Regular expressions: derivations

1. Which of the following statements is true? If it is true, give a derivation; if not, explain.

   (a) $a \in \mathsf{L}(a+b)$

   Solution:

   $$\dfrac{\dfrac{}{a \in \mathsf{L}(a)}\,\text{Char}}{a \in \mathsf{L}(a+b)}\,\text{Left}$$

   (b) $ab \in \mathsf{L}((a+b))$

   Solution: There is no derivation: $a+b$ can only match one character, where the string ab has two.

   (c) $ab \in \mathsf{L}((a+b)(a+b))$

   Solution:

   $$\dfrac{\dfrac{\dfrac{}{a \in \mathsf{L}(a)}\,\text{Char}}{a \in \mathsf{L}(a+b)}\,\text{Left} \qquad \dfrac{\dfrac{}{b \in \mathsf{L}(b)}\,\text{Char}}{b \in \mathsf{L}(a+b)}\,\text{Right}}{ab \in \mathsf{L}((a+b)(a+b))}$$

   (d) $aa \in \mathsf{L}(a+a)$

   Solution: There is once again no solution: aa has two characters whereas $a+a$ matches at most one.

   (e) $\varepsilon \in \mathsf{L}(b*)$

   Solution:

   $$\dfrac{}{\varepsilon \in \mathsf{L}(b*)}\,\text{Stop}$$

   (f) $b \in \mathsf{L}(b*)$

   Solution:

   $$\dfrac{\dfrac{}{b \in \mathsf{L}(b)}\,\text{Char} \qquad \dfrac{}{\varepsilon \in \mathsf{L}(b*)}\,\text{Stop}}{b \in \mathsf{L}(b*)}\,\text{Step}$$

   (g) $bb \in \mathsf{L}(b*)$

   Solution:

   $$\dfrac{\dfrac{}{b \in \mathsf{L}(b)}\,\text{Char} \qquad \dfrac{\dfrac{}{b \in \mathsf{L}(b)}\,\text{Char} \qquad \dfrac{}{\varepsilon \in \mathsf{L}(b*)}\,\text{Stop}}{b \in \mathsf{L}(b*)}\,\text{Step}}{bb \in \mathsf{L}(b*)}$$

Regular expressions: properties

Two regular expressions r and r' are equivalent if forall xs, $xs \in \mathsf{L}(r)$ if and only if $xs \in \mathsf{L}(r')$.

Prove the following regular expressions are equivalent, for all regular expression a, b, c.

Clearly state how the proof is constructed, either by using rule induction or applying rules. When using rule induction, state the cases and hypotheses available at every step:

(a) a and $a + 0$

> **Solution:** We do need to prove the implication in two directions:
>
> - Suppose $xs \in \mathsf{L}(a)$. From the Left rule, we can prove $a + 0 \in \mathsf{L}()$.
> - Suppose $xs \in \mathsf{L}(a + 0)$. We do rule induction on our hypothesis and distinguish two cases:
>   - If our hypothesis was built using the Left rule, we may assume $xs \in \mathsf{L}(a)$ – which is exactly what we aiming to prove.
>   - If our hypothesis was built using the Right rule, we may assume $xs \in \mathsf{L}(0)$ – but no such derivation can exist. From this false hypothesis, we can conclude our goal.

(b) $a + a$ and a

> **Solution:** We do need to prove the implication in two directions:
>
> - Suppose $xs \in \mathsf{L}(a)$. From the Left rule, we can prove $a + a \in \mathsf{L}()$.
> - Suppose $xs \in \mathsf{L}(a + a)$. We do rule induction on our hypothesis and distinguish two cases:
>   - If our hypothesis was built using the Left rule, we may assume $xs \in \mathsf{L}(a)$ – which is exactly what we aiming to prove.
>   - If our hypothesis was built using the Right rule, we may assume $xs \in \mathsf{L}(a)$ – which is exactly what we aiming to prove.

(c) $a + b$ and $b + a$

> **Solution:** We do need to prove the implication in two directions:
>
> - Suppose $xs \in \mathsf{L}(a + b)$. We do rule induction on our hypothesis and distinguish two cases:
>   - If our hypothesis was built using the Left rule, we may assume $xs \in \mathsf{L}(a)$. By applying the Right rule, we show $xs \in \mathsf{L}(a + b)$.
>   - If our hypothesis was built using the Right rule, we may assume $xs \in \mathsf{L}(b)$. By applying the Left rule, we show $xs \in \mathsf{L}(a + b)$.
> - The other case is entirely symmetrical.

(d) $a + (b + c)$ and $(a + b) + c$

> **Solution:** This proof follows the previous one closely: rule induction on the hypothesis, followed by applications of Left and/or Right.

(e) $1a$ and a

Solution:

- Suppose $xs \in \mathsf{L}(1a)$. By rule induction, this proof is necessarily built from the Seq rule. Hence, we learn $ys \in \mathsf{L}(1)$ and $zs \in \mathsf{L}(a)$ for some ys and zs that satisfy $ys +\!\!+ zs = xs$. As there is only one rule for 1, we know that $ys = \varepsilon$ and hence also $zs = xs$. So our second hypothesis is now $xs \in \mathsf{L}(a)$ – which is exactly what we needed to prove.
- Suppose $xs \in \mathsf{L}(a)$. Applying the Seq and One rules, we can show $xs \in \mathsf{L}(1a)$.

(f) $(a^*)^*$ and a^*

Evaluation of lambda terms

Given the following definitions:

$$\begin{aligned} I &= \lambda x.x \\ K &= \lambda xy.x \\ S &= \lambda xyz.(xz)(yz) \end{aligned}$$

Given a derivation of following terms to a normal form, using the rules presented in class:

1. Ia

Solution:

$$(\lambda x.x)(a) \to a$$

2. $KIab$

Solution:

$$\begin{aligned} (((\lambda xy.x)(\lambda x.x))a)b &\quad \to \\ ((\lambda y.(\lambda x.x))a)b &\quad \to \\ (\lambda x.x)b &\quad \to \\ b & \end{aligned}$$

3. $(IK)(II)$

Solution:

$$\begin{aligned} ((\lambda x.x)(\lambda xy.x))((\lambda x.x)(\lambda x.x)) &\quad \to \\ (\lambda xy.x)((\lambda x.x)(\lambda x.x)) &\quad \to \\ (\lambda xy.x)(\lambda x.x) &\quad \to \\ (\lambda yx.x) & \end{aligned}$$

4. $S(K(Ka))(Kb)c$

Solution: I've sketched the solution below, but haven't unfolded the definitions as I did for the exercises above.

$$
\begin{aligned}
S(K(Ka))(Kb)c &\quad \to \\
(K(Ka)c)(Kbc) &\quad \to \\
(Ka)(Kbc) &\quad \to \\
(Ka)b &\quad \to \\
&\quad a
\end{aligned}
$$

The typed lambda calculus

Let Γ be an environment including:

- one : N
- isEven : $N \to B$
- not : $B \to B$
- add : $N \to N \to N$

Give typing derivations for the following terms using the rules presented in class:

1. isEven one

Solution:

$$
\dfrac{\dfrac{}{\Gamma \vdash \text{isEven} : N \to B}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash \text{one} : N}\,\text{Var}}{\Gamma \vdash \text{isEven} : N \to B}\,\text{App}
$$

2. add one one

Solution:

$$
\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \text{add} : N \to N \to N}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash \text{one} : N}\,\text{Var}}{\Gamma \vdash \text{add one} : N \to N}\,\text{App} \qquad \dfrac{}{\Gamma \vdash \text{one} : N}\,\text{Var}}{\Gamma \vdash \text{add one one} : N}\,\text{App}
$$

3. $\lambda x : B.\, \text{not}(\text{not}\ x)$

Solution:

$$
\dfrac{\dfrac{\dfrac{}{\Gamma, x : B \vdash \text{not} : B \to B}\,\text{Var} \qquad \dfrac{\dfrac{}{\Gamma, x : B \vdash \text{not} : B \to B}\,\text{Var} \qquad \dfrac{}{\Gamma, x : B \vdash x : B}}{\Gamma, x : B \vdash \text{not x} : B}\,\text{App}}{\Gamma, x : B \vdash \text{not}(\text{not}\ x) : B}\,\text{App}}{\Gamma \vdash \lambda x : B.\ \text{not}(\text{not}\ x) : B \to B}\,\text{Lam}
$$

4. $\lambda x : N.\,\text{one}$

Solution:

$$\dfrac{\dfrac{}{\Gamma, x : N \vdash \text{one} :\ N}\ \text{Var}}{\Gamma \vdash \lambda x : N.\ \text{one} : N \to N}\ \text{Lam}$$

5. $\lambda x : N.\lambda y : N.\text{isEven}\ x$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, x : N, y : N \vdash \text{isEven} : N \to B}\ \text{Var} \qquad \dfrac{}{\Gamma, x : N, y : N \vdash x : N}\ \text{Var}}{\Gamma, x : N, y : N \vdash \text{isEven}\ x : B}\ \text{App}}{\Gamma, x : N \vdash \lambda y : N.\ \text{isEven}\ x : N \to B}\ \text{Lam}}{\Gamma \vdash \lambda x : N \lambda y : N.\ \text{isEven}\ x : N \to N \to B}\ \text{Lam}$$

6. $\lambda x : (N \to N).\text{not}$

Solution:

$$\dfrac{\dfrac{}{\Gamma, x : N \to N \vdash \text{not} : B \to B}\ \text{Var}}{\Gamma \vdash \lambda x : (N \to N).\ \text{not}\ (N \to N) \to (B \to B)}\ \text{Lam}$$