Towards an enstrophy-conserving deep-atmosphere quasi-hydrostatic dynamical core

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Atmosphere dynamics of our planet is quite well described by traditional primitive equations based on the so-called shallow atmosphere approximation. However, to model planetary atmospheres, we can not make the shallow approximation anymore because of the low planet radius (such as Titan) or the depth of their atmospheres (such as Jupiter or Saturne). The full Coriolis force needs then to be taken into account, in addition to all metric terms [3]. Non-traditional terms have little-known dynamical effects [8] and are rarely integrated into general circulation dynamical cores [4].

The goal of the present work is to incorporate the quasi-hydrostatic equations into a primitive-equation dynamical core while preserving the discrete conservation of potential vorticity [1, 2]. For this we derive the vector-invariant form of the equations in general, time-dependant curvilinear coordinates. This suggests to use absolute angular momentum instead of relative velocity as a prognostic variable. Furthermore the modification of the hydrostatic balance requires a mass-based vertical coordinate [5]. This formulation leads to a straightforward generalization of Sadourny's discretization [1], reusing most of the primitive-equation dynamical core.

Motivations

White and Bromley [3] pointed out the fact that the traditional equations are dynamically consistent when the following hypothesis are both made : the shallow atmosphere approximation ($r \to R$) AND the $\cos\phi$ Coriolis force can be neglected.

$$D_t u - (2\Omega + \frac{u}{r\cos\phi})(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\partial_\lambda p = 0$$

$$D_t v - (2\Omega + \frac{u}{r\cos\phi})u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\partial_\phi p = 0$$

$$-\left(2\Omega + \frac{u}{r\cos\phi}\right)u\cos\phi - \frac{v^2}{r} + g + \frac{1}{\rho}\partial_r p = 0$$

That approximation is relevant to investigate the climate on Earth because of its low ratio between atmosphere thickness and Earth's radius, but it's not for Titan, the largest moon of Saturn (FIG. 1).

	Earth	Titan
Ratio	$\sim 1.5\%$	$\sim 25\%$
g variability	$\sim 2.8\%$	$\sim 53.4\%$

FIG. 1 Data from NASA website

The last line corresponds to variation of equatorial gravity between the surface and the top of the mesosphere.

The present work is based around the general circulation model (GCM) LMD-Z [6] where the traditional primitive equations are discretized on a longitude/latitude horizontal grid and with a pressure-based coordinate. The spatial discretization is derived from a finite difference scheme of the vector-invariant form of the primitive equations.

Then, the potential vorticity is exactly conserved [1, 2] and it's considered as an essential property of a robust GCM. That's why we generalize Sadourny's scheme [1, 2] to derive the non traditional 3D quasi-hydrostatic equations to get an enstrophy-conserving deep atmosphere dynamical core.

Part 1: Derivation from vector-invariant form in general, time-dependant curvilinear coordinates

In a general, time-dependant, curvilinear coordinates system $(a_1, a_2, a_3, t) \to \underline{x}(a_1, a_2, a_3, t)$, the 3D quasi-hydrostatic momentum equations are equivalent to the following vector-invariant form :

$$\partial_t \tilde{\mathbf{v}} + (\nabla \times \tilde{\mathbf{v}}) \times (\hat{\mathbf{u}} - \hat{\mathbf{w}}) + \nabla(K + \Phi - \mathbf{v}.\mathbf{w}) + \frac{1}{\rho}\nabla p = 0$$

Primal basis (covariant)	Dual basis (contravariant)
$\mathbf{e}_i = \partial_i \mathbf{x}$	$\mathbf{f}_i = \dfrac{\mathbf{e}_{i+1} \times \mathbf{e}_{i-1}}{J}$
and $\tilde{u}_i = \mathbf{e}_i \cdot \mathbf{u}$	and $\hat{u}_i = \mathbf{f}_i.\mathbf{u}$

Jacobian $J = \mathbf{e}_i \cdot \mathbf{f}_i$

On the sphere : λ-longitude/ ϕ-latitude horizontal grid and η-hybrid vertical coordinate

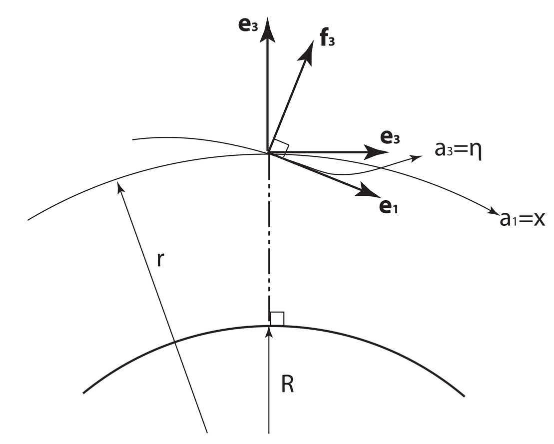

FIG. 2 Curvilinear coordinates

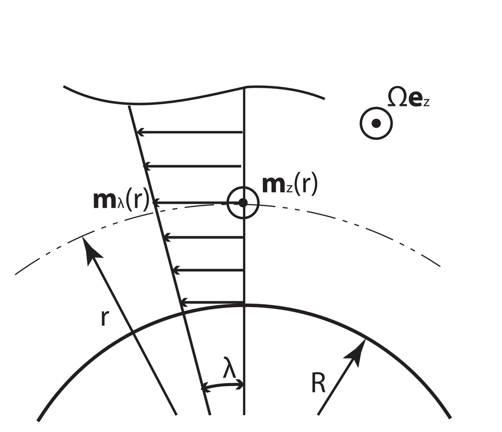

FIG. 3 Vertical variation of angular momentum

▷ **Spherical geometry**
Mapping : $(\lambda, \phi, r, t) \to (x(\lambda), y(\phi), \eta(\lambda, \phi, r), t)$
$c_u = r\cos\phi\,\partial_x\lambda$, $c_v = r\partial_y\phi$ and $J = r^2\partial_x\lambda\partial_y\phi\partial_\eta r\cos\phi$

▷ **The absolute horizontal velocity covariant part is the ABSOLUTE ANGULAR MOMENTUM** :

$$\tilde{v}_1 = c_u(u + \Omega r\cos\phi) = m_z \text{ and } \tilde{v}_2 = c_v v = m_\lambda$$

With that formulation, the metric terms - which are time-dependent - are incorporate in the covariant velocity component. By taking into account these terms we actually consider vertical variation of angular momentum (FIG. 3).

▷ **The relative horizontal velocity contravariant part is the RELATIVE ANGULAR VELOCITY** :

$$\hat{u}_1 = \frac{u}{c_u} = \dot{\lambda} \text{ and } \hat{u}_2 = \frac{v}{c_v} = \dot{\phi}$$

Then the horizontal momentum equations are equivalent to :

$$\partial_t\tilde{v}_1 + \frac{W}{m}\partial_3\tilde{v}_1 - Vq + \partial_x(\Phi + K) + \theta\partial_x\Pi = 0$$

$$\partial_t\tilde{v}_2 + \frac{W}{m}\partial_3\tilde{v}_2 + Uq + \partial_y(\Phi + K) + \theta\partial_y\Pi = 0$$

mass per element	$m = \rho c_u c_v \partial_\eta r$
horizontal mass fluxes	$U = m\hat{u}_1, V = m\hat{u}_2$
vertical mass flux	$W = m(\hat{u}_3 - \hat{w}_3)$
conserved potential vorticity	$q = \dfrac{\partial_y\tilde{v}_2 - \partial_x\tilde{v}_1}{m}$

Part 2 : Modification of hydrostatic balance

In the shallow case, pressure p at a given altitude is the weight per unit area of the hydrostatic atmosphere column above while for a deep atmosphere, because of the additional lateral forces, it's not (FIG. 4).

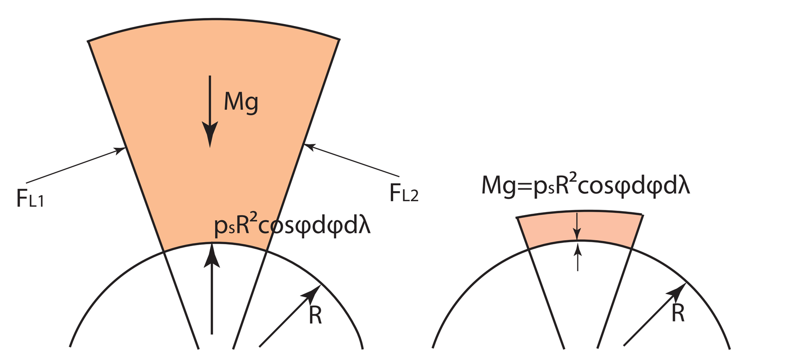

FIG. 4 Surface pressure for deep vs shallow case

In the deep case, because of the spherical geometry and the modification of hydrostatic balance, we can't use a pressure-based vertical coordinate anymore. Wood and Stanisforth [4] used a mass-based coordinate $M = \int_\eta^{top}\rho J d\eta' = \int_\eta^{top} m d\eta'$.

$$m = \rho J = \rho r^2\cos\phi\partial_x\lambda\partial_y\phi\partial_\eta r \;\Rightarrow\; \partial_\eta r^3 = \frac{3m}{\rho(p,\theta)\cos\phi} = \frac{3mR\theta p^{\kappa-1}}{p_r^\kappa\cos\phi}$$

The quasi-hydrostatic equation is :

$$\partial_\eta p = \frac{m}{r^2\cos\phi}\left(-g + \frac{u^2 + v^2}{r} + 2u\Omega\cos\phi\right)$$

The previous equations are coupled by density ρ which is pressure dependent and they have to be solved simultaneously by an iteration processing. The pressure r is integrated from the top ($p = 0$) to the bottom while the vertical position r is integrated from the bottom ($r = R$) to the top of the atmosphere.

Part 3 : Horizontal discretization

We based the vertical discretization on a mass coordinate $M(\eta)$ which is processed as the pressure p on the interface of each vertical layer. Other dynamical quantities are positioned at the mid-layers. The horizontal mesh is a longitude/latitude C-grid (FIG. 5) so that Exner function Π, geopotential Φ, and potential temperature are processed on the primal mesh and velocity and potential vorticity on the dual mesh.

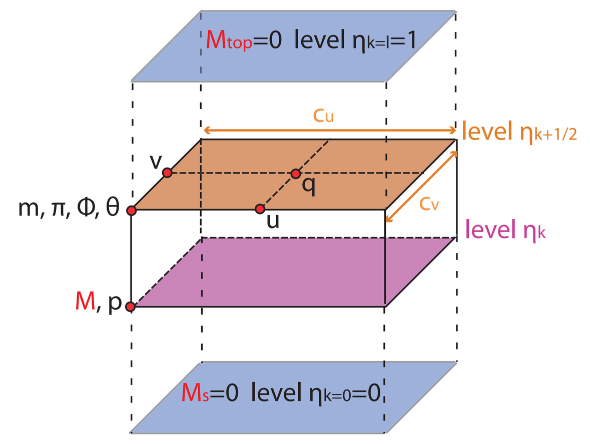

FIG. 5 Horizontal C-grid and vertical discretizations

Algorithm

▷ **Initialization**
− vertical discretization : $a(\eta)$, $b(\eta)$,
− geometry : $c_u = r\delta_x\lambda$, $c_v = r\cos\phi\delta_y\phi$,
− prognostic variables : $\tilde{v}_1 = c_u(u + \Omega\cos\phi r)$, $\tilde{v}_2 = c_v v$, $m\theta$ and M_s.

▷ **Mass coordinate and diagnostic of p and r : iterative resolution**
$M = a + bM_s$
$m = \delta_k M$

$$\begin{cases} \delta_k p = \dfrac{m_k}{r_k^2\cos\phi}\left(-g + \dfrac{u_k^2 + v_k^2}{r_k} + 2u_k\Omega\cos\phi\right) \\ \delta_\eta r_k^3 = \dfrac{3m_k}{\rho_k(p_k, \theta_k)\cos\phi} \end{cases}$$

▷ **Diagnostic for W, Π, Φ, K, Z**

$$\delta_t M_s = -\textstyle\sum_k(\delta_x U_k + \delta_y V_k)$$
$$\delta_k W = -\delta_x U_k - \delta_y V_k - \delta_k b\delta_t M_s$$
$$\Phi_k = gr_k$$
$$\delta_k\Pi = \frac{\kappa\Pi_k\delta_k p}{p_k}$$
$$Z_k = \frac{\delta_x\tilde{v}_2 - \delta_y\tilde{v}_1}{m_k}$$

▷ **Prognostic for $m\theta$, $\tilde{v}_1$, $\tilde{v}_2$, M_s**

$$\delta_t(m_k\theta_k) = -\delta_x(U_k\theta_k) - \delta_y(V\theta_k) - \delta_k(W\theta)$$
$$\delta_t\tilde{v}_1 = -\frac{W_k}{m_k}\delta_k\tilde{v}_1 + Z_k V_k - \delta_x(\Phi_k + K_k) - \theta_k\delta_x\Pi$$
$$\delta_t\tilde{v}_2 = -\frac{W_k}{m_k}\delta_k\tilde{v}_2 - Z_k U_k - \delta_y(\Phi_k + K_k) - \theta_k\delta_y\Pi$$

Conclusions and future work

We derived the vector-invariant form of the non-traditional (deep) primitive equations which incorporate the metric terms and the whole Coriolis force into the prognostic variables. We got preliminary results obtained on a prototype implementation of the method into the GCM LMD-Z and performed baroclinic test case [3] with no physics and on the Earth (no fundamental differences with the shallow case as expected) but the work is still under way to adapt the GCM for deep atmospheres.

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