Session 4: Normal Forms

- Normal Forms
  - Disjunctive Normal Form
  - Conjunctive Normal Form

Normal Forms

- Converting arbitrary propositional statements into canonical form
- Useful for automated proofing of theorems
  - If two expressions have the same normal form they are equivalent
- Basis for the formulation of some of the most central problems in complexity theory
- Applications for circuit design

Disjunctive Normal Form (DNF)

- A propositional formula is in **disjunctive normal form** if
  - it consists of a disjunction of compound expressions
  - where each compound expressions consists of a conjunction of a set of propositional variables or their negation

Examples

$$\left(p \wedge \neg q\right) \vee \left(\neg p \wedge q\right)$$

$$\left(p \wedge \neg q\right) \vee \neg\left(\neg p \wedge q\right)$$

$$p \vee \left(\neg p \wedge q\right)$$

$$(p \vee q) \wedge (\neg p \vee q)$$

$$\neg\left(\neg p \vee q\right)$$

Example

Find the Disjunctive Normal Form (DNF) of

$$(p \vee q) \rightarrow \neg r$$

This proposition is true when r is false or when both p and q are false.

Construction of Disjunctive Normal Form

Every compound proposition can be put in disjunctive normal form

1. Construct the truth table for the proposition
2. Select rows that evaluate to **T**
3. For the propositional variables in the selected rows add a conjunct which includes the positive form of the propositional variable if the variable is assigned **T** in that row and the negated form if the variable is assigned **F** in that row

The resulting proposition can be further simplified using the equivalence

$(p \land q) \lor (p \land \neg q) \equiv p$

Example

$(p \vee q) \rightarrow \neg r$

p	q	r	¬r	p ∨ q	(p ∨ q) → ¬r
T	T	T	F	T	F
T	**T**	**F**	**T**	**T**	**T**
T	F	T	F	T	F
T	**F**	**F**	**T**	**T**	**T**
F	T	T	F	T	F
F	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**F**	**F**	**T**
F	**F**	**F**	**T**	**F**	**T**

Conjunctive Normal Form (CNF)

- A compound proposition is in *Conjunctive Normal Form* (CNF)
  - if it consists of a conjunction of compound expressions
  - where each compound expressions consists of a disjunction of a set of propositional variables or their negation
- Every proposition can be put in an equivalent CNF.
- Conjunctive Normal Form (CNF) can be obtained by
  - eliminating implications
  - moving negation inwards and
  - using the distributive and associative laws.

Examples

$$\left(p \wedge \neg q\right) \vee \left(\neg p \wedge q\right)$$

$$\left(p \wedge \neg q\right) \vee \neg\left(\neg p \wedge q\right)$$

$$p \vee \left(\neg p \wedge q\right)$$

$$(p \vee q) \wedge (\neg p \vee q)$$

$$\neg\left(\neg p \vee q\right)$$

Example

Put the following into CNF: $\neg(p \to q) \vee (r \to p)$

Complexity of DNF and CNF

- Both DNF and CNF can be much larger than the original proposition

- More precisely: there exist cases of propositions with **n** clauses for which the CNF (DNF) has $\mathbf{2^n}$ clauses

Summary

- Normal Forms
  - Disjunctive Normal Form
  - Conjunctive Normal Form
  - Every proposition can be converted to DNF and CNF
  - The resulting proposition can explode in size