Deformation of L_∞-Algebras

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Abstract

In this paper, deformations of L_∞-algebras are defined in such a way that the bases of deformations are L_∞-algebras, as well. A universal and a semiuniversal deformation is constructed for L_∞-algebras, whose cotangent complex admits a splitting. The paper also contains an explicit construction of a minimal L_∞-structure on the homology H of a differential graded Lie algebra L and of an L_∞-quasi-isomorphism between H and L.

Contents

Introduction

L_∞-algebras (see Section 1) play a crucial role in deformation theory. They are natural generalizations of differential graded Lie algebras (DGLs). Deformation problems can always be described by DGLs (see [7], for instance). The importance of L_∞-algebras in deformation theory comes from the fact that two different deformation problems are equivalent, if the corresponding DGLs are equivalent as L_∞-algebras. This was one ingredient of Kontsevich's [7] prove that deformation quantization works on each Poisson manifold. L_∞-algebras also build a bridge from algebra to geometry. A simple shift of degrees makes a formal DG-manifold out of an L_∞-algebra (see Section 1). This observation is also due to Kontsevich. If a deformation problem is governed by a DGL L, then the (formal) local muduli space, if it exists, is an analytic subspace of the formal DG-manifold corresponding to L.

In the other way, to each DGL L, one can define an abstract deformation functor Def_L. In the classical theory Def_L is a set-valued functor on the category of (Artinian, local) algebras. Recent studies in mirror symmetry ([8], [17]) led to an extension of this functor first to graded and then differential graded Artinian algebras. The aim of this extension is to produce smooth (in a sense) formal moduli spaces with tangent space, isomorphic to the whole cohomology of L. But sometimes it is not evident (or not even possible) to give an algebraic or geometric meaning to the objects obtained by the extended deformation functor. (Sometimes this is possible. For the classical deformations of associative algebras, the extended deformation functor produces the more general A_∞-algebras.)

The deformation theory of L_∞-algebras (or in geometric terms, of formal DG-manifolds) presented in this paper is in fact an extended deformation theory of (formal) singularities. Instead of working with deformation functors, we present a completely geometric (extended) deformation theory of formal DG manifolds. The bases of deformations are formal DG manifolds as well. The theory is developed analogous to "embedded deformations" of singularities. The deformations of a given formal DG-manifold $M = (M, Q^M)$ are governed by the DGL L of formal vectorfields on M (see Section 2). The degree 1 shift of L is again a formal DG manifold, denoted by U. There are two nice observations. The first is that the going over $M \mapsto U$ doesn't change the category. (This one is trivial.) The second (Theorem 2.9) is that U is the base of a universal deformation of M. For the construction of a semiuniversal deformation of M, we have to construct an L_∞-structure on the homology H of (L, d), such that H and L are equivalent as L_∞-algebras. H with such an L_∞-structure is called a minimal model for L.

Hence, the essence of this paper is the following general recipe for the construction of (formal) analytic moduli spaces: Take a minimal representative in the class of L_∞-algebras modulo L_∞-equivalence of the DGL controlling the deformation problem. This recipe had been discovered before (see [10], [16]) and was rediscovered independently by the author.

The contents of this paper: In Section 1 we remind the definitions of L_∞-algebras and of their correspondence with differential graded coalgebras. We state the conditions for a sequence of maps, to define an L_∞-morphism. We will prove those conditions in the Appendix, since they are hard to find in the literature. Then we remind Kontsevich's geometric point of view (=formal DG manifolds) of L_∞-algebras. In Section 2, we define deformations of formal DG manifolds with formal DG bases and morphisms of those. Our definition generalizes the one of Fialowski and Penkava [2]. We show that for an arbitrary formal DG manifold M, the differential graded Lie algebra $\mathrm{Coder}(S(M), S(M))$ (which we call tangent complex of M) is a base of a universal deformation of M. In Section 3, we give an ad-hoc combinatorial introduction to binary trees. In a sense, binary trees contain the algebraic structure of L_∞-algebras (see [21]). In Section 4, they are used to define an L_∞-structure μ_* on the homology H of a differential graded Lie algebra $L = (L, d, [\cdot, \cdot])$ (admitting a splitting). Furthermore, again with the help of binary trees, in Section 4, we construct explicitly an L_∞-quasi-isomorphism between (H, μ_*) and $(L, d, [\cdot, \cdot])$. Similar constructions

in the A_∞-context are due to Gugeheim/Stasheff [3], Merkulov [18] and Kontsevich/Soibelman [8]. In Section 5, we prove that $(L, d, [\cdot, \cdot])$ is as L_∞-algebra isomorphic to the direct sum of the L_∞-algebras (H, μ_*) and (F, d), where F is the complement of H in L. As consequence, we can show that for each formal DG manifold M such that $L = \mathrm{Coder}(S(M), S(M))$ splits, the shift V of (H, μ_*) is the base of a semi-universal deformation of M.

1 L_∞-Algebras and Coalgebras

In this paper we shall always work over a ground ring k of characteristic zero.

1.1 Graded Symmetric and Exterior Algebras

For a graded module W, the graded symmetric algebra $S(W)$ is defined as the tensor algebra $T(W) = \oplus_{n \geq 0} W^{\otimes n}$ modulo the relations $w_1 \otimes w_2 - (-1)^{w_1 w_2} w_2 \otimes w_1 = 0$. We denote the graded symmetrical product by $\odot$. The algebra $T(W)$ (resp.$S(W)$) is bigraded. The graduation on $T(W)$ (resp. $S(W)$), defined by $g(w_1 \otimes ... \otimes w_n) = g(w_1) + ... + g(w_n)$ (resp. $g(w_1 \odot ... \odot w_n) = g(w_1) + ... g(w_n)$), where g is the graduation of W, will be called **linear graduation**. The one defined by $g(w_1 \otimes ... \otimes w_n) = n$ (resp.$g(w_1 \odot ... \odot w_n) = n$), will be called **polynomial graduation**. Set $S_+(W) := \oplus_{n \geq 1} W^{\odot n}$. On $S_+(W)$, there is a natural k-linear comultiplication $\Delta^+ : S_+(W) \longrightarrow S_+(W) \otimes S_+(W)$, given by

$$w_1 \odot ... \odot w_n \mapsto \sum_{j=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(j,n)} \epsilon(\sigma, w_1, ..., w_n) w_{\sigma(1)} \odot \ldots \odot w_{\sigma(j)} \otimes w_{\sigma(j+1)} \odot \ldots \odot w_{\sigma(n)}.$$

Here $\epsilon(\sigma) := \epsilon(\sigma, w_1, ..., w_n)$ is defined such that $w_{\sigma(1)} \odot \ldots \odot w_{\sigma(n)} = \epsilon(\sigma) w_1 \odot \ldots \odot w_n$. Note that we have $\mathrm{Kern}\, \Delta^+ = W$. On $S(W)$, there is a k-linear comultiplication Δ, defined by $\Delta(1) := 1 \otimes 1$ and $\Delta(\underline{w}) := \underline{w} \otimes 1 + \Delta'(\underline{w}) + 1 \otimes \underline{w}$, for $\underline{w} \in S_+(W)$. Note that Δ is injective.

For a graded module L, the graded exterior algebra $\bigwedge^+ L$ without unit is defined as the tensor algebra $T_+(L) = \oplus_{n \geq 1} L^{\otimes n}$ modulo the relations $a_1 \otimes a_2 + (-1)^{a_1 a_2} a_2 \otimes a_1 = 0$. We denote the graded exterior product by $\wedge$. $L[1]$ denotes the graded module with $L[1]^i = L^{i+1}$ and $\downarrow$ the canonical map $L \longrightarrow L[1]$ of degree -1. Set $\uparrow := \downarrow^{-1}$. Remark that, for each $n \geq 1$, there is an isomorphism

$$\downarrow^n : \bigwedge^n L \longrightarrow L[1]^{\odot n}$$

$$a_1 \wedge ... \wedge a_n \mapsto (-1)^{(n-1) \cdot a_1 + ... + 1 \cdot a_{n-1}} \downarrow a_1 \odot \ldots \odot \downarrow a_n$$

Its inverse map is given by $(-1)^{\frac{n(n-1)}{2}} \uparrow^n$. As we shall always do, we just have applied the Koszul sign convention, i.e. for homogeneous graded morphisms f, g of graded modules, we set $(f \otimes g)(a \otimes b) := (-1)^{ga} f(a) \otimes g(b)$. In the exponent, a always means the degree of an homogeneous element (or morphism) a and ab means the product of degrees, not the degree of the product.

For $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in L$, we define the sign $\chi(\sigma) := \chi(\sigma, a_n, \ldots, a_n)$ in such a way that

$$a_{\sigma(1)} \wedge ... \wedge a_{\sigma(n)} = \chi(\sigma) a_1 \wedge ... \wedge a_n.$$

We have the following correlation between χ and ϵ:

Remark 1.1. For $a_1, ..., a_n \in L$, we have

$$\chi(\sigma, a_1, \ldots, a_n) = (-1)^{(n-1)(a_1 + a_{\sigma(1)}) + ... + 1 \cdot (a_{n-1} + a_{\sigma(n-1)})} \epsilon(\sigma, \downarrow a_1, \ldots, \downarrow a_n).$$

For a graded module V, we define two different actions of the symmetric group Σ_n on $V^{\otimes n}$: The first one is given by

$$\Sigma_n \times V^{\otimes n} \longrightarrow V^{\otimes n}$$
$$(\sigma, v_1 \otimes ... \otimes v_n) \mapsto \epsilon(\sigma, v_1, ..., v_n) v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}$$

Here, the application of a σ commutes with the canonical projection $V^{\otimes n} \longrightarrow V^{\odot n}$. The second one is given by

$$\Sigma_n \times V^{\otimes n} \longrightarrow V^{\otimes n}$$
$$(\sigma, v_1 \otimes ... \otimes v_n) \mapsto \chi(\sigma, v_1, ..., v_n) v_{\sigma(1)} \otimes ... \otimes v_{\sigma(n)}$$

Here, the application of a σ commutes with the canonical projection $V^{\otimes n} \longrightarrow \wedge^n V$. When we work with symmetric powers, we use the first action; when we work with exterior powers, we use the second one. Since the context shall always be clear, we don't distinguish both actions by different notations. We will use the anti-symmetrisation maps:

$$\alpha_n := \sum_{\sigma \in \Sigma_n} \sigma. : V^{\otimes n} \longrightarrow V^{\otimes n}.$$

When $\sigma.$ denotes the first action, α_n can be seen as map $V^{\odot n} \longrightarrow V^{\otimes n}$; when $\sigma.$ denotes the second action, α_n can be seen as map $\wedge^n V \longrightarrow V^{\otimes n}$. Furthermore, for both cases, we will use the maps

$$\alpha_{k,n} := \sum_{\sigma \in \mathrm{Sh}(k,n)} \sigma. : V^{\otimes n} \longrightarrow V^{\otimes n}.$$

For the natural projection $\pi : W^{\otimes n} \longrightarrow W^{\odot n}$ (resp. $V^{\otimes n} \longrightarrow \bigwedge^n V$), we have

$$\pi \circ \alpha = n! \, \mathrm{Id} \, .$$

1.2 Free Differential Graded Coalgebras

Let (C_1, Δ_1) and (C_2, Δ_2) be coalgebras. Remember that a module homomorphism $F : C_1 \longrightarrow C_2$ is a coalgebra map, iff the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\;F\;} & C_2 \\ \downarrow{\scriptstyle \Delta_1} & & \downarrow{\scriptstyle \Delta_2} \\ C_1 \otimes C_1 & \xrightarrow{F \otimes F} & C_2 \otimes C_2 \end{array} \qquad (1.1)$$

commutes. Each coalgebra morphism $F : (S(W), \Delta) \longrightarrow (S(W'), \Delta)$ satisfies $F(1) = 1$. The restriction $F \mapsto F|_{S_+(W)}$ gives is a 1:1-correspondence between coalgebra morphisms $(S(W), \Delta) \longrightarrow (S(W'), \Delta)$ and coalgebra morphisms $F : (S_+(W), \Delta^+) \longrightarrow (S_+(W'), \Delta^+)$.

The next proposition gives a one-to-one correspondence between coalgebra maps $F : S(W) \longrightarrow S(W')$ and sequences of linear maps $F_n : S_n(W) \longrightarrow W'$, $n \geq 1$. We fix the following notations:

$$\hat{F}_n := F|_{W^{\odot n}} : W^{\odot n} \longrightarrow S(W')$$
$$F_{k,l} := \mathrm{pr}_{W'^{\odot l}} \circ \hat{F}_k : W^{\odot k} \longrightarrow W'^{\odot l}$$
$$F_n := F_{n,1} : W^{\odot n} \longrightarrow W'$$

Sometimes, we shall consider the maps F_n as antisymmetric maps $W^{\otimes n} \longrightarrow W$ instead of maps $W^{\odot n} \longrightarrow W$. For each multi-index $I = (i_1, ..., i_k) \in \mathbb{N}^k$, we set $I! := i_1! \cdot ... \cdot i_k!$ and $|I| := i_1 + ... + i_k$ and

$$F_I := \frac{1}{I!k!}(F_{i_1} \odot ... \odot F_{i_k}) \circ \alpha_n.$$

Here, by $F_{i_1} \odot ... \odot F_{i_k}$, we mean the composition of $F_{i_1} \otimes ... \otimes F_{i_k}$ and the natural projection $W'^{\otimes k} \longrightarrow W'^{\odot k}$.

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Proposition 1.2. For $n \geq 1$, we have that

$$\hat{F}_n = \sum_{k=1}^{n} \sum_{\substack{I \in \mathbb{N}^k}}^{|I|=n} F_I. \tag{1.2}$$

The proof can be found in the appendix.

A coalgebra homomorphism $F : S(W) \longrightarrow S(W')$ is called **strict**, if $F_n = 0$ for each $n \geq 2$.

For a coalgebra (C, Δ), remember that a module homomorphism $Q : C \longrightarrow C$ is a coderivation, iff the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \ Q\ \ } & C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ C \otimes C & \xrightarrow{Q \otimes 1 + 1 \otimes Q} & C \otimes C \end{array} \tag{1.3}$$

commutes. By the next proposition, there is a one-to-one correspondence between coderivations $Q : S(W) \longrightarrow S(W)$ of degree $+1$ and sequences of linear maps $Q_n : S_n(W) \longrightarrow W$ of degree $+1$. We fix the following notations:

$$\hat{Q}_n := Q|_{W^{\odot n}} : W^{\odot n} \longrightarrow S(W)$$
$$Q_{k,l} := \mathrm{pr}_{W^{\odot l}} \circ \hat{Q}_k : W^{\odot k} \longrightarrow W^{\odot l}$$
$$Q_n := Q_{n,1} : W^{\odot n} \longrightarrow W$$

Proposition 1.3. Let Q be a coderivation of degree $+1$ on the coalgebra $(S(W), \Delta)$. Then, $Q(1) = Q_0(1) \in W$ and for $n \geq 1$ and $w_1, ..., w_n \in W$, we have

$$\hat{Q}_n(w_1, ..., w_n) = \sum_{l=0}^{n} \sum_{\sigma \in \mathrm{Sh}(l,n)} \epsilon(\sigma) Q_l(w_{\sigma(1)}, \ldots, w_{\sigma(l)}) \odot w_{\sigma(l+1)} \odot \ldots \odot w_{\sigma(n)}, \tag{1.4}$$

where the $l = 0$ term must be interpreted as $Q_0(1) \odot w_1 \odot \ldots \odot w_n$.

The proof can be found in the appendix. Remark that there is a 1:1-correspondence between coderivations of degree $+1$ on $(S_+(W), \Delta^+)$ and coderivations of Q degree $+1$ on $(S(W), \Delta)$ with $Q(1) = 0$.

Corollary 1.4. Let Q be a coderivation of degree $+1$ on the coalgebra $S(W)$, Q' a coderivation of degree $+1$ on the coalgebra $S(W)$ and $F := S(W) \longrightarrow S(W')$ a morphism of coalgebras. Then, for $n \geq 1$ and $1 \leq l \leq n+1$, $1 \leq k \leq n$, we have

$$Q_{n,l} = (Q_{n-l+1} \otimes 1 \otimes ... \otimes 1) \circ \alpha_{n-l+1,n}.$$

and

$$F_{n,k} = \sum_{i_1 + ... + i_k = n} F_I.$$

F respects the coderivations Q and Q' iff $F(Q(1)) = Q'(1)$ and for each $n \geq 1$ we have

$$\sum_{k=1}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} Q'_k \circ F_I = \sum_{k+l=n+1} F_l \circ (Q_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n} \tag{1.5}$$

On the right hand - side, the sum is over all $l \geq 1$ and $k \geq 0$. $(Q_0 \otimes 1 \otimes \ldots \otimes 1)(w_1 \otimes \ldots \otimes w_n)$ must be interpreted as $Q_0(1) \otimes w_1 \otimes \ldots \otimes w_n$.

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1.3 L_∞-Algebras

Remember that a module L with a sequence of maps $\mu_n : \bigwedge^n L \longrightarrow L$ of degree $2 - n$, for $n \geq 0$, is called an L_∞-**algebra** iff the coderivation Q (of degree $+1$) on $S(W)$, defined by the maps

$$Q_n := (-1)^{n(n-1)/2} \downarrow \circ \mu_n \circ \uparrow^n \colon W^{\odot n} \longrightarrow W$$

is a codifferential, i.e. $Q^2 = 0$.

Remark 1.5. The condition $Q^2 = 0$ just means that for each n and $w_1, \ldots, w_n \in W$ the term

$$(Q^2)_n(w_1, \ldots, w_n) = \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma) Q_l(Q_k(w_{\sigma(1)}, \ldots, w_{\sigma(k)}), w_{\sigma(k+1)}, \ldots, w_{\sigma(n)})$$

disappears. This conditions can easily be translated in the following ones

$$\sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} (-1)^{k(l-1)} \chi(\sigma) \mu_l(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}) = 0 \qquad (1.6)$$

for each $n \geq 0$ and $a_1, \ldots, a_n \in L$.

In the literature μ_0 is mostly assumed to be trivial. If this is the case, (L, μ_1) is a DG module.

Definition 1.6. An L_∞-algebra $(L, \mu_n)_{n \geq 1}$ is called **minimal**, if $\mu_1 = 0$. It is called **linear**, if $\mu_i = 0$ for $i \geq 2$.

Now let (L, μ_*) and (L', μ'_*) be L_∞-algebras. Set $W := L[1]$, $W' := L'[1]$ and denote the induced codifferentials on $S(W)$ and $S(W')$ by Q and Q'. A sequence of maps $f_n : \bigwedge^n L \longrightarrow L'$; $n \geq 0$ of degree $1 - n$ is called L_∞-**morphism** iff the maps $F_n := W^{\odot n} \longrightarrow W'$ induced by f_n (explicitly: $F_n = (-1)^{n(n-1)/2} \downarrow \circ f_n \circ \uparrow^n$) define a map $F : S(W) \longrightarrow S(W')$ of differential graded coalgebras. Rewrite condition (1.5) into terms of f_n and μ_n:

$$\mu'_1 \circ f_n - \sum_{i+j=n} \frac{(-1)^i}{2} \mu'_2(f_i, f_j) \circ \alpha_{i,j} + \sum_{k=3}^n \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} (-1)^{k(k-1)/2 + i_1(k-1) + \ldots + i_{k-1} \cdot 1} \mu'_k \circ f_I$$

$$= \sum_{k+l=n+1} (-1)^{k(l-1)} f_l \circ (\mu_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n}$$

For the case where L' is a differential graded Lie-algebra, i.e. $\mu'_k = 0$ for $k = 0$ and $k \geq 3$, set $d := \mu'_1$ and $[\, , \,] := \mu'_2$. Then we get the following conditions for the maps f_n to define an L_∞-morphism (see Definition 5.2 of [11]):

$$df_n(a_1, \ldots, a_n) -$$

$$- \sum_{i+j=n} \sum_\sigma \chi(\sigma)(-1)^{i+(j-1)(a_{\sigma(1)} + \ldots + a_{\sigma(i)})} [f_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), f_j(a_{\sigma(i+1)}, \ldots, a_{\sigma(n)})]$$

$$= \sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} (-1)^{k(l-1)} \chi(\sigma) f_l(\mu_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(n)}),$$

where $a_1, \ldots, a_n \in L$ and the second sum goes over all σ in $\mathrm{Sh}(i, n)$ such that $\sigma(1) < \sigma(i+1)$.

Definition 1.7. A morphism $f : (L, \mu_n)_{n \geq 1} \longrightarrow (L', \mu'_n)_{n \geq 1}$ of L_∞-algebras is called L_∞-**quasi-isomorphism**, if f_1 is a quasi-isomorphism of differential graded modules.

1.4 L_∞-Algebras and Formal DG Manifolds

In this subsection, we explain briefly the geometric point of view of L_∞-algebras, as proposed by Kontsevich [7]. First, recall the definition of **pointed modules** (see Section II.6 of [1]): A pointed module is a pair $(M, *)$ of a module M and an element $* \in M$. We restrict ourselves to the case where $*$ is just the zero element. For modules M and N, a **homogeneous polynomial of degree** p on M with values in N is a mapping $\tilde{f} : M \longrightarrow N$ of the form $f \circ \Delta^{(p)}$, where f is a p-multilinear form $M \times ... \times M \longrightarrow N$ and $\Delta^{(p)}$ is the diagonal $m \mapsto (m, ..., m)$. The polarization formula (Lemma II.6.2 of [1]) says that $f \mapsto \tilde{f}$ is a 1:1-correspondence between symmetrical p-multilinear forms $M \times ... \times M \longrightarrow N$ and homogeneous polynomials of degree p on M with values in N. For pointed modules $M = (M, 0)$ and $N = (N, 0)$, a formal map $f : M \longrightarrow N$ is a formal sum $f = \sum_{p \geq 1} \tilde{f}_p$, where $\tilde{f}_p$ is a homogeneous polynomial of degree p. Pointed modules, together with formals maps form a category. By the polarization formula and Proposition 1.2, there is a 1:1-correspondence between formal maps $f : M \longrightarrow N$ and morphisms $S(M) \longrightarrow S(N)$ of (non-graded free) coalgebras.

For the definition of formal supermanifolds, we replace modules by $\mathbb{Z}$-graded modules and symmetric multilinear forms by graded symmetric multilinear forms.

Definition 1.8. A **formal supermanifold** is a pair $M = (M, 0)$ of a $\mathbb{Z}$-graded module M and its zero element. A **formal map** $f : M \longrightarrow N$ of degree j of formal supermanifolds is a sequence $(f_p)_{p \geq 1}$, where f_p is a graded symmetric multilinear form $M \times ... \times M \longrightarrow N$ of linear degree j. The **composition** $f \circ g$ of formal maps $g : L \longrightarrow M$ and $f : M \longrightarrow N$ is defined as the sequence $(g_p)_{p \geq 1}$ with

$$g_p = \sum_{k=1}^{p} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=p}} f_k \circ g_I.$$

A **morphism** of formal supermanifolds is a formal map of degree zero.

It is clear by this definition that the category of formal supermanifolds is equivalent to the category of free, graded coalgebras with coalgebra maps of degree zero.

Definition 1.9. A **vectorfields** of degree j on a formal supermanifold M is a coderivation of degree j on $S(M)$.

By Proposition 1.3, a vectorfields on M can be interpreted as formal map $M \longrightarrow M$. The M on the right hand-side of the arrow should be considered as tangent space of M. The graded commutator defines the structure of a graded Lie algebra on $\mathrm{Coder}(S(M), S(M))$. Therefore, there is a bracket $[\cdot, \cdot]$ of vectorfields.

Let (M, Q^M) and (N, Q^N) be formal supermanifolds with vectorfields. A formal map $f : M \longrightarrow N$ is called Q-**equivariant**, if the induced map $S(M) \longrightarrow S(N)$ of coalgebras commutes with Q^M and Q^N. Remark that in the case where M and N are non-graded free k-modules of finite dimension, this definitions coincide with the classical definitions and the Q-equivariance just means that

$$Q^N \circ f = Df \circ Q^M.$$

Definition 1.10. A **formal DG manifold** is a pair (M, Q^M) of a formal supermanifold M and a vectorfield Q^M of degree 1 such that $[Q^M, Q^M] = 0$. Morphisms of formal DG manifolds are Q-equivariant maps of formal supermanifolds (sometimes we call them L_∞-morphisms). Denote the category of formal DG manifolds by DG-Manf.

By the previous subsection, the lifting $L \mapsto L[1]$ gives a 1:1-correspondence between L_∞-algebras and formal DG manifolds and the functor $M \mapsto S(M)$ gives a 1:1-correspondence between formal DG manifolds and free differential graded coalgebras.

We use the following superscripts to denote full subcategories of DG-Manf:

L ("local"): the subcategory of all (M, Q^M) in DG-Manf such that $Q_0^M = 0$;

M ("minimal"): the subcategory of all (M, Q^M) in DG-Manf^L such that $Q_1^M = 0$;

G ("g-finite"): the subcategory of all (M, Q^M) in DG-Manf^L such that $H(M, Q_1^M)$ is g-finite;

C ("convergent"): the subcategory of all (M, Q^M) in DG-Manf$^G M$ such that the mapping $M_0 \longrightarrow M_1$ induced by Q^M converges.

2 Deformation of L_∞-Algebras

Fialowski and Penkava [2] have defined a deformation theory of L_∞-algebras such that the base of a deformation is an algebras with augmentation. The new approach here is to take L_∞-algebras also as bases of deformations. Since the geometric language is more elegant, we will talk about formal DG manifolds instead of L_∞-algebras. Thus, the objects that we deform are DG structures, i.e. degree 1 vectorfields Q with $Q^2 = 0$ on formal supermanifolds.

In our setting, not every "fiber" of a deformation of a DG structure on M gives DG structure on M but in general only a degree 1 vectorfield. But it is easy to find those points of the basis B of a deformation of M for which the associated deformation of Q^M is again a DG structure. They just correspond to the zero locus of the vectorfield Q^B.

A very nice fact for this deformation theory is that we get a universal deformation for free: The deformations of a DG manifold M are governed by the differential graded Lie algebra of vectorfields on M, i.e. the DGL L of coderivations on $S(M)$ with graded commutator as bracket $[\cdot, \cdot]$ and differential $d = [\cdot, Q^M]$. In contrast to Fialowski/ Penkava, we use the linear grading on L (see Section 1). Set $U := L[1]$ and denote the vectorfield corresponding to the DGL structure of L by Q^U. We will see that (U, Q^U) is the base of a universal deformation of M.

2.1 Definitions

Definition 2.1. Let (M, Q^M) in DG-Manf and (B, Q^B) in DG-Manf^L be formal DG manifolds. A **deformation** of M with base B, or more exactly a deformation of the DG structure Q^M, is a degree 1 vectorfield Q on $B \times M$ with $Q_0 = 0$, such that

(i) $Q|_{\{0\} \times M} = 0$.

(ii) $\tilde{Q} := Q^M + Q^B + Q$ is a DG structure on $B \times M$.

(iii) The projection $B \times M \longrightarrow B$ is a homomorphism of formal DG manifolds.

We denote deformations of (M, Q^M) as triples (B, Q^B, Q). Remark that condition (i) is equivalent to the condition that the inclusion $M \longrightarrow B \times M$ is a morphism of formal DG manifolds. Condition (iii) is equivalent to the condition

$$\text{im}(Q) \subseteq \{0\} \times M.$$

A deformation is trivial, if the projection $B \times M \longrightarrow M$ respects the DG structures $\tilde{Q}$ and Q^M.

Definition 2.2. A **morphism** of deformations (B, Q^B, Q) and $(B', Q^{B'}, Q')$ of (M, Q^M) is a pair (F, f), where F is a morphism of formal DG manifolds $(B \times M, \tilde{Q} := Q^B + Q^M + Q)$ and $(B' \times M, \tilde{Q}' := Q^B + Q^M + Q')$ and f is a morphism of formal DG manifolds (B, Q^B) and $(B', Q^{B'})$ such that the diagram

$$
\begin{array}{ccc}
B \times M & \longrightarrow & B' \times M \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
$$

is cartesian and the diagram

$$
\begin{array}{ccc}
M & & \\
\downarrow & \searrow & \\
B \times M & \longrightarrow & B' \times M
\end{array}
$$

commutes.

Definition 2.3. Two deformations are called **equivalent**, if there exist homomorphisms in both senses.

Proposition 2.4. (Base change) Suppose that $(B', Q^{B'}, Q')$ is a deformation of (M, Q^M) and $f : B \longrightarrow B'$ a homomorphism of formal DG manifolds with $B = (B, Q^B)$ in DG-Manf^L. Then, via

$$
Q_n(b_1, ..., b_r, m_1, ..., m_s) := \sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I| = r}} Q'_{s+t}(f_I(b_1, ..., b_r), m_1, ..., m_s)
$$

for $r \geq 1$ with $r + s = n$ and $b_i \in B'$, $m_j \in M$, we can define a deformation (B, Q^B, Q) of (M, Q^M) and $(f \times \mathrm{Id}, f)$ is a morphism of deformations.

Proof: We have to show that $(Q^2)_n(b_1, ..., b_r, m_1, ..., m_s) = 0$, for $b_1, ..., b_r \in B$ and $m_1, ..., m_s \in M$. First let $s \geq 1$. Then

$$
(Q^2)_n(b_1, ..., b_r, m_1, ..., m_s) =
$$

$$
\sum_{k=r}^{n} \sum_{\sigma \in \mathrm{Sh}(k-r,s)} \epsilon(\sigma) Q^M_{n-k+1}(Q_k(b_1, ..., b_r, m_{\sigma(1)}, ..., m_{\sigma(k-r)}), m_{\sigma(k-r+1)}, ..., m_{\sigma(s)}) +
$$

$$
\sum_{k=1}^{r} \sum_{\sigma \in \mathrm{Sh}(k,r)} \epsilon(\sigma) Q_{n+1-k}(Q^B_k(b_{\sigma(1)}, ..., b_{\sigma(k)}), b_{\sigma(k+1)}, ..., b_{\sigma(r)}, m_1, ..., m_s) +
$$

$$
\sum_{k=0}^{s} \sum_{\sigma \in \mathrm{Sh}(k,s)} \epsilon(\sigma) Q_{n+1-k}(b_1, ..., b_r, Q^M_k(m_{\sigma(1)}, ..., m_{\sigma(k)}), m_{\sigma(k+1)}, ..., m_{\sigma(s)}) +
$$

$$
\sum_{k+l=n+1} \sum_{p=1}^{r-1} \sum_{\sigma \in \mathrm{Sh}(p,r)} \sum_{\tau \in \mathrm{Sh}(k+p-r,s)} \epsilon(\sigma)\epsilon(\tau) Q_l(b_{\sigma(1)}, ..., b_{\sigma(p)}, Q_k(b_{\sigma(p+1)}, ..., b_{\sigma(r)},
$$

$$
m_{\tau(1)}, ..., m_{\tau(k+p-r)}), m_{\sigma(k+p-r+1)}, ..., m_{\tau(s)}).
$$

Using the definition of Q and the assumption that f is a DG morphism, after changing the order of summation, this sum takes the form

$$
\sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I| = r}} \sum_{p=0}^{s} \sum_{\sigma \in \mathrm{Sh}(p,s)} \epsilon(\sigma) Q^M_{s-p+1}(Q'_{p+t}(f_I(b_1, ..., b_r), m_{\sigma(1)}, ..., m_{\sigma(p)}), m_{\sigma(p+1)}, ..., m_{\sigma(s)}) +
$$

$$
\sum_{p=1}^{r} \sum_{t=1}^{p} \sum_{I', I'', u} \sum_{\sigma \in \mathrm{Sh}(p,r)} \epsilon(\sigma) Q'_{s+u+1}(Q^B_t(f_{I'}(b_{\sigma(1)}, ..., b_{\sigma(p)})), f_{I''}(b_{\sigma(p+1)}, ..., b_{\sigma(r)}), m_1, ..., m_s) +
$$

$$
\sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I| = r}} \sum_{p=0}^{s} \sum_{\sigma \in \mathrm{Sh}(p,s)} \epsilon(\sigma) Q'_{s-p+t+1}(f_I(b_1, ..., b_r), Q^M_p(m_{\sigma(1)}, ..., m_{\sigma(p)}), m_{\sigma(p+1)}, ..., m_{\sigma(s)}) +
$$

$$
\sum_{p=1}^{r-1} \sum_{q=0}^{s} \sum_{t=1}^{p} \sum_{I', I'', u} \sum_{\sigma \in \mathrm{Sh}(p,r)} \sum_{\tau \in \mathrm{Sh}(q,s)} \epsilon(\sigma)\epsilon(\tau) Q'_{t+s-q+1}(f_{I'}(b_{\sigma(1)}, ..., b_{\sigma(p)}),
$$

$$
Q'_u(f_{I''}(b_{\sigma(p+1)}, ..., b_{\sigma(r)}), m_{\tau(1)}, ..., m_{\tau(q)}), m_{\tau(q+1)}, ..., m_{\tau(s)}),
$$

9

where in the second and forth term, the sum is taken over all $I' \in \mathbb{N}^t$ such that $|I'| = p$, over all $u = 1, ..., r - p$ and all $I'' \in \mathbb{N}^u$ such that $|I''| = r - p$. But this sum equals

$$\sum_{t=1}^{r} \sum_{\substack{I \in \mathbb{N}^t \\ |I| = r}} (\tilde{Q})_{s+t}^2 (f_I(b_1, ..., b_r), m_1, ..., m_s),$$

which is zero. The case $s = 0$ goes in the same manner.

Now, $f \times \text{Id}$ is a map of formal DG manifolds and for a diagram

one can show that $j := g \times (\text{pr}_M \circ h)$ is a DG morphism completing the diagram commutatively. Hence, the quadratic diagram is cartesian and the pair $(f \times \text{Id}, f)$ is a morphism of deformations. □

Corollary 2.5. *If (F, f) is a morphism $(B, Q^B, Q) \longrightarrow (B', Q^{B'}, Q')$ of deformations and f an isomorphism, then (F, f) is also an isomorphism.*

Proof: The deformation (B, Q^B, Q) is natural isomorphic to the deformation, obtained by base change. For the latter one, the statement is clear. □

2.2 A Universal Deformation

Definition 2.6. A deformation (U, Q^U, Q) of (M, Q^M) is called **universal**, if for each deformation (B, Q^B, Q') there exists a morphism $(F, f) : (B, Q^B, Q') \longrightarrow (U, Q^U, Q)$, where f is uniquely defined. A deformation (V, Q^V, Q) is called **semi-universal**, if for each deformation (B, Q^B, Q') there exists a homomorphism $(B, Q^B, Q') \longrightarrow (V, Q^V, Q)$ of deformations and if (V, Q^V) is minimal (in the sense of Definition 1.6).

Let L be the differential graded Lie algebra $\text{Coder}(S(M), S(M))$ with bracket

$$[s, t] = s \circ t - (-1)^{st} t \circ s,$$

for homogeneous s, t and differential $d(s) := (-1)^s [s, Q^M]$.

Definition 2.7. The complex (L, d) will be called **tangent complex** of M.

Set $U := L[1]$ and denote the vectorfield corresponding to the DGL structure on L by Q^U. There is a canonical construction of a deformation Q of M with base U:

Define multilinear maps $q_n : U \otimes M^{\otimes n-1} \longrightarrow M$ of degree $+1$ by

$$u \otimes m_1 \otimes ... \otimes m_{n-1} \mapsto (\uparrow u)(m_1 \odot ... \odot m_{n-1})$$

and denote the symmetrisation of the map $(U \times M)^{\otimes n} \longrightarrow U \times M$, induced by $\frac{1}{n!} q_n$ by Q_n. Hence, we get a vectorfield Q of degree $+1$ on $U \times M$, such that $Q|_{\{0\} \times M} = 0$.

Proposition 2.8. $\tilde{Q} := Q^M + Q^U + Q$ is an DG vectorfield on $U \times M$ and the projection $U \times M \longrightarrow U$ respects the DG structures $\tilde{Q}$ and Q^U.

Proof: Remember that we have

$$(\tilde{Q}^2)_n = \sum_{k+l=n+1} \tilde{Q}_l \circ (\tilde{Q}_k \otimes 1 \otimes ... \otimes 1) \circ \alpha_{k,n}.$$

Since Q^M and Q^U are L_∞-structures, we have $(\tilde{Q}^2)_n(a_1,...,a_n) = 0$ if all a_i belong to M or if all a_i belong to U. Hence, it is enough to show that $(\tilde{Q}^2)_n$ is zero on products of the form

(a) $w \odot m_2 \odot ... \odot m_n$, for $n \geq 2$, $w \in U$ and $m_2,...,m_n \in M$,

(b) $w_1 \odot w_2 \odot m_3 \odot ... \odot m_n$, for $n \geq 3$, $w_1, w_2 \in U$ and $m_3,...,m_n \in M$.

We use the abbreviations $m'_i := m_{i+1}$ and $m''_i := m_{i+2}$. For products of the form (a), we have

$$(\tilde{Q}^2)_n(w, m_2, ..., m_n) =$$

$$\sum_{\substack{k+l=n+1 \\ k>1}} \sum_{\substack{\sigma \in \mathrm{Sh}(k,n) \\ \sigma(1)=1}} \epsilon(\sigma, w, m_2, ..., m_n) Q_l^M(Q_k(w, m_{\sigma(2)}, ..., m_{\sigma(k)}), m_{\sigma(k+1)}, ..., m_{\sigma(n)})$$

$$+ \sum_{\substack{k+l=n+1 \\ l>1}} \sum_{\substack{\sigma \in \mathrm{Sh}(k,l) \\ \sigma(1)=k+1}} \epsilon(\sigma, w, m_2, ..., m_n) Q_l(Q_k^M(m_{\sigma(1)}, ..., m_{\sigma(k)}), w, m_{\sigma(k+2)}, ..., m_{\sigma(n)})$$

$$+ Q_n(Q_1^U(w), m_2, ... m_n) =$$

$$- \sum_{k'+l=n} \sum_{\sigma' \in \mathrm{Sh}(k',n-1)} \epsilon(\sigma', m'_1, ..., m'_{n-1}) Q_l^M(\uparrow w(m'_{\sigma'(1)}, ..., m'_{\sigma'(k')}), m'_{\sigma(k'+1)}, ..., m'_{\sigma(n-1)})$$

$$- \sum_{k+l'=n} \sum_{\sigma' \in \mathrm{Sh}(k,n-1)} (-1)^w \epsilon(\sigma', m'_1, ..., m'_{n-1})(\uparrow w)(Q_k^M(m'_{\sigma'(1)}, ..., m'_{\sigma'(k)}), m'_{\sigma(k+1)}, ..., m'_{\sigma(n-1)})$$

$$+ (Q^M \circ \uparrow w + (-1)^w \uparrow w \circ Q^M)(m'_1, ..., m'_{n-1})$$

$$= 0.$$

For products of the form (b), we have

$$(\tilde{Q}^2)_n(w_1, w_2, m_3, ..., m_n) = Q_{n-1}(Q_2^U(w_1, w_2), m_3, ..., m_n) +$$

$$\sum_{\substack{k+l=n+1 \\ k\geq 2}} \sum_{\substack{\sigma \in \mathrm{Sh}(k,n) \\ \sigma(1)=1, \sigma(k+1)=2}} \epsilon(\sigma, w_1, w_2, m_3, ..., m_n) Q_l(Q_k(w_1, m_{\sigma(2)}, ..., m_{\sigma(k)}), w_2, m_{\sigma(k+2)}, ..., m_{\sigma(n)}) +$$

$$\sum_{\substack{k+l=n+1 \\ k\geq 2}} \sum_{\substack{\sigma \in \mathrm{Sh}(k,n) \\ \sigma(1)=2}} \epsilon(\sigma, w_1, w_2, m_3, ..., m_n) Q_l(Q_k(w_2, m_{\sigma(2)}, ..., m_{\sigma(k)}), w_1, m_{\sigma(k+2)}, ..., m_{\sigma(n)})$$

For a $\sigma \in \mathrm{Sh}(k,n)$ such that $\sigma(1) = 1$ (resp. $\sigma(1) = 2$) and $\sigma(k+1) = 2$ (resp. $\sigma(k+1) = 1$), define $\sigma' \in \mathrm{Sh}(k-1, n-2)$ by $\sigma'(1) = \sigma(2) - 2, ..., \sigma'(k-1) := \sigma(k) - 2$ and $\sigma'(k) := \sigma(k+2) - 2, ..., \sigma'(n-2) = \sigma(n) - 2$. Then, we have

$$\epsilon(\sigma, w_1, w_2, m_3, ..., m_n) = (-1)^{w_2(m_{\sigma(2)}+...+m_{\sigma(k)})} \epsilon(\sigma', m''_1, ..., m''_{n-2})$$

(resp.

$$\epsilon(\sigma, w_1, w_2, m_3, ..., m_n) = (-1)^{w_1(m_{\sigma(2)}+...+m_{\sigma(k)})} \epsilon(\sigma', m''_1, ..., m''_{n-2}). \quad)$$

Hence, the above sum takes the form

$$-(-1)^{w_1}(\uparrow w_1 \circ \uparrow w_2)(m_3, ..., m_n) - (-1)^{w_2+w_1 w_2}(\uparrow w_2 \circ \uparrow w_1)(m_3, ..., m_n)$$

$$+ \sum_{k'+l'=n-1} \sum_{\sigma' \in \mathrm{Sh}(k',n-2)} (-1)^{w_2(w_1+1)} \epsilon(\sigma')(\uparrow w_2)((\uparrow w_1)(m''_{\sigma'(1)}, ..., m''_{\sigma'(k')}), m''_{\sigma(k'+1)}, ..., m''_{\sigma'(n-2)})$$

$$+ \sum_{k'+l'=n-1} \sum_{\sigma' \in \mathrm{Sh}(k',n-2)} (-1)^{w_1} \epsilon(\sigma')(\uparrow w_1)((\uparrow w_2)(m''_{\sigma'(1)}, ..., m''_{\sigma'(k')}), m''_{\sigma(k'+1)}, ..., m''_{\sigma(n-2)})$$

$$= 0,$$

where $\epsilon(\sigma')$ stands for $\epsilon(\sigma', m''_1, ..., m''_{n-2})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Hence, (U, Q^U, Q) is a deformation of (M, Q^M).

Theorem 2.9. (U, Q^U, Q) *is a universal deformation of* Q^M. *More precisely, the mapping* $Q' \mapsto f$, *where*

$$(\uparrow f_n(b_1 \odot \ldots \odot b_n))_k(m_1, ..., m_k) := Q'_{n+k}(b_1, ..., b_n, m_1, ..., m_k)$$

defines a 1:1-correspondence between deformations of M *with base* (B, Q^B) *and morphisms* $B \longrightarrow U$ *of formal DG manifolds.*

Proof: We have to show that $(Q^M + Q^B + Q')^2 = 0$, iff the family $(f_n)_n$ defines a map $f : S(B) \longrightarrow S(U)$ of differential graded coalgebras, i.e. iff for each n, and $b_1, ..., b_n \in B$, the equation

$$Q_1^U(f_n(b_1, ..., b_n)) + \frac{1}{2} \sum_{i+j=n} \sum_{\sigma \in \mathrm{Sh}(i,n)} \epsilon(\sigma, b) Q_2^U(f_i(b_{\sigma(1)}, ..., b_{\sigma(i)}) \otimes f_j(b_{\sigma(i+1)}, ..., b_{\sigma(n)})) =$$

$$\sum_{k+l=n+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma, b) f_l(Q_k^B(b_{\sigma(1)}, ..., b_{\sigma(k)}), b_{\sigma(k+1)}, ..., b_{\sigma(n)}) \quad (2.7)$$

holds. In equation (2.7), we apply both sides on terms $m_1 \odot \ldots \odot m_r \in M^{\odot r}$ and use the definition of f. Then, the condition on f is equivalent to the condition that the following term is zero:

$$\sum_{k+l=r+1} \sum_{\tau \in \mathrm{Sh}(k,r)} \epsilon(\tau, m) Q_l^M(Q'_{n+k}(b_1, ...b_n, m_{\tau(1)}, ..., m_{\tau(k)}), m_{\tau(k+1)}, ..., m_{\tau(r)}) +$$

$$(-1)^{b_1+...+b_n} \sum_{k+l=r+1} \sum_{\tau \in \mathrm{Sh}(k,r)} \epsilon(\tau, m) Q'_{n+l}(b_1, ...b_n, Q_k^M(m_{\tau(1)}, ..., m_{\tau(k)}), m_{\tau(k+1)}, ..., m_{\tau(r)})$$

$$+ \sum_{\substack{i+j=n \\ i,j \geq 1}} \sum_{\sigma \in \mathrm{Sh}(i,n)} \epsilon(\sigma, b)(-1)^{b_{\sigma(1)}+...+b_{\sigma(i)}} \sum_{k=0}^{n} \sum_{\tau \in \mathrm{Sh}(k,r)} \epsilon(\tau, m) Q'_{i+l}(b_{\sigma(1)}, ..., b_{\sigma(i)},$$

$$Q'_{j+k}(b_{\sigma(i+1)}, ...b_{\sigma(n)}, m_{\tau(1)}, ..., m_{\tau(k)}), m_{\tau(k+1)}, ..., m_{\tau(r)})$$

$$+ \sum_{k+l=r+1} \sum_{\sigma \in \mathrm{Sh}(k,n)} \epsilon(\sigma, b) Q'_{n+r}(Q_k^B(b_{\sigma(1)}, ..., b_{\sigma(k)}), b_{\sigma(k+1)}, ..., b_{\sigma(n)}, m_1, ..., m_r)$$

But this term just equals

$$\sum_{\substack{k+l=n+r+1 \\ k \geq n}} \sum_{\substack{\sigma \in \mathrm{Sh}(k,n+r) \\ \sigma(1)=1,\ldots,\sigma(n)=n}} \epsilon(\sigma,u) Q_l^M(Q_k'(b_1,...,b_n,u_{\sigma(n+1)},...,u_{\sigma(k)}),u_{\sigma(k+1)},...,u_{\sigma(n+r)})$$

$$+ \sum_{k+l=n+r+1} \sum_{\substack{\sigma \in \mathrm{Sh}(k,n+r) \\ \sigma(1)>n}} \epsilon(\sigma,u) Q_l'(Q_k^M(u_{\sigma(1)},...,u_{\sigma(k)}),b_1,...,b_n,u_{\sigma(k+n+1)},...,u_{\sigma(n+r)})$$

$$+ \sum_{k+l=n+r+1} \sum_{\substack{\sigma \in \mathrm{Sh}(k,n+r) \\ \sigma(k)\leq n}} \epsilon(\sigma,u) Q_l'(Q_k^B(b_1,...,b_k),b_{\sigma(k+1)},...,b_{\sigma(n)},m_1,...,m_r)$$

$$+ \sum_{k+l=n+r+1} \sum_{\substack{\sigma \in \mathrm{Sh}(k,n+r) \\ \sigma(1)\leq n, \sigma(k+1)\leq n}} \epsilon(\sigma,u) Q_l'(Q_k'(u_{\sigma(1)},...,u_{\sigma(k)}),u_{\sigma(k+1)},...,u_{\sigma(n+r)}),$$

which is

$$(Q^M + Q^B + Q')^2(b_1,...,b_n,m_1,...,m_r).$$

Here, we have set $(u_1,...,u_{n+r}) := (b_1,...,b_n,m_1,...,m_r)$. This proves the second part of the theorem.

To prove its first part, we will show that the map $F := (f \times \mathrm{Id}) : B \times M \longrightarrow U \times M$ respects the DG structures, i.e. that for $n \geq 0$ the following equality holds:

$$\tilde{Q}_1 F_n + \frac{1}{2} \sum_{i+j=n} \tilde{Q}_2 \circ F_i \otimes F_j \circ \alpha_{i,n} + \sum_{k=3}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \frac{1}{I!k!} \tilde{Q}_k \circ (F_{i_1} \otimes ... \otimes F_{i_k}) \circ \alpha_n =$$

$$\sum_{k+l=n+1} F_l \circ (\tilde{Q}_k' \otimes 1 \otimes ... \otimes 1) \circ \alpha_{k,n} \quad (2.8)$$

Remark that F_n takes the following values on products $b_1 \odot \ldots \odot b_r \odot m_1 \odot \ldots \odot m_{n-r}$, with $r < n$, $b_i \in B$ and $m_j \in M$:

$$F_n(b_1,...,b_r,m_1,...,m_{n-r}) = 0 \qquad \text{for} \qquad 0 < r < n$$
$$F_n(b_1,...,b_n) = f_n(b_1,...,b_n)$$
$$F_n(m_1,...,m_n) = 0 \qquad \text{for} \qquad m > 1$$
$$F_1(m_1) = m_1$$

Applying the left hand-side of equation (2.8) on $b_1 \odot \ldots \odot b_r \odot m_1 \odot \ldots \odot m_{n-r}$, we only get the term

$$Q_{1+n-r}(f_r(b_1,...,b_r),m_1,...,m_r).$$

Applying the right hand-side of equation (2.8) on $b_1 \odot \ldots \odot b_r \odot m_1 \odot \ldots \odot m_{n-r}$, we only get the term

$$Q_n'(b_1,...,b_r,m_1,...,m_r).$$

By our construction, both terms coincide. $\qquad \square$

At the end of Section 5 we will be able to construct a semiuniversal deformation of an L_∞-algebra with split tangent complex (see Theorem 5.13).

3 Trees

Trees were used by Gugeheim/Stasheff [3], Merkulov [18], Kontsevich/Soibelman [8] and others to construct infinity structures. We define binary trees (in a slightly different manner as usual) and assign several invariants to them, which are important to get good signs, later.

3.1 Definitions

Definition 3.1. A **tree** with n leaves is a pair $\phi = (\phi, V)$ consisting of a set $V = \{K_0, \ldots, K_{n-2}\}$ of **ramifications** such that for each $i = 0, \ldots, n-2$, we have:

(i) $\phi^{-1}(K_i)$ contains at most 2 elements.

(ii) There is an $n \geq 0$ such that $\phi^n(K_i) = K_0$.

K_0 is called **root** of ϕ.

There is a tree with one leaf and no ramification, which will always be denoted by τ.

Definition 3.2. An **orientation** of a tree (ϕ, V) is a family $\pi = (\pi_K)_{K \in V}$ of inclusions $\pi_K : \phi^{-1}(K) \longrightarrow \{1, 2\}$. The triple $\phi = (\phi, V, \pi)$ is called an **oriented tree.**

Remark 3.3. For each oriented tree (ϕ, V, π), there is a natural ordering on the set V: For $K \in V \setminus K_0$, suppose that $\phi^m(K) = K_0$. We set

$$ v(K) := \frac{\pi_{\phi(K)}(K)}{3^m} + \frac{\pi_{\phi^2(K)}(\phi(K))}{3^{m-1}} + \ldots + \frac{\pi_{\phi^m(K)}(\phi^{m-1}(K))}{3}. $$

Set $v(K_0) := 0$. Then $v : V \longrightarrow \mathbb{R}$ is invective, hence it induces an ordering on V.

When we write down the value $v(K)$ of a ramification K in its 3-ary decomposition, we just get an algorithm, how to get from the root K_0 to K. For example 0.1121 means "go (in the driving direction) right-right-left-right". When (ϕ, V, π) is an oriented tree with n leaves, we can extend the map ϕ to a map $\tilde{\phi} : V \setminus K_0 \cup \{1, ..., n\} \longrightarrow V$, such that

- For $1 \leq i < j \leq n$ we have $\tilde{\phi}(i) \leq \tilde{\phi}(j)$.

- For each $K \in V$, $\tilde{\phi}^{-1}(K)$ has exactly 2 elements.

The numbers $1, ..., n$ stand for the leaves of ϕ. Furthermore, we can extend the map $v : K \longrightarrow (0, 1)$ on $\tilde{V} := V \cup \{1, ..., n\}$ in such a way that the 3-ary decomposition of $v(i)$ describes the way from the root to the i-th leaf of ϕ, for $i = 1, ..., n$. Then we have $v(i) < v(j)$ for $1 \leq i < j \leq n$. In consequence, we have an ordering on $\tilde{V}$.

Definition 3.4. Two trees (ϕ, V) and (ϕ', V') are called **equivalent** if there is a bijection $f : V \longrightarrow V'$ of the ramification sets such that $f \circ \phi = \phi' \circ f$. Two oriented trees (ϕ, V, π) and (ϕ', V', π') are called **oriented equivalent** if there is a bijection $f : V \longrightarrow V'$ of the ramification sets such that $f \circ \phi = \phi' \circ f$ and $\pi' \circ f = \pi$.

When we draw oriented trees, we shall put elements K' of $\phi^{-1}(K)$ down left of K if $\pi_K(K') = 1$ and down right of K if $\pi_K(K') = 2$.

Example 3.5. The following trees with three leaves are equivalent but not oriented equivalent:
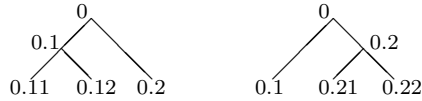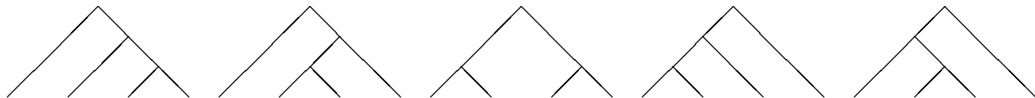

For each ramification and each leaf we have indicated its value.

Set $\mathrm{Ot}(n)$ to be the set of equivalence classes of oriented trees with n leaves.

Example 3.6.   (i) $\mathrm{Ot}(2)$ contains just one element. It will always be denoted by β.

(ii) $\mathrm{Ot}(4)$ contains just the following elements:

Remark 3.7. For a tree (ϕ, V) and $K \in V$, there is a tree $\phi|_K$ with root K and ramifications $\{K' \in V : \phi^n(K') = K \text{ for an } n \geq 0\}$.

We have to introduce several invariants:

For a tree ϕ with $n > 1$ leaves and $1 \leq i \leq n$, set $w_\phi(i)$ to be the difference of the number $s_\phi(i)$ of ramifications of ϕ which are smaller than i and $i - 1$. ($i - 1$ is the number of leaves of ϕ, smaller than i.) For $K \in V$ set $w_\phi(K) := w_{\phi - \phi|_K}(K)$, where on the right hand-side K is considered as leaf of $\phi - \phi|_K$.

Remark 3.8. For $K \in V$, the integer $w_\phi(K)$ is just the number of 1's arising in the 3-ary decomposition of $v(K)$.

Now, for each tree ϕ with at least 2 leaves, set $e(\phi) := (-1)^{w_\phi(1) + \ldots + w_\phi(n)}$. Set $e(\tau) := 1$

Example 3.9. (i) $e(\beta) = -1$

(ii) For the first tree in Example 3.5, we have $e(\phi) = -1$; For the second tree in Example 3.5, we have $e(\phi) = +1$;

Now, let L be a graded module, ϕ an oriented tree with n leaves and $B = (b_K)_{K \in V}$ a family of bilinear maps $L \otimes L \longrightarrow L$. Recursively, we want to define a multilinear map

$$\phi(B) : L^{\otimes n} \longrightarrow L.$$

- If ϕ has one leaf, i.e. B is empty, we set $\phi(B) := \mathrm{Id}$.

- If ϕ has only two leaves, i.e. $V = \{K_0\}$, for a bilinear map $b_0 : L \otimes L \longrightarrow L$, we set $\phi(b_0) := b_0$.

- If $\phi^{-1}(K_0)$ contains exactly one element, say K_1, and $\pi_{K_0}(K_1) = 1$, we set

$$\phi(B) := b_0 \circ (\phi|_{K_1}((b_K)_{K \in V \setminus K_0}) \otimes 1).$$

- If $\phi^{-1}(K_0)$ contains exactly one element, say K_1, and $\pi_{K_0}(K_1) = 2$, we set

$$\phi(B) := b_0 \circ (1 \otimes \phi|_{K_1}((b_K)_{K \in V \setminus K_0})).$$

- If $\phi^{-1}(K_0) = \{K_1, K_2\}$ with $\phi_{K_0}(K_1) = 1$ and $\phi_{K_0}(K_2) = 2$, we set

$$\phi(B) := b_0 \circ (\phi|_{K_1}((b_K)_{K \in V_1}) \otimes \phi|_{K_2}((b_K)_{K \in V_2})).$$

Here, V_1 denotes the ramification set of $\phi|_{K_1}$ and V_2 the ramification set of $\phi|_{K_2}$.

3.2 Operations on trees

Addition Let (ϕ, V, π) and (ϕ', V', π') be oriented trees with disjoint ramification sets. Let R be a point in neither one of them. Set $V'' := V \cup V' \cup \{R\}$. We define a map $\psi : V'' \setminus R \longrightarrow V''$ by $\psi|_{V \setminus K_0} := \phi$, $\psi|_{V' \setminus K_0'} := \phi'$ and $\psi(K_0) := \psi(K_0') := R$.

There is a family $(\pi_K'')_{K \in W}$ of inclusions $\pi_K'' : \psi^{-1}(K) \longrightarrow \{0, 1\}$ with $\pi_K'' = \pi_K$ for $K \in V$, $\pi_K'' = \pi_K'$ for $K \in V'$ and $\pi_R''(K_0) = 0$ and $\pi_R''(K_0') = 1$. Now, we set

$$(\phi, V, \pi) + (\phi', V', \pi') := (\psi, V'', \pi'').$$

It is obvious, how to define the addition of non-oriented trees. The addition of oriented trees is not commutative. The addition of non-oriented trees is commutative.

Example 3.10. $\tau + \tau = \beta$. Furthermore, each tree can be reconstructed by addition out of copies of τ.

Subtraction Let (ϕ, V) be a tree with n leaves and $K \in V$. Let l be the number of leaves of $\phi|_K$. Then the definition of a tree $\phi - \phi|_K$ with $n - l + 1$ leaves is quite obvious.

Composition Let (ϕ, V, π) be an oriented tree with n leaves and let $(\psi^{(1)}, V^{(1)}, \pi^{(1)}), \ldots, (\psi^{(n)}, V^{(n)}, \pi^{(n)})$ be oriented trees. Let W be the disjoint union of V and all $V^{(i)}$. For $K \in V$ set $n(K) := 2 - |\phi^{-1}(K)|$. (This is the number of leaves belonging to K.) Let $K_1 < \ldots < K_l$ all elements K of V with $n(K) > 0$. Then we define a map $\Phi : W \setminus K_0 \longrightarrow W$ as follows: For $K \in V \setminus K_0$ set $\Phi(K) := \phi(K)$. For $K \in V^{(i)} \setminus K_0^{(i)}$ set $\Phi(K) := \psi^{(i)}(K)$. And define the values of Φ on the $K^{(i)}$, setting

$$(\Phi(K_0^{(1)}), \ldots, \Phi(K_0^{(n)})) := (\underbrace{K_1, \ldots, K_1}_{n_1 \text{ times}}, \ldots, \underbrace{K_l, \ldots, K_l}_{n_l \text{ times}}).$$

Then, (Φ, W) is a tree with a canonical orientation π', given as follows: For each i, $K \in V^{(i)}$ and $K' \in \Phi^{-1}(K)$, we set $\pi'(K') := \pi^{(i)}(K')$. For $K \in V$ and $K' \in \Phi^{-1}(K) \cap V$, we set $\pi'(K') := \pi(K')$. It remains to define π'_{K_i} on elements of $\Phi^{-1}(K_i) \setminus V$, for $i = 1, \ldots, l$. So, if $n(K_i)$ equals 2, then $\Phi^{-1}(K_i) \setminus V$ has two elements, say $K_0^{(j)}$ and $K_0^{(k)}$ with $j < k$. Set $\Phi_{K_i}(K_0^{(j)}) := 1$ and $\Phi_{K_i}(K_0^{(k)}) := 2$. If $n(K_i)$ equals 1, then $\Phi^{-1}(K_i)$ has one element in V, say K and one element which is not in V, say K'. Set $\Phi_{K_i}(K') := 1$ if $\phi_{K_i}(K) = 2$ and $\Phi_{K_i}(K') := 2$ if $\phi_{K_i}(K) = 1$.

We will denote this decomposition by $\Phi = \phi \circ (\psi^{(1)}, ..., \psi^{(n)})$.

Remark 3.11. In this situation, suppose that there is a family $B = (b_K)_{K \in W}$ of of bilinear maps $L \otimes L \longrightarrow L$. Set $B^{(0)} := (b_K)_{K \in V}$ and $B^{(i)} := (b_K)_{K \in V^{(i)}}$ for $i = 1, \ldots, n$. Then we have

$$\phi \circ (\psi^{(1)}, \ldots, \psi^{(n)})(B) = (-1)^{\text{exponent}} \phi(B^{(0)}) \circ (\psi^{(1)}(B^{(1)}) \otimes \ldots \psi^{(n)}(B^{(i)})),$$

where the exponent is the sum $(\sum_{K \in V^{(1)}} b_K)(\sum_{K \in V}^{V > 1} b_K) + \ldots + \sum_{K \in V^{(n-1)}} b_K)(\sum_{K \in V}^{V > n-1} b_K)$. We remind that $V > i$ means that the value $v(V)$ is greater than the value $v(i)$ of the i-th leaf of ϕ.

4 L_∞-equivalence of L and $H(L)$

Let $L = (L, d, [\cdot, \cdot])$ be a differential graded Lie-algebra, where the differential d is of degree $+1$. Suppose that there is a splitting η, i.e. a map of degree -1 such that $d\eta d = d$. Furthermore, suppose that $\eta^2 = 0$ and $\eta d\eta = \eta$. When we use a Lie bracket on $\text{Hom}(L, L)$, we mean the graded commutator.
In this section, we want to construct an L_∞-algebra structure μ_* on $H := H(L, d)$ with $\mu_1 = 0$, such that $(L, d, [\cdot, \cdot])$ and (H, μ_*) are L_∞-equivalent. The multilinear forms μ_n will be constructed using trees as in the last section. A similar construction for A_∞-algebras can be found in [3] and [18].
We have to make some preparations. First of all, there is the following easy but important remark:

Remark 4.1. Let $n \geq 3$ be a natural number. There is a 1:1-correspondence between triples (Φ, K, σ), where $\Phi = (\Phi, V, \pi)$ is an oriented tree with n leaves, K is a ramification in V, σ a permutation in Σ_n and 6-tuples $(k, \phi, \psi, \rho, \gamma, \delta)$, where k is a natural number with $2 \leq k \leq n - 1$, ϕ is a tree in $\text{Ot}(k)$, ψ is a tree in $\text{Ot}(l)$ where $l := n + 1 - k$, ρ is a shuffle in $\text{Sh}(k, n)$ and $\gamma \in \Sigma_l$, $\delta \in \Sigma_k$ are permutations.
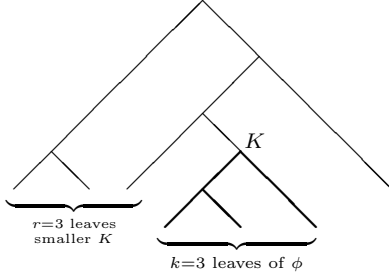
Example:
The fine lines represent the tree ψ and the fat lines the tree ϕ. In the sequel, the first r leaves of Φ will be associated to the indexes $\sigma(1) = \rho(\gamma(1) + k - 1), ..., \sigma(r) = \rho(\gamma(r) + k - 1)$, the following k leaves to the indexes $\sigma(r + 1) = \rho(\delta(1)), ..., \sigma(r + k) = \rho(\delta(k))$ and the remaining leaves to the indexes $\sigma(r + k + 1) = \rho(\gamma(r + 2) + k - 1), ..., \sigma(n) = \rho(\gamma(l) + k - 1)$.

Here-fore, to the triple (Φ, K, σ), we associate the following data: Set k to be the number of leaves of $\Phi|_K$, $\phi := \Phi|_K$, $\psi := \Phi - \phi$. Let r be the number of leaves F of Φ with $F < K$. ρ is chosen in such a way that $\{\rho(1), ..., \rho(k)\} = \{\sigma(r + 1), ..., \sigma(r + k)\}$. δ is defined by $\delta(i) := \rho^{-1}(\sigma(r + i))$ for $i = 1, ..., k$ and γ is defined in the following way:

$$\gamma(i) := \begin{cases} \rho^{-1}(\sigma(i)) - k + 1 & \text{for} \quad i = 1, ..., r \\ 1 & \text{for} \quad i = r + 1 \\ \rho^{-1}(\sigma(i + k - 1)) - k + 1 & \text{for} \quad i = r + 2, ..., l \end{cases}$$

In the other way, to the 6-tuple $(k, \phi, \psi, \rho, \gamma, \delta)$, we associate the following data: Set $r := \gamma^{-1}(1) - 1$. Then Φ is the composition

$$\Phi = \psi \circ (\underbrace{\tau, ..., \tau}_{r \text{ times}}, \phi, \tau, ..., \tau),$$

where τ again stands for the tree with one leaf. K is the root of ϕ, considered as ramification of Φ and σ is given by

$$\sigma(i) := \begin{cases} \rho(\gamma(i) + k - 1) & \text{for} \quad i = 1, ..., r \\ \rho(\delta(i - r)) & \text{for} \quad i = r + 1, ..., r + k \\ \rho(\gamma(i - (k - 1)) + k - 1) & \text{for} \quad i = r + k + 1, ..., n. \end{cases}$$

Now suppose that such corresponding tuples $(\Phi, \hat{K}, \sigma)$ and $(k, \phi, \psi, \rho, \gamma, \delta)$ are given. Let V' be the ramification set of ψ and V'' the ramification set of ϕ. Then $V := V' \cup V''$ is the ramification set of Φ. Again, set $r := \gamma^{-1}(1) - 1$. Remark that the ordering on V depends on γ. We define a permutation $\tilde{\gamma} \in \Sigma_{l-1}$ by

$$\tilde{\gamma}(i) := \begin{cases} \gamma(i) - 1 & \text{for} \quad i = 1, ..., r \\ \gamma(i + 1) - 1 & \text{for} \quad i = r + 1, ..., l - 1. \end{cases}$$

Lemma 4.2. *We keep all notations from above. Let $B = (b_K)_{K \in V}$ be a family of homogeneous bilinear forms $L \otimes L \longrightarrow L$. Denote the subfamilies $(b_K)_{K \in V'}$ and $(b_K)_{K \in V''}$ by B' and B''. Set W to be the set of all ramifications $K \in V$ such that $K > \hat{K}$. Then we have*

$$\psi(B') \circ \gamma \circ (\phi(B'') \circ \delta \otimes 1 \otimes ... \otimes 1) \circ \rho$$

$$= (-1)^{r + rk} \psi(B') \circ (\underbrace{1 \otimes ... \otimes 1}_{r \text{ times}} \otimes \phi(B'') \otimes 1 \otimes ... \otimes 1) \circ \sigma$$

$$= (-1)^{r + rk + \sum_{K \in W} b_k \cdot B''} \Phi(B) \circ \sigma.$$

Proof: Let $a_1, ..., a_n$ be homogeneous elements of L. Then we have

$$(\psi(B') \circ \gamma \circ (\phi(B'') \circ \delta \otimes 1 \otimes ... \otimes 1) \circ \rho)(a_1 \otimes ... \otimes a_n) =$$

$$= \chi(\rho, a_1, ..., a_n)\chi(\delta, a_{\rho(1)}, ..., a_{\rho(k)}) \cdot (\psi(B') \circ \gamma)(\underbrace{\phi(B'')(a_{\rho(\delta(1))}, ..., a_{\rho(\delta(k))})}_{=:u_1} \otimes \underbrace{a_{\rho(k+1)}}_{u_2} \otimes ... \otimes \underbrace{a_{\rho(n)}}_{u_l})$$

$$= \chi(\rho, a_1, ..., a_n)\chi(\delta, a_{\rho(1)}, ..., a_{\rho(k)})\chi(\gamma, u_1, ..., u_l)\psi(B')(u_{\gamma(1)}, ..., u_{\gamma(l)}).$$

Using the following three formulas

$$\chi(\sigma, a_1, ..., a_n) = (-1)^{kr+(a_{\sigma(1)}+...+a_{\sigma(r)})(a_{\rho(1)}+...+a_{\rho(k)})}\chi(\rho, a_1, ..., a_n) \cdot$$
$$\chi(\delta, a_{\rho(1)}, ..., a_{\rho(k)})\chi(\tilde{\gamma}, a_{\rho(k+1)}, ..., a_{\rho(n)}),$$

$$u_{\gamma(1)} \otimes ... \otimes u_{\gamma(l)} = (-1)^{B''(a_{\sigma(1)}+...+a_{\sigma(r)})}(1 \otimes ... \otimes 1 \otimes \phi(B'') \otimes 1... \otimes 1)(a_{\sigma(1)} \otimes ... \otimes a_{\sigma(n)}),$$

$$\chi(\gamma, u_1, ..., u_l) = (-1)^{r+u_1(u_{\gamma(1)}+...+u_{\gamma(r)})}\chi(\tilde{\gamma}, u_2, ..., u_l),$$

this expression is just

$$(-1)^{kr+r}\chi(\sigma, a_1, ..., a_n)\psi(B')((1 \otimes ... \otimes 1 \otimes \phi(B'') \otimes 1 \otimes ... \otimes 1)(a_{\sigma(1)} \otimes ... \otimes a_{\sigma(n)})).$$

The second equality of this Lemma is just a special case of Remark 3.11. $\qquad\square$

We turn to the construction of an L_∞-structure on $H(L)$.

Remark 4.3. $[d, \eta] = d\eta + \eta d$ is a projection, i.e. $[d, \eta]^2 = [d, \eta]$. And $H := \text{Kern}[d, \eta]$ is, as module, isomorphic to $H(L)$. The bracket on L induces a Lie-bracket on $H(L)$ and the induced bracket on H (via the isomorphism $H \longrightarrow H(L)$) is just given by $(1 - d\eta)[\cdot, \cdot] = (1 - [d, \eta])[\cdot, \cdot]$.

For simplicity, we set $g := \eta[\cdot, \cdot]$.

Theorem 4.4. *The following graded anti-symmetric maps $\mu_n : H^{\otimes n} \longrightarrow H$ of degree $2 - n$ define the structure of an L_∞-algebra on H:*

$$\mu_1 := 0$$
$$\mu_2 := (1 - d\eta)[\cdot, \cdot]$$
$$\vdots$$
$$\mu_n := (\frac{-1}{2})^{n-1} \sum_{\phi \in \text{Ot}_n} e(\phi)\phi((1 - [d, \eta])[\cdot, \cdot], g, \ldots, g) \circ \alpha_n$$

Proof: We must show that

$$\sum_{k+l=n+1} (-1)^{k(l-1)}\mu_l \circ (\mu_k \otimes 1 \otimes ... \otimes 1) \circ \alpha_{k,n} = 0. \qquad (4.9)$$

Up to the factor $(-1)^{n-1}$ this sum has the form

$$\sum_k \sum_{\phi, \psi} \sum_{\rho, \gamma, \delta} (-1)^{k(l-1)} e(\phi)e(\psi)\psi((1 - [d, \eta])[\cdot, \cdot], g, ..., g) \circ \gamma \circ$$

$$\circ (\phi((1 - [d, \eta])[\cdot, \cdot], g, ..., g) \circ \delta \otimes 1 \otimes ... \otimes 1) \circ \rho, \quad (4.10)$$

where k ranges from from 2 to $n - 1$, $l = n + 1 - k$, ϕ and ψ vary in $\text{Ot}(k)$ and $\text{Ot}(l)$, ρ in $\text{Sh}(k, n)$, γ and δ in Σ_l and Σ_k. For corresponding tuples $(k, \phi, \psi, \rho, \gamma, \delta)$ and $(\Phi, \hat{K}, \sigma)$ as in Remark 4.1,

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we denote as usual $r := \gamma^{-1}(1) - 1$ and by t the number of ramifications of ψ, greater than $r+1$. Using

$$e(\Phi) = (-1)^{w_\Phi(\hat{K})(k-1)} e(\phi) e(\psi)$$
$$w_\Phi(\hat{K}) = l - 1 - r - t$$

and Lemma 4.2, the expression 4.10 can be expressed as

$$\sum_{\Phi \in \mathrm{Ot}(n)} \sum_{\hat{K} \in V \setminus K_0} e(\Phi)(-1)^{r + w_\Phi(\hat{K})} \Phi(B) \circ \alpha_n,$$

where $B = (B_K)_{K \in V}$ is the family with $b_{K_0} = b_{\hat{K}} = (1 - [d, \eta])[\cdot, \cdot]$ and $b_k = \eta[\cdot, \cdot]$ for $K \neq K_0, \hat{K}$. To show that the last term is zero, it is enough to show the following two conditions:

- $\sum_{\Phi \in \mathrm{Ot}(n)} \sum_{K \in V \setminus K_0} \sum_{\sigma \in \Sigma_n} (-1)^{r + w_\Phi(K)} e(\Phi) \Phi((1 - [d, \eta])[\cdot, \cdot], g, ..., g, \underbrace{[\cdot, \cdot]}_{\text{pos. } K}, g, ..., g) \circ \sigma = 0.$

- For each tree Φ, we have

$$\sum_{K \in V \setminus K_0} (-1)^{r + w_\Phi(K)} e(\Phi) \Phi((1 - [d, \eta])[\cdot, \cdot], g, ..., g, \underbrace{[d, \eta][\cdot, \cdot]}_{\text{position } K}, g, ..., g) \circ \sigma = 0. \qquad (4.11)$$

The first condition follows by the Jacobi-identity and an easy combinatorial argument. In equation (4.11) the term annihilate each other since the differential d trickles down the branches of Φ:

Initiation of the trickling: Suppose that $\Phi^{-1}(K_0)$ contains an element K' with $\pi_{K_0}(K') = 1$. Then we have the following picture:

Here, we only have drawn the top of the tree Φ for the case where $\Phi^{-1}(K_0)$ consists of two elements K', K'' and the corresponding bilinear forms. It is quite evident how this goes when $\Phi^{-1}(K_0)$ has only one element, since $d|_H = 0$.

Going-on of the trickling at a ramification $K \in V$:
We illustrate the case, where $\Phi^{-1}(K)$ has two elements K', K'' with $\pi_K(K') = 1$.

Iterating the trickling down to the leaves and using $d|_H = 0$, we see that all terms in the sum are annihilated. $\qquad \square$

Theorem 4.5. *The following anti-symmetric maps* $f_n : H^{\otimes n} \longrightarrow L$ *of degree* $n - 1$ *define an*

L_∞-equivalence $H \longrightarrow L$ (i.e. an L_∞-quasi-isomorphism).

$$f_1 := inclusion$$
$$f_2 := -g$$
$$\vdots$$
$$f_n := -(\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi(g, \ldots, g) \circ \alpha_n.$$

Proof: For $n \geq 0$, we have to prove the equation

$$df_n - \sum_{i+j=n} \frac{(-1)^i}{2}[f_i, f_j]\alpha_{i,n} = \sum_{k+l=n+1} (-1)^{k(l-1)} f_l \circ (\mu_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n}$$

For $l = 1$, the right hand-side is just μ_n. Since

$$df_n = (\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi(-d\eta[\cdot, \cdot], g, \ldots, g) \circ \alpha_n,$$

it is sufficient to show the following three identities:

$$-\sum_{i+j=n} \frac{(-1)^i}{2}[f_i, f_j]\alpha_{i,n} = (\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi([\cdot, \cdot], g, \ldots, g) \circ \alpha_n \qquad (4.12)$$

$$f_l \circ (\phi([\cdot, \cdot], g, \ldots, g) \circ \alpha_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n} = 0 \text{ for } l > 1, k + l = n + 1. \qquad (4.13)$$

$$(\frac{-1}{2})^{n-1} \sum_{\phi \in \mathrm{Ot}(n)} e(\phi)\phi(\eta d[\cdot, \cdot], g, \ldots, g) \circ \alpha_n = - \sum_{\substack{k+l=n+1 \\ l \geq 2}} (-1)^{k(l-1)} \sum_{\phi \in \mathrm{Ot}(k)} (\frac{-1}{2})^{k-1} e(\phi)$$

$$f_l \circ (\phi(d\eta + \eta d)[\cdot, \cdot], g, \ldots, g) \circ \alpha_k \otimes 1 \otimes \ldots \otimes 1) \circ \alpha_{k,n}. \quad (4.14)$$

Proof of equation (4.14) The right-hand side of equation (4.14) is

$$(\frac{-1}{2})^{n-1} \sum_{k+l=n+1}^{k,l \geq 2} \sum_{\phi, \psi} \sum_{\gamma, \delta, \rho} (-1)^{k(l-1)} e(\phi)e(\psi)\psi(g, \ldots, g) \circ \gamma \circ (\phi([d, \eta][\cdot, \cdot], g, \ldots, g) \circ \delta \otimes 1 \otimes \ldots \otimes 1) \circ \rho.$$

As in the proof of Theorem 4.4, this expression takes the form

$$(\frac{-1}{2})^{n-1} \sum_{\Phi \in \mathrm{Ot}(n)} \sum_{\hat{K} \in V} \sum_{\sigma \in \Sigma_n} (-1)^{r+w_\Phi(\hat{K})} e(\Phi)\Phi(B)\sigma,$$

where $B = (b_K)_{K \in V}$ is the family with $b_{\hat{K}} = [d, \eta][\cdot, \cdot]$ and $b_K = \eta[\cdot, \cdot]$ for $K \neq \hat{K}$.

Hence to show equation (4.14), it is enough to show that for each tree ϕ, we have

$$\Phi(\eta d[\cdot, \cdot], g, \ldots, g) = \sum_{K \in V \setminus K_0} (-1)^{r+w_\Phi(K)} \Phi(B).$$

This is true by the same trickling argument as in Theorem 4.4.

Proof of equation (4.13): This is again the Jacobi-identity and some combinatorics.

Proof of equation (4.12):

$$\sum_{i+j=n} \frac{(-1)^i}{2}[\cdot,\cdot] \circ (f_i \otimes f_j) \circ \alpha_{i,n} =$$

$$= \sum_{i+j=n} (-\frac{1}{2})^{i-1+j-1} \sum_{\phi \in \mathrm{Ot}(i), \psi \in \mathrm{Ot}(j)} \frac{(-1)^i}{2} e(\phi)e(\psi)(\phi+\psi)([\cdot,\cdot],g,...,g) \circ \alpha_n$$

$$= -(-\frac{1}{2})^{n-1} \sum_{\Phi \in \mathrm{Ot}(n)} e(\Phi)\Phi([\cdot,\cdot],g,...,g) \circ \alpha_n.$$

$\square$

5 Decomposition Theorem for Differential Graded Lie Algebras

In this section, we want to realize two things: (a) the construction of an inverse map of the quasi-isomorphism $f : (H, \mu_*) \longrightarrow (L, d, [\cdot, \cdot])$, constructed in Section 4; (b) the construction of a semi-universal deformation (V, Q^V, Q') for a given formal DG manifold. As consequence of (a), we get the following decomposition theorem $(L, d, [\cdot, \cdot]) \cong (H, \mu_*) \oplus (F, d, 0)$ for DGLs, where F is the complement of H in L and the sum is taken in the category of L_∞-algebras. The existence of such a decomposition, was already stated by Kontsevich (see [7]) and an A_∞-analogue was proved by Kadeishvili (see [5]). In fact, each L_∞- (resp. A_∞-algebra) over a field is isomorphic to the direct sum of a minimal and a linear contractible one. Our Proposition 5.6 is analogue to the corresponding statement for A_∞-algebras, which was proved by Lefevre (see [12]). The proof here is almost a transcription of Lefevre's proof.

5.1 Obstructions

Consider the formal DG manifolds (W, Q) and (W', Q'). For any $n \geq 0$, there is a differential δ of degree $+1$ on the graded module $\mathrm{Hom}(W^{\otimes n}, W')$, given by $\delta(g) = Q_1' \circ g - (-1)^g g \circ Q_1^{(n)}$. Now, let $f : W \longrightarrow W'$ be a morphism of formal supermanifolds. Set

$$r(f_1, ..., f_{n-1}) := \sum_{\substack{k+l=n+1 \\ k \geq 2}} f_l \circ Q_k^{(n)} - \sum_{k=2}^{n} \sum_{i_1+...+i_k=n} Q_k' \circ f_I.$$

Recall that f is an L_∞-homomorphism iff, for each $n \geq 1$, we have $\delta(f_n) = r(f_1, ..., f_n)$. If this condition is satisfied only for $n \leq m$, we will call f (or the family $(f_1, ..., f_m)$) an L_m-**homomorphism**.

Lemma 5.1. *Suppose that f is an L_{n-1}-homomorphism. Then $\delta(r(f_1, ..., f_{n-1})) = 0$.*

The proof is done in the appendix.

Remark 5.2. Let $e : W \longrightarrow W'$ and $f : V' \longrightarrow V$ be strict L_∞-morphisms and let $g : V \longrightarrow W$ be any L_∞-morphism. Then

(i) $r((gf)_1, ..., (gf)_{n-1}) = r(g_1, ..., g_{n-1}) \circ f_1^{\otimes n}$.

(ii) $r((eg)_1, ..., (eg)_{n-1}) = e_1 \circ r(g_1, ..., g_{n-1})$.

5.2   Constructions

Proposition 5.3. Let $f : M \longrightarrow M'$ be a morphism of formal supermanifolds. Suppose, there is a module homomorphism $g' : M' \longrightarrow M$ such that $g' \circ f_1 = \mathrm{Id}_M$. Then, there is a morphism $g : M' \longrightarrow M$ of formal supermanifolds such that $g_1 = g'$ and $gf = \mathrm{Id}_M$. If f_1 is an isomorphism with inverse g and if f is Q-equivariant and if g' respects $Q_1^{M'}$ and Q_1^M, then g can be chosen Q-equivariant, as well.

Proof: One can check directly that the sequence of maps defined by

$$g_n := -\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} g_1 \circ f_k \circ (g_I),$$

for $n \geq 2$, define a morphism of formal supermanifolds with the desired property. $\square$

Lemma 5.4. *Let $f : V \longrightarrow W$ be a morphism of formal DG manifolds.*

  (i) *If f_1 is split injective, then there is a formal DG manifold W' and an L_∞-isomorphism $\kappa : W \longrightarrow W'$ such that $\kappa \circ f$ is strict.*

  (ii) *If f_1 is split surjective, then there is a formal DG manifold V' and an L_∞-isomorphism $\kappa : V' \longrightarrow V$ such that $f \circ \kappa$ is strict.*

Proof: (i) As module, set $W' := W$. We have to construct an isomorphism $\kappa : S(W) \longrightarrow S(W')$ of graded coalgebras and then, we can define the DG structure on W' via $Q^{W'} := \kappa \circ Q^W \circ \kappa^{-1}$. Set $\kappa_1 := \mathrm{Id}$. Inductively, we define maps $\kappa_n : W^{\odot n} \longrightarrow W'$ such that for $2 \leq m \leq n$, we have

$$(\kappa \circ f)_m = \sum_{k=1}^{m} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \kappa_k \circ f_I = 0.$$

Let $g : W \longrightarrow V$ be a module homomorphism with $g \circ f_1 = \mathrm{Id}_V$. When $\kappa_1, ..., \kappa_n$ is already constructed, set

$$\kappa_{n+1} := -\sum_{k=1}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n+1}} \kappa_k \circ f_I \circ g^{\odot n+1}.$$

Obviously, $(\kappa \circ f)_m = 0$ for $2 \leq m \leq n+1$. (ii) goes in a similar way. $\square$

For our situation, we have the following, more explicit statement:

Lemma 5.5. *Let $f : H \longrightarrow L$ be the L_∞-quasi-isomorphism, constructed in Section 4. Consider the applications $\kappa_n : L^{\otimes n} \longrightarrow L$, defined by $\kappa_1 := \mathrm{Id}$ and $\kappa_n := -f_n \circ \mathrm{pr}_H^{\otimes n}$, for $n \geq 2$. Then, κ is an L_∞-morphism and $\kappa \circ f$ is strict. Furthermore, $(\kappa^{-1})_1 = \mathrm{Id}$ and $(\kappa^{-1})_n = f_n \circ \mathrm{pr}_H^{\otimes n}$ for $n \geq 2$.*

Proof: For $n \geq 2$, we have

$$(\kappa \circ f)_n = \sum_{k=1}^{n} \sum_{\substack{I \in \mathbb{N}^K \\ |I|=n}} \kappa_k \circ f_I = f_n - f_n = 0.$$

The second statement is as easy to prove. $\square$

The following important proposition says that the quadruple (category of L_∞-algebras; class of L_∞-quasi-isomorphisms; class of those L_∞-morphisms f such that f_1 is split injective; class of those L_∞-morphism f such that f_1 is split surjective) satisfies one of Quillen's axioms for model categories.

Proposition 5.6. Let

$$
\begin{array}{ccc}
A & \xrightarrow{\,c\,} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle e} \\
B & \xrightarrow{\,d\,} & D
\end{array}
$$

a commutative diagram of L_∞-algebras. Suppose that f is split injective and that e is split surjective and that or f or e is an L_∞-quasi-isomorphism. Then, there is an L_∞-morphism $g : B \longrightarrow C$, such that the complete diagram

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & {\nearrow}^{\,g} & \downarrow \\
B & \longrightarrow & D
\end{array}
$$

commutes.

Proof: By Lemma 5.4, we may suppose that e and f are strict. Inductively, we will construct morphisms $g_n : B^{\odot n} \longrightarrow C$, such that

(i) $\delta(g_m) + r(g_1, ..., g_{m-1}) = 0$,

(ii) $g_m \circ f_1^{\otimes m} = c_m$,

(iii) $e_1 \circ g_m = d_m$,

for each $m \leq n$. Choose maps $u : (D, Q_1^D) \longrightarrow (C, Q_1^C)$ and $v : (B, Q_1^B) \longrightarrow (A, Q_1^A)$ of DG-modules, such that $v \circ f_1 = \mathrm{Id}_A$ and $e_1 \circ u = \mathrm{Id}_D$. A candidate for g_1 can easily be found. Suppose that $g_1, .., g_{n-1}$ are already constructed. Then

$$
\beta := c_n v^{\odot n} + u d_n - u e_1 c_n v^{\odot n}
$$

satisfies conditions (i) and (ii). By Lemma 5.2, we get

$$
(\delta(\beta) + r(g_1, ..., g_{n-1})) \circ f_1^{\odot n} =
$$
$$
\delta(\beta \circ f_1^{\odot n}) + r((gf)_1, ..., (gf)_{n-1}) =
$$
$$
\delta(c_n) + r(c_1, ..., c_{n-1}) = 0.
$$

On the other side, again by Lemma 5.2, we have

$$
e_1 \circ (\delta(\beta) + r(g_1, ..., g_{n-1})) =
$$
$$
\delta(e_1\beta) + r((eg)_1, ..., (eg)_{n-1}) =
$$
$$
\delta(d_n) + r(d_1, ..., d_{n-1}) = 0.
$$

Hence, $\delta(\beta) + r(g_1, ..., g_{n-1})$ has a factorization

$$
B^{\otimes n} \xrightarrow{\,p\,} \mathrm{Cokern}(f_1^{\odot n}) \xrightarrow{\,q\,} \mathrm{Kern}(e_1) \xrightarrow{\,i\,} C,
$$

where i is the natural inclusion and p the natural epimorphism. By Lemma 5.1, $\delta(\beta)+r(g_1, ..., g_{n-1})$ is a cycle, so $\delta(q) = 0$, i.e. q is a map of complexes. Now, or $\mathrm{Cokern}(f_1^{\odot n})$ or $\mathrm{Kern}(e_1)$ is contractible. Hence $q = \delta(h)$, for a morphism $h : \mathrm{Cokern}(f_1^{\odot n}) \longrightarrow \mathrm{Kern}(e_1)$ of graded modules. Then $g_n := \beta - i \circ h \circ p$ satisfies the conditions (i)-(iii). $\qquad\square$

Corollary 5.7. *There is a map* $g : (L, d, [\cdot, \cdot]) \longrightarrow (H, \mu_*)$ *of* L_∞-*algebras such that* $g \circ f = \mathrm{Id}_H$.

Corollary 5.8. *Let* M *be an* L_∞-*manifold and* (B, Q^B, Q) *a deformation of* M, *such that* (B, Q_1^B) *is contractible and* $Q_1 = 0$. *Then* (B, Q^B, Q) *is a trivial deformation.*

Proof: There is a commutative diagram

$$
\begin{array}{ccc}
M & \longrightarrow & (B \times M, Q^B + Q^M + Q) \\
\downarrow & & \downarrow \\
(B \times M, Q^B + Q^M) & \longrightarrow & B
\end{array}
$$

where the vertical left arrow induces an injective quasi-isomorphism of DG-modules and the vertical right arrow induces a surjective map of DG modules. By Proposition 5.6, there is a map $q : (B \times M, Q^B + Q^M) \longrightarrow (B \times M, Q^B + Q^M + Q)$ with $q_1 = \mathrm{Id}$, completing the diagram commutatively. In particular, q establishes an isomorphism of the given deformation and of the trivial deformation of M with base B. $\hfill\square$

Proposition 5.9. There exists

(i) a homomorphism $\iota : (F, d, 0) \longrightarrow (L, d, [\cdot, \cdot])$ of L_∞-algebras such that ι_1 is the natural inclusion;

(ii) a homomorphism $p : (L, d, [\cdot, \cdot]) \longrightarrow (F, d, 0)$ of L_∞-algebras such that $p \circ \iota = \mathrm{Id}_F$.

Proof: (a) Suppose, that there are already homomorphisms $\iota_m : F^{\otimes m} \longrightarrow L$, for $m \le n - 1$, which form an L_{n-1}-homomorphism. We have to find an ι_n such that $\delta(\iota_n) = r(\iota_1, ..., \iota_{n-1})$. Since (F, d) is contractible, $\mathrm{Hom}(F^{\otimes n}, L)$ is acyclic, so the existence of ι_n follows by Lemma 5.1.
(b) By Lemma 5.4, we can assume that ι is strict. Set $p_1 := \mathrm{pr}_F = [d, \eta]$. Now assume that $p_1, ..., p_{n-1}$ are already constructed such that they define an L_{n-1}-homomorphism $p' : L \longrightarrow F$ such that $(p' \circ \iota)_m = 0$, for $m \le n - 1$. We have to find $p_n : L^{\otimes n} \longrightarrow F$ such that

$$
\delta(p_n) + r(p_1, ..., p_{n-1}) = 0,
$$
$$
p_n \circ \iota^{\otimes n} = 0.
$$

We may chose $p_n := \eta r(p_1, ..., p_{n-1})$. Then, since $r(p_1, ..., p_{n-1}) \in \mathrm{Kern}\,\delta$, we have $\delta(p_n) = [\eta, d] \circ r(p_1, ..., p_{n-1}) = r$, and again by Remark 5.2, we get

$$
p_n \circ \iota^{\otimes n} = \eta \circ r(p_1, ..., p_{n-1}) \circ \iota^{\otimes n} = 0.
$$

So inductively, the map p can be constructed. $\hfill\square$

As consequence, we get the expected decomposition theorem for differential graded Lie algebras admitting a splitting:

Theorem 5.10. *We have an isomorphism of L_∞-algebras*

$$
f \times \iota : H \times F \longrightarrow L.
$$

Corollary 5.11. *If (L, μ_*) and (L', μ'_*) are L_∞-algebras such that $(L; \mu_1)$ and $(L'; \mu'_1)$ are split, then, for each L_∞-quasi-isomorphism $f : L \longrightarrow L'$, there exists an L_∞-morphism $g : L' \longrightarrow L$, such that f_1 and g_1 are inverse maps on the homology. In particular, if k is a field, then L_∞-quasi-isomorphism is an equivalence relation.*

5.3   A Semiuniversal Deformation

Remark 5.12.   (i) Let $M = (M, Q^M)$ be a formal DG manifold and N a Q^M-closed submodule of M, i.e. for all $j \ge 1$ and $n_1, ..., n_j \in N$, we have $Q_j^M(n_1, ..., n_j) \in N$. Then, $(N, Q^M|_N)$ is a formal DG manifold and the inclusion $N \longrightarrow M$ is a morphism in DG-Manf.

(ii) Let (B, Q^B, Q) be a deformation of (M, Q^M). Suppose that (B, Q^B) is a direct sum of formal DG manifolds $(B', Q^{B'})$ and $(B'', Q^{B''})$. Then, $(B'', Q^{B''}, Q|_{B'' \times M})$ is also a deformation of M and the canonical map

$$(B'' \times M, Q^{B''} + Q^M + Q|_{B'' \times M}) \longrightarrow (B \times M, Q^B + Q^M + Q) \qquad (5.15)$$

defines a morphism of deformations.

(iii) If in the situation of (ii), $(B', Q_1^{B'})$ is contractible, then the map (5.15) is an equivalence of deformations.

Proof: The statements (i) and (ii) are easy to see. To show (iii), we apply Proposition 5.6 to the commutative diagram

$$
\begin{array}{ccc}
(B'' \times M, Q^{B''} + Q^M + Q|_{B'' \times M}) & \longrightarrow & (B \times M, Q^B + Q) \\
\downarrow & & \downarrow \\
(B' \times (B'' \times M), Q^{B'} + Q^{B''} + Q^M + Q|_{B'' \times M}) & \longrightarrow & B''
\end{array}
$$

of formal DG manifolds. We get an isomorphism

$$(B, Q^B, Q|_{B'' \times M}) \longrightarrow (B, Q^B, Q)$$

of deformations with base B. Obviously, the left one is equivalent to $(B'', Q^{B''}, Q|_{B'' \times M})$. $\qquad \square$

For the rest of this subsection, we work in the setting of Section 2. Thus L is the DGL $\mathrm{Coder}(S(M), S(M))$ for some formal DG manifold (M, Q^M). Again, we must assume that the complex (L, d) has a splitting η. Equip the homology H of (L, d) with the L_∞-structure μ_*, constructed in Section 4. Set $U := L[1]$, $V := H[1]$ and denote the morphism $V \longrightarrow U$, induced by the quasi-isomorphism $H \longrightarrow L$, constructed in Section 4 again by f.

Let (U, Q^U, Q) be the universal deformation of M (see Section 2). Again, set $\tilde{Q} := Q^U + Q^M + Q$. By base change $f : V \longrightarrow U$, we get a deformation (V, Q^V, Q') of (M, Q^M). Explicitly, on products $v_1 \odot \ldots \odot v_r \odot m_1 \odot \ldots \odot m_s$ with $r, s \geq 1$ and $n = r + s$, the perturbation Q' is given by

$$Q'_n(v_1, ..., v_r, m_1, ..., m_s) = (\uparrow f_r(v_1, ..., v_r))_s(m_1, ..., m_s).$$

Set $\tilde{Q}' := Q^V + Q^M + Q'$.

Theorem 5.13. *The deformation (V, Q^V, Q') is semi-universal.*

Proof: Since (H, μ_*) is minimal and (U, Q^U, Q) is universal, we only have to show, that there exists a morphism of deformations from (U, Q^U, Q) to (V, Q^V, Q'). This is a consequence of Theorem 5.10 and Remark 5.12. $\qquad \square$

A Appendix

A.1 Some Calculations

Proof of Proposition 1.2: Induction on n
The case $n = 1$ follows by the commutativity of diagram 1.1 and $\mathrm{Kern}(\Delta^+) = W'$. Now suppose

that the formula is proved for all $m \leq n - 1$. Then we have

$$(F \otimes F \circ \Delta^+)(w_1, \ldots, w_n) =$$

$$\sum_{j=1}^{n-1} \sum_{\tau \in \mathrm{Sh}(j,n)} \epsilon(\tau, w_1, \ldots, w_n) \hat{F}_j(w_{\tau(1)}, \ldots, w_{\tau(j)}) \otimes \hat{F}_{n-j}(w_{\tau(j+1)}, \ldots, w_{\tau(n)}) =$$

$$\sum_{j=1}^{n-1} \sum_{k,k'} \sum_{I,I'} \frac{1}{k!k'!I!I'!} [(F_{i_1} \odot \ldots \odot F_{i_k}) \circ \alpha_j \otimes (F_{i'_1} \odot \ldots \odot F_{i'_{k'}}) \circ \alpha_{n-j}] \circ \alpha_{j,n}(w_1 \odot \ldots \odot w_n),$$

where k ranges over $1, \ldots, j$; k' over $1, \ldots, n-j$, I takes all values in $\mathbb{N}^k$, such that $|I| = j$ and I' takes all values in $\mathbb{N}^{k'}$, such that $|I'| = n - j$. The last expression equals

$$\sum_{k=2}^{n} \sum_{l=1}^{k-1} \sum_{\substack{I \in \mathbb{N}^k}}^{|I|=n} \frac{1}{I!l!(k-l)!} [(F_{i_1} \odot \ldots \odot F_{i_l}) \otimes (F_{i_{l+1}} \odot \ldots \odot F_{i_k})] \circ \alpha_n(w_1, \ldots, w_n) \tag{1.16}$$

On the other side, we have

$$\Delta^+ \left(\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma) F_{i_1}(w_{\sigma(1)}, \ldots, w_{\sigma(i_1)}) \odot \ldots \odot F_{i_k}(w_{\sigma(n-i_k+1)}, \ldots, w_{\sigma(n)}) \right) =$$

$$\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma) \sum_{l=1}^{k-1} \sum_{\tau \in \mathrm{Sh}(l,k)} \epsilon(\sigma, F_{i_1}(\lozenge), \ldots, F_{i_k}(\lozenge)) F_{i_{\tau(1)}}(\lozenge) \odot \ldots \odot F_{i_{\tau(l)}}(\lozenge) \otimes$$

$$F_{i_{\tau(l+1)}}(\lozenge) \odot \ldots \odot F_{i_{\tau(k)}}(\lozenge) =$$

$$\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} \sum_{l=1}^{k-1} \binom{k}{l} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma, w_1, \ldots, w_n) F_{i_1}(\lozenge) \odot \ldots \odot F_{i_l}(\lozenge) \otimes$$

$$F_{i_{l+1}}(\lozenge) \odot \ldots \odot F_{i_k}(\lozenge)$$

Here, we have set $F_{i_m}(\lozenge) := F_{i_m}(w_{\sigma(i_1+\ldots+i_{m-1}+1)}, \ldots, w_{\sigma(i_1+\ldots+i_m)})$. Since $\binom{k}{l} \cdot \frac{1}{k!} = \frac{1}{l!(k-l)!}$, we see that both sides coincide. Hence, by the commutativity of diagram (1.1), the difference

$$F_n(w_1, \ldots, w_n) - \sum_{k=2}^{n} \sum_{i_1+\ldots+i_k=n} \sum_{\sigma \in \Sigma_n} \frac{1}{I!k!} \epsilon(\sigma) F_{i_1}(w_{\sigma(1)}, \ldots, w_{\sigma(i_1)}) \odot \ldots \odot F_{i_k}(w_{\sigma(n-i_k+1)}, \ldots, w_{\sigma(n)})$$

belongs to $\mathrm{Kern}(\Delta^+) = W'$. Thus it is just the term $F_n(w_1, \ldots, w_n)$, and the induction step is done. $\qquad \square$

Proof of Proposition 1.3: Induction on n. Set $q_0 := Q_{0,1}(1)$. (1) By the commutativity of diagram (1.3), comparing terms of polynomial degree zero, we have that $Q_{i,0} = 0$, for each $i \geq 0$. (2) By the commutativity of diagram (1.3), comparing terms of polynomial degree i and linear degree $+1$, we have that $Q_{0,i} = 0$, for $i \neq 1$. (3) By similar arguments, we see that for $w \in W$, we have $Q_{1,2}(w) = q_0 \odot w$ and $Q_{1,i} = 0$, for $i \geq 3$. Thus the cases $n = 0, 1$ are done. Now suppose that the statement is proved for all $m \leq n-1$. Then $(Q \otimes 1 + 1 \otimes Q)(\Delta(w_1, \ldots, w_n))$

can be written in the form

$$\sum_{l=0}^{n-1}\sum_{k=1}^{n-l}\sum_{\sigma\in\mathrm{Sh}(k+l-1,n)}\sum_{\tau\in\mathrm{Sh}(l,k+l-1)}\epsilon(\sigma)\epsilon(\tau,w_{\sigma(1)},\ldots,w_{\sigma(k+l-1)})Q_l(w_{\sigma(\tau(1))},\ldots,w_{\sigma(\tau(l))})\odot$$

$$\odot\,w_{\sigma(\tau(l+1))}\odot\ldots\odot w_{\sigma(\tau(k+l-1))}\otimes w_{\sigma(k+l)}\odot\ldots\odot w_{\sigma(n)}+$$

$$+\sum_{l=0}^{n-1}\sum_{k=1}^{n-l}\sum_{\sigma\in\mathrm{Sh}(k,n)}\sum_{\tau\in\mathrm{Sh}(l,n-k)}\epsilon(\sigma)\epsilon(\tau,w_{\sigma(k+1)},\ldots,w_{\sigma(n)})w_{\sigma(1)}\odot\ldots\odot w_{\sigma(k)}\otimes$$

$$\otimes\,Q_l(w_{\sigma(k+\tau(1))},\ldots,w_{\sigma(k+\tau(l))})\odot w_{\sigma(k+l+1)}\odot\ldots\odot w_{\sigma(n)}$$

This is just the sum over k and l of the following expression:

$$\sum_{\sigma\in\mathrm{Sh}(l,k+l-1,n)}\epsilon(\sigma)Q_l(w_{\sigma(1)},\ldots,w_{\sigma(l)})\odot w_{\sigma(l+1)}\odot\ldots\odot w_{\sigma(k+l-1)}\otimes w_{\sigma(k+l)}\odot\ldots\odot w_{\sigma(n)}+$$

$$+\sum_{\sigma\in\mathrm{Sh}(k,k+l,n)}\epsilon(\sigma)(-1)^{w_{\sigma(1)}+\ldots+w_{\sigma(k)}}\epsilon(\tau,w_{\sigma(1)},\ldots,w_{\sigma(k+l-1)})w_{\sigma(1)}\odot\ldots\odot w_{\sigma(k)}\otimes$$

$$\otimes\,Q_l(w_{\sigma(k+1)},\ldots,w_{\sigma(k+l)})\odot w_{\sigma(k+l+1)}\odot\ldots\odot w_{\sigma(n)}.$$

Here, by $\mathrm{Sh}(l,m,n)$ we mean the set of all permutations $\sigma\in\Sigma_n$, such that $\sigma(1)<\ldots<\sigma(l)$ and $\sigma(l+1)<\ldots<\sigma(m)$ and $\sigma(m+1)<\ldots<\sigma(n)$. On the other side,

$$\Delta(\sum_{l=0}^{n-1}\sum_{\sigma\in\mathrm{Sh}(l,n)}\epsilon(\sigma)Q_l(w_{\sigma(1)},\ldots,w_{\sigma(l)})\odot w_{\sigma(l+1)}\odot\ldots\odot w_{\sigma(n)})$$

can be written as sum over k and l of expressions of the form

$$\sum_{\sigma\in\mathrm{Sh}(l,n)}\sum_{\tau\in\mathrm{Sh}(k,l-n+1)}\epsilon(\sigma)\epsilon(\tau,u_1,\ldots,u_{n-l+1})u_{\tau(1)}\odot\ldots\odot u_{\tau(k)}\otimes u_{\tau(k+1)}\odot\ldots\odot u_{\tau(n-l+1)},$$

where we have set

$$(u_1,\ldots,u_{n-l+1}):=(Q_l(w_{\sigma(1)},\ldots,w_{\sigma(l)}),w_{\sigma(l+1)},\ldots,w_{\sigma(n)}).$$

We see easily that on both sides we have the same sums. This finishes the induction step. $\square$

Proof of Lemma 5.1: By our hypothesis, for each $m\leq n-1$, we have

$$Q_1\circ f_m-f_m\circ Q_1^{(m)}=\sum_{\substack{k+l=m+1\\k\geq 2}}f_l\circ Q_k^{(n)}-\sum_{k=2}^{m}\sum_{\substack{I\in\mathbb{N}^k\\|I|=m}}Q_k\circ f_I. \tag{1.17}$$

Furthermore, we can generalize the fact that Q is an L_∞-structure to the the following equations: Let m,k,l be natural numbers such that $m\geq 1$ and $k+l=m+1$. Then

$$Q_1^{(l)}\circ Q_k^{(m)}+Q_k^{(m)}\circ Q_1^{(m)}+\sum_{\substack{r+s=k+1\\r,s\geq 2}}Q_r^{(m+1-s)}\circ Q_s^{(m)}=0. \tag{1.18}$$

Of course, they are also correct for Q'. Now, we apply δ on the first summand of $r(f_1,\ldots,f_{n-1})$.

Using equations (1.18) and (1.17), it takes the following form:

$$\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} (Q'_k \circ f_I \circ Q_1^{(n)} - Q'_1 \circ Q'_k \circ f_I) =$$

$$\sum_{k} \sum_{I} Q'_k \circ Q_1'^{(k)} \circ f_I + \sum_{k} \sum_{I} \sum_{\substack{r+s=k+1 \\ r,s \geq 2}} Q'_r \circ Q_s'^{(k)} \circ f_I + \sum_{K} \sum_{I} Q'_k \circ f_I \circ Q_1^{(n)} =$$

$$-\sum_{k=2}^{n-1} \sum_{I} \sum_{\nu=1}^{k} \sum_{r=2}^{i_\nu} \sum_{j_1+...+j_r=i_\nu} Q'_k \circ (1^{\otimes \nu-1} \otimes Q'_r \otimes 1^{\otimes k-\nu}) \circ f_{(i_1,...,i_{\nu-1},j_1,...,j_r,i_{\nu+1},...,i_k)} +$$

$$\sum_{\bar{k}=2}^{n} \sum_{\substack{|\bar{I}| \in \mathbb{N}^{\bar{k}} \\ |\bar{I}|=n}} \sum_{\substack{\bar{r}+\bar{s}=\bar{k}+1 \\ \bar{r},\bar{s} \geq 2}} \sum_{\bar{u}+\bar{v}=\bar{r}-1} Q'_{\bar{r}} \circ (1^{\otimes \bar{u}} \otimes Q'_{\bar{s}} \otimes 1^{\bar{v}}) \circ f_{\bar{I}} +$$

$$\sum_{k=2}^{n} \sum_{I} \sum_{\nu=1}^{k} \sum_{\substack{s+t=i_\nu+1 \\ t \geq 2}} Q'_k \circ f_{(i_1,...,i_{\nu-1},s,i_{\nu+1},...,i_k)} \circ (1^{\otimes i_1+...+i_{\nu-1}} \otimes Q_t^{(i_\nu)} \otimes 1^{\otimes i_{\nu+1}+...+i_k}).$$

The first and second summand annihilate each other, since we have the following 1:1 - correspondence of index sets:

$$(k, I, \nu, r, J) \mapsto (\bar{k} = k+r-1, \bar{I} = (i_1,...,i_{\nu-1},j_1,...,j_r,i_{\nu+1},...,i_k), \bar{r} = k, \bar{u} = \nu-1),$$

$$(k = \bar{r}, I = (\bar{i}_1,...,\bar{i}_{\bar{u}}, \bar{i}_{\bar{u}+\bar{s}+1},...,\bar{i}_{\bar{k}}), \nu + \bar{u} + 1, r = \bar{s}, J = (\bar{i}_{\bar{u}+1},...,\bar{i}_{\bar{u}+\bar{s}})) \leftarrow (\bar{k}, \bar{I}, \bar{r}, \bar{u}).$$

We apply δ on the second summand of $r(f_1,...,f_{n-1})$:

$$\delta(\sum_{k=2}^{n} \sum_{\substack{I \in \mathbb{N}^k \\ |I|=n}} Q'_k \circ f_I) = \sum_{k} \sum_{I} (Q'_k \circ f_I \circ Q_1^{(n)} - Q'_1 \circ Q'_k \circ f_I) =$$

$$\sum_{k} \sum_{I} Q'_k \circ f_I \circ Q_1^{(n)} - \sum_{k} \sum_{I} Q'_k \circ Q_1'^{(k)} \circ f_I - \sum_{k} \sum_{I} \sum_{\substack{u+v=k+1 \\ u,v \geq 2}} Q'_u \circ Q_v'^{(u)} \circ f_I =$$

$$\sum_{k=2}^{n-1} \sum_{I} \sum_{\nu=1}^{k} \sum_{r=2}^{i_\nu} \sum_{j_1+...+j_r=i_\nu} Q'_k \circ (1^{\otimes \nu-1} \otimes Q'_r \otimes 1^{\otimes k-\nu}) \circ f_{(i_1,...,i_{\nu-1},j_1,...,j_r,i_{\nu+1},...,i_k)}$$

$$-\sum_{k} \sum_{I} \sum_{\substack{s+t=i_\nu+1 \\ t \geq 2}} Q'_k \circ f_{(i_1,...,i_{\nu-1},s,i_{\nu+1},...,i_k)} \circ (1^{\otimes i_1+...+i_{\nu-1}} \otimes Q_t^{(i_\nu)} \otimes 1^{\otimes i_{\nu+1}+...+i_k})$$

$$-\sum_{\bar{k}} \sum_{\bar{I}} \sum_{\substack{u+v=\bar{k}+1 \\ u,v \geq 2}} \sum_{c+d=u-1} Q'_u \circ (1^{\otimes c} \otimes Q'_v \otimes 1^{\otimes d}) \circ f_I.$$

The first and third summand annihilate each other, since we have the following 1:1 - correspondence of index sets:

$$(k, I, \nu, r, J) \mapsto (\bar{k} = k-1+r, \bar{I} = (i_1,...,i_{\nu-1},j_1,...,j_r,i_{\nu+1},...,i_k), u = k, c = \nu-1)$$

and

$$(k = u, I = (\bar{i}_1,...,\bar{i}_c, \bar{i}_{c+v},...,\bar{i}_{\bar{k}}), \nu = c+1, r = v, J = (\bar{i}_{c+1},...,\bar{i}_{c+v-1})).$$

The second term is just the remaining term above. So the statement is proved. □

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