# Session 2: Implication and Compound Propositions

- Logical connectives
  - Implication
  - Biconditional
- Compound propositions
  - Precendence
  - Tautologies, Contradictions, and Contingencies
  - Truth tables

# Implication

Let $p$ and $q$ be propositions. The *conditional statement* $p \to q$ is the proposition "if $p$, then $q$." The conditional statement $p \to q$ is false when $p$ is true and $q$ is false, and true otherwise. In the conditional statement $p \to q$, $p$ is called the *hypothesis* (or *antecedent* or *premise*) and $q$ is called the *conclusion* (or *consequence*).

**Example**

p := "The earth is round" (T)

q := "The moon is round" (T)

r := "The moon is made of green cheese" (F)

# Understanding Implication

Implication does not require any connection between the antecedent and the consequent

**Example:** These implications are perfectly fine, but would not be used in our daily conversation

- "If the moon is made of green cheese, then I have more money than Bill Gates. "
- "If the moon is made of green cheese, then I'm on welfare."

# Understanding Implication

- One way to view the logical conditional is to think of an obligation or contract
    - Politician: “If I am elected, then I will lower taxes.”

- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge.
- If the politician is not elected, no one cares …

# Understanding Implication

- If p is false, a conditional statement $p \to q$ is always true
  - “If the moon is made of green cheese, then ….”

- If q is true, a conditional statement $p \to q$ is always true
  - “If …., then 1+1 = 2”

# Properties of Implication

p → q is different from q → p: this is a common logical fallacy

- "If the moon is made of green cheese, then the earth is round"
- "If the earth is round, then the moon is made of green cheese"

# Implication in Natural Language

- Conditional statements are at the heart of any logical reasoning
- Therefore you find many way how they are expressed

"if $p$, then $q$"
"if $p$, $q$"
"$p$ is sufficient for $q$"
"$q$ if $p$"
"$q$ when $p$"
"a necessary condition for $p$ is $q$"
"$q$ unless $\neg p$"

"$p$ implies $q$"
"$p$ only if $q$"
"a sufficient condition for $q$ is $p$"
"$q$ whenever $p$"
"$q$ is necessary for $p$"
"$q$ follows from $p$"
"$q$ provided that $p$"

# Implication in Mathematical Language

- In mathematics $p \rightarrow q$ is often formulated as

- A **necessary condition** for p is q
- A **sufficient condition** for q is p

- What if p is a necessary and sufficient condition for q?

# Biconditional

Let $p$ and $q$ be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "$p$ if and only if $q$." The biconditional statement $p \leftrightarrow q$ is true when $p$ and $q$ have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

**Example**

p := "The earth is round" (T)

q := "The moon is round" (T)

# Expressing the Biconditional

- Some alternative ways “*p* if and only if *q*” is expressed in English:

  - *p* **is necessary and sufficient for** *q*
  - **if** *p* **then** *q* **, and conversely**
  - *p* **iff** *q*

# Biconditional in Natural Language

- In natural language the biconditional is often implicit

“If you finish your meal, then you can have a dessert”

# Precendence in Compound Propositions

- We can compose complex logical expressions from simpler ones

$$\neg(p \vee q)$$

$$(\neg p) \vee q$$

- To simplify notation **precedence** is used
- Examples

$$\neg p \vee q$$

$$p \rightarrow q \vee \neg r$$

**TABLE 8**
**Precedence of Logical Operators.**

| *Operator* | *Precedence* |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\rightarrow$ | 4 |
| $\leftrightarrow$ | 5 |

# Tautologies, Contradictions, and Contingencies

- A **tautology** is a proposition which is always true
  - Example: $p \vee \neg p$
- A **contradiction** is a proposition which is always false
  - Example: $p \wedge \neg p$
- A **contingency** is a proposition which is neither a tautology nor a contradiction
  - Example: $p$

# Propositional Satisfiability

- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that make it true
  - Either a tautology of a contingency
- When no such assignments exist, the compound proposition is **unsatisfiable**.
  - A compound proposition is unsatisfiable if and only if its negation is a tautology
- Modeling a problem as propositional statement and asking for satisfiability corresponds to asking: is there a solution?

# Examples

Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

# Truth Tables For Compound Propositions

- Construction of a truth table
  - A row for every possible combination of values for the atomic propositions
  - A column for the compound proposition (usually at far right)
  - A column for the truth value of each sub-expression that occurs in the compound proposition; this includes the atomic propositions

# Example

**TABLE 7** The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

| $p$ | $q$ | $\neg q$ | $p \vee \neg q$ | $p \wedge q$ | $(p \vee \neg q) \rightarrow (p \wedge q)$ |
|---|---|---|---|---|---|
| | | | | | |

# Summary

- Implication
- Biconditional
- Precendence
- Tautologies, Contradictions, and Contingencies
- Truth tables for Compound Propositions

Next: Logical Equivalences