Particle methods for geophysical flow on the sphere

Peter Bosler[1], Christiane Jablonowski[2], and Robert Krasny[1]

[1]*Department of Mathematics* [2]*Department of Atmospheric, Oceanic and Space Sciences*

University of Michigan, Ann Arbor, MI

Equations of motion : Lagrangian form

α : Lagrangian parameter

$\boldsymbol{x}$: physical position

t : time

$\boldsymbol{u}(\boldsymbol{x},t)$: velocity

$$\left.\begin{aligned} \frac{D}{Dt}\boldsymbol{x}(\alpha,t) &= \boldsymbol{u}(\boldsymbol{x}(\alpha,t),t) \\ \boldsymbol{x}(\alpha,0) &= \boldsymbol{\alpha} \end{aligned}\right\} \text{Flow map}$$

Discretizing the sphere

$$\mathcal{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = 1\}$$

$\boldsymbol{y}_i = \boldsymbol{y}(\boldsymbol{\alpha}_i, t)$: panel vertices $\quad i = 1, \ldots, M$

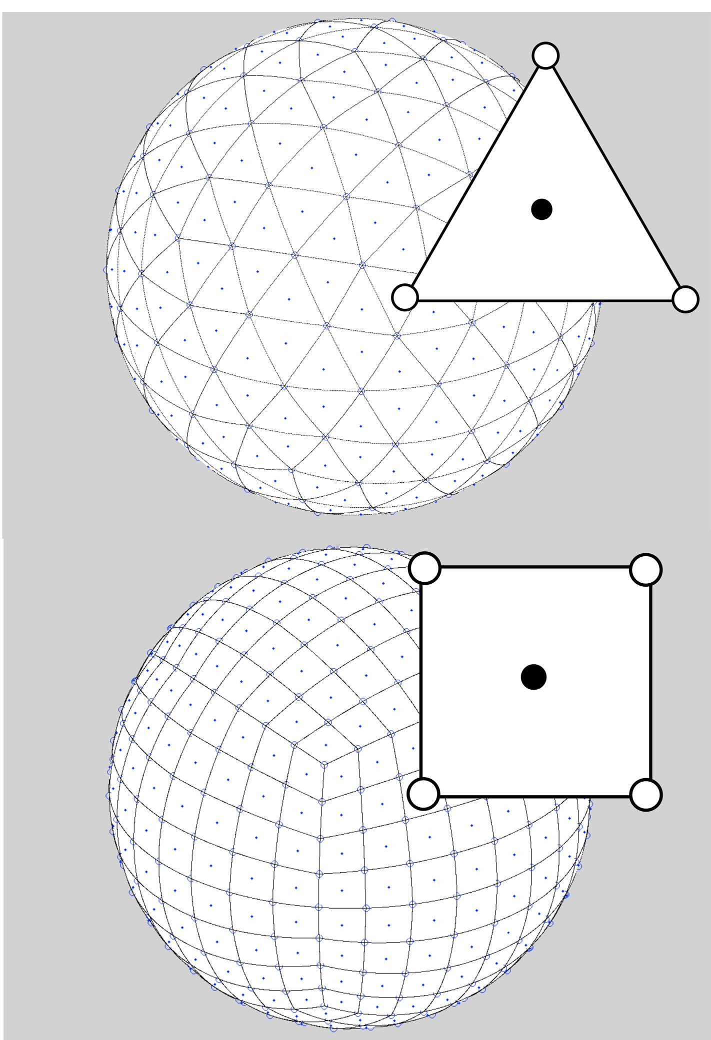
$\boldsymbol{x}_j = \boldsymbol{x}(\boldsymbol{\alpha}_j, t)$: panel centers $\quad j = 1, \ldots, N$

ϕ_j : passive tracer

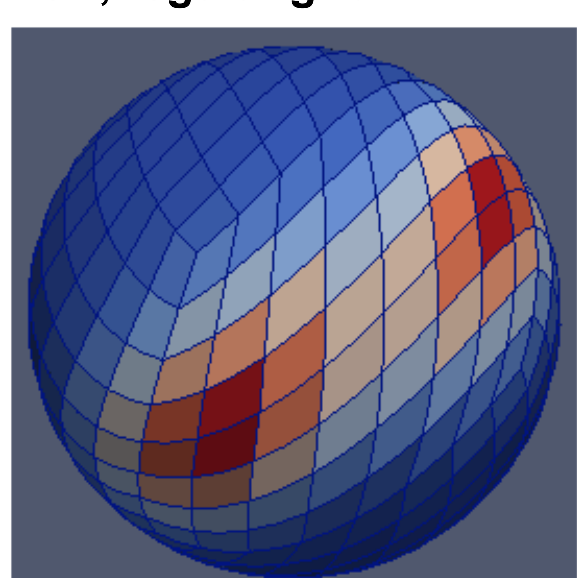
ζ_j : relative vorticity

ω_j : absolute vorticity

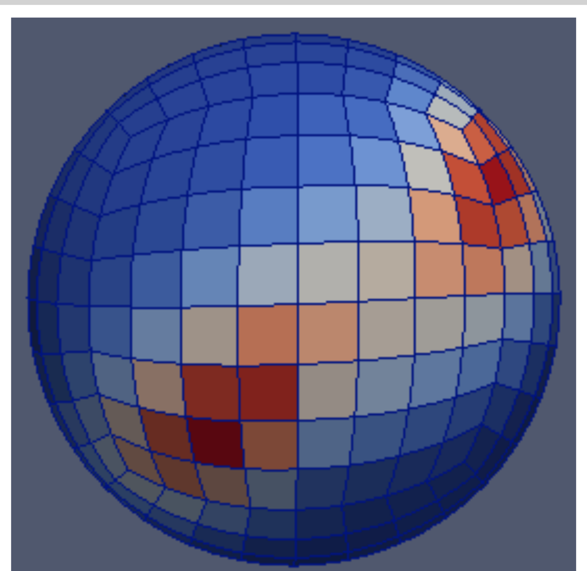
A_j : area of panel j



Lagrangian grids encounter the problem of mesh distortion, as the particles move with the fluid velocity. To maintain accuracy, we remesh at regular intervals by interpolating the Lagrangian parameter, α, from a distorted mesh to a new, regular grid.



Distorted mesh



Regular mesh

Advection equation

$$\boldsymbol{u}(\boldsymbol{x},t) = F(\boldsymbol{x},t)$$

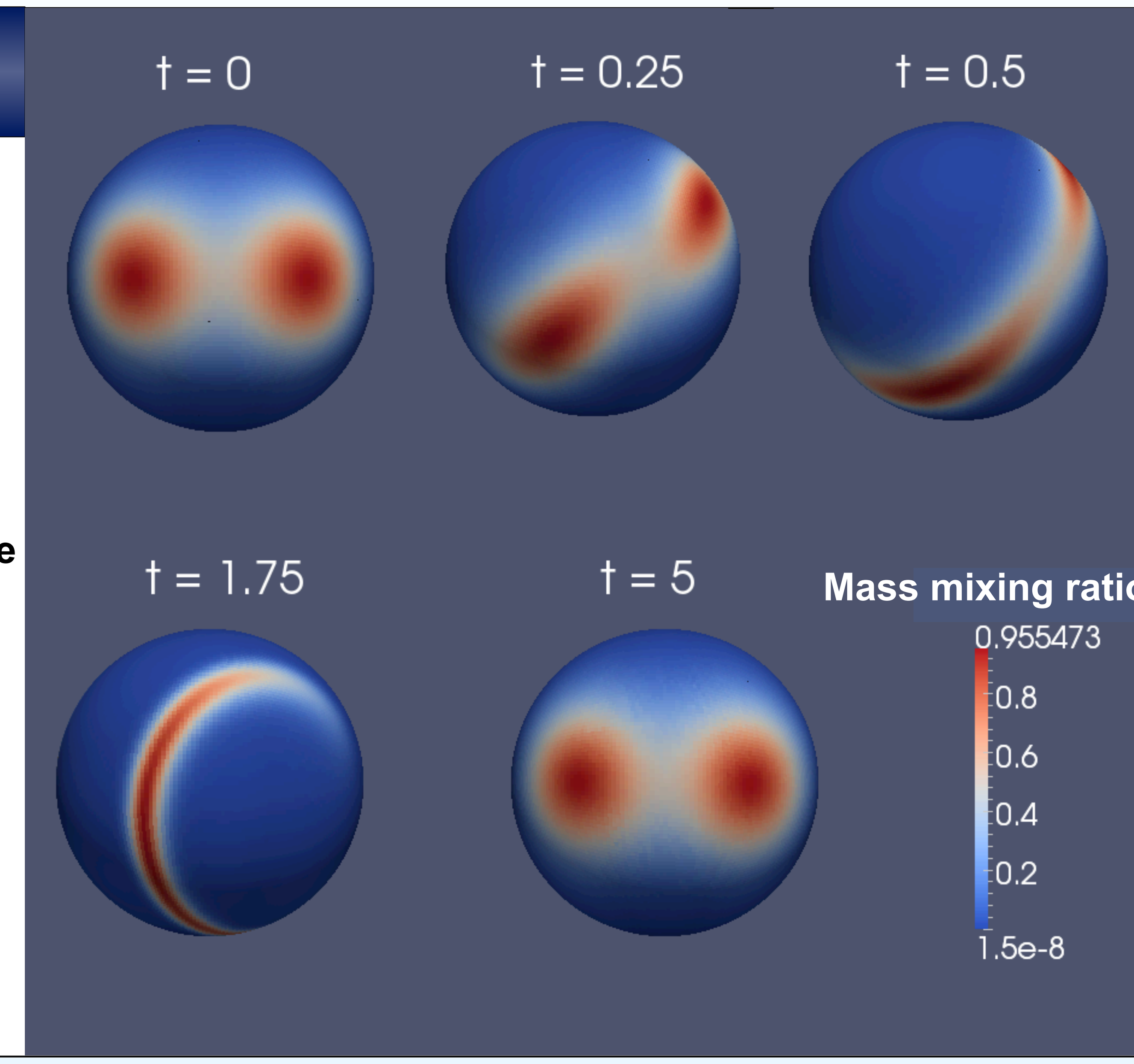
$$\frac{D\phi}{Dt} = 0$$

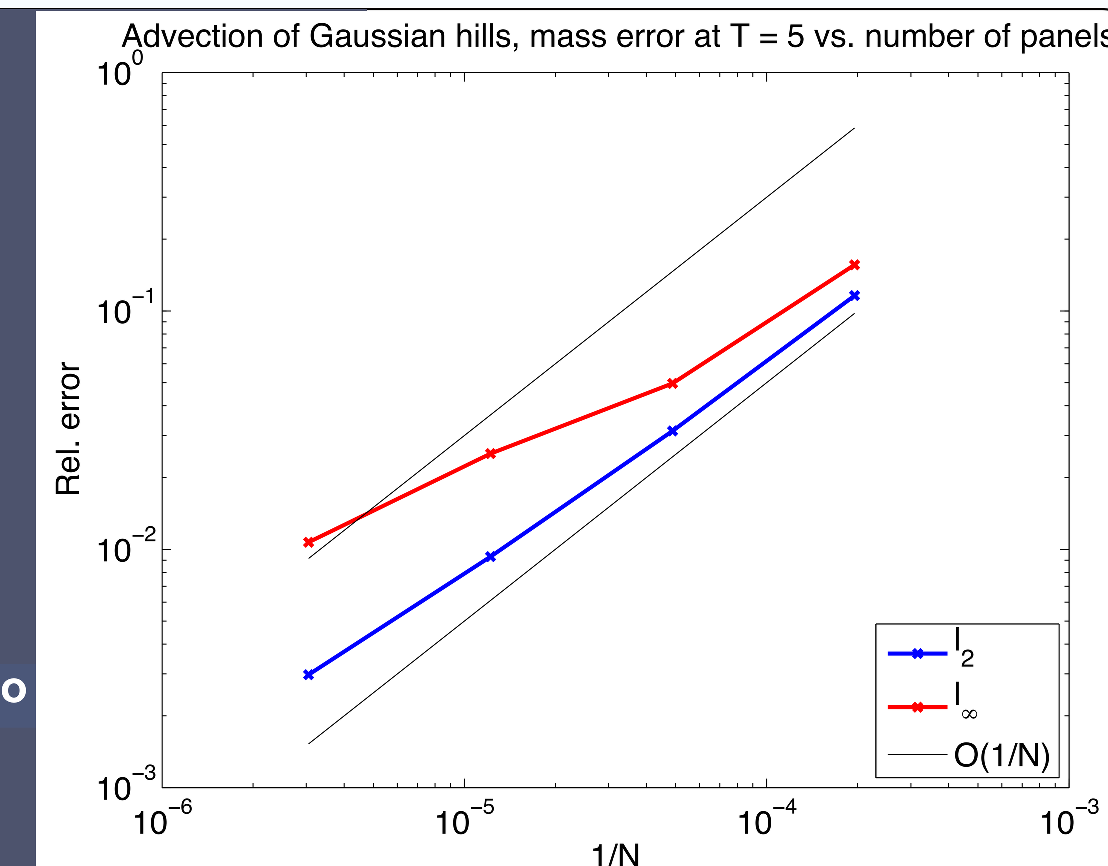
In the advection equation, velocity is prescribed from a known function, *F*, as in Lauritzen et al. (2012), where *F* is designed to test a scheme's mass conservation, or to simulate a realistic flow. A mass distribution, ϕ, is conserved materially.

$$\frac{D}{Dt}\boldsymbol{x}_{i,j}(\boldsymbol{\alpha}_{i,j},t) = F(\boldsymbol{x}_{i,j},t)$$

$$\phi_{i,j} = \phi(\boldsymbol{\alpha}_{i,j})$$

The Lagrangian parameter, α, is independent of time, which allows us to treat mass concentrations $\phi_{i,j}$ at each particle as constants.





Advection of a set of Gaussian hills. This test is a reversible flow with period T = 5, so that the mass distribution at t = 5 should be identical to that at t = 0 . Figures depict a quadrilateral mesh with N = 24,576. Remesh applied at t = 0.25, t = 0.5, t = 0.75,... Time integrated with RK4 and dt = 0.00125.

Barotropic vorticity equation

$$\boldsymbol{u}(\boldsymbol{x},t) = -\frac{1}{4\pi}\int_{\mathcal{S}^2} \frac{\boldsymbol{x}\times\tilde{\boldsymbol{x}}}{1-\boldsymbol{x}\cdot\tilde{\boldsymbol{x}}}\zeta(\tilde{\boldsymbol{x}})\,dA(\tilde{\boldsymbol{x}})$$

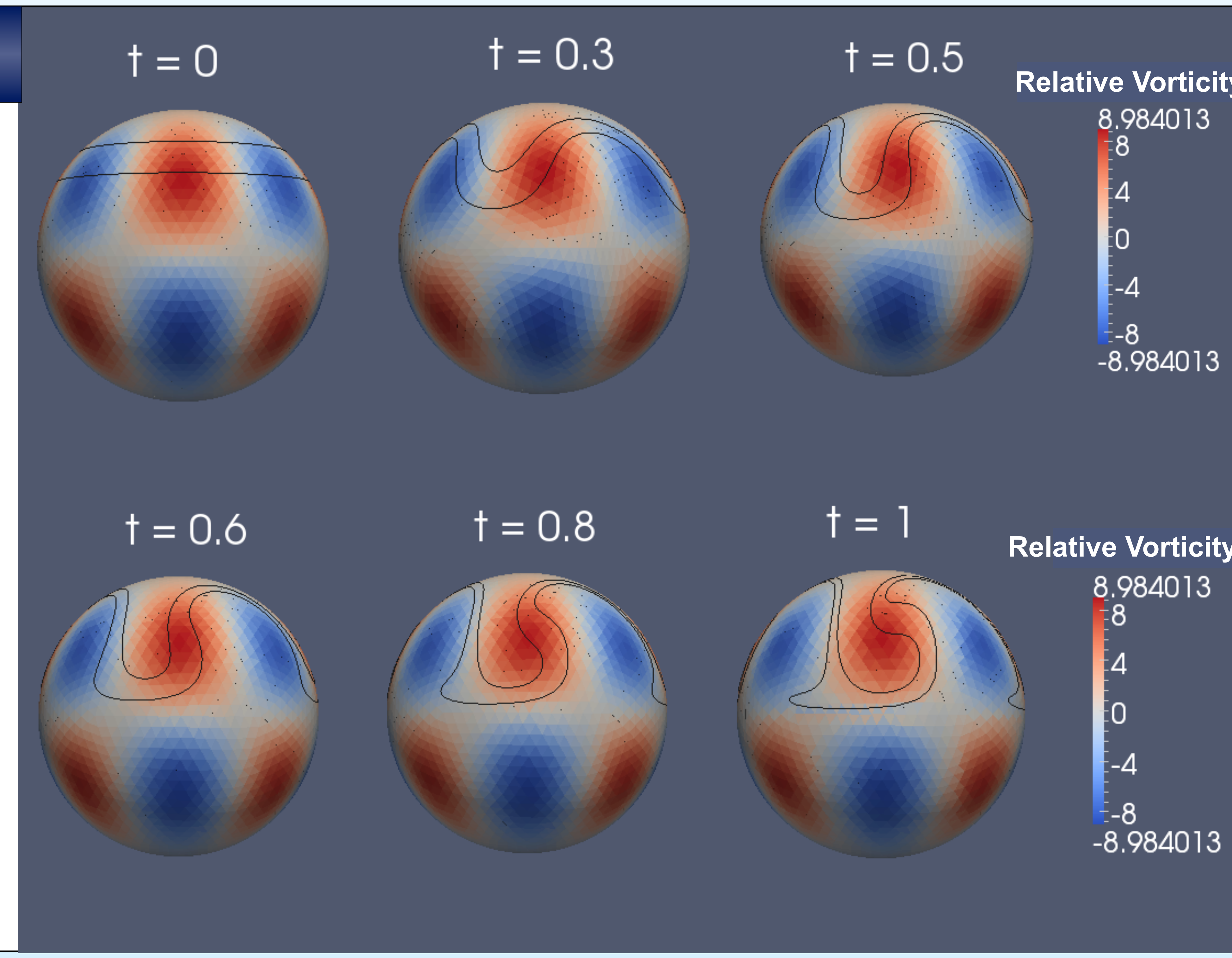
$$\frac{D\zeta}{Dt} = -2\Omega\frac{Dz}{Dt}$$

In the barotropic vorticity equation, the equations represent point vortices; velocity is given by a Biot-Savart integral (Bogomolov, 1977). Relative vorticity, ζ, is modified by the background rotation of the sphere, which has angular velocity Ω.

$$\frac{D\boldsymbol{x}_j}{Dt} = -\frac{1}{4\pi}\sum_{\substack{k=1\\k\neq j}}^{N}\frac{\boldsymbol{x}_j\times\boldsymbol{x}_k}{1-\boldsymbol{x}_j\cdot\boldsymbol{x}_k}\zeta_k A_k \quad j = 1,\ldots,N$$

$$\frac{D\boldsymbol{y}_i}{Dt} = -\frac{1}{4\pi}\sum_{k=1}^{N}\frac{\boldsymbol{y}_i\times\boldsymbol{x}_k}{1-\boldsymbol{y}_i\cdot\boldsymbol{x}_k}\zeta_k A_k \quad i = 1,\ldots,M$$

$$\frac{D\zeta_j}{Dt} = -2\Omega\frac{Dz_j}{Dt}$$



Solutions of the barotropic vorticity equation for the problem of a stationary Rossby-Haurwitz wave (Haurwitz, 1940). The red-blue pattern of vorticity should remain fixed, even while fluid particles travel in the velocity field induced by the wave. Black lines depict material curves, initially located on latitude contours.

Triangular mesh with N = 5,120 panels; RK4 with dt = 0.01; $\Omega = 2\pi$ remesh at t = 0.2, 0.4, 0.6, ...

Bogomolov, V.A. 1977. "Dynamics of vorticity at the sphere." *Fluid Dynamics*, 6 : 863 - 870.

Haurwitz, B. 1940. "The motion of atmospheric disturbances on a spherical Earth." *Journal of Marine Research*, 3 : 254-267.

Lauritzen, P.H., Skamarock, W.C., Prather, M.J., and Taylor, M.A. 2012. "A standard test case suite for two-dimensional linear transport on the sphere." *Geoscientific Model Development Discussions*, 5 : 189-228.

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