Institut Pierre Simon Laplace
Sciences de l'environnement

École Polytechnique
Université Paris-Saclay

DYNAMICO

Dynamical core on icosahedral mesh

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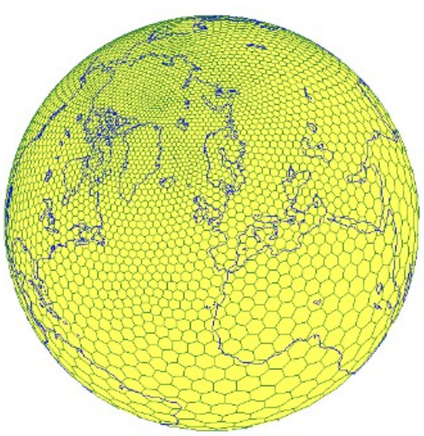
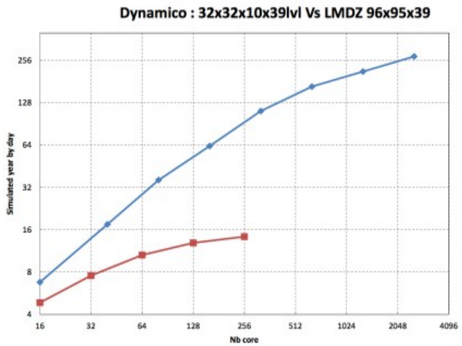
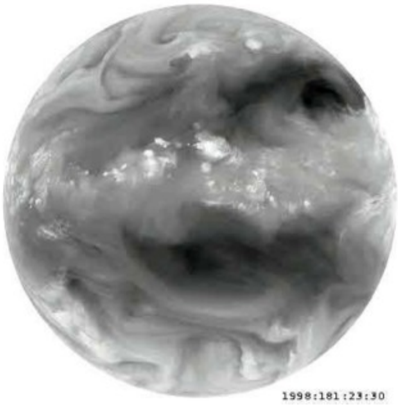
Yann Meurdesoif (LSCE/IPSL), Evaggelos Kritsikis (LAGA/Paris XIII), ...

$$\delta \int \mathcal{L} \mathrm{d}t = 0$$

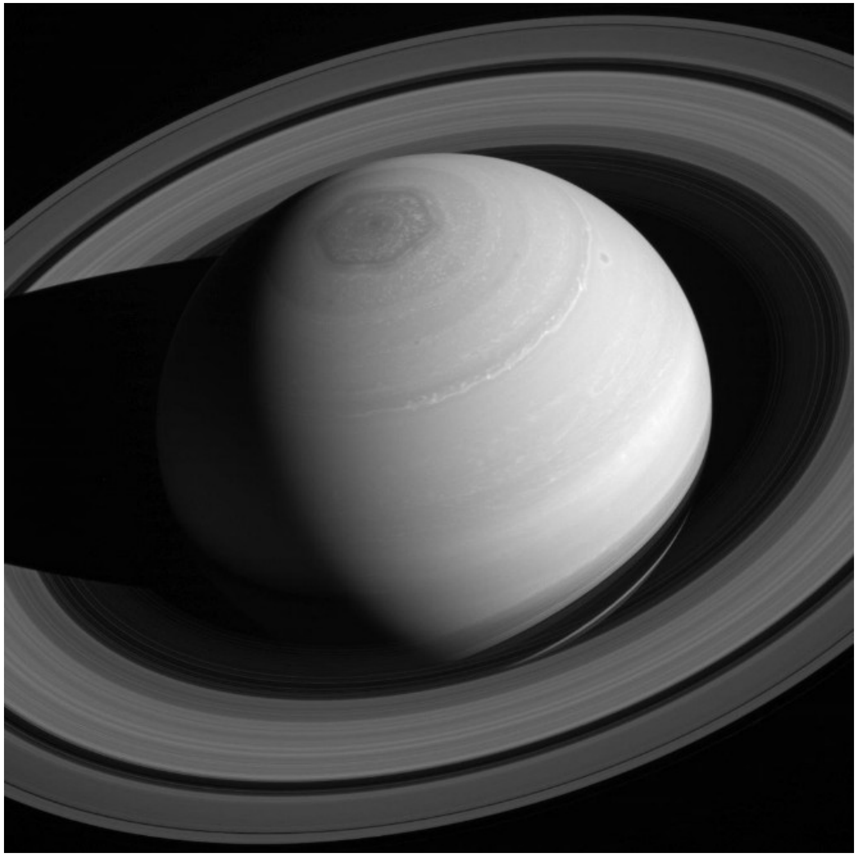






DYNAMICO

Equations of motion	*shallow-water* *shalllow-atmosphere, hydrostatic* *ongoing : deep-atmosphere, fully compressible*
Conservation properties	*Mass (air and species)* *Energy*
Formulation	*Mass : flux-form* *Momentum :* ***Hamiltonian vector-invariant form*** *(a.k.a curl form, Crocco's theorem, Carter-Lichnerowicz equation)*
Vertical coordinate	*Terrain-following mass-based* *(often conflated with pressure-based)*
Numerics	*Mass : finite volume* *Momentum : low-order* ***mimetic finite difference*** *Mesh :* ***icosahedral-hexagonal C-grid, Lorenz*** *Time : (additive) Runge-Kutta (HEVI)*
Computing	*MPI / OpenMP* *Scales at least to O(10^4), including* ***I/O***



Saturn GCM simulations

Grid

- Horizontal resolution: $1/2°$ (+ tests $1/4°$ & $1/8°$)
- Vertical levels: 32 levels from 3 bars to 1 mbar

Boundary conditions

- Initial: steady-state temperature from 1D run, no winds
- Dissipation (SGS): very strong ($\mathcal{D}^+$), strong ($\mathcal{D}^=$), regular ($\mathcal{D}^-$)
- Type 1: 64 levels + sponge layer on uppermost 4 levels
- Type 2: Bottom drag $|\lambda| > 33°$: $\tau = 100$d ($\mathcal{F}^+$), 1000d ($\mathcal{F}^-$) [Liu and Schneider JAS 2010]

Machinery

- MPI+openMP code run on Occigen cluster in CINES
- cores: 1200 ($1/2°$), 9000 ($1/4°$), 11520-30000 ($1/8°$)
- Results are shown after 20000 Saturn days integrations

1 Sat day = 10 h ; 1 Sat year = 30 Earth yr

Physical parameterizations ⇒ 1D computations of forcings on each grid point

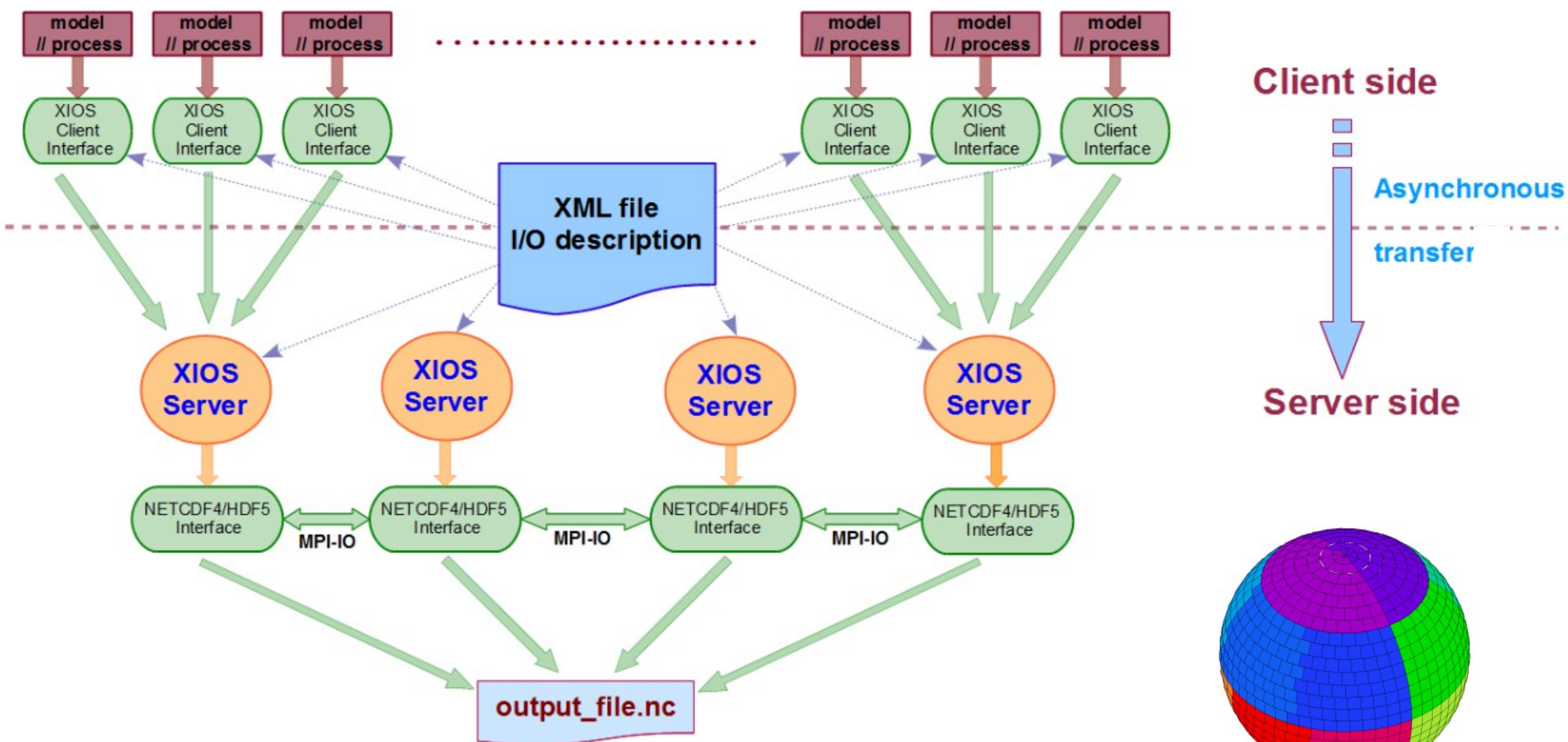
☞ Radiative transfer ⇒ Guerlet et al. Icarus 2014

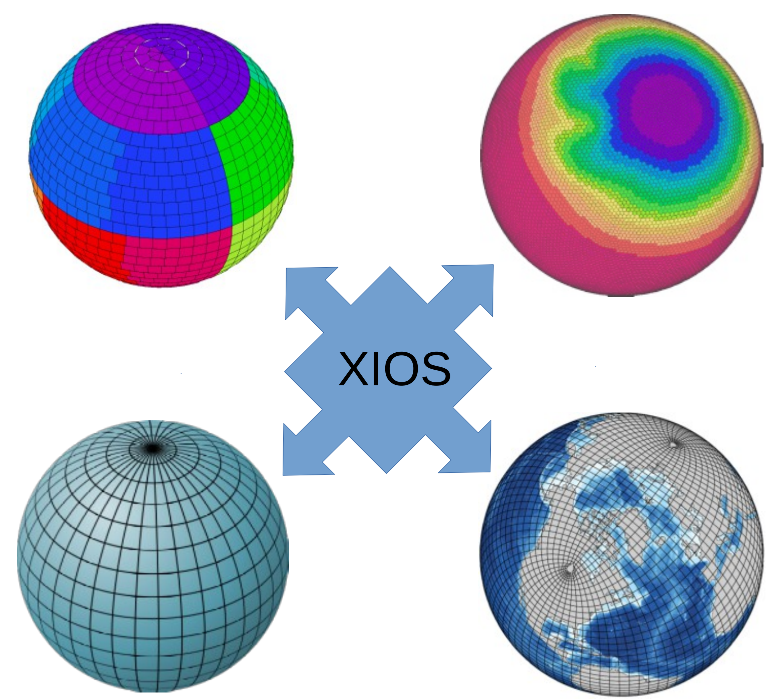
- correlated-k scheme for IR and VIS heating rates [Wordsworth et al. 2010]
- gases CH_4, C_2H_6, C_2H_2 with optimized spectral discretization
- HITRAN 2012 database + Karkoschka and Tomasko 2010 for CH_4 around 1μm
- collision-induced absorption H_2-H_2 and H_2-He [Wordsworth et al. 2012]
- Rayleigh scattering H_2, He
- simple two-layer aerosol model [constrained by Roman et al. 2013]
  - tropospheric haze layer $180-660$ mbar / $\tau \sim 8$ / $r = 2\mu$m
  - stratospheric haze layer $1-30$ mbar / $\tau \sim 0.1$ / $r = 0.1\mu$m
- free bottom surface with internal heat flux
- incoming flux: ring shadowing, oblateness

☞ Turbulent diffusion + dry convective adjustment [Hourdin et al. 1993]

XIOS (Y. MEURDESOIF) : XML I/O SERVER

PARALLEL ASYNCHRONOUS I/O - ONLINE POST-PROCESSING LIBRARY AND SERVER

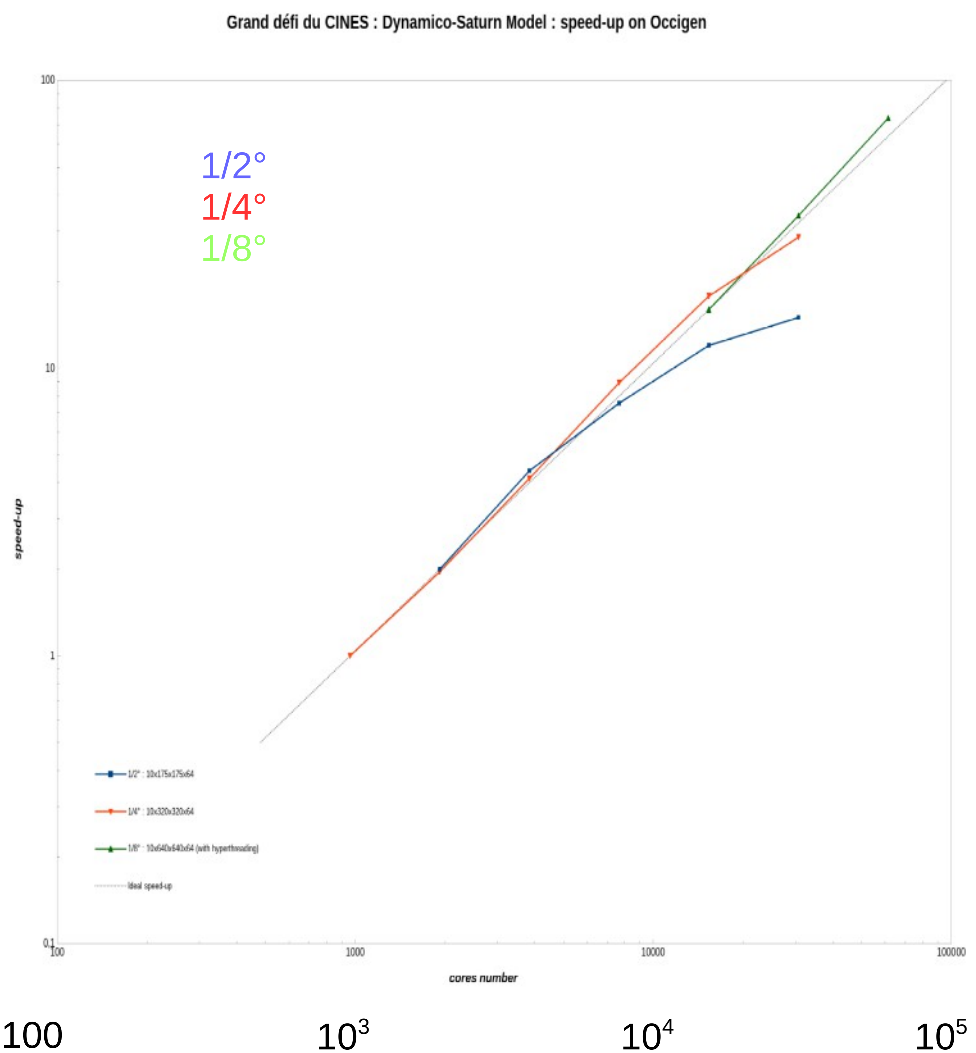




- Throughput on OCCIGEN (dycore only) for 60 vertical levels :

  1° : 500 cores ~ 40yr/day

  1/4° : 8000 cores ~10yr/day, 2Mh/century
- LMDZ CMIP6 physics now coupled, aquaplanet evaluation under way
- expect at least a few yr/day at 1/4 ° for full GCM



Post-doctoral position offer

Development of the high resolution atmospheric model LMDZ with the DYNAMICO dynamical core

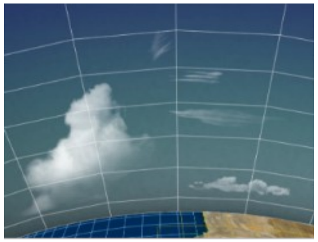
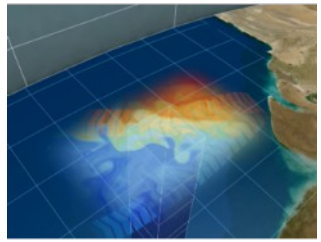
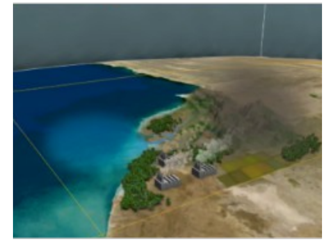
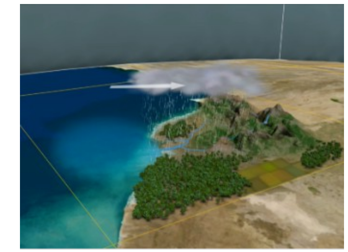
Description of the work:
The aim of the post-doctoral position is twofold. Firstly it is to asses and improve the LMDZ6 model with the new dynamical core, to identify and to understand the differences with the previous grid-point, latitude-longitude dynamical core. Secondly, it is to increase the horizontal resolution up to 0.5° (or finer), to assess the performance of the model and adjust the parameterizations as needed to the finer grid. In particular, **scale-aware stochastic triggering of convection** will be used and improved. The analysis may be focused on specific phenomena, like cyclones, depending on the interest of the candidate. The final version of this model will be used to perform prescribed SST simulations that will contribute to the HighResMIP project.

http://www.lists.rdg.ac.uk/archives/met-jobs/2016-05/pdfZwfuIwImhL.pdf

DYNAMICO

- Context : climate modelling at IPSL
- Background
  - Equations of motion : kinematics vs dynamics
  - Vertical coordinate
  - Curl-form and Hamiltonian formulation
- Numerics
  - Hydrostatic primitive equations
  - From hydrostatic to non-hydrostatic dynamics

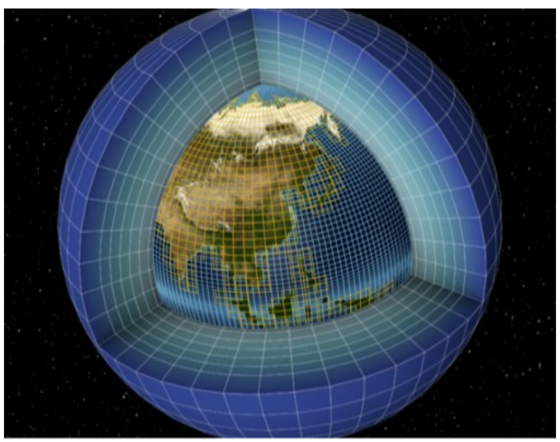
Earth System Modelling at IPSL

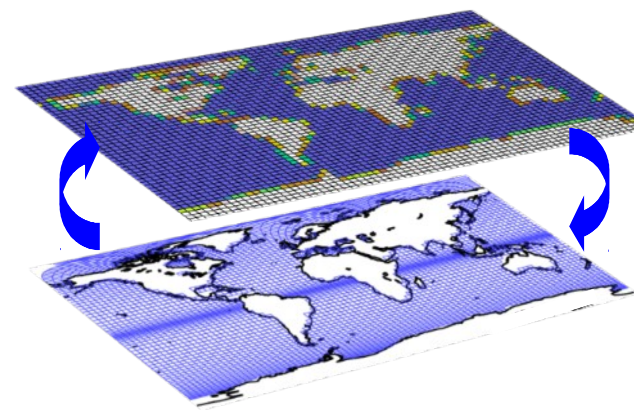
INCA / REPROBUS
(chimie atmosphérique)
(aérosol)

ORCHIDEE
(surfaces continentales)
(végétation)

LMDZ
(atmosphère)



OASIS
(coupleur)



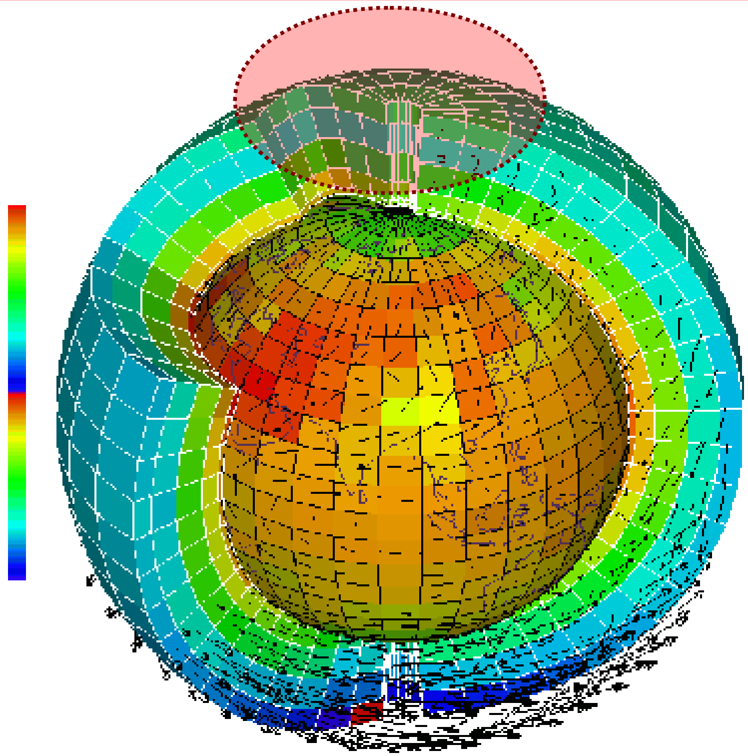
OPA
(océan)

LIM
(glace de mer)

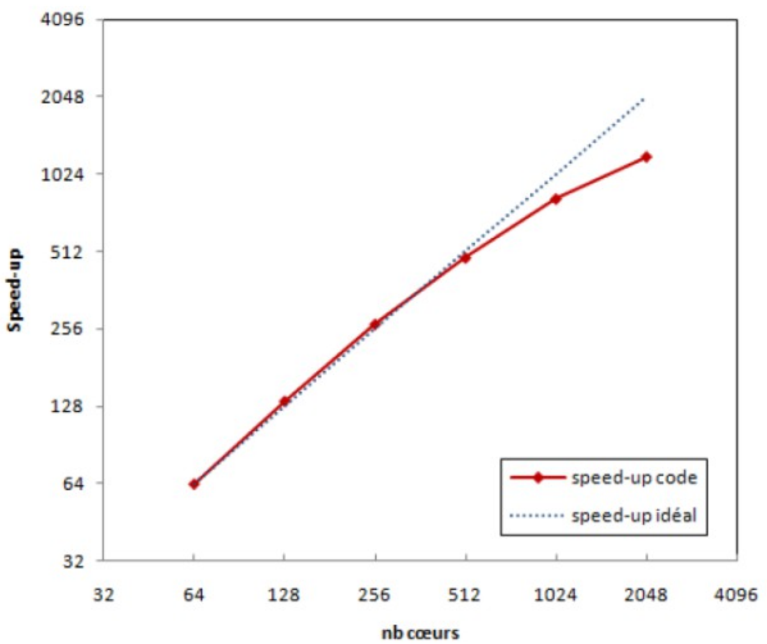
NEMO

PISCES
(biogéochimie marine)

The pole problem : FFT filters around the pole for stability => global dependency (Williamson, 2007)



LMD-Z lon-lat core



Y. Meurdesoif (2010, 1/4 degree)

Enstrophy-conserving finite differences on lon-lat C-grid (Sadourny, 1975)

Positive definite finite-volume transport (Hourdin & Armengaud, 1999)

Why would a numerical model want to conserve energy ?

density — specific volume — specific entropy

$$\mathcal{H} = \int \rho \left(\underbrace{\frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}}_{\text{kinetic}} + \underbrace{e\left(\frac{1}{\rho}, s\right)}_{\text{internal}} + \underbrace{gz}_{\text{potential}} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Energy minimum given total mass and entropy ?

$$\mathcal{M} = \int \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad \mathcal{S} = \int \rho s \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\delta\left(\mathcal{H} - G_0\mathcal{M} - T_0\mathcal{S}\right) = 0$$

$$\Updownarrow$$

$u = v = w = 0$ *Resting*

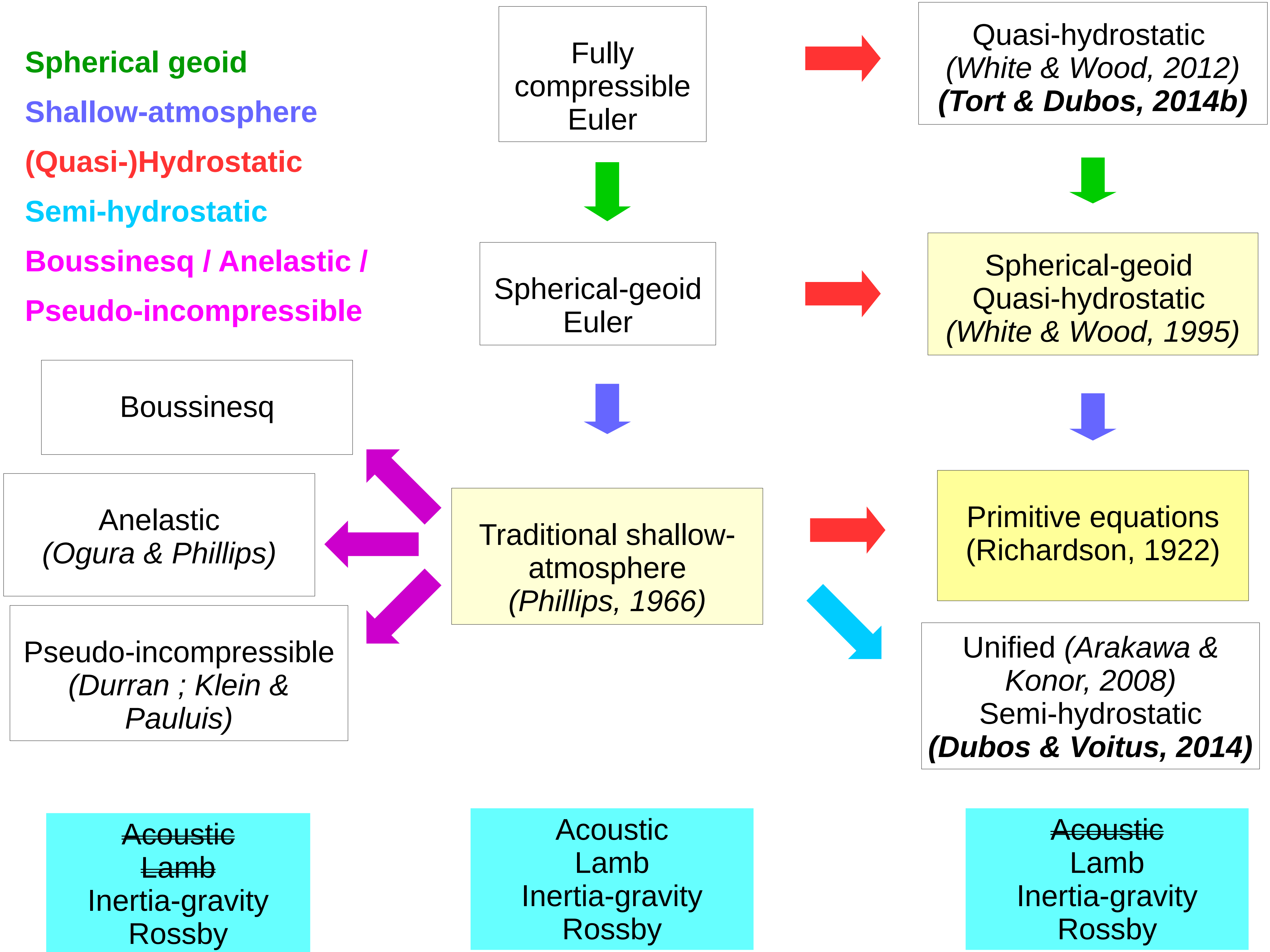
$T = T_0$ *Isothermal*

$e + \frac{p}{\rho} - Ts + gz = G_0$ *Hydrostatic*

(discrete) conservation limits dynamically accessible states
=> stability

numerics conserve energy, mass , entropy

=> isothermal state of rest is stable

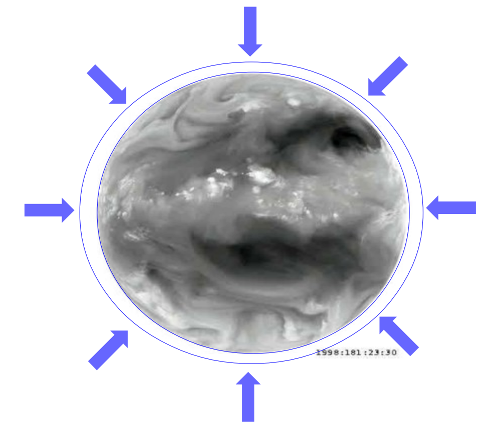


Spherical geoid
Shallow-atmosphere
(Quasi-)Hydrostatic
Semi-hydrostatic
Boussinesq / Anelastic /
Pseudo-incompressible
Fully compressible Euler
Quasi-hydrostatic *(White & Wood, 2012)* ***(Tort & Dubos, 2014b)***
Spherical-geoid Euler
Spherical-geoid Quasi-hydrostatic *(White & Wood, 1995)*
Boussinesq
Anelastic *(Ogura & Phillips)*
Pseudo-incompressible *(Durran ; Klein & Pauluis)*
Traditional shallow-atmosphere *(Phillips, 1966)*
Primitive equations (Richardson, 1922)
Unified *(Arakawa & Konor, 2008)* Semi-hydrostatic ***(Dubos & Voitus, 2014)***
~~Acoustic~~ ~~Lamb~~ Inertia-gravity Rossby
Acoustic Lamb Inertia-gravity Rossby
~~Acoustic~~ Lamb Inertia-gravity Rossby

Adiabatic equations of motion imply **conservation laws** because they derive from a least **action principle**

inertia | Coriolis | pressure | gravity

$$\frac{D\dot{\mathbf{x}}}{Dt} + 2\boldsymbol{\Omega}\times\dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$



$$\delta\int\mathcal{L}\,\mathrm{d}t = 0$$

$$\Updownarrow$$

$$\frac{D}{Dt}\frac{\partial L}{\partial\dot{\mathbf{x}}} = \frac{1}{\rho}\nabla\left(\rho^2\frac{\partial L}{\partial\rho}\right) + \frac{\partial L}{\partial\mathbf{x}}$$

$$L(\mathbf{x},\dot{\mathbf{x}},\rho,s) = \frac{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}}{2} + (\boldsymbol{\Omega}\times\mathbf{x})\cdot\dot{\mathbf{x}} - gz - e\left(\frac{1}{\rho},s\right)$$

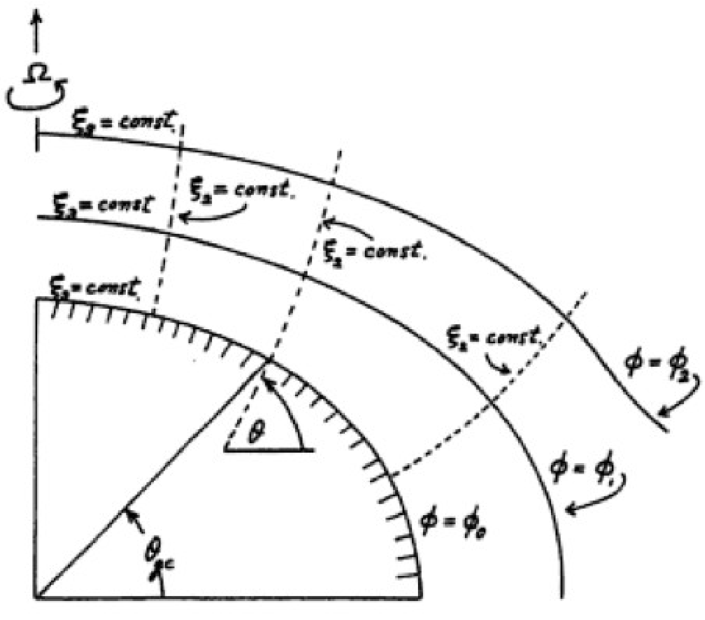
$$\mathcal{L} = \mathcal{K} + \mathcal{C} - \mathcal{P} = \int L(\mathbf{x},\,\dot{\mathbf{x}},\,\rho,\,s)\mathrm{d}m$$

$$\mathcal{K} = \frac{1}{2}\int\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\mathrm{d}m$$ Kinetic energy

$$\mathcal{C} = \int(\boldsymbol{\Omega}\times\mathbf{x})\cdot\dot{\mathbf{x}}\mathrm{d}m$$ Planetary velocity

$$\mathcal{P} = \int\left(e\left(\frac{1}{\rho},s\right) + \Phi(\mathbf{x})\right)\mathrm{d}m$$ Internal energy, Potential energy

Least action principle in curvilinear coordinates



Surfaces of constant ξ_2 and constant ξ_3

$$(x,\, y,\, z) \rightarrow (\xi^1,\, \xi^2,\, \Phi)$$

$$\dot{x} \nearrow u^i = \frac{D\xi^i}{Dt}$$

$$\dot{x} \searrow v_i = \frac{\partial L}{\partial u^i}$$

contravariant relative velocity

$$D/Dt = \partial_t + u^i \partial_i$$

covariant absolute momentum

$$v_i = G_{ij} u^j + R_j$$

$$\delta \int \mathcal{L} \mathrm{d}t = 0$$

$$\Updownarrow$$

$$\frac{D}{Dt}\frac{\partial L}{\partial u^i} = \frac{1}{\hat{\rho}} \partial_i \left(\hat{\rho}^2 \frac{\partial L}{\partial \hat{\rho}} \right) + \frac{\partial L}{\partial \xi^i}$$

$$\mathcal{L} = \int L\left(\xi^i, u^i, \hat{\rho}\right) \mathrm{d}m$$

$$\mathcal{K} = \frac{1}{2} \int G_{ij} u^i u^j \mathrm{d}m$$ Kinetic energy

$$\mathcal{C} = \int R_j u^j \mathrm{d}m$$ Planetary velocity

$$\mathcal{P} = \int \left(e\left(\frac{J}{\hat{\rho}},\, s \right) + \Phi(\xi^3) \right) \mathrm{d}m$$ Internal energy Potential energy

Kinematics / dynamics separation

Mass budget in shallow-atmosphere vs deep-atmosphere geometry :

$$\frac{\partial \rho}{\partial t} + \frac{1}{a\cos\phi}\frac{\partial}{\partial \lambda}\rho u + \frac{1}{a\cos\phi}\frac{\partial}{\partial \phi}\rho\cos\phi v + \frac{\partial}{\partial r}\rho w = 0 \qquad \begin{array}{l} u = a\cos\phi\dot{\lambda},\ v = a\dot{\phi},\ w = \dot{r} \\ \mu = a^2\cos\phi\rho \end{array}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r\cos\phi}\frac{\partial}{\partial \lambda}\rho u + \frac{1}{r\cos\phi}\frac{\partial}{\partial \phi}\rho\cos\phi v + r^{-2}\frac{\partial}{\partial r}\rho r^2 w = 0 \qquad \begin{array}{l} u = r\cos\phi\dot{\lambda},\ v = r\dot{\phi},\ w = \dot{r} \\ \mu = r^2\cos\phi\rho \end{array}$$

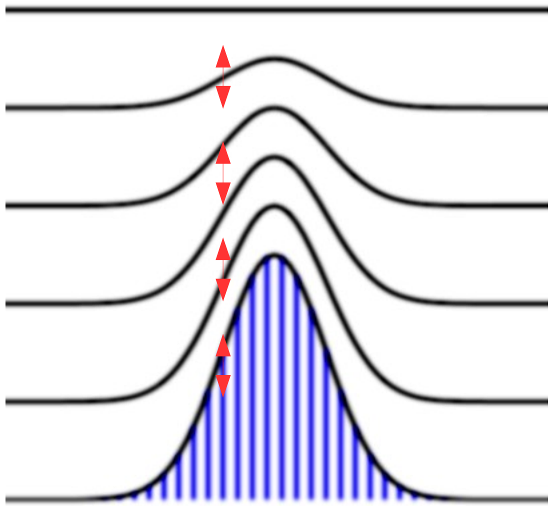
$$\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial \lambda}\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial \mathbf{x}}(\mu\dot{\mathbf{x}}) + \frac{\partial}{\partial \eta}(\mu\dot{\eta}) = 0 = 0 \qquad \begin{array}{c} (\lambda,\phi) \to \mathbf{x} \in S^2 \\ r \to \eta \end{array}$$

$$\frac{\partial}{\partial t}\mu\theta + \frac{\partial}{\partial \mathbf{x}}(\theta\mu\dot{\mathbf{x}}) + \frac{\partial}{\partial \eta}(\theta\mu\dot{\eta}) = 0$$

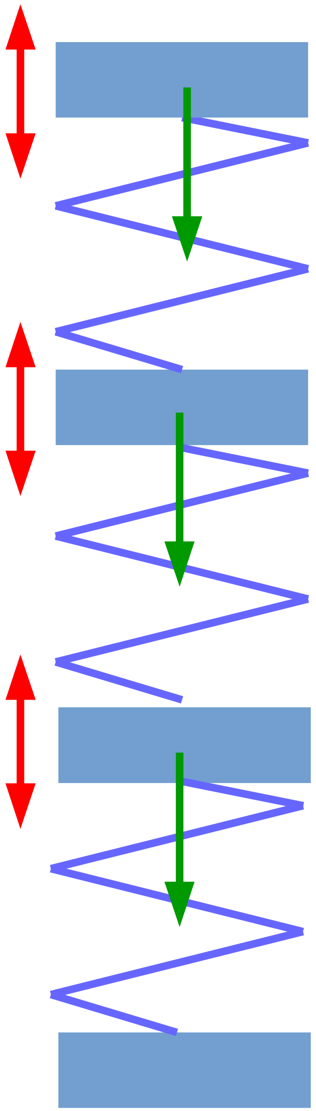
- Transport works in a metric-independent computational space $(\mathbf{x},\ \eta) \in S^2 \times [0,1]$
- Metric is needed only when converting between covariant and contravariant
- Prognosing momentum (covariant) and diagnosing flux (contravariant) is the business of the dynamics

Since transport is formally coordinate-independent, we may exploit the freedom to choose a non-Eulerian vertical coordinate :

- Isentropic/isopycnal
- based on mass / hydrostatic pressure



- In that case we must prognose/diagnose altitude (geopotential)
- Do certain vertical coordinates offer a benefit ?

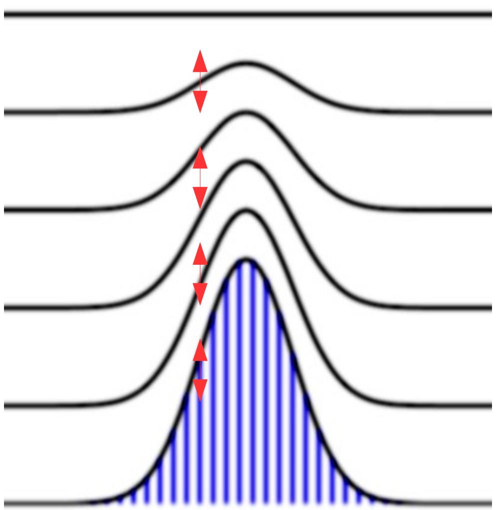


Floating vertical coordinates offer a natural representation of **hydrostatic adjustment**

Energy / vorticity conserving schemes and the curl (vector-invariant) form

$$\partial_t \mathbf{v} + (\nabla \times \mathbf{v}) \times \mathbf{u} + \nabla (K + \Phi) + \theta \nabla \pi = 0$$

- curl-form standard starting point to devise energy/vorticity-conserving schemes
- intimately connected to Hamiltonian formulation (Salmon, 1983 ; Gassmann, 2012)
- for **compressible flow,** Hamiltonian formulation well-established if coordinate system is **Eulerian**



- We want to handle non-Eulerian vertical coordinates ...

Hamiltonian formulation for generalized vertical coordinates :
Dubos & Tort (MWR 2014)

Energy / vorticity conserving schemes and the curl (vector-invariant) form

$$\partial_t \mathbf{v} + (\nabla \times \mathbf{v}) \times \mathbf{u} + \nabla (K + \Phi) + \theta \nabla \pi = 0$$

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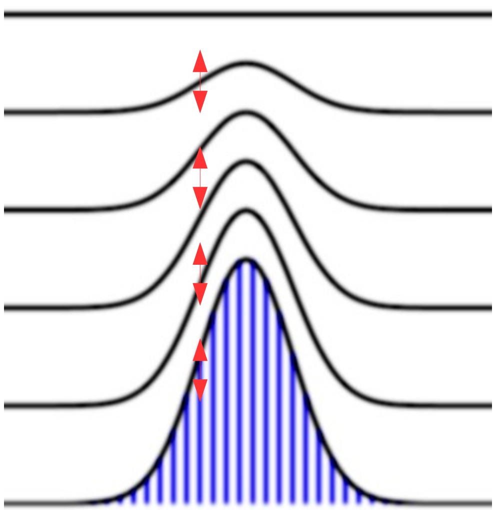


- We want to handle non-Eulerian vertical coordinates ...

Hamiltonian formulation for generalized vertical coordinates :
Dubos & Tort (MWR 2014)

Hamiltonian formulation in generalized vertical coordinates

(Dubos & Tort, MWR 2014)

$$
\begin{aligned}
\partial_t \mu + \partial_\eta\left(\mu\dot\eta\right) + \partial_i \frac{\delta\mathcal{H}}{\delta v_i} &= 0, \\
\partial_t \Theta + \partial_\eta\left(\Theta\dot\eta\right) + \partial_i \left(\theta\frac{\delta\mathcal{H}}{\delta v_i}\right) &= 0, \\
\partial_t v_i + \dot\eta\partial_\eta v_i + v_3\partial_i\dot\eta + \frac{\partial_j v_i - \partial_i v_j}{\mu}\frac{\delta\mathcal{H}}{\delta v_j} + \partial_i\frac{\delta\mathcal{H}}{\delta\mu} + \theta\partial_i\left(\frac{\delta\mathcal{H}}{\delta\Theta}\right) &= 0, \\
\partial_t V_3 + \partial_\eta\left(V_3\dot\eta\right) + \frac{\delta\mathcal{H}}{\delta\Phi} &= 0, \\
\partial_t \Phi + \dot\eta\partial_\eta\Phi - \frac{\delta\mathcal{H}}{\delta V_3} &= 0.
\end{aligned}
$$

Integration by parts
+ invariance w.r.t. vertical coordinate
=> conservation of energy

Hamiltonian formulation in generalized vertical coordinates

(Dubos & Tort, MWR 2014)

$$
\begin{aligned}
\partial_t \mu + \partial_\eta (\mu \dot\eta) + \partial_i \frac{\delta \mathcal{H}}{\delta v_i} &= 0, \\
\partial_t \Theta + \partial_\eta (\Theta \dot\eta) + \partial_i \left(\theta \frac{\delta \mathcal{H}}{\delta v_i} \right) &= 0, \\
\partial_t v_i + \dot\eta \partial_\eta v_i + v_3 \cancel{\partial_i \eta} + \frac{\partial_j v_i - \partial_i v_j}{\mu} \frac{\delta \mathcal{H}}{\delta v_j} + \partial_i \frac{\delta \mathcal{H}}{\delta \mu} + \theta \partial_i \left(\frac{\delta \mathcal{H}}{\delta \Theta} \right) &= 0, \\
\cancel{\partial_t V_3} + \cancel{\partial_\eta (V_3 \dot\eta)} + \frac{\delta \mathcal{H}}{\delta \Phi} &= 0, \\
\cancel{\partial_t \Phi} + \cancel{\dot\eta \partial_\eta \Phi} - \cancel{\frac{\delta \mathcal{H}}{\delta V_3}} &= 0.
\end{aligned}
$$

Hydrostatic $v_3 \equiv \frac{\partial L}{\partial u^3} = 0$

Least action principle

density, *entropy*, *velocity*

$$\delta \int L(\mathbf{x}, \rho, s, \mathbf{u})\mathrm{d}m\mathrm{d}t = 0$$

L=kinetic+Coriolis
-internal-potential

Eulerian coordinates in planetary frame

density, *entropy*, *velocity*

Least action principle

$$\delta \int L(\mathbf{x}, \rho, s, \mathbf{u}) \mathrm{d}m \mathrm{d}t = 0$$

L=kinetic+Coriolis -internal-potential

$$(x, y, z) \to (\xi^1, \xi^2, \xi^3) \qquad \mathbf{u}$$

$$u^i = D\xi^i / Dt$$

contravariant relative velocity

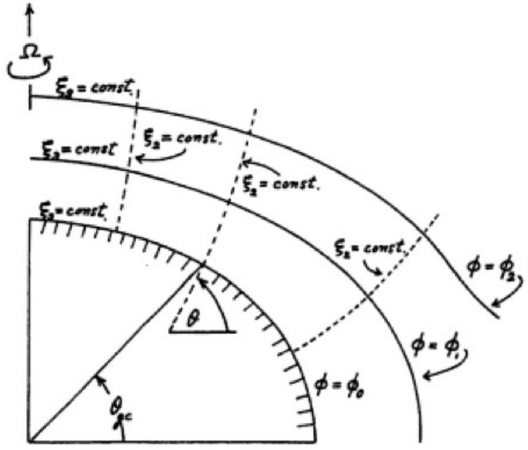
$$D/Dt = \partial_t + u^i \partial_i$$

$$v_i = \partial L / \partial u^i$$

covariant absolute momentum

$$v_i = G_{ij} u^j + R_j$$

Eulerian coordinates in planetary frame



Surfaces of constant ξ_2 and constant ξ_3

Eulerian coordinates in planetary frame

Least action principle

density, *entropy*, *velocity*

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Euler-Lagrange equations

Tort & Dubos, JAS 2014

$$\frac{Dv_i}{Dt} + \frac{\partial L}{\partial \xi^i} = \frac{1}{\mu}\partial_i \left(\mu^2 \frac{\partial L}{\partial \mu}\right)$$

Hamiltonian formulation

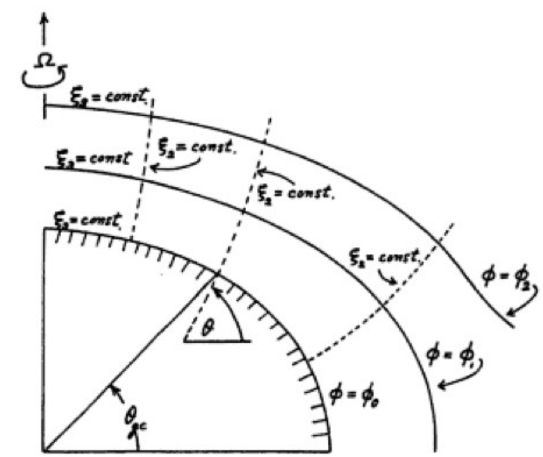
Dubos & Tort, MWR 2014

Curl form

$$\frac{\partial v_i}{\partial t} = \dots$$

Flux form

$$\frac{\partial}{\partial t}(\mu v_i) = \dots$$



Surfaces of constant ξ_2 and constant ξ_3

Generalized vertical coordinate

density, *entropy*, *velocity*

Least action principle

$$\delta \int L(\mathbf{x}, \rho, s, \mathbf{u})\mathrm{d}m\mathrm{d}t = 0$$

L=kinetic+Coriolis -internal-potential

$$(x, y, z) \to (\xi^1, \xi^2, \xi^3) \qquad \mathbf{u}$$

$$u^i = D\xi^i/Dt$$

contravariant relative velocity

$$D/Dt = \partial_t + u^i\partial_i$$

$$v_i = \partial L/\partial u^i$$

covariant absolute momentum

$$v_i = G_{ij}u^j + R_j$$

Euler-Lagrange equations

Tort & Dubos, JAS 2014

$$\frac{Dv_i}{Dt} + \frac{\partial L}{\partial \xi^i} = \frac{1}{\mu}\partial_i\left(\mu^2\frac{\partial L}{\partial \mu}\right)$$

Hamiltonian formulation

Dubos & Tort, MWR 2014

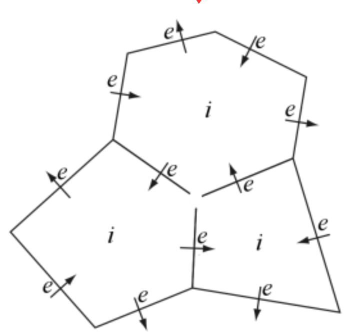
Curl form

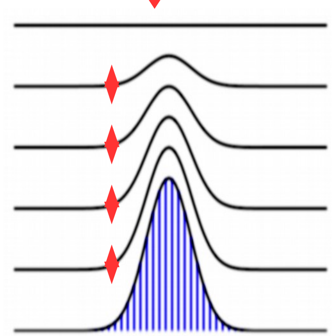
$$\frac{\partial v_i}{\partial t} = \dots$$

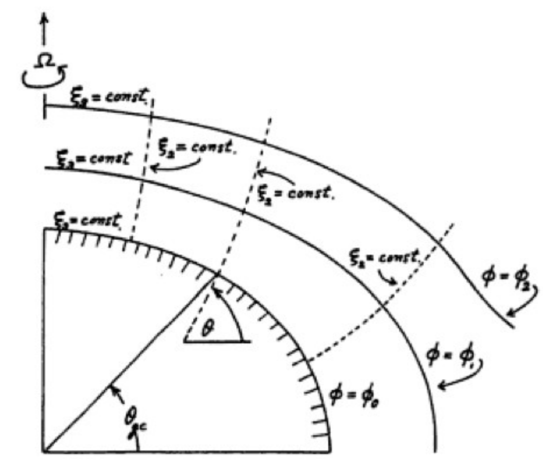
Flux form

$$\frac{\partial}{\partial t}(\mu v_i) = \dots$$

Discretize !







Surfaces of constant ξ_2 and constant ξ_3

Eulerian coordinates in planetary frame

Generalized vertical coordinate

Physical space

$$\lambda, \varphi, \Phi, g$$

$$u, v, w$$

$$\alpha = 1/\rho,\ s,\ r$$

Thermodynamics

$$p, T, \chi$$

Geopotential Coordinates

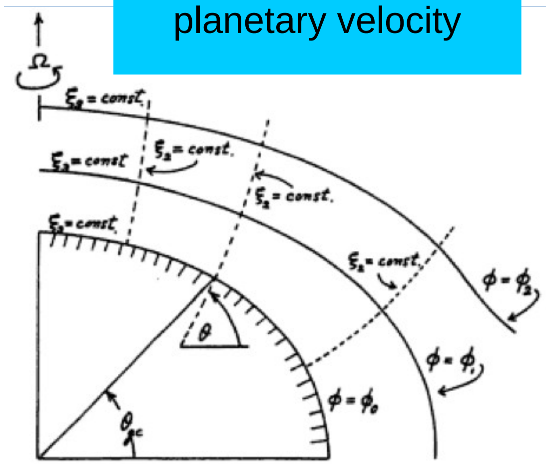
Metric, gravity, planetary velocity

Computational space $S^2 \times [0,1]$

$$\mathbf{v} = \mathbf{G}(\mathbf{n}, \Phi) \cdot \dot{\mathbf{n}} + \mathbf{R}(\mathbf{n}, \Phi)$$

$$\Phi(\mathbf{n}, \eta, t), \mu,$$

$$\mathbf{n} \qquad \eta$$



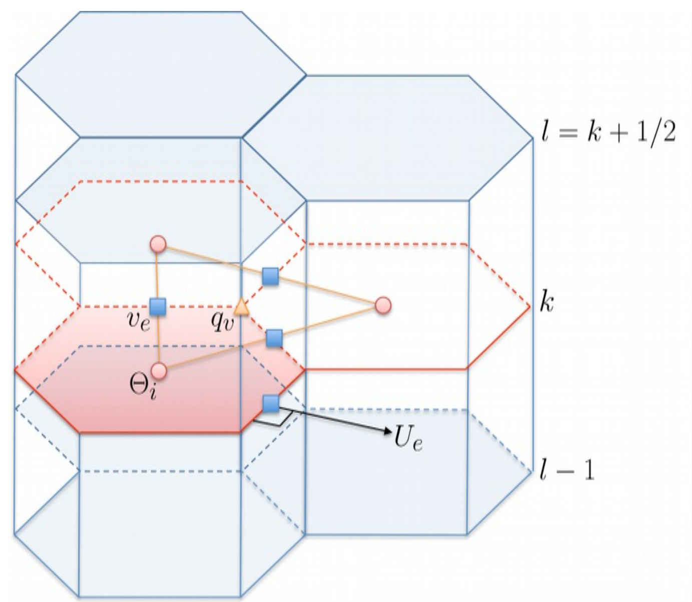
Surfaces of constant ξ_2 and constant ξ_3

Horizontal mesh

Icosahedral C-grid

Vertical mesh

Lorenz



Discrete space

$$m_{ik} = \int\int\int \mu \mathrm{d}\mathbf{n}\mathrm{d}\eta$$

$$W_{il} = \int\int \mu\dot{\eta}\mathrm{d}\mathbf{n}$$

$$v_{ek} = \int \mathbf{v} \cdot \mathrm{d}\mathbf{n}$$

$$\alpha_{ik} = \alpha(p_{ik}, s_{ik}),$$

Physical space

$$\lambda, \varphi, \Phi, g$$

$$u, v, w$$

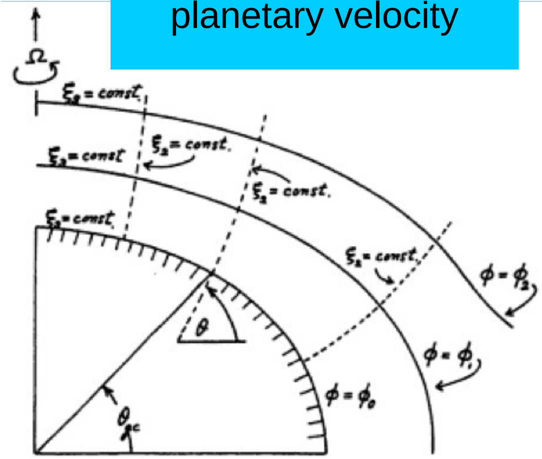
$$\alpha = 1/\rho,\ s,\ r$$

Thermodynamics

$$p, T, \chi$$

Geopotential Coordinates

Metric, gravity, planetary velocity



Surfaces of constant ξ₂ and constant ξ₃

Computational space $S^2 \times [0,1]$

$$\mathbf{v} = \mathbf{G}(\mathbf{n}, \Phi) \cdot \dot{\mathbf{n}} + \mathbf{R}(\mathbf{n}, \Phi)$$

$$\Phi(\mathbf{n}, \eta, t), \mu,$$

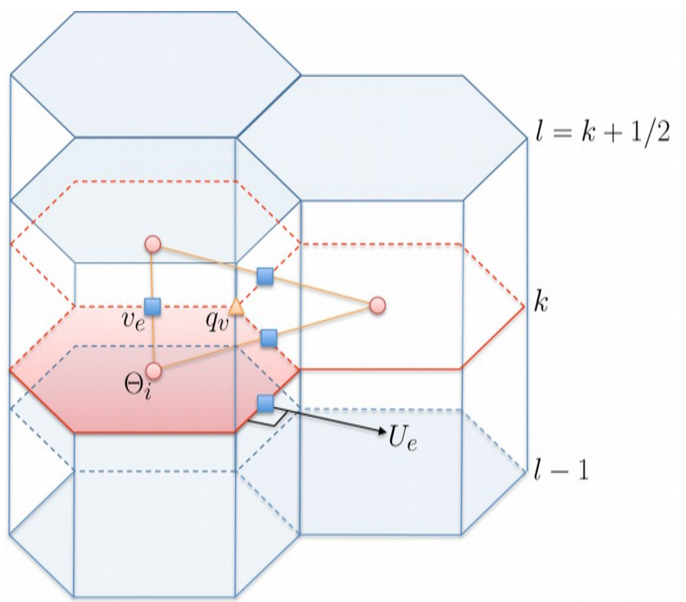
$\mathbf{n}$ η

Horizontal mesh

Icosahedral C-grid

Vertical mesh

Lorenz



Discrete space

$$m_{ik} = \int\int\int \mu \mathrm{d}\mathbf{n} \mathrm{d}\eta$$

$$W_{il} = \int\int \mu \dot{\eta} \mathrm{d}\mathbf{n}$$

$$v_{ek} = \int \mathbf{v} \cdot \mathrm{d}\mathbf{n}$$

$$\alpha_{ik} = \alpha(p_{ik}, s_{ik}),$$

- Discrete integration by parts (Bonaventura & Ringler, 2005 ; Taylor, 2010)
- Energy- and vorticity- conserving Coriolis discretization (TRiSK : Thuburn et al., 2009 ; Ringler et al., 2010)

Energy-conserving 3D core

Hydrostatic primitive equations : discrete

(Lagrangian vertical coordinate – extra terms for vertical transport when mass-based)

TRiSK

shallow-atmosphere metric

planetary velocity

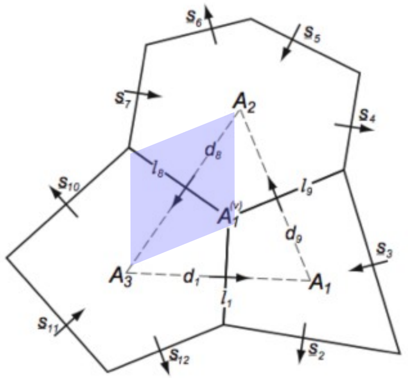
mass-weighted potential temperature

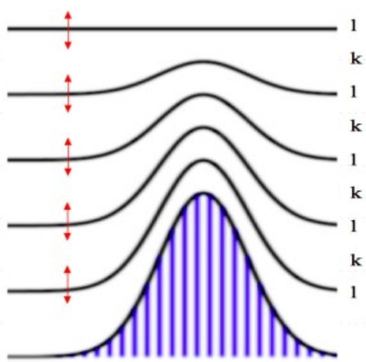
$$H = K(m_{ik},\, v_{ek}) + P(m_{ik},\, \Theta_{ik},\, \Phi_{il})$$

$$K = \sum_{ike} m_{ik} \frac{A_{ie}}{A_i} \left(\frac{v_{ek} - R_e}{a d_e} \right)^2$$

normal velocity

$$P_{HPE} = \sum_{ik} m_{ik} \left[\overline{\Phi_i}^k + e\left(\alpha_{ik} = \frac{A_i \delta_k (a^2 \Phi_i / g)}{m_{ik}},\, \theta_{ik} = \frac{\Theta_{ik}}{m_{ik}} \right) \right]$$





mass flux

$$\partial_t m_{ik} + \delta_i \frac{\partial H}{\partial v_{ek}} = 0$$

centered/upwind flux

$$\partial_t \Theta_{ik} + \delta_i \left(\theta^*_{ek} \frac{\partial H}{\partial v_{ek}} \right) = 0$$

$$\partial_t v_{ek} + \left(\frac{\delta_v v_k}{\overline{m_k}^v} \frac{\partial H}{\partial v_{ek}} \right)^\perp + \delta_e \frac{\partial H}{\partial m_{ik}} + \theta^*_{ek} \delta_e \frac{\partial H}{\partial \Theta_{ik}} = 0$$

SW pot. Vort. Bernoulli function Exner function

Discrete energy budget : Lagrangian vertical coordinate

mass flux

centered/upwind

$$\partial_t m_{ik} + \delta_i \frac{\partial H}{\partial v_{ek}} = 0 \qquad \partial_t \Theta_{ik} + \delta_i \left(\theta^*_{ek} \frac{\partial H}{\partial v_{ek}} \right) = 0$$

$$\partial_t v_{ek} + \left(\frac{\delta_v v_k}{\overline{m_k}^v} \frac{\partial H}{\partial v_{ek}} \right)^\perp + \delta_e \frac{\partial H}{\partial m_{ik}} + \theta^*_{ek} \delta_e \frac{\partial H}{\partial \Theta_{ik}} = 0$$

SW pot. Vort.

Bernoulli function

Exner function

discrete div

discrete integration by parts

discrete grad

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \frac{\partial H}{\partial m_{ik}} \partial_t m_{ik} + \frac{\partial H}{\partial \Theta_{ik}} \partial_t \Theta_{ik} + \frac{\partial H}{\partial v_{ek}} \partial_t v_{ek} + \frac{\partial H}{\partial \Phi_{il}} \partial_t \Phi_{il} = 0$$

Discrete energy budget : Lagrangian vertical coordinate

mass flux

centered/upwind

$$\partial_t m_{ik} + \delta_i \frac{\partial H}{\partial v_{ek}} = 0 \qquad \partial_t \Theta_{ik} + \delta_i \left(\theta^*_{ek} \frac{\partial H}{\partial v_{ek}} \right) = 0$$

$$\partial_t v_{ek} + \left(\frac{\delta_v v_k}{\overline{m_k}^v} \frac{\partial H}{\partial v_{ek}} \right)^{\perp} + \delta_e \frac{\partial H}{\partial m_{ik}} + \theta^*_{ek} \delta_e \frac{\partial H}{\partial \Theta_{ik}} = 0$$

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$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \frac{\partial H}{\partial m_{ik}} \partial_t m_{ik} + \frac{\partial H}{\partial \Theta_{ik}} \partial_t \Theta_{ik} + \frac{\partial H}{\partial v_{ek}} \partial_t v_{ek} + \frac{\partial H}{\partial \Phi_{il}} \partial_t \Phi_{il} = 0$$

Hydrostatic balance

$$\Rightarrow p_{ik} \Rightarrow \alpha_{ik} \Rightarrow \Phi_{il}$$

Discrete energy budget : Lagrangian vertical coordinate

mass flux

$$\partial_t m_{ik} + \delta_i \frac{\partial H}{\partial v_{ek}} = 0$$

centered/upwind

$$\partial_t \Theta_{ik} + \delta_i \left(\theta^*_{ek} \frac{\partial H}{\partial v_{ek}} \right) = 0$$

$$\partial_t v_{ek} + \left(\frac{\delta_v v_k}{\overline{m_k}^v} \frac{\partial H}{\partial v_{ek}} \right)^\perp + \delta_e \frac{\partial H}{\partial m_{ik}} + \theta^*_{ek} \delta_e \frac{\partial H}{\partial \Theta_{ik}} = 0$$

SW pot. Vort.

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$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \frac{\partial H}{\partial m_{ik}} \partial_t m_{ik} + \frac{\partial H}{\partial \Theta_{ik}} \partial_t \Theta_{ik} + \frac{\partial H}{\partial v_{ek}} \partial_t v_{ek} + \frac{\partial H}{\partial \Phi_{il}} \partial_t \Phi_{il} = 0$$

Hydrostatic balance

$$\Rightarrow p_{ik} \Rightarrow \alpha_{ik} \Rightarrow \Phi_{il}$$

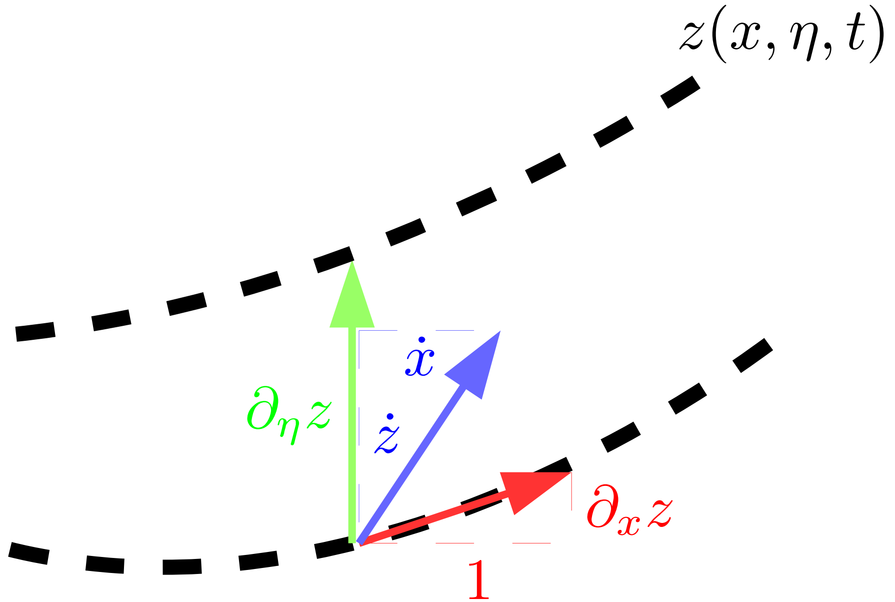
Non-Lagrangian vertical coordinate : also possible to cancel additional contributions from vertical transport (Tort et al., QJRMS 2015)

What is new in NH (fully compressible) compared to hydrostatic ?

- More *energy* : vertical kinetic energy K_V not neglected any more
- More *prognostic* variables : geopotential, vertical momentum

Example : Cartesian (x,z) slice, non-rotating

$$v_1 = \dot{x} + \dot{z}\partial_x z$$
$$v_3 = \dot{z}\partial_\eta z$$
$$W = \mu\dot{z}$$



$$H = \int \left[\frac{\left(v_1 - \frac{W}{\mu}\partial_x z\right)^2 + \left(\frac{W}{\mu}\right)^2}{2} + e\left(\frac{\partial_\eta z}{\mu}, \frac{\Theta}{\mu}\right) + gz \right] \mu \mathrm{d}x\mathrm{d}\eta$$

What is new in NH (fully compressible) compared to hydrostatic ?

- More *energy* : vertical kinetic energy K_V not neglected any more
- More *prognostic* variables : geopotential, vertical momentum $\Phi,\ W=\mu w$

$$H = K_H[\mu,\Phi,\mathbf{v}] + P[\mu,\Theta,\Phi]$$

$$\Downarrow$$

$$H = K_H[\mu,\Phi,\mathbf{v},W] + K_V[\mu,\Phi,W] + P[\mu,\Theta,\Phi]$$

- But not more *interesting* degrees of freedom (see Dubos & Voitus, JAS 2015)
- The new prognostic variables support acoustic waves and should be *slaved* to the «meteorological» DOFs through *implicit* time-stepping

Need an appropriate *fast-slow splitting*

- In HEVI horizontal transport is considered slow (cf e.g Weller et al. 2013)
- In a **mass-based coordinate**, vertical mass flux is deduced from horizontal mass flux ; therefore **3D transport is slow ; vertical transport is explicit**
- **Hamiltonian splitting** : rather than terms in the equations of motions, split Hamiltonian in fast-slow contributions (and the splitting in the equations of motion follows)

slow

fast

$$H = K_H[\mu, \Phi, \mathbf{v}, W] + K_V[\mu, \Phi, W] + P[\mu, \Theta, \Phi]$$

slow *fast*

$$H = K_H[\mu, \Phi, \mathbf{v}, W] + K_V[\mu, \Phi, W] + P[\mu, \Theta, \Phi]$$

$$\partial_t \mu + \partial_{\mathbf{x}} \cdot \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\mu \dot{\eta}\right) = 0$$

$$\partial_t \Theta + \partial_{\mathbf{x}} \cdot \theta \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\theta \mu \dot{\eta}\right) = 0$$

slow fast

$$H = K_H[\mu, \Phi, \mathbf{v}, W] + K_V[\mu, \Phi, W] + P[\mu, \Theta, \Phi]$$

$$\partial_t \mu + \partial_{\mathbf{x}} \cdot \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\mu \dot{\eta}\right) = 0$$

$$\partial_t \Theta + \partial_{\mathbf{x}} \cdot \theta \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\theta \mu \dot{\eta}\right) = 0$$

$$\partial_t \mathbf{v} + \dot{\eta} \left(\partial_\eta \mathbf{v} - \partial_{\mathbf{x}} w\right) + \frac{\partial_{\mathbf{x}} \times \mathbf{v}}{\mu} \times \frac{\delta H}{\delta \mathbf{v}} + \partial_{\mathbf{x}} \frac{\delta K_H}{\delta \mu}$$
$$+ \partial_{\mathbf{x}} \frac{\delta (K_V + P)}{\delta \mu} + \theta \partial_{\mathbf{x}} \frac{\delta P}{\delta \Theta} = 0$$

slow fast

$$H = K_H[\mu, \Phi, \mathbf{v}, W] + K_V[\mu, \Phi, W] + P[\mu, \Theta, \Phi]$$

$$\partial_t \mu + \partial_{\mathbf{x}} \cdot \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\mu \dot{\eta}\right) = 0$$

$$\partial_t \Theta + \partial_{\mathbf{x}} \cdot \theta \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\theta \mu \dot{\eta}\right) = 0$$

$$\partial_t \mathbf{v} + \dot{\eta} \left(\partial_\eta \mathbf{v} - \partial_{\mathbf{x}} w\right) + \frac{\partial_{\mathbf{x}} \times \mathbf{v}}{\mu} \times \frac{\delta H}{\delta \mathbf{v}} + \partial_{\mathbf{x}} \frac{\delta K_H}{\delta \mu}$$

$$+ \partial_{\mathbf{x}} \frac{\delta (K_V + P)}{\delta \mu} + \theta \partial_{\mathbf{x}} \frac{\delta P}{\delta \Theta} = 0$$

$$\partial_t \Phi + \dot{\eta} \partial_\eta \Phi - \frac{\delta K_H}{\delta W} - \frac{\delta K_V}{\delta W} = 0$$

$$\partial_t W + \partial_\eta \left(\dot{\eta} \Phi\right) + \frac{\delta K_H}{\delta \Phi} + \frac{\delta (K_V + P)}{\delta \Phi} = 0$$

slow *fast*

$$H = K_H[\mu, \Phi, \mathbf{v}, W] + K_V[\mu, \Phi, W] + P[\mu, \Theta, \Phi]$$

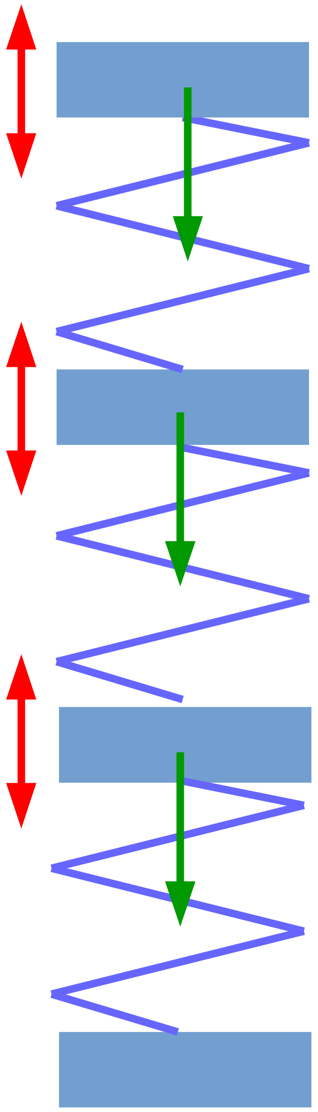
$$\partial_t \mu + \partial_{\mathbf{x}} \cdot \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\mu \dot{\eta}\right) = 0$$

$$\partial_t \Theta + \partial_{\mathbf{x}} \cdot \theta \frac{\delta H}{\delta \mathbf{v}} + \partial_\eta \left(\theta \mu \dot{\eta}\right) = 0$$

$$\begin{aligned}\partial_t \mathbf{v} + \dot{\eta}\left(\partial_\eta \mathbf{v} - \partial_{\mathbf{x}} w\right) + \frac{\partial_{\mathbf{x}} \times \mathbf{v}}{\mu} \times \frac{\delta H}{\delta \mathbf{v}} + \partial_{\mathbf{x}} \frac{\delta K_H}{\delta \mu} \\ + \partial_{\mathbf{x}} \frac{\delta (K_V + P)}{\delta \mu} + \theta \partial_{\mathbf{x}} \frac{\delta P}{\delta \Theta} = 0\end{aligned}$$

$$\partial_t \Phi + \dot{\eta} \partial_\eta \Phi - \frac{\delta K_H}{\delta W} - \frac{\delta K_V}{\delta W} = 0$$

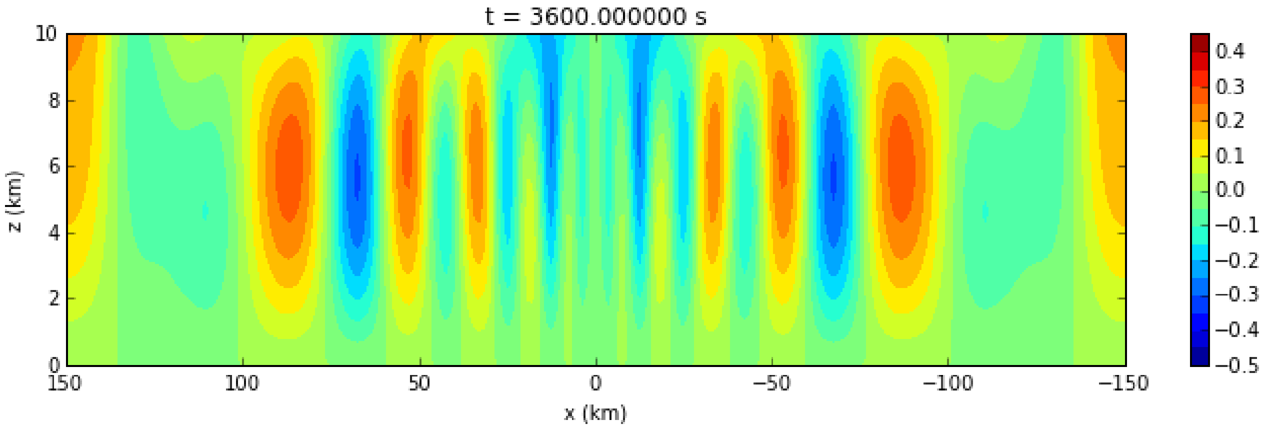
$$\partial_t W + \partial_\eta \left(\dot{\eta} \Phi\right) + \frac{\delta K_H}{\delta \Phi} + \frac{\delta (K_V + P)}{\delta \Phi} = 0$$



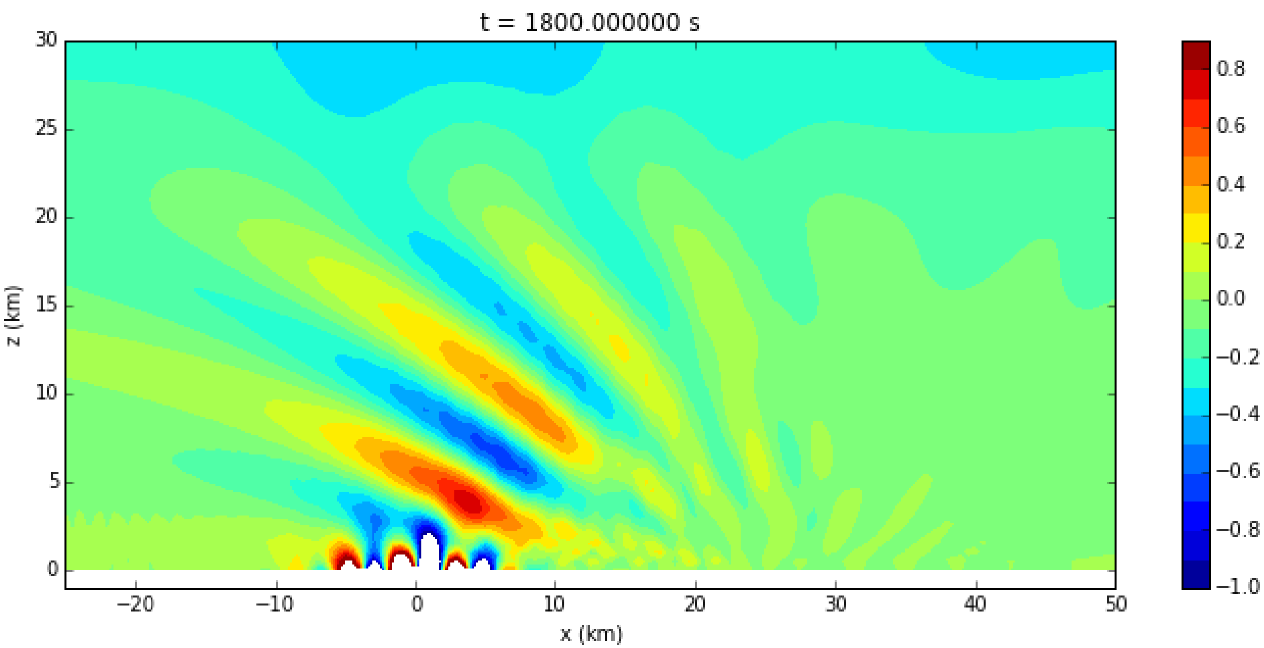
- The implicit problem only couples vertical position and vertical momentum
- eliminate W and obtain a scalar tridiagonal implicit problem for Φ

x-z slice prototype :

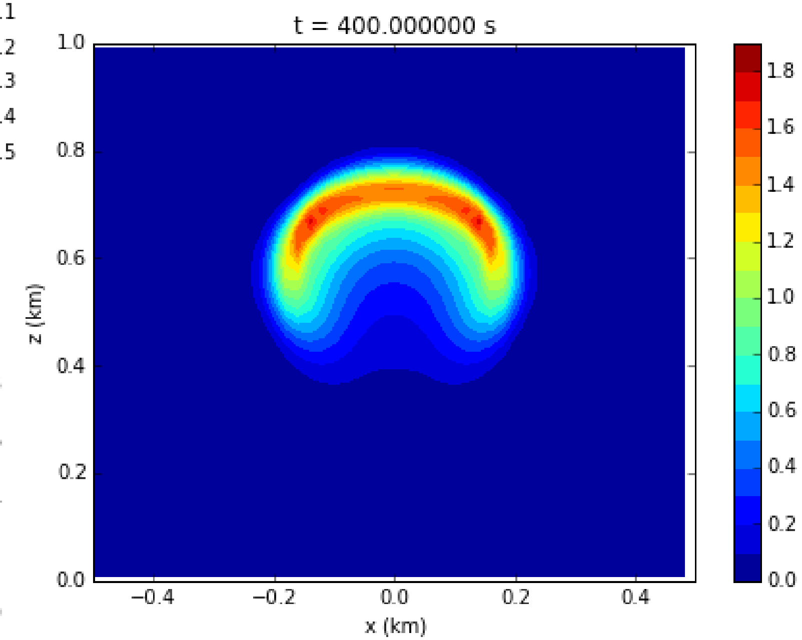
- *Horizontal : C-grid, 2nd order centered FD*
- *Vertical : Lorentz, 2nd order centered FD*
- *Time : 3-stage, 2nd order ARK scheme (Giraldo et al, 2013)*



dx=1000m dt=4s



dx=500m dt=2.5s



dx=20m dt=0.1s

DYNAMICO

Equations of motion	*shallow-water* *shalllow-atmosphere, hydrostatic* *ongoing : deep-atmosphere, fully compressible*
Conservation properties	*Mass (air and species)* *Energy*
Formulation	*Mass : flux-form* *Momentum : **Hamiltonian vector-invariant form*** *(a.k.a curl form, Crocco's theorem, Carter-Lichnerowicz equation)*
Vertical coordinate	*Terrain-following mass-based* *(often conflated with pressure-based)*
Numerics	*Mass : finite volume* *Momentum : low-order **mimetic finite difference*** *Mesh : **icosahedral-hexagonal C-grid, Lorenz*** *Time : (additive) Runge-Kutta (HEVI)*
Computing	*MPI / OpenMP* *Scales at least to O(10^4), including **I/O***