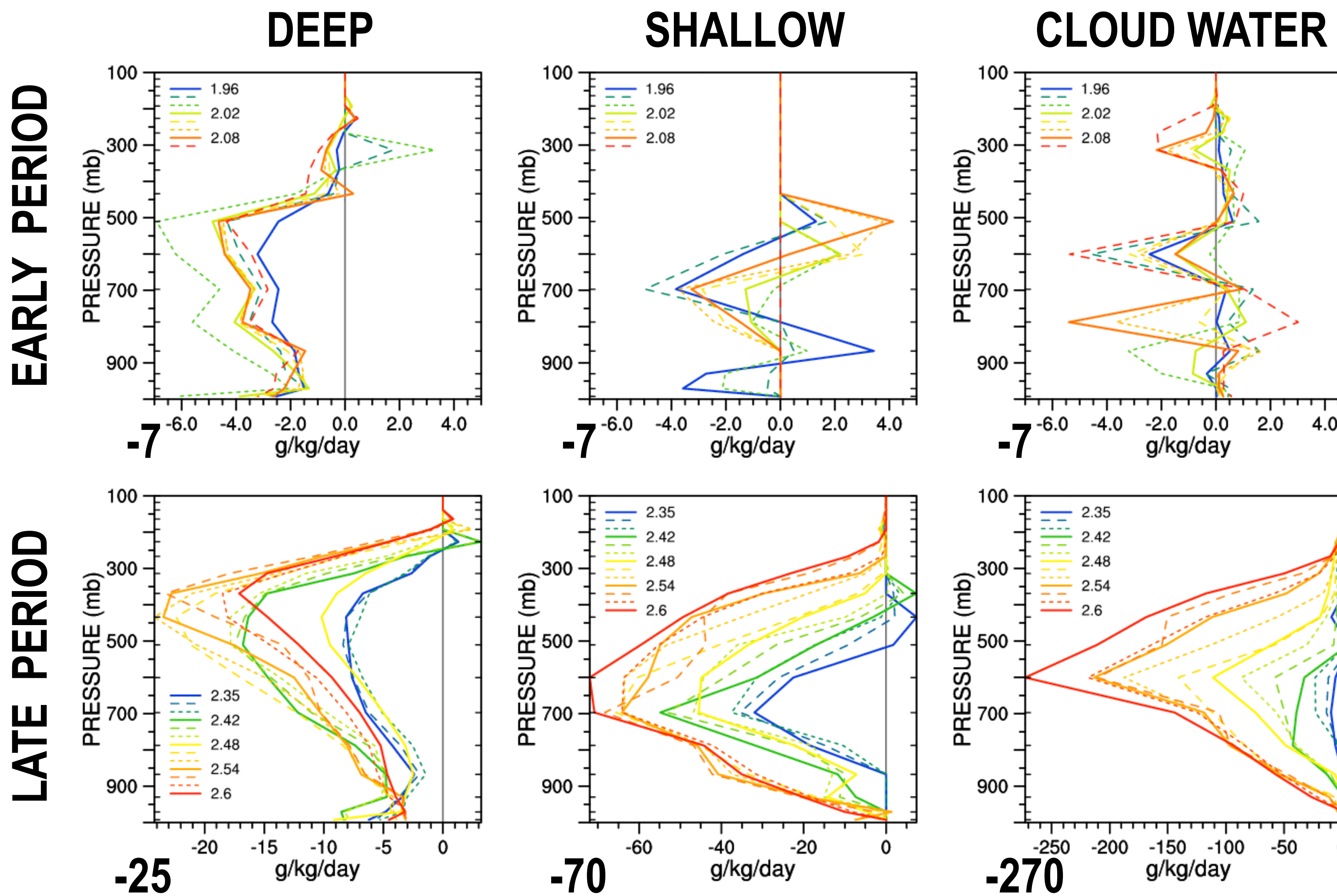
$$\alpha = \frac{1}{2} \text{ hour}^{-1}$$



OMEGA AT 600 mb (mb / day) --- SOLID LINE
PRECIPITATION (mm / day) --- DASHED LINE



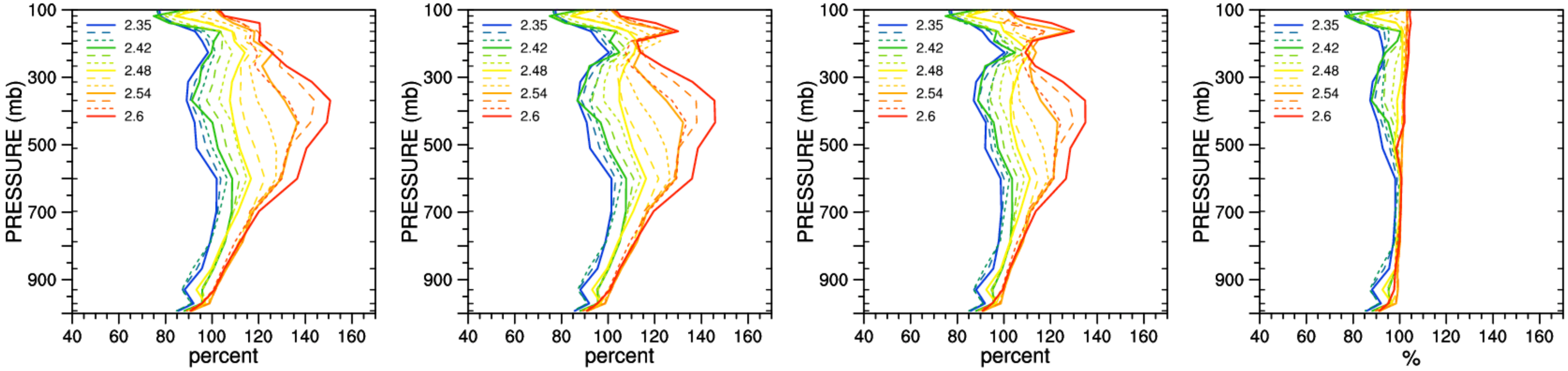
SPECIFIC HUMIDITY PARAMETERIZATION TENDENCIES



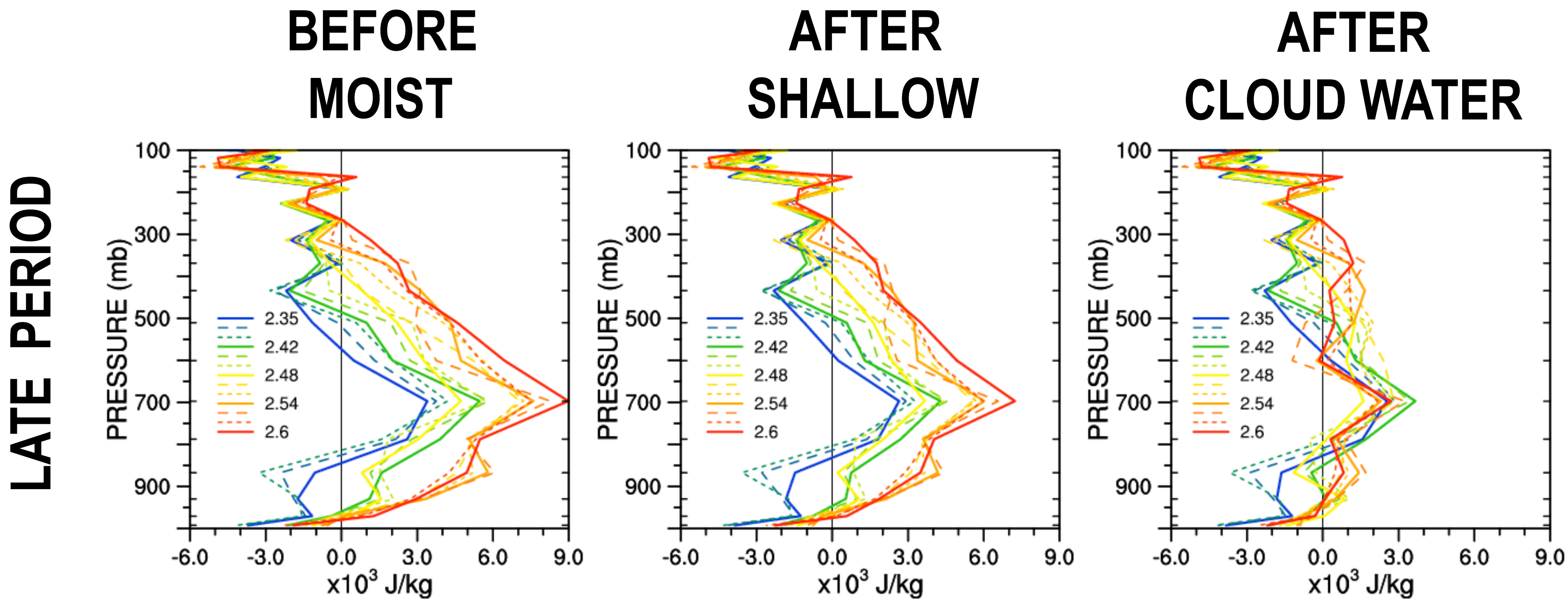
RELATIVE HUMIDITY

BEFORE MOIST | AFTER DEEP | AFTER SHALLOW | AFTER CLOUD WATER

LATE PERIOD



$$h_{k+1} - h_k^*$$

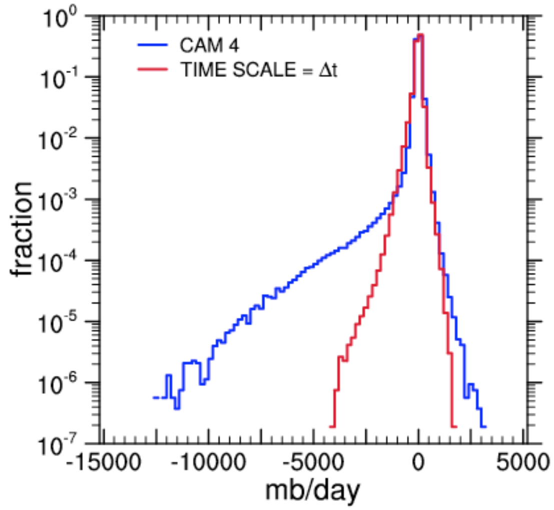


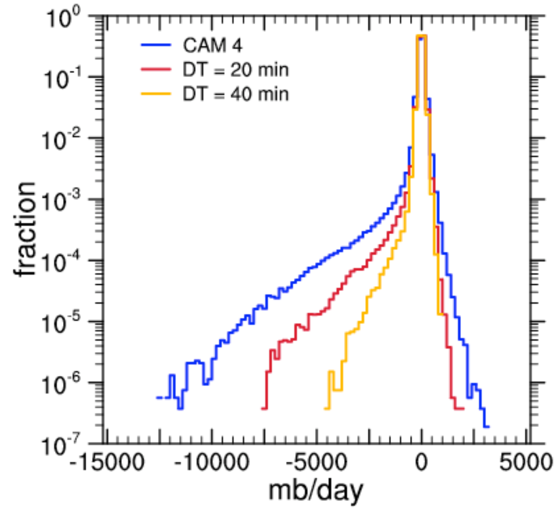
$$\text{UNSTABLE WHEN}: \quad h_{k+1} + pert > h_k^*$$

$$\text{MOIST STATIC ENERGY}: \quad h = C_pT + gz + Lq$$

$$\text{SATURATED MOIST STATIC ENERGY}: \quad h^* = C_pT + gz + Lq^*$$

OMEGA AT 600 mb (mb / day)





Problem arises because some individual parameterizations
  do not produce atmospheric-like state
    because constrained by assumed time-scale
Other unconstrained parameterizations
  work in unintended ways

As time step goes to zero
  convection parameterizations become less active
  large scale condensation takes over

When time scales are shortened or
  time step is lengthened
    strong storms do not form

Partition of the total tendency into individual process tendencies should not depend on the time step

In the limit of small time steps there should be a reasonable distribution between parameterized processes

Parameterizations should complete their processes in the time step e.g. remove any instability introduced in that time step

On what spatial scales should parameterizations be calculated?

Historically calculated on the dynamical core grid.

Parameterizations should be coupled to dynamics
    by applying them to scales that are larger
        than smallest scales resolved by dynamics

Parameterized processes should be calculated on the scale
    that the model can handle properly

Smallest scales not calculated accurately by dynamics
    They should not be forced directly nor used directly
    They should be left to deal with the enstrophy cascade
        and effects of truncation

Lander and Hoskins, 1997, Mon. Wea. Rev., 125, 292-303

Map to coarser grids for parameterization calculation

Map forcing back to dynamics grid

Laprise (1992) and Lander and Hoskins define
well resolved for spectral models
Pielke (1991) does so for grid point models

Lander and Hoskins, 1997, Mon. Wea. Rev., 125, 292-303
Laprise, 1992, Bull. Amer. Meteor. Soc., 73, 1453-1454
Pilke ,1991, Bull. Amer. Meteor. Soc., 72, 1941

Problem – model blew up after five or six months

Caused by nonlinear surface exchange
- **Surface stress should damp**
- **Surface stress amplified small-scale local structures**
  - **not seen by coarse resolution stress**

Stabilized by applying linear stress to all scales in dynamics
- **and removing it from larger scales in parameterizations**

Nonlinear stress on parameterization resolution
Linear stress on additional scales in dynamics

Williamson, 1999, Tellus, 51A, 663-673

DYNAMICAL CORE AND PARAMETERIZATION SUITE ON DIFFERENT VERTICAL GRIDS

$$\frac{\partial \psi}{\partial t} = D(\psi) + P(\psi)$$

$$\psi_P^* = \psi_P^n + \Delta t P_P(\psi_P^*, \psi_P^n)$$

$$\psi_D^{n+1} = \psi_D^n + \Delta t D_D(\psi_D^{n+1}, \psi_D^n) + \Delta t I_{P\to D}\left[P_P(\psi_P^*, \psi_P^n)\right]$$

$$\psi_P^{**} = \psi_P^* + \Delta t I_{D\to P}\left[D_D(\psi_D^{n+1}, \psi_D^n)\right]$$

$$\psi_P^{n+1} = \psi_P^{**} + \gamma\left[I_{D\to P}\left(\psi_D^{n+1}\right) - \psi_P^{**}\right]$$

Molod, 2009, Tellus, 61A, 381-393

Need to consider more carefully coupling of parameterizaton suite
into dynamical core

It is time to become less cavalier in dealing with the interactions
within parameterization suite itself

Need to consider the interaction of the components more carefully
- reduce splitting errors within parameterization suite
- reformulate parameterizations with this in mind