A High Order Characteristic Discontinuous Galerkin Scheme for the Advection of Tracers in MPAS-Ocean

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Context

Modern global climate models involve the advection of dozens of tracers within the contexts of atmospheric, oceanic, sea ice and land ice flows. This can become an extremely computationally intensive process. The purpose of the current work is to develop an advection scheme with relaxed time stepping constraints that scales sub-linearly with an increasing number of tracers. This characteristic discontinuous Galerkin (CDG) scheme is based on the advection of both element vertices and test functions along velocity characteristics in order to determine the "pre-image" of the edges at previous time levels so as to calculate the material fluxes. These pre-images are the same for all tracers, and hence much of the computation may be re-used for the calculation of fluxes for additional tracers. The scheme may be run to high order via the exact integration of the basis functions of a given order and is suitable for implementation on unstructured meshes. The CDG scheme is being developed for use in the MPAS-Ocean model [3, 4].

Formulation

The conservative advection of a thickness weighted tracer field $q(\vec{x},t)$ may be described as

$$\frac{\partial hq}{\partial t} + \nabla\cdot(\vec{u}hq) = 0 \tag{1}$$

where $h(\vec{x},t)$ is the thickness of a fluid layer and $\vec{u}(\vec{x},t)$ is the velocity field. Note that the mass conservation equation is given by an advection equation for which $q = 1$ everywhere. We begin by multiplying hq by a set of i piecewise smooth modal test functions for each element k

$$\sum_i \phi_{k,i}(\vec{x},t) = 1 + \frac{1}{\Delta x}(x-\bar{x}) + \frac{1}{\Delta y}(y-\bar{y}) + \frac{1}{2\Delta x^2}(x^2-\bar{x}^2) + \frac{1}{\Delta x\Delta y}(x-\bar{x})(y-\bar{y}) + \frac{1}{2\Delta y^2}(y^2-\bar{y}^2) + \dots \tag{2}$$

where the overbars denote averages which are removed such that the higher order terms remain massless. This allows for the use of a slope limiter to preserve monotonicity without loss of conservation. Expanding using the chain rule we have

$$\frac{\partial\phi_{k,i}hq}{\partial t} + \nabla\cdot(\phi_{k,i}\vec{u}hq) = hq\Big(\frac{\partial\phi_{k,i}}{\partial t} + \vec{u}\cdot\nabla\phi_{k,i}\Big) = hq\frac{D\phi_{k,i}}{Dt} \tag{3}$$

where D/Dt is the material derivative. Integrating over the element volume Ω_k and between time levels n and $n+1$ and applying Gauss' theorem we have

$$\int_{\Omega_k}(\phi_{k,i}hq)^{n+1} - (\phi_{k,i}hq)^n\mathrm{d}\Omega_k + \int_{t^n}^{t^{n+1}}\int_{\partial\Omega_k}\phi_{k,i}\vec{u}hq\cdot\mathrm{d}\vec{s}\mathrm{d}t = \int_{t^n}^{t^{n+1}}\int_{\Omega_k}hq\frac{D\phi_{k,i}}{Dt}\mathrm{d}\Omega_k\mathrm{d}t \tag{4}$$

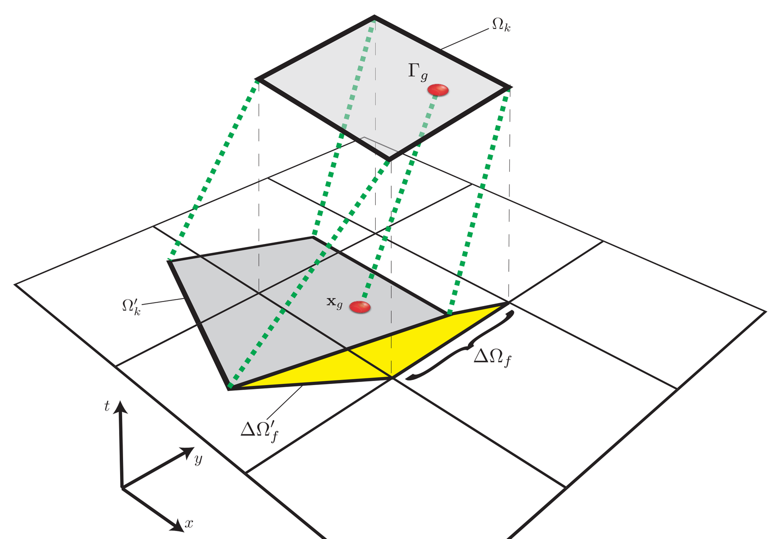
where the superscripts denote the time levels and $\partial\Omega_k$ is the boundary of element k. The right hand side of the above expression may be taken as zero if the test functions are themselves held to be constant along flow characteristics such that

$$\frac{D\phi_{k,i}}{Dt} = 0. \tag{5}$$

The tracer field may be approximated by a series of trial functions as

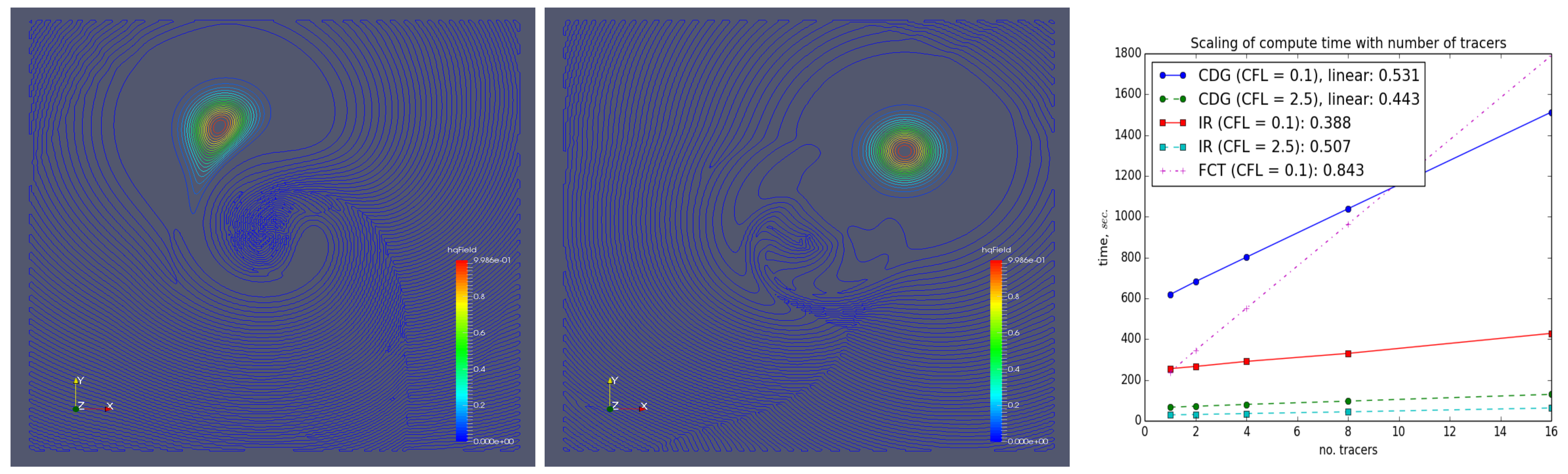
$$hq(\vec{x},t) = c^n_{k,j}\phi_j(\vec{x},t) \tag{6}$$

where $c^n_{k,j}$ is the coefficient for the j^{th} trial function in the k^{th} element at the n^{th} time level, and double indexing implies summation.

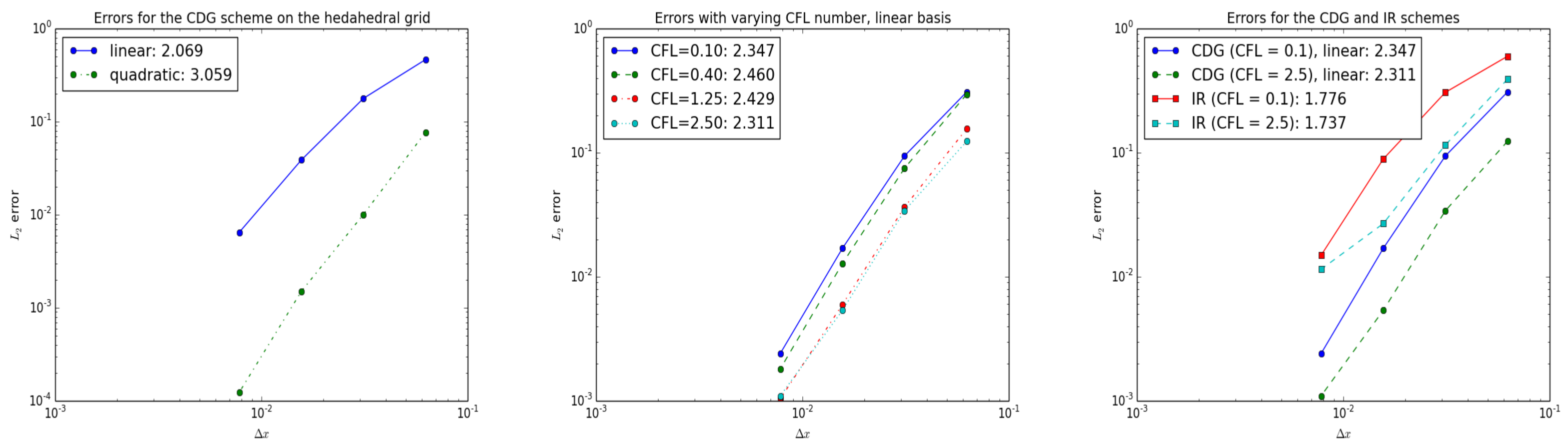


Schematic of the CDG algorithm; the vertices of each edge $\Delta\Omega_f$ are traced back in time and the intersection of the area swept by the edge and its pre-image and the Eulerian mesh is integrated to determine the flux across the edge. Quadrature points are traced forwards in time in order to preserve the value of the test functions along flow characteristics.

Results (plane)



Tracer field at $t = T/4$ (left) $t = T/2$ (center) on the hexahedral planar mesh subject to the velocity field $u_\theta = \frac{4\pi r}{T}\left(1-\cos\left(\frac{2\pi t}{T}\right)\frac{1-(4r)^6}{1+(4r)^6}\right)$, $u = u_\theta\sin(\theta)$, $v = -u_\theta\cos(\theta)$, $T = 30,000s$ [2], and scaling of compute time with number of tracers for the linear CDG, Incremental Remap (IR) [1] and Flux Corrected Transport (FCT) schemes with varying CFL numbers (right). The CDG scheme scales more poorly than the IR scheme and is more efficient than the FCT scheme for approximately 10 tracers, however its performance is greatly improved for large CFL numbers for which the FCT scheme is not stable.



Error convergence rates for the CDG scheme with linear and quadratic basis (left), with varying CFL number (center) and in comparison to the IR scheme (right). Both performance and accuracy are improved with increased CFL number, and the linear CDG scheme is approximately as accurate as the IR scheme with twice the horizontal resolution at a given CFL number due to the compact higher order basis within each element.

Formulation (cont.)

Assuming that the test functions follow flow characteristics and expanding equation 4 we have

$$\int_{\Omega_k}(\phi_{k,i}\phi_{k,j})^{n+1}\mathrm{d}\Omega_k c^{n+1}_{k,j} = \int_{\Omega_k}(\phi_{k,i}hq)^n\mathrm{d}\Omega_k - \int_{t^n}^{t^{n+1}}\int_{\partial\Omega_k}\phi_{k,i}\vec{u}hq\cdot\mathrm{d}\vec{s}\mathrm{d}t \tag{7}$$

Equation 7 represents a system of linear equations that may be solved at each time level for the coefficients of hq. Note that as the basis functions may be considered to "arrive" at the Eulerian mesh elements at the new time level, the $\int_{\Omega_k}\phi_{k,i}\phi_{k,j}\mathrm{d}\Omega_k$ matrix inverse may be precomputed at the static quadrature points and stored for each element.

Within the dynamical core, the thickness field, h, must also be prognosed using equation 1 with $q = 1$ everywhere, using either the existing scheme or CDG. The Galerkin basis representation of the tracer field itself, $q(\vec{x},t) = d^n_{k,j}\phi_j(\vec{x})$ may be extracted from h and hq by solving the following linear system in each element

$$\int_{\Omega_k}\phi_i\phi_j h^{n+1}\mathrm{d}\Omega_k d_{k,j} = \int_{\Omega_k}\phi_i hq^{n+1}\mathrm{d}\Omega_k \tag{8}$$

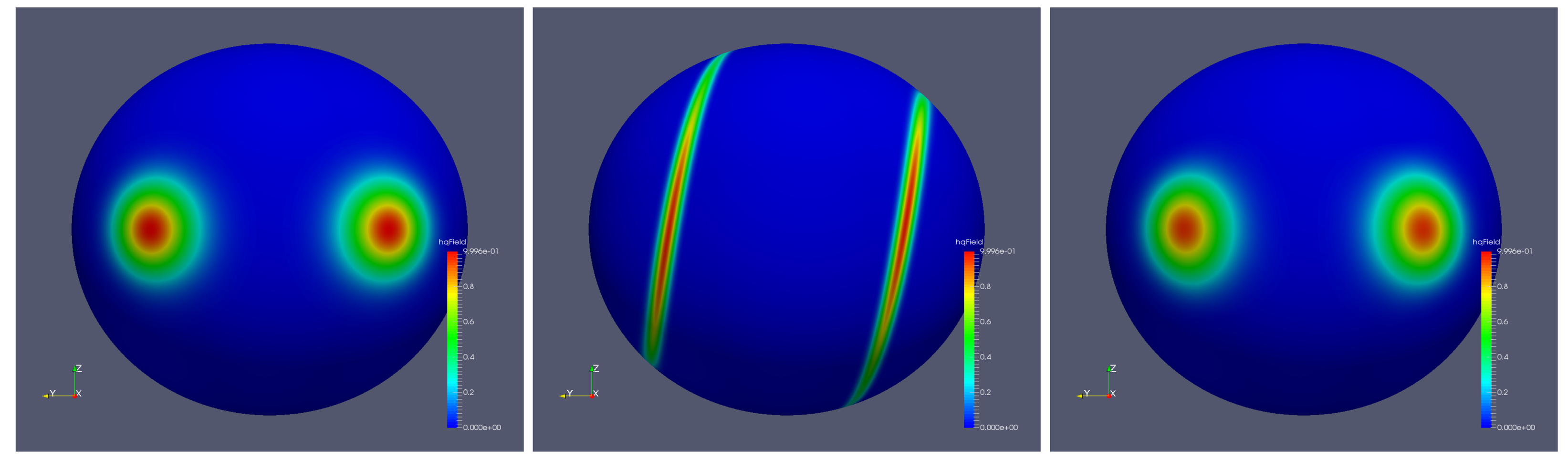
This system must be assembled for each element at each time step, however the left hand side matrix is the same for all tracers at a given time level, so it only need be assembled and inverted once for all tracers.

Note that for a divergence free velocity field the conservative advection of both hq and h implies that q should also be conserved. However in practice numerical divergence due to the discrete approximation of the velocity field means that q as determined from equation 8 will not be exactly conservative.

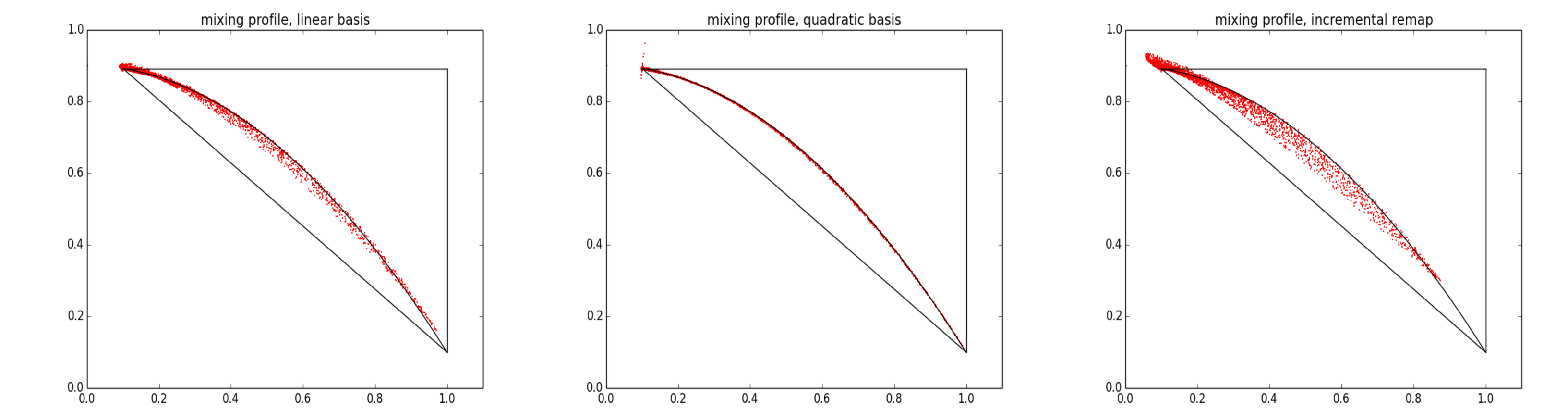
Spherical Implementation

- Swept-region/element intersections are performed on the sphere assuming great circle arcs.
- Trial and test functions are evaluated on the plane by projecting from the sphere onto a plane tangent to the center of each element.
- Vertices and quadrature points are integrated along characteristics on the sphere.

Results (sphere)



Thickness weighted tracer field hq at $t = 0$ days (left), $t = 6$ days (center), and $t = 12$ days (right) on the unit sphere with $\Delta x = 0.00875m$ subject to the time varying velocity field of $u = \frac{10R}{T}\sin^2(\theta')\sin(2\phi)\cos\left(\frac{\pi t}{T}\right) + \frac{2\pi R}{T}\cos(\phi)$, $v = \frac{10R}{T}\sin(2\theta')\cos(\phi)\cos\left(\frac{\pi t}{T}\right)$ [7].



Spurious numerical mixing of two tracers at $t = 6$ days correlated as $hq_2 = -0.8hq_1^2 + 0.9$ [6] for the the CDG scheme with linear (left) and quadratic (center) bases and the IR scheme (right) on the unit sphere with $\Delta x = 0.03503m$. The exact correlation between the tracers is given by the quadratic curve, while diffusive mixing is represented by points that lie between the linear and quadratic lines, anti-diffusion by points above the quadratic curve and overshoots by points outside the bounding box.

Challenges for Implementation within MPAS

The following issues must be addressed before the CDG scheme may be successfully implemented within the MPAS-Ocean framework:

- The MPAS-Ocean model [3, 4] uses an explicit finite volume dynamical core with the various forcing terms expressed as tendencies. This is contrast to the CDG scheme, which uses a weak form representation to update the tracer values via the solution of a linear system. If the CDG scheme is to be successfully implemented within MPAS, then some form of operator splitting approach may be required [8], with the horizontal advection performed using the CDG scheme in weak form, and then the vertical advection and diffusivities applied as tendencies consistent with the standard MPAS implementation. Otherwise the use of a Lagrangian vertical coordinate may be applied to mitigate the splitting error.
- Since the thickness field h is prognosed by the dynamics, this may differ somewhat from the h field determined by the CDG scheme, leading to loss of consistency between the tracer advection and the dynamics. One potential solution to this problem would be to scale the area of the swept region determined by the CDG scheme by the mass flux across the edge derived from the dynamics. The CDG tracer coefficients could then be determined from this new consistent swept region.

References

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