Session 3: Logical Equivalences

- Showing Logical Equivalence
- Important Logical Equivalences
- Contrapositive, Converse and Inverse
- Equivalence Proofs

Logical Equivalence

Two compound propositions p and q are **logically equivalent** if $p \leftrightarrow q$ is a tautology

We write this as $p \equiv q$ where p and q are compound propositions

Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree

Example

Using a truth table we show that $\neg p \vee q \equiv p \rightarrow q$

De Morgan’s Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Using a truth table show that De Morgan’s Second Law holds

Equivalences with Basic Connectives

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law

$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

Equivalences with Implications

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Contrapositive, Converse and Inverse

$\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$, $\quad p \rightarrow q \equiv \neg q \rightarrow \neg p$

From $p \rightarrow q$ we can form the following conditional statements

- $q \rightarrow p$ is the **converse** of $p \rightarrow q$
- $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$ (and the contrapositive of $q \rightarrow p$)

They are equivalent to each other, but not equivalent to $p \rightarrow q$

Applying Logical Equivalences

The propositions in a known equivalence can be replaced by any compound proposition

Example: since we know that

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

we also know that, for example

$$\left(p_1 \vee p_2\right) \rightarrow \left(q_1 \wedge q_2\right) \equiv \neg \left(q_1 \wedge q_2\right) \rightarrow \neg \left(p_1 \vee p_2\right)$$

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements
- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B
- This is called an **equivalence proof**

Example: Equivalence Proofs

Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Example: Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Summary

- Showing Logical Equivalence
- De Morgan’s Laws
- Many Logical Equivalences
- Contrapositive, Converse and Inverse
- Equivalence Proofs

- Next: Normal Forms