Modeling the Growth of Attributed Networks

Harshay Shah, Suhansanu Kumar, Hari Sundaram
Department of Computer Science
University of Illinois at Urbana-Champaign
{hrshah4,skumar56,hs1}@illinois.edu

ABSTRACT

We propose a network growth model based on local processes that jointly explains the emergence of key structural properties of real-world attributed directed networks: heavy-tailed indegree distribution, attribute mixing patterns, high local clustering and degree-clustering correlation. In real-world networks, individuals form edges under constraints of limited network access and partial information. However, well-known growth models that preserve multiple structural properties do not incorporate these resource constraints. Conversely, models with resource-constrained edge formation mechanisms cannot jointly preserve multiple structural properties of real networks. Furthermore, most growth models disregard the effect of homophily on edge formation and global network structure.

Our Attributed Random Walk (ARW) model explains how structural & content-based properties of real-world networks jointly arise from individual preferences & edge formation under constraints of limited information and network access. In our model, each node that joins the network selects a seed node from which it initiates a biased random walk to concurrently explore the network and link to existing nodes. Our experimental results against seven well-known growth models indicate significant improvement (2.5-10x) in accurately preserving global structural properties and local mixing patterns content-based properties of seven large scale real-world networks.

KEYWORDS

Network evolution, Network growth models, Attributed networks, Homophily

1 INTRODUCTION

We develop a resource constrained model of network growth that explains the emergence of key structural properties. The problem is important for several reasons. Individuals form real-world networks acting under resource constraints and while using local information. These networks that individuals form exhibit rich structural properties. However, we lack an understanding of mechanisms that are resource constrained and which use local information explain the emergence of these structural related properties.

Classic models of network growth, make unrealistic assumptions about what agents who form edges do. Consider as a simple stylized example, the process of finding the a set of papers to cite when writing an article. In the preferential attachment model [3] of network growth, a node making m citations would pick a paper uniformly at random from *all* papers in the domain, and either cite it or copy one of its references. We would repeat this process, till we've exhausted our budget of m references. Notice that the process assumes access to the entire dataset, and that one would pick papers uniformly at random. An equivalent formulation of this copying model is to cite papers from the entire dataset in proportion to their in degree. The latter formulation assumes that agent making citations know the entire in-degree distribution. While preferential attachment models explains the emergence of the power-law degree distribution, the attachment model is an unrealistic representation of how agents make decisions on edge formation.

The problem of developing a model of network growth, where agents act under resource constraints, including access to only local information is hard. The problem lies in identifying simple mechanisms, with few parameters, where the agents only use local information and *jointly* preserve the properties related structure.

We propose a random walk based model of network growth that jointly explains the emergence of the following properties: heavy-tailed in-degree distributions, local clustering and clustering-degree relationships. In the growth model, an incoming node picks a recent node as the seed. It will link to this node with some constant linking probability. Then, it follows the outgoing link or the incoming link of this seed node and arrives at a new node. At each new node, it decides to link with the same constant linking probability. Then it has to decide whether to jump back to the seed node, or following incoming or outgoing links. The process repeats until the agent has exhausted its budget for linking. To summarize, new nodes concurrently acquire information and form edges by exploring the local neighborhoods of existing nodes, without access to the entire network.

Our main contributions are as follows:

- We propose a model of network growth using a local edge formation mechanism that incoporates the resource constraints that influence individuals' edge formation mechanisms in real-world networks.
- We propose a model that jointly explains multiple structural properties, including in-degree distribution, clustering, degree clustering relationship and edge densification.

We conducted extensive experimental results, against state of the art baselines, on large citation network datasets. We show that our growth model outperforms that best competing model in jointly and accurately preserving multiple structural properties—degree distribution, clustering and degree-clustering relationship—by a significant margin.

The rest of the paper is organized as follows. In Section 9, we describe the related work. Then, in **??**, we define key structural properties and introduce the datasets. We formally state the goal of the paper in Section 2. In **??** and Section 5, we report prominent structural characteristics of citation networks and propose a network growth model respectively. This is followed by Section 6, where we validate our model against multiple baselines.

2 PROBLEM STATEMENT

Consider an attributed directed network $G = (V, E, B)$, where V & E are sets of nodes & edges and each node has an attribute value $b \in B$. The goal is to develop a directed network growth model that preserves structural and attribute based properties observed in G. The growth model should be normative, accurate and parsimonious:

(1) **Normative**: The model should account for normative behavior: In real-world networks, individuals form edges with limited global information and under resource constraints (e.g. partial network access).
(2) **Accurate**: The model should preserve key structural and attribute based properties such as heavy tailed degree distribution, skewed local clustering, negatively correlated degree-clustering relationship and attribute mixing patterns.
(3) **Parsimonious**: The model should be simple but expressive enough to generate networks with varying structural properties.

3 DATASETS

We consider six large-scale citation networks from diverse sources: research articles, utility patents and judicial cases. We study the structural and content properties of these networks in Section 4 and empirically validate the effectiveness of the proposed model using these network datasets in Section 6.

We focus on citation networks for three reasons. First, nodes in citation networks form all outgoing edges to existing nodes at the time of joining the network; Nodes do not form or delete edges at a later time. This allows us to analyze the edge formation mechanisms of new nodes that join the network form edges. Second, citation network datasets include the time (e.g., publication year of academic papers) at which nodes join the network. As a result, local edge formation processes and global structural properties can be better understood by studying network snapshots at different stages of the growth process. Third, the citation networks are large networks that tend to have one or more nodal attributes (e.g. category of patents) and span multiple decades. As a result, the structural and content properties of the citation networks considered are well-defined.

Network	$\|V\|$	$\|E\|$	T	A	$\|A\|$
USSC	30,288	216,738	1754-2002	-	-
HEP-PH	34,546	421,533	1992-2002	-	-
Semantic	7,706,506	59,079,055	1991-2016	-	-
ACL	18,665	115,311	1965-2016	VENUE	50
APS	577,046	6,967,873	1893-2015	JOURNAL	13
Patents	3,923,922	16,522,438	1975-1999	CATEGORY	6

Table 1: Network summary statistics: number of nodes $|V|$ and edges $|E|$, time period T, categorical attribute A and number of attribute values $|A|$ of seven citation networks.

Now, we briefly describe the datasets considered in this paper. Three of the six network datasets have nodal attribute data; That is, each node has a categorical attribute value. Table 1 provides summary statistics of the following networks:

(1) **Association of Computational Linguistics** (ACL) [39] is an attributed academic citation network that consists of papers published in ACL conferences, journals and workshops. The attribute value of each paper is the name of the venue where it was published.
(2) **U.S. Supreme Court Cases** (USSC) [14] is a judicial citation network of U.S. Supreme Court cases. There is an edge from case i to case j if and only if case i cites case j in its majority opinion.
(3) **ArXiv HEP-PH** (HEP-PH) [15] is an academic citation network of HEP-PH (high energy physics phenomenology) papers in the ArXiv e-print.
(4) **APS Journals** (APS) [1] is an attributed academic citation network maintained by the American Physical Society (APS). The attribute value of each paper is the APS journal in which it was published.
(5) **U.S Utility Patents** (Patents) [26] is an attributed citation network of U.S. utility patents maintained by the National Bureau of Economic Research (NBER). The attribute value of each patent is an NBER patent category.
(6) **Semantic Scholar** (Semantic) [2] is an academic citation network of Computer Science and Neuroscience papers, released in June 2017 by Semantic Scholar.

In this section, we outlined the citation network datasets that we use in our analysis and experiments. Next, we discuss common factors that affect edge formation mechanisms and identify common global structural properties of real networks.

4 EMPIRICAL ANALYSIS

In this section, we describe key factors that impact edge formation in real networks and analyze global structural properties of real networks, a cumulative effect of edge formation mechanisms over time. Then, we briefly contrast findings of empirical studies in sociology to common assumptions in network modeling.

4.1 Factors Influencing Edge Formation

We describe three factors — preferential attachment, triadic closure & homophily — that influence how individuals form links in real networks. Compact statistical descriptors of global network properties [33] — degree distribution, local clustering coefficient & attribute assortativity — quantify the cumulative effect of these factors on global network structure.

In the preferential attachment process [3, 42], nodes with higher degree receive links at a faster rate because incoming nodes tend to link to well-connected nodes that have more visibility. As a result, initial differences in node connectivity get reinforced over time, giving rise to a rich-get-richer effect. This phenomenon cumulatively leads to a heavy tailed degree distribution, in which a small but significant fraction of nodes turn into well-connected hubs.

In the triadic closure phenomenon [34, 41], nodes with one or more common neighbors have an increased likelihood of forming a connection. The local clustering coefficient of a node measures the prevalence of triadic closure in its neighborhood; It is the probability that two randomly chosen neighbors of the node i are connected: In directed networks, there are multiple definitions of a node's

[1] https://journals.aps.org/datasets

neighborhood: The neighborhood of node i can refer to the set of nodes that link to i, set of nodes that i links to or the union of both sets. We define neighborhood to be the set of all nodes that link to node i.

Real attributed networks tend to exhibit homophily [28], the phenomenon in which similar nodes are more likely to be connected than dissimilar nodes. The assortativity coefficient [35] $r \in [-1, 1]$, measures the level of homophily (or heterophily) in an attributed network with categorical nodal attribute $B = \{b_1 ... b_l\}$. It is defined as the ratio between the observed modularity and the maximum possible modularity with respect to attribute B Intuitively, it compares the observed fraction of edges between nodes with the same attribute value to the expected fraction of edges between nodes with same attribute value if the edges were rewired randomly. High assortativity implies that nodes with the same attribute value are more likely to be connected than nodes with different attribute values.

Preferential attachment, triadic closure and homophily not only effect how individuals form connections at the local level but also have a cumulative effect on the global structural properties of real networks.

4.2 Observations from Network Data

In this subsection, we analyze global structural properties of the bibliographic network datasets listed in Section 3. We identify regularities in key global network properties: indegree distribution, network growth rate, clustering and attribute mixing patterns.

Citation networks exhibit highly skewed, heavy tailed indegree distributions. This implies that most papers receive zero or a few citations, but a small but significant percent of the nodes turn into popular hubs that receive many citations. Lognormal fits describe the indegree distribution of all network datasets, well consistent with Broido & Clauset's [9] observation that most real-world scale free networks are rare; The parameters of the lognormal fits are listed in table 2.

The average outdegree of nodes that join real-world networks tends to increase as functions of network size and time. This phenomenon densifies networks and shrinks their diameter over time; Leskovec et al. [26] show that densification in many real networks exhibit a power law relationship between the number of edges $e(t)$ and nodes $n(t)$ at time t: $e(t) \propto n(t)^{\alpha}$. Table 2 lists the densification power law exponent α in the network datasets. In our proposed model, we increase the outdegree of incoming nodes at a linear or superlinear rate to account for the accelerated network growth observed in real networks.

Real-world networks tend to exhibit high average local clustering, as shown in Table 2. Local clustering quantifies the extent to which triadic closure influences underlying edge formation mechanisms. As shown in Figure 1, the distribution over local clustering in these networks is skewed. While models [19, 22] that rely on triangle closing mechanisms preserve *average* local clustering, our experiments indicate that richer edge formation mechanisms are necessary to preserve the skewed clustering distributions of real networks accurately.

The bivariate relationships between indegree and clustering in fig. 1 show that average local clustering decreases as a function

Network Dataset	LN (μ, σ)	DPL α	Avg. LCC	AA r
USSC	(1.19, 1.18)	2.32	0.12	-
HEP-PH	(1.32, 1.41)	1.67	0.12	-
Semantic	(1.78, 0.96)	1.58	0.06	-
ACL	(1.93, 1.38)	1.43	0.07	0.07
APS	(1.62, 1.20)	1.26	0.11	0.44
Patents	(1.10, 1.01)	1.94	0.04	0.72

Table 2: Network properties: lognormal (LN) degree distribution mean & standard deviation (μ, σ), densification power law (DPL) exponent α, average local clustering coefficient (LCC) and attribute assortativity (AA) coefficient of six network datasets.

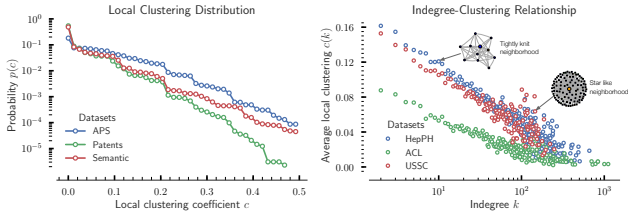

Figure 1: Local clustering in real networks have common characteristics: skewed local clustering distribution (left subplot) and a negatively correlated relationship between indegree and average local clustering (right subplot).

of indegree. Low indegree nodes have small, tightly knit neighborhoods and high indegree nodes tend have large, star-shaped neighborhoods.

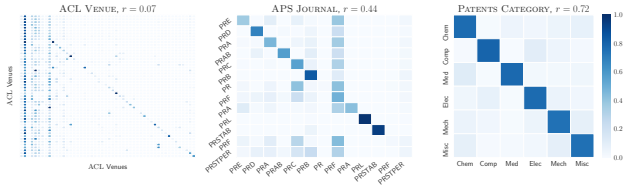

Figure 2: Attributed networks exhibit varying levels of homophily. The subplots illustrate the mixing patterns in APS and Patents w.r.t. attributes Journal ($r = 0.44$) and Category ($r = 0.72$) respectively.

Attributed networks such as ACL, APS & Patents exhibit varying level of homophily, as shown in Figure 2, with assortativity ranging from 0.07 to 0.72. The magnitude of the attribute assortativity signifies the extent to which attribute similarity influences edge formation. Indeed, homophilic preferences at the local level can lead to networks that have relatively dense clusters of similar nodes. We model edge formation as a function of attribute similarity to generate networks with varying attribute mixing patterns.

4.3 Insights from Sociological Studies

Sociological studies on network formation provide information about factors that influence how individuals form edges in real-world networks. Empirical studies [6, 23] that investigate the interplay between triadic closure and homophily in evolving networks indicate that *both* structural proximity and homophily are

statistically significant factors that influence edge formation. While homophilic preferences [28] induce edges between similar nodes, structural factors (e.g. network distance) act as constraints that restrict edge formation to structurally proximate nodes (e.g. friend of a friend). Furthermore, extensive work [16, 27, 43] on individual decision making establish that individuals are *boundedly* rational actors. That is, individuals make decisions under constraints of limited information, cognitive capacity and time. Boundedly rational actors employ simple rules to form edges under constraints of limited nodal information and partial network access. For example, a researcher cites academic papers without knowledge of or access to the entire literature in her or his field.

Based on these observations, an faithful characterization of edge formation in real-world networks necessitates bias towards nodes that are similar, proximate or well-connected under constraints of limited information and network access. Existing preferential attachment or fitness-based models [3, 13, 21, 44] make two assumptions that are inconsistent with studies on edge formation formation in evolving networks. First, by assuming that successive edge formations are independent, these model disregard the effect of triadic closure and structural proximity. Second, they implicitly require incoming nodes to have complete network access (e.g., connect to any node) or explicit knowledge of one or more properties (e.g., fitness) of

To summarize, citation networks tend to be homophilic networks that undergo accelerated network growth and exhibit regularities in structural properties: tailed indegree distributions, skewed local clustering distributions, negatively correlated degree-clustering relationship and varying mixing patterns. These global properties are a cumulative effect of individuals' edge formation decisions under resource constraints.

Next, we propose a growth model that unifies multiple sociological phenomena to explain how *local* factors that affect edge formation lead to the emergence of global structural properties observed in real networks.

5 ATTRIBUTED RANDOM WALK MODEL

We propose an Attributed Random Walk (ARW) model to explain the emergence of key structural properties of real networks through *entirely local* edge formation mechanisms. First, we explain how ARW intuitively incorporates multiple sociological phenomena. Second, we describe the edge formation mechanisms underlying ARW. Finally, we briefly discuss the methods required to fit the model to data.

5.1 Unifying Sociological Phenomena

ARW grows a directed network over time as new nodes join the network. The mechanism that incoming nodes use to form edges intuitively corresponds to how we expect researchers to conduct a literature survey and cite relevant work. First, the researcher broadly identifies *relevant* papers, possibly with the help of external information sources. Then, under time and information constraints, the individual navigates a chain of references to identify *similar* papers that either support or address the problem in hand. Next, through careful analysis, he or she decides to cite a subset of these papers; Heuristics such as number of citations maybe used in the decision making process. Similarly, an incoming node selects a seed node and initiates a random walk to explore the network by navigating through neighborhoods of existing nodes. It halts the random walk after connecting to a few visited nodes.

The Attributed Random Walk model unifies multiple well-known sociological phenomena into its edge formation mechanism:

Phenomenon 1. *(Limited Resources) Individuals are boundedly rational [16, 27, 43] actors that form edges under constraints of limited information, partial network access and finite cognitive capacity.*

In ARW, we use random walks to incorporate constraints of limited information and partial network access. A new node u selects a seed node from which it initiates a biased random walk. At each step of the walk, the new node either jumps back to the seed node or traverses an outgoing or incoming edge to visit another node. It links to each visited node with some probability and halts after forming a few edges, as shown in fig. 3. Random walks inherently account for limited information and partial network access as they only require information about the 1-hop neighborhood of visited nodes.

Phenomenon 2. *(Structural Constraints) Structural factors such as network distance act as constraints that limit edge formation to proximate nodes. [23]*

We incorporate structural constraints into ARW using the jump parameter p_j. The jump parameter p_j is the probability which new nodes jump back to their seed node after every step of the random walk. This implies that the probability with which the new node is at most k steps from its seed node is $1 - p_j^k$; As a result, the jump parameter p_j controls the extent to which new nodes' random walks explore the network to form edges.

Phenomenon 3. *(Triadic Closure) Nodes with common neighbors have an increased likelihood of forming a connection. [41]*

We control the effect of triadic closure on edge formation using the rate parameter α. A new node u, which visit nodes using random walks, link to each visited node with probability proportional to α. As a result, the probability with which node u closes a triad by linking to a visited node and its neighbor is proportional to α^2.

Phenomenon 4. *(Attribute Homophily) Nodes that have similar attributes are more likely to form a connection. [28]*

We incorporate attribute homophily into the edge formation process via attribute parameter p_a. New node u links to visited node v with probability $\alpha \cdot p_a$ if they share the same attribute value. Otherwise, u connects to v with probability $\alpha \cdot (1 - p_a)$. The attribute parameter p_a effectively controls the global assortativity coefficient.

Phenomenon 5. *(Preferential Attachment) Nodes tend to link to high degree nodes that have more visibility. [3]*

Individuals cannot link to high degree nodes *directly* under constraints of limited information and partial network access. In the absence of global information, we induce preferential attachment in ARW by adding structural bias to random walk traversals. We utilize the positive correlation between node age and node degree to adjust bias towards visiting old nodes that tend to have high degree. Indeed, random walks that traverse outgoing edges only

eventually visit old nodes that tend to have high indegree. Similarly, random walks that traverse incoming edges only visit recently joined noes that tend to have low indegree. We use out parameter p_o, the probability with which nodes choose outgoing edges in their random walks, to adjust the effect of preferential attachment on edge formation.

ARW unifies five well-known sociological phenomena into a single edge formation mechanism based on random walks. Random walks inherently account for limited information and partial network access. Furthermore, the jump parameter p_j, attribute parameter p_a, rate parameter α and out parameter p_o incorporate the effect of structural constraints, homophily, triadic closure and preferential attachment respectively.

5.2 Model Details

The Attributed Random Walk (ARW) model grows a directed network $\{G_t\}_{t=1}^{T}$. More formally, at every discrete time step t, a new node u, with attribute value $B(u)$, joins the network $G_t = (V_t, E_t, B_t)$, where V_t, E_t and B_t are sets of nodes, edges and unique attribute values at time t. After joining the network, node u forms $m(t)$ edges to existing nodes. Outdegree of incoming nodes increases over time to reflect the nonlinear growth and densification of real networks. We discuss the issue of initializing G_0, sampling attribute values of inomcing nodes and modeling densification in Subsection 5.3.

The edge formation mechanism consists of two components: Select-Seed and Random-Walk. A new node u with attribute value $B(u)$ that joins the network at time t first selects a seed node s_u using Select-Seed:

Select-Seed

(1) With probability p_a, randomly select s_u from the set of existing nodes that have attribute value $B(u)$.

(2) Otherwise, with probability $1 - p_a$, randomly select s_u from the set of existing nodes that do *not* have attribute value $B(u)$.

Seed-Select accounts for homophilic preferences of incoming nodes using attribute parameter p_a. Note that Seed-Select only requires attribute information of the sampled nodes. As show in fig. 3, after selecting the seed s_u, node u initiates a random walk using

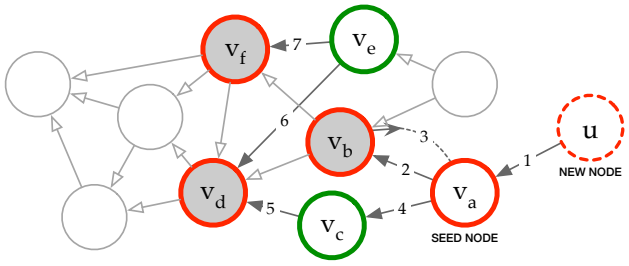

Figure 3: Edge formation in ARW. A new node u joins the attributed network with outdegree $m = 3$ and attribute value $B(u) =$ RED $\in$ {RED, BLUE}. It uses Select-Seed to select seed node v_a and initiates a Random-Walk. As denoted by the labeled edges, node u starts from v_a, moves to v_b, jumps back to seed v_a and visits v_c, v_d, v_e and v_f. It links to three nodes — v_b, v_d & v_f — that have the same attribute value.

Random-Walk to form $m(t)$ links. The Random-Walk mechanism consists of four parameters - α & p_a parameterize edge formation decisions and p_j & p_o characterize random walk traversals:

Random-Walk

(1) At each step of the walk, new node u visits node v_i.
- If $B(u) = B(v_i)$, u links to v_i with probability $\alpha \cdot p_a$
- Otherwise, u links to v_i with probability $\alpha \cdot (1 - p_a)$

(2) Then, with probability p_j, u jumps back to seed s_u.

(3) Otherwise, with probability $1 - p_j$, u continues to walk. It picks an outgoing edge with probability p_o *or* an incoming edge with probability $1 - p_o$ to visit a neighbor of v_i.

(4) Steps 1-3 are repeated until u links to $m(t)$ nodes.

When attribute data is absent, the attribute parameter p_a is not required. Then, Seed-Select simply selects an existing node uniformly at random and the probability of edge formation in Random-Walk is equal to the rate parameter α only.

Next, we explain how each parameter is necessary to conform to normative behavior of individuals in evolving networks.

Harshay: move To summarize, the Attributed Random Walk (ARW) model intuitively describes how individuals form edges under resource constraints. ARW uses four parameters — α, p_a, p_j, p_o — to incorporate individuals' biases towards similar, proximate and high degree nodes.

5.3 Model Fitting

We now briefly describe methods to estimate ARW parameters, initialize $\hat{G}$ at time $t = 0$, densify $\hat{G}$ over time and sample incoming nodes' attribute values.

Parameter Estimation. The parameter estimation task consists of finding the set of parameters values for (α, p_a, p_j, p_o) that best explain the structural properties of an observed network $G = (V, E, A)$. We use a straightforward grid search method to estimate the four parameters. Other derivative-free optimization methods such as the Nelder-Mead [32] method can be used to quicken parameter estimation.

Initialization. The edge formation mechanism in ARW is sensitive to a large number of weakly connected components (WCCs) in the initial network $\hat{G}_0$ because incoming nodes can only form edges to nodes in the same WCC. To ensure that $\hat{G}_0$ is weakly connected, we perform an undirected breadth-first search on the observed, to-be-fitted network G that starts from the oldest node and terminates after visiting 0.1-1% of the nodes. The initial network $\hat{G}_0$ is the small subgraph induced from the set of visited nodes.

Densification. We incorporate densification into ARW by increasing the outdegree of new nodes that *sequentially* join the network $\hat{G}$. Each incoming node u that joins $\hat{G}$ at time t corresponds to some node that joins the observed network G in year $y(t)$; The number of edges $m(t)$ that u forms is equal to the average outdegree of nodes that join G in year $y(t)$. Therefore, the rate of growth in $\hat{G}$ coarsely reflects the rate of growth in G.

Sampling Attribute Values. In real networks $G = (V, E, A)$, the distribution over the set of attribute values $P_G(A)$ changes over time. For instance, the attribute distribution over journals in the APS citation network changes over time as old journals decay in

popularity and new journals gain traction. To incorporate this phenomenon into ARW, we sample the attribute value $A(u)$ of node u, that joins $\hat{G}$ at time t, from $P_G(A \mid \text{year} = y(t))$, the observed attribute distribution conditioned on the corresponding year of node u.

In this section, we described the Attributed Random Walk model and discussed methods related to model fitting. Next, our experiments in section 6 show that ARW accurately preserves *multiple* structural properties of real networks

6 MODELING NETWORK STRUCTURE

In this section, we evaluate the effectiveness of our model in preserving structural properties of six real-world networks described in section 3. Our experiments compare ARW to six well-known growth models. In Subsection 6.1, we describe these growth models and the evaluation metrics used in the experiments. In Subsection 6.2 , we discuss our experimental results.

6.1 Experiment Setup

We first briefly summarize the existing models used in the experiments. Then, we describe the evaluation metrics used to quantify the extent to which growth models preserve structural properties of real networks.

Existing Growth Models. We compare ARW to six well-known growth models that are representative of the key edge formation mechanisms. We consider two preferential attachment models, two attributed network growth models and two random walk models:

(1) **Dorogovtsev-Mendes-Samukhin model** [13] (DMS) is a preferential attachment model in which the probability of linking to a node is proportional to its indegree and "initial attractiveness."
(2) **Holme-Kim model** [19] (HK) is a preferential attachment model which uses a triangle-closing mechanism to generate scale-free, clustered networks.
(3) **Kim-Altmann model** [21] (KA) is a fitness-based model that defines fitness as the product of degree and attribute similarity. It can generate *attributed* networks with assortative mixing and heavy tailed degree distribution.
(4) **Social Attribute Network model** [17] (SAN) generates attributed networks with heavy tailed degree distribution, clustering and homophily using attribute-augmented preferential attachment and triangle closing mechanisms.
(5) **Relay Linking model** [44] (RL) propose a set of preferential attachment models that use relay linking to explain the change in node popularity over time. We use the iterated preferential relay-cite (IPRC) variant, which best fits real-world network properties.
(6) **Herera-Zufiria model** [40] (SK) is a random walk model that tunes the length of random walks to generate clustered networks with power law degree distributions.
(7) **Saramaki-Kaski** [18] (HZ) is a random walk model that generates scale-free networks with tunable average local clustering.
(8) **Forest Fire model** [26] (FF) is a recursive random walk model that generates directed networks which exhibit decreasing diameter over time, heavy-tailed degree distribution and high clustering.

To ensure a fair comparison, we modify these models in three ways. First, models that do not have an explicitly defined initial graph use the initialization method described in Subsection 5.3. Second, we extend models that use constant node outdegree to incorporate densification using the method described in Subsection 5.3. Third, we adjust models that generate undirected networks to create directed edges and thereby generated directed networks.

Evaluation. A network model fit should generate a network $\hat{G}$ that preserves the global network structure of the observed network G. We evaluate the fit by comparing four important global network properties of G and $\hat{G}$: degree distribution, local clustering distribution, degree-clustering relationship and attribute assortativity.

We use the Kolmogorov-Smirnov (KS) statistic to compare the univariate degree & local clustering distributions. We compare the bivariate degree-clustering relationship in G and $\hat{G}$ using Weighted Relative Error (WRE). The evaluation metric WRE aggregates the relative error between the average local clustering $c(k)$ and $\hat{c}(k)$ of nodes with indegree k in G and $\hat{G}$ respectively; The weight of each relative error term equals the fraction of nodes with indegree k in G.

Jointly preserving multiple structural properties is a multi-objective optimization problem; Model parameters that accurately preserve the degree distribution (i.e. low KS) may not preserve the clustering distribution. Therefore, we use grid search to select the model parameters that minimize the ℓ^2-norm of the aforementioned evaluation metrics. Since the evaluation metrics have different scales, we normalize the metrics before computing the ℓ^2-norm to prevent any bias towards a particular metric. We note that the sensitivity of the Forest Fire (FF) model necessitates a manually guided grid search method.

6.2 Experiment Results

Our experiment results test the efficacy of ARW in *jointly* modeling multiple structural properties relative to seven well-known models outlined in subsection 6.1. We evaluate the network models on six network datasets outlined in section 3.

We evaluate the performance of each network model as follows. We begin by fitting the model to each real-world network dataset G. Then, we compare the structural properties of network dataset G and network $\hat{G}$ generated by the fitted model using metrics outlined in subsection 6.1. We evaluate multiple instances of $\hat{G}$ to average out fluctuations and acquire data to conduct statistical tests.

Table 4 lists the evaluation metrics for every pair of model and dataset; The metrics measure the accuracy with which these models preserve key global network properties: degree distribution, local clustering distribution and indegree-clustering relationship. We do not explicitly compare the extent to which these models preserve attribute assortativity because the attribute related model parameters can be tuned to obtain arbitrary precision. Instead, models that preserve assortativity up to two decimal places — KA, SAN and ARW — have green ticks (✓) in table 4. We use a nonparametric exact significance test to evaluate the relative performance of our

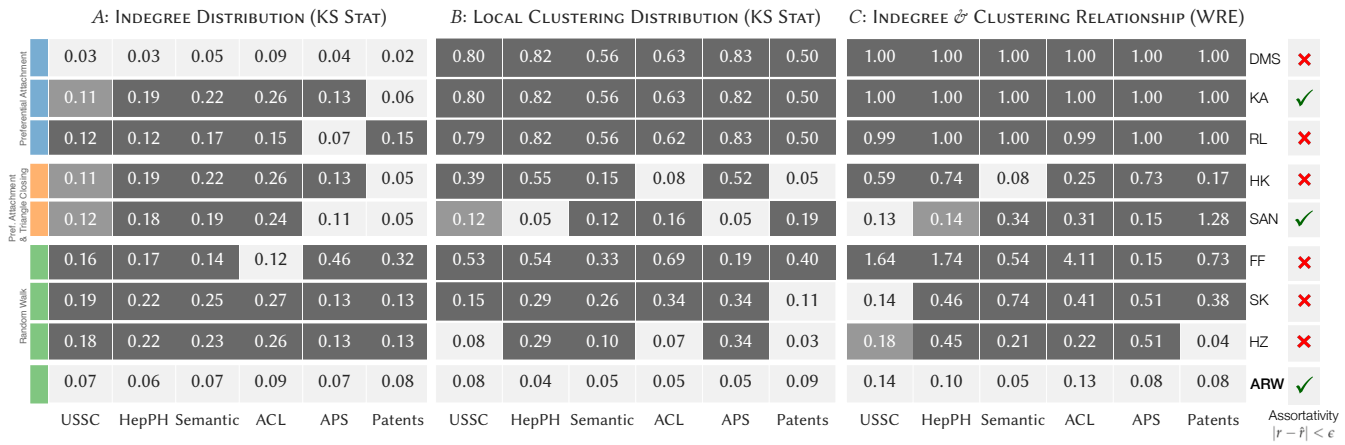

Figure 4: Modeling network structure. We assess the extent to which network models fit key structural properties of six real-world networks. Tables A, B and C measure the accuracy of seven models in fitting the indegree distribution, local clustering distribution, indegree-clustering relationship respectively and global attribute assortativity. Existing models tend to underperform because they either disregard the effect of factors such as triadic closure and/or homophily or are unable to generate networks with varying structural properties. Our model, ARW, jointly preserves all three properties accurately and often performs considerably better than existing models: The cells are shaded gray or dark gray if the proposed model ARW performs better at significance level $\alpha = 0.01$ or $\alpha = 0.001$ respectively.

model ARW. If ARW performs better than a model on a dataset with significance level $\alpha = 0.01$ or $\alpha = 0.001$, the corresponding cells in table 4 are shaded gray or dark gray boxes respectively.

A common characteristic of existing models outlined in subsection 6.1 is that they fail to accurately preserve *multiple* structural properties. For instance, the preferential attachment model DMS can accurately fit the heavy-tailed degree distributions but does not account for local clustering.

Preferential attachment models —DMS, RL and KA—preserve heavy tailed degree distributions but disregard clustering. DMS outperforms other models in accurately modeling degree distribution (table 4A) because its "initial attractiveness" parameter in can be tuned to adjust preference towards low degree nodes. Unlike KA, however, DMS cannot preserve global assortativity. By assuming that successive edge formations are independent, both models disregard the effect of triadic closure and do not preserve local clustering. (tables 4B & 4C).

HK and SAN are preferential attachment models that use triangle closing mechanisms to preserve local clustering in addition to heavy tailed degree distributions. SAN preserves assortative mixing patterns as well. Note that HK and KA fit degree distributions with the same KS statistic (table 4A) because they lack parameters that can generate varying degree distributions. While triangle closing leads to considerable improvement over DMS and KA in modeling local clustering, HK and SAN are not flexibe enough to preserve local clustering in *all* datasets (see tables 4B & 4C).

Existing random walk models FF, SK and HZ are not flexible enough to accurately preserve network structure observed in real networks datasets. The recursive approach in FF, wherein nodes perform a probabilistic breadth-first search and link to *all* visited nodes, considerably overestimates local clustering. In SK and HZ, nodes perform a single random walk and link to each visited node with some probability μ; The parameter μ indirectly controls the effect on triadic closure on edge formation. While tuning μ in SK and HZ leads to some improvement over FF in preserving local clustering distribution (table 4B) and indegree-clustering relationship (table 4), the improvements are not substantial when compared to our model ARW. Furthermore, existing random walk models disregard attribute homophily and do not model attribute mixing patterns.

The experiment results in table 4 validate the effectiveness of the proposed model ARW in *jointly* preserving multiple global network properties. ARW can generate networks with varying degree distribution by adjusting nodes' preference towards high degree nodes using out parameter p_o. As a result, ARW accurately preserves degree distribution (table 4A), often significantly better than all models except DMS. Similarly, ARW matches the local clustering distribution (table 4B) and indegree-clustering relationship (table 4C) observed in real networks with high accuracy; This is because the jump parameter p_j and link parameter p_l in ARW effectively control the effect of triadic closure on edge formation. Edge formation in ARW depends on attribute similarity via attribute parameter p_a, which can be tuned to match the attribute assortativity coefficient of attributed network datasets up to arbitrary precision.

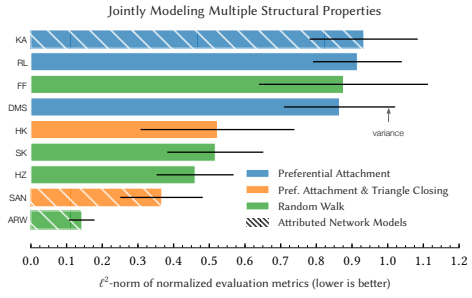

Figure 5: Jointly modeling multiple network properties: ARW outperforms existing network models by a margin of 2.5-10x.

In Figure 5, we show that ARW performs significantly better than six well-known, representative network models in jointly modeling degree distribution, local clustering distribution and indegree-clustering relationship. ARW improves upon the average ℓ^2-norm of

the second best performing model, SAN by a margin of 2.5x. This is because ARW unifies different factors that influence edge formation, described in subsection 5.1, into a single edge formation mechanism.

7 MODELING LOCAL MIXING PATTERNS

The global assortativity coefficient r, defined in subsection 4.1, quantifies the level of homophily or heterophily in an attributed network. It sheds light on the average propensity of links to occur between similar nodes by capturing the attribute mixing pattern across the entire network. However, global assortativity is not a representative summary statistic of heterogeneous mixing patterns observed in large-scale networks. It does not quantify anomalous mixing patterns and fails to measure how mixing varies across a network.

We use local assortativity [38] to measure varying mixing patterns in an attributed network $G = (V, E)$ with attribute values $B = \{b_1 \ldots b_h\}$. Unlike global assortativity that counts all edges between similar nodes, local assortativity of node i, $r_l(i)$, captures mixing pattern in the local neighborhood of node i by using a locality biased weight distribution w_i; The distribution w_i reweighs edges between similar nodes based on how local they are to node i. Peel et al. [38] prescribe a personalized pagerank weight distribution, which is prohibitively expensive to compute for all nodes in large graphs; Large network datasets necessitate efficient weighting schemes. Therefore, we define w_i as a uniform distribution over $N(i)$, the set of nodes that are at most 1 hop away from node i. More formally, the local assortativity coefficient $r_l(i)$ of node i, with outdegree $m(i)$ and attribute value $b(i)$ is defined as follows:

$$r_l(i) = \frac{\overbrace{\frac{1}{|N(i)|}\sum_{j\in N(i)}^{m(j)>0}\sum_{k\in V}\frac{\mathcal{I}\{(j,k)\in E \wedge b(j)=b(k)\}}{m(i)}}^{\text{obs}} - \overbrace{\sum_{b\in B} e_{b.}e_{.b}}^{\text{rnd}}}{\underbrace{\frac{1}{\max(\text{obs})}}_{} - \underbrace{\sum_{b\in B} e_{b.}e_{.b}}_{\text{rnd}}}$$

Intuitively, $r_l(i)$ compares the observed fraction of edges between similar nodes in the local neighborhood of node i (obs) to the expected fraction if the edges are randomly rewired (rnd).

The local assortativity distributions of ACL, APS and Patents reveal anomalous, skewed and heterophilic mixing patterns that are not easily inferred via the global assortativity coefficient:

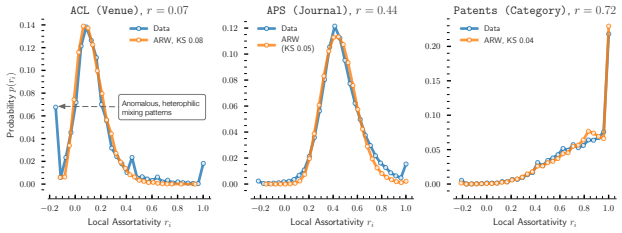

Figure 6: Modeling local mixing patterns: ARW preserves local assortativity distributions with high accuracy, but does not account for nodes with extreme heterophilic or homophilic preferences.

As shown in fig. 6, ARW can preserve diverse local assortativity distributions with high accuracy even though nodes share the same attribute preference parameter p_a. This is because ARW incorporates multiple sources of stochasticity through its edge formation proceses. As a result, incoming nodes with fixed homophilic preferences can end up having variable local assortativity by (a) selecting a seed node in a region with too few (or too many) similar nodes or (b) exhausting all its links before visiting similar (or dissimilar) nodes. However, ARW is not expressive enough to accurately model anomalous mixing patterns. Richer mechanisms such as sampling p_a from a mixture of Bernoulli distributions are necessary to account for anomalous mixing patterns.

8 DISCUSSION

In this section, we discuss the importance of measuring distributional network properties, insufficiency of the well-known triangle closing mechanism and the limitations of our model ARW.

8.1 Dissecting the Triangle Closing Mechanism

Network models (e.g., SAN [17]) commonly use triangle closing mechanisms to generate networks with tunable local clustering. However, our experimental results in subsection 6.2 show that models that rely on triangle closing cannot model the local clustering distribution or bivariate degree-clustering relationship satisfactorily. To understand why, we examine the degree-clustering relationship in the APS network:

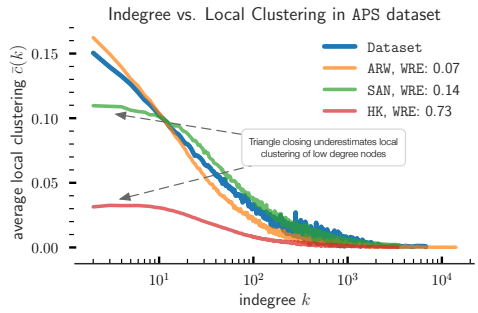

Figure 7: Triangle closing mechanisms in SAN and HK fail to model average local clustering of low indegree nodes. In contrast, the random walk mechanism in ARW visits low indegree nodes and "closes triangles" in their neighborhoods to preserve local clustering with high accuracy.

As annotated in fig. 7, models based on triangle closing mechanisms, SAN and HK, considerably underestimate the local clustering of nodes that have low indegree. This is because incoming nodes in SAN and HK tend to close triangles in the neighborhood of high indegree nodes to which they connect via preferential attachment; Indeed, local clustering plateaus as indegree decreases because triangle closing along with preferential attachment fail to form connections in neighborhoods of low indegree nodes. In contrast, ARW accurately models the degree-clustering relationship because incoming nodes initiate random walks in neighborhoods of seeds nodes that tend to have low indegree.

8.2 Measurement of Global Network Properties

Despite their widespread usage, summary statistics of global network properties such as global assortativity and average clustering

have limited representative power. Unlike point estimates, distributional properties reveal variance, skewness and anomalies in network data.

Notably, understanding local processes via distributional network properties guided the development of ARW, which consists of entirely *local* processes that do not rely on global information (e.g. fitness values of all nodes). For instance, the *skewed* clustering distribution and the relationship between clustering and degree necessitated the jump parameter p_j in our model. The structural constraints imposed by the jump parameter amplify the effect of triadic closure and preserve high clustering observed in neighborhoods of low degree nodes.

To summarize, we believe that the analysis and evaluation of *distributional* network properties is crucial to accurately model network structure.

8.3 ARW Limitations

We discuss two limitations of our work. First, ARW does not preserve the average path length distribution of real-world networks. This is because the random walk mechanism is inherently local and does not form long-range connections to bridge distant regions in the network. Our experiment results on forming "structural bridges" by initiating multiple random walks per node indicate a tradeoff between modeling small average path length and high local clustering. Second, we only consider citation network datasets to focus on edge formation mechanisms of incoming nodes that form all edges at once. We can adapt ARW to other kinds of networks: attributed random walks that pause and resume intermittently can jointly model edge formation processes between new and existing nodes in social networks; Similarly, metapath based random walks can model interactions between nodes of different types in heterogeneous information networks.

9 RELATED WORK

Network growth models can be broadly categorized by their edge formation mechanism:

Preferential Attachment & Fitness In preferential attachment and fitness-based models [4, 5, 10, 29], a new node u links to an existing node v with probability proportional to the attachment function $f(k_v)$, a function of either degree k_v or fitness ϕ_v of node v; Node fitness is defined as a dimensionless measure of node attractiveness. For instance, linear preferential attachment functions [3, 13, 24] lead to power law degree distributions and small diameter [8] and attachment functions of degree & node age [46] can preserve realistic temporal dynamics. Extensions of preferential attachment [31, 47, 48] that incorporate the limitated information and partial network access constraints disregard network properties other than power law degree distribution and small diameter. Additional mechanisms are necessary to explain network properties such as clustering and attribute mixing patterns.

Triangle Closing A set of models [19, 22, 25] incorporate triadic closure using triangle closing mechanisms, which increase *average* local clustering by forming edges between nodes with one or more common neighbors. However, as explained in subsection 8.1, models based on preferential attachment and triangle closing do not preserve the local clustering of low degree nodes.

Attributed network models Attribute network growth models [12, 17, 20, 49] account for the effect of attribute homophily on edge formation and preserve mixing patterns. Existing models can be broadly categorized as (a) fitness-based model that define fitness as a function of attribute similarity and (b) microscopic models of network evolution that require complete temporal information about edge arrivals & deletion. Our experiment results in subsection 6.2 show that well-known attributed network models SAN and KA preserve assortative mixing patterns, degree distribution to some extent, but not local clustering and degree-clustering correlation.

Random walk models first introduced by Vazquez [45], random walk models are inherently local. Models [7] in which new nodes only link to terminal nodes of short random walks generate networks with power law degree distributions [11] and small diameter [30] but do not preserve clustering. Models such as SK [40] and HZ [18], in which new nodes probabilistically link to each visited nodes incorporate triadic closure but are not flexible enough to preserve *skewed* local clustering of real-world networks, as shown in subsection 6.2. We also observe that recursive random walk models such as FF [26] preserve temporal properties such as shrinking diameter but considerably overestimate local clustering and degree-clustering relationship of real-world networks. Furthermore, existing random walk models disregard the effect of homophily and do not model attribute mixing patterns.

Recent Work Pálovics et al. [37] use preferential & uniform attachment to model the decreasing power law exponent of real-world, undirected networks in which average degree increases over time. Singh et al. [44] (RL) augment preferential attachment to explain the shift in popularity of nodes over time via the concept of relay linking. Both models do not incorporate mechanisms to preserve clustering, attribute mixing patterns and resource constraints that affect how individuals form edges in real-world networks.

To summarize, network growth models seek to explain a subset of structural properties observed in real networks. Existing models do not explain how resource constrained and local processes *jointly* preserve multiple global network properties of attributed networks. To the best of our knowledge, ARW is the first model that unifies multiple sociological phenomena into an entirely local process to jointly model network structure *and* attribute mixing patterns. We point the reader to extensive surveys [1, 36] of network growth models for more information.

10 CONCLUSION

In this paper, we model resource-constrained network growth model in which nodes use a random walk process to form edges under constraints of limited information and network access constraints. The problem is important because edge formation in real networks is usually a local process. In typical network growth scenarios, nodes in the network either have limited information about the other nodes in the network or the system allows access to only restricted portion of the existing network. It therefore becomes imperative to model how the local processes of link formation gives rise to network characteristics. In this work, we show that multiple structural properties of real networks can arise from the local

process of exploration and link formation. Our results indicate significant improvement over the next best competing model HZ [18] by a significant margin.

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11 TODO

- Dataset dist + model comparison plot (discussion)
- random walk sequence of snapshots evolution diagram
- model parameters section with table on fitted values?
- SAN microscopic simplifications (experiment setup)
- theory and dataset bias justification (discussion)
- Implement raising graphs from randomness and relay linking model (baselines)