

The LLL Algorithm: Lattice Basis Reduction and applications to Approximate Shortest Vector Problem

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Recap: Euclidean Space and Inner Product

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We consider a real finite-dimensional vector space \mathbb{R}^n equipped with the standard **Euclidean inner product**:

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This inner product induces the **Euclidean norm**:

$$\|\mathbf{u}\|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\sum_{i=1}^n \mathbf{u}_i^2}$$

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- **Discrete:** For every $\mathbf{x} \in \mathcal{L}$, there exists $\varepsilon > 0$ such that

$$\mathcal{B}(\mathbf{x}, \varepsilon) \cap \mathcal{L} = \{\mathbf{x}\}$$

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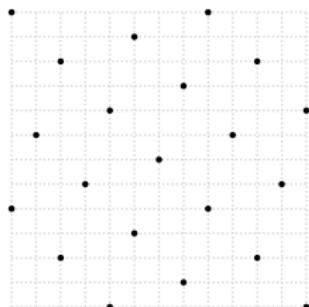


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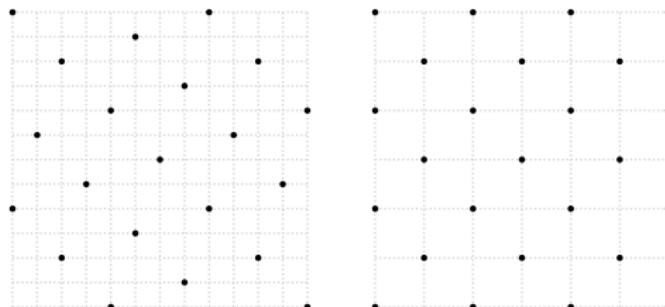


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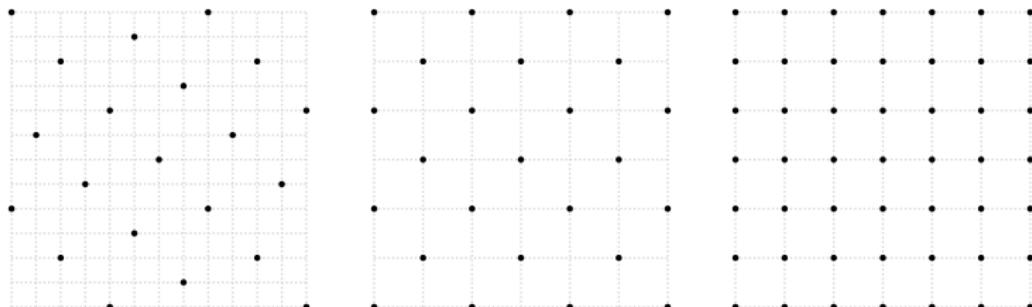


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Any lattice $\mathcal{L} \subseteq \mathbb{R}^n$ admits a maximal \mathbb{Z} -linearly independent family $(\mathbf{b}_i)_{1 \leq i \leq m}$, with $m \leq n$ such that:

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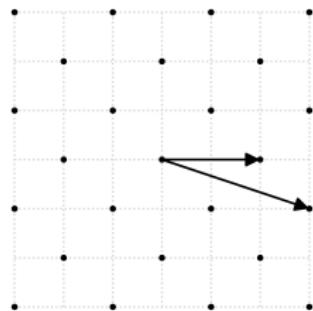


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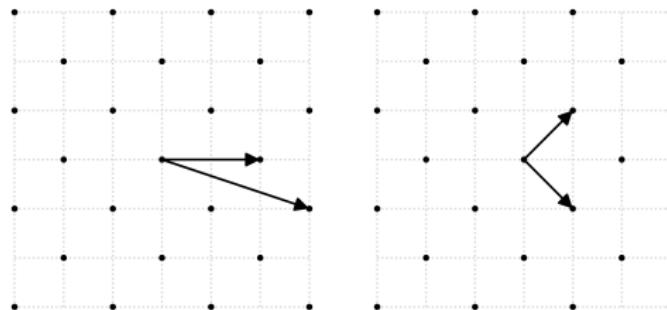


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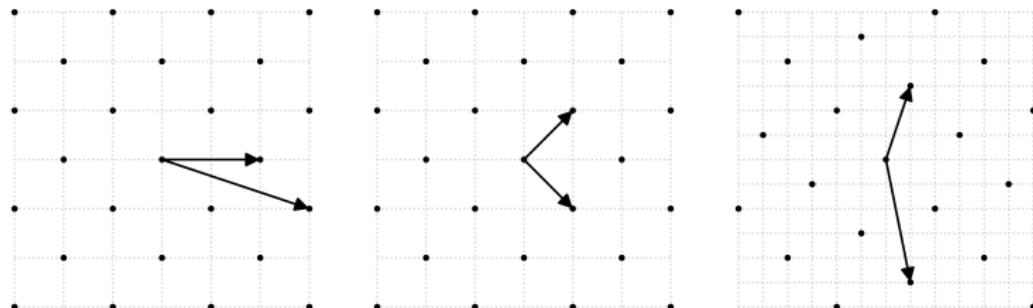
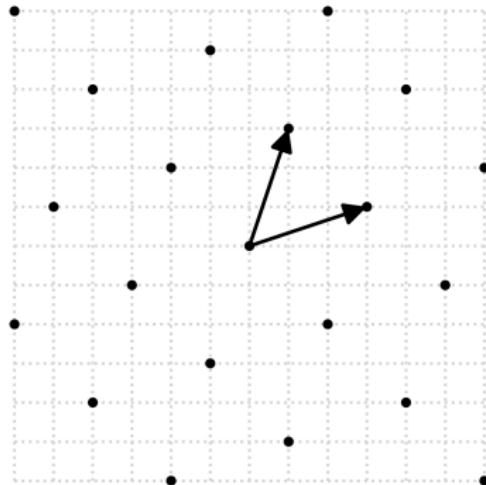


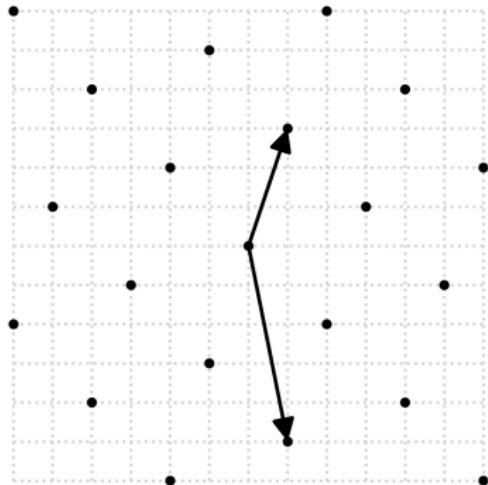
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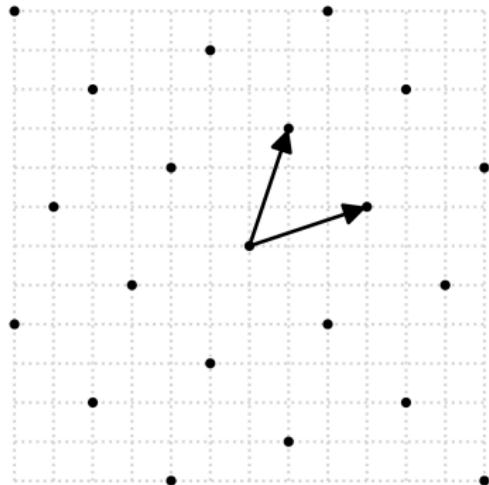


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looks good



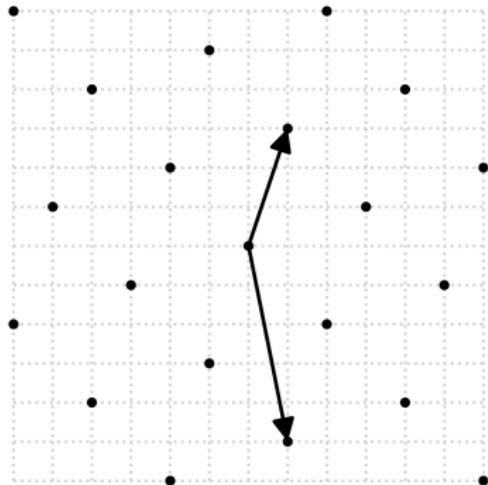
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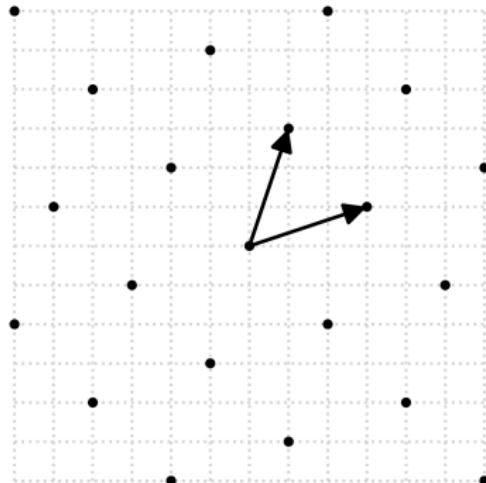
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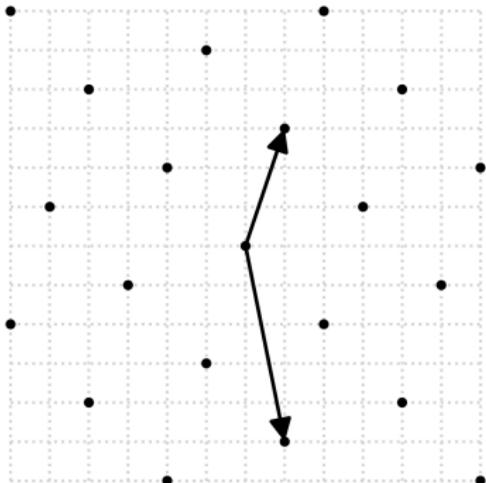
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→ notion of **quasi-orthogonal** (or **reduced**) bases.



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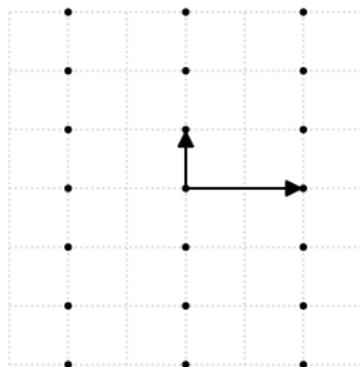


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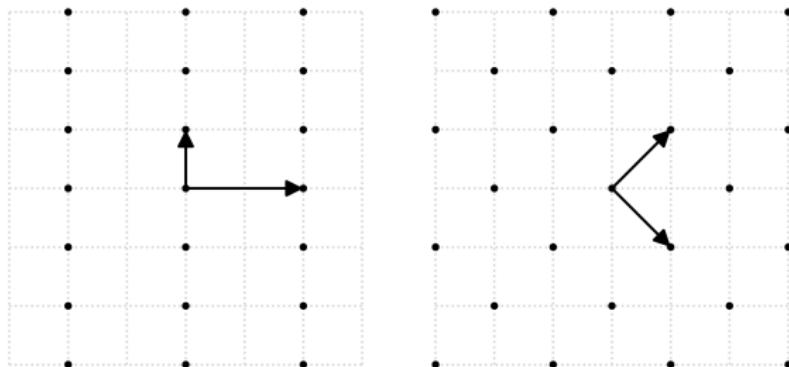


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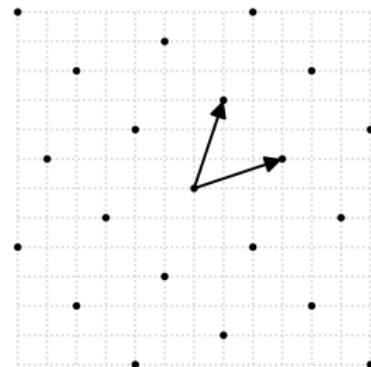
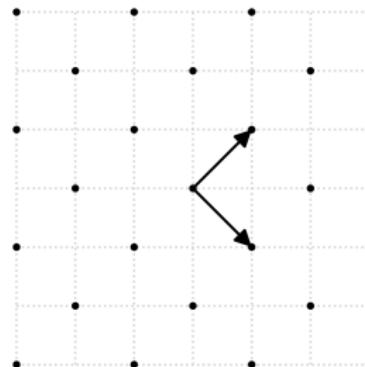
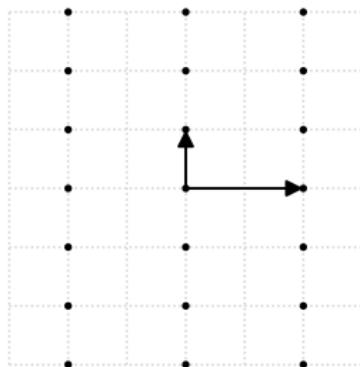


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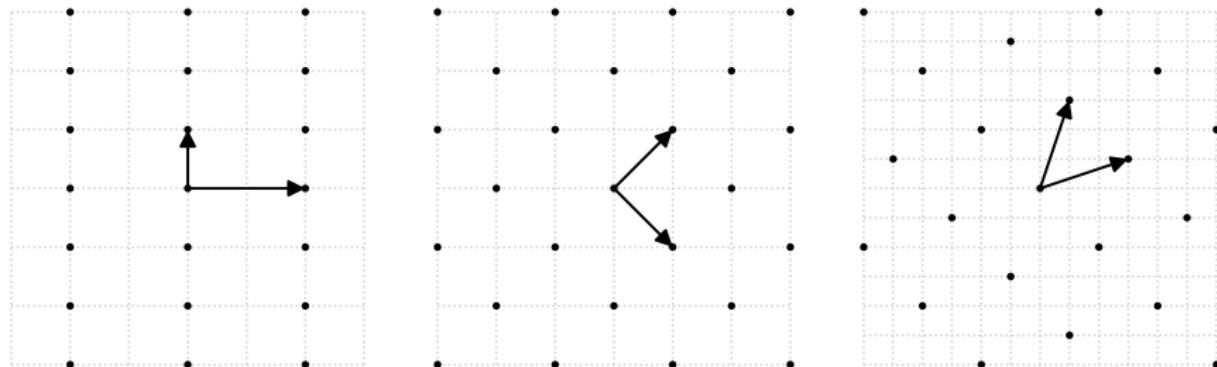


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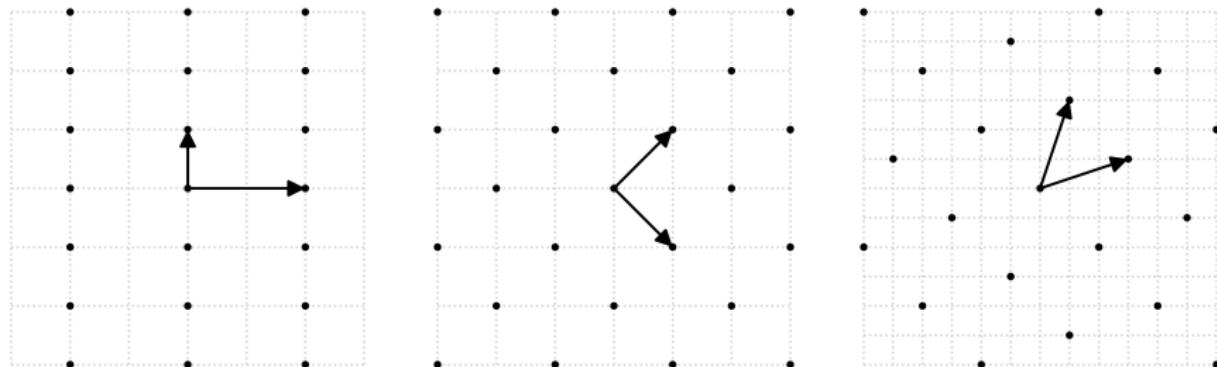


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→ **Gram-Schmidt orthogonalization process**

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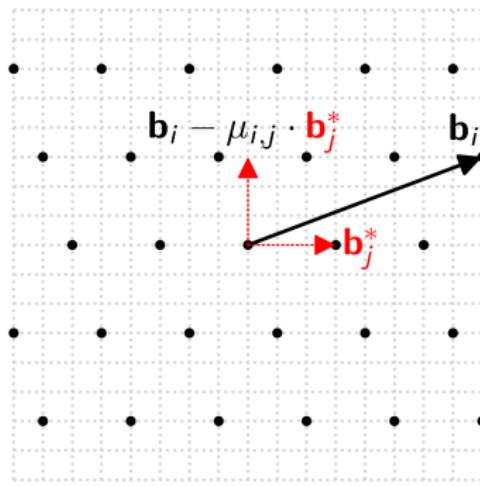
Let $(\mathbf{b}_i)_{1 \leq i \leq n}$ be a basis of \mathbb{R}^n . The associated orthogonal basis $(\mathbf{b}_i^*)_{1 \leq i \leq n}$ is constructed via the **Gram–Schmidt orthogonalization process**:

$$\mathbf{b}_1^* := \mathbf{b}_1, \quad \mathbf{b}_i^* := \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{i,j} \mathbf{b}_j^*, \quad \mu_{i,j} := \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\|\mathbf{b}_j^*\|^2}.$$

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The coefficients $\mu_{i,j}$ are called **Gram–Schmidt coefficients**.

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mu_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mu_{n,1} & \cdots & \mu_{n,n-1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \mathbf{b}_2^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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The resulting family $(\mathbf{b}_i^*)_{1 \leq i \leq n}$ is orthogonal.

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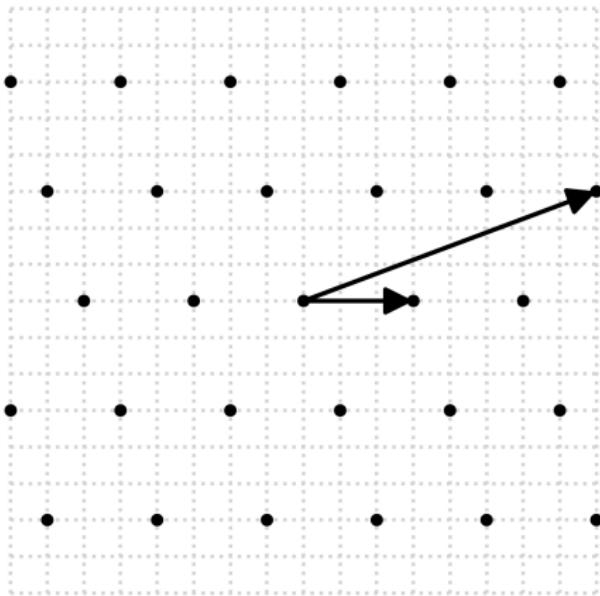
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Problem: The Gram–Schmidt orthogonal basis of B is generally not a basis of the lattice $\mathcal{L}(B)$.

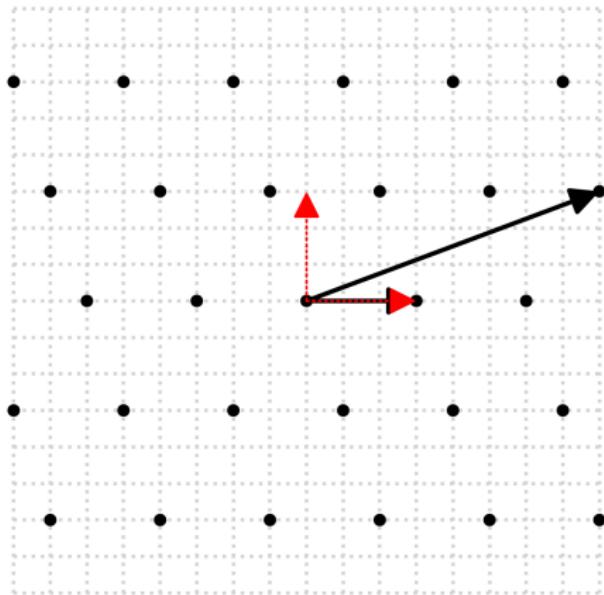
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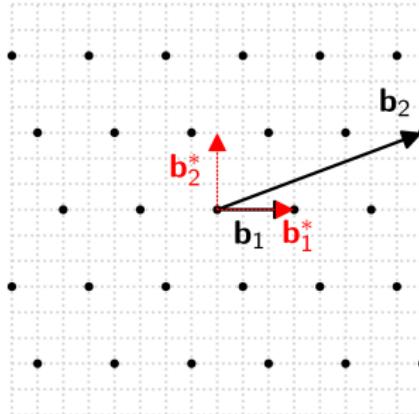
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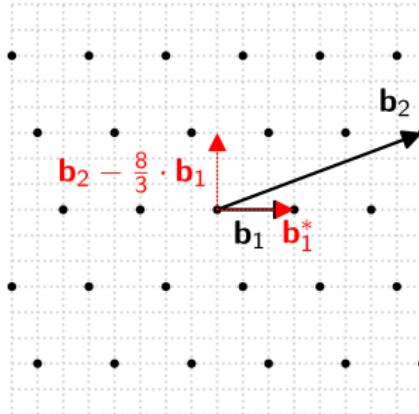
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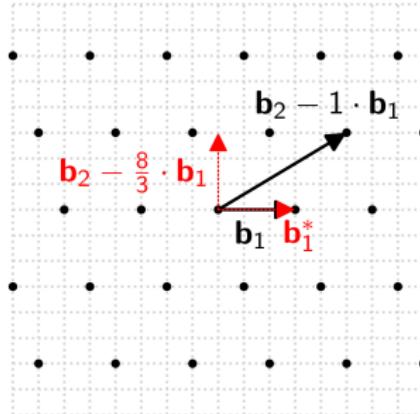
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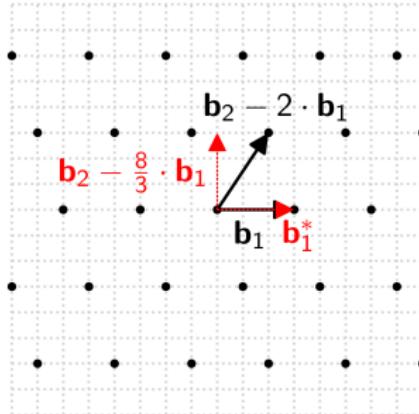
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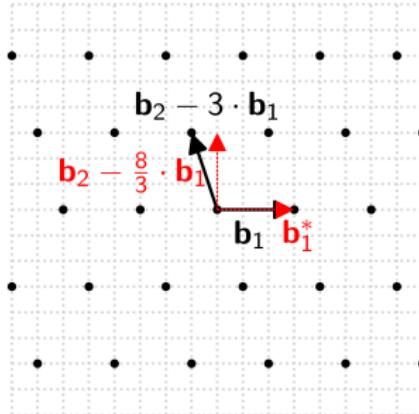
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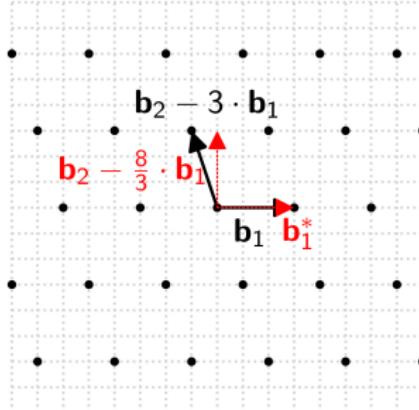
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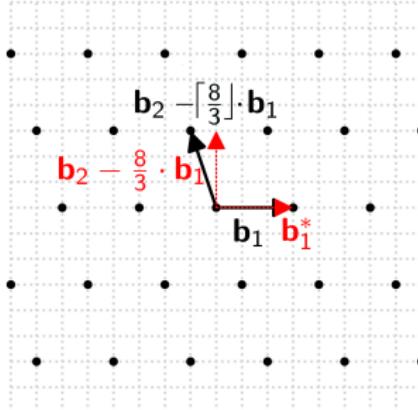
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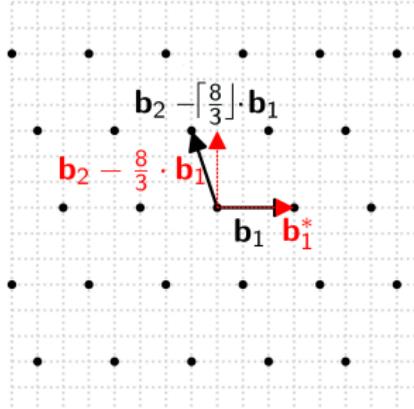
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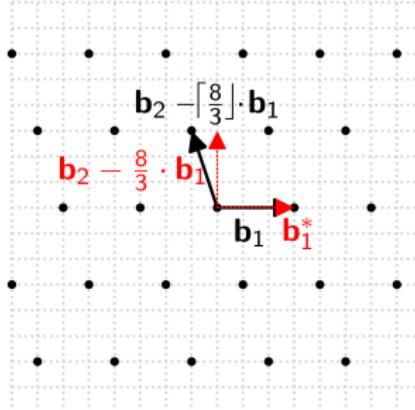


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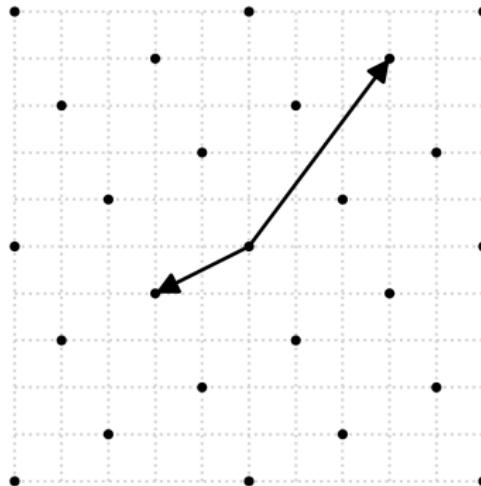
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Definition: A basis is said to be **size-reduced** if:

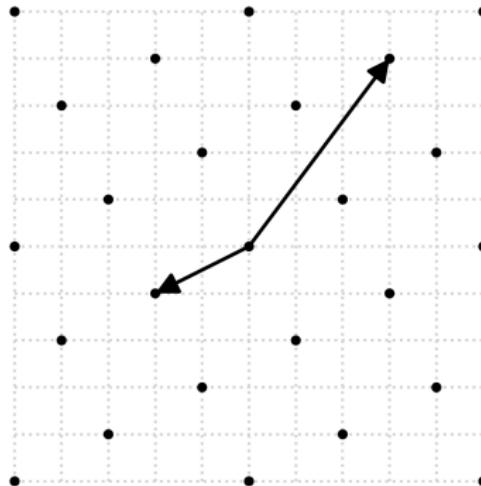
$$\max_{1 \leq j < i \leq n} |\mu_{i,j}| \leq \frac{1}{2}$$

Why Size Reduction is Not Enough



A size-reduced basis.

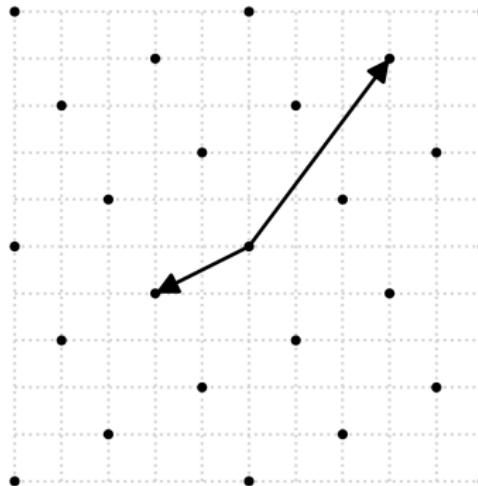
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Length reduction alone **does not imply** almost-orthogonality!

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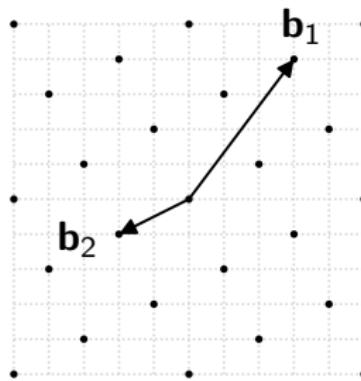
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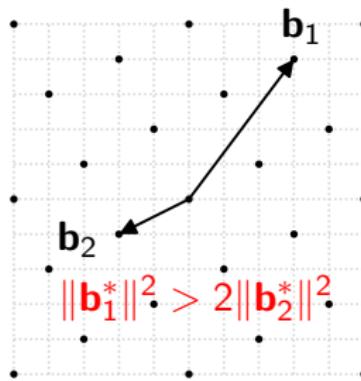
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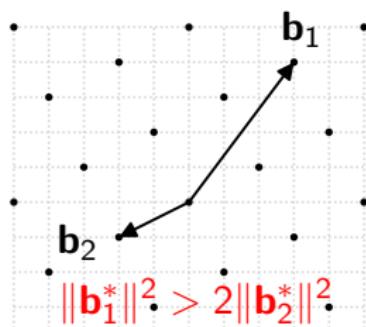
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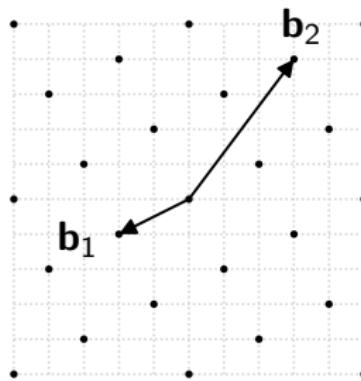
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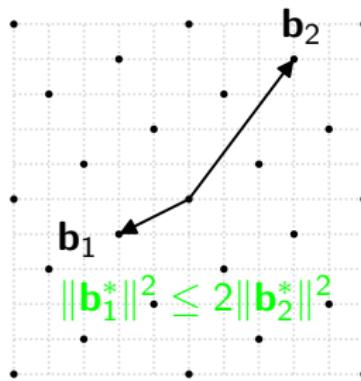
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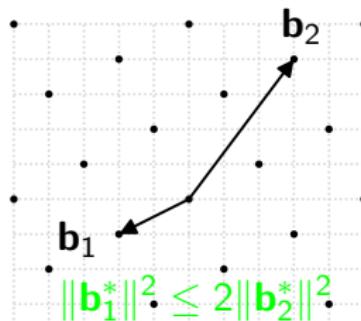
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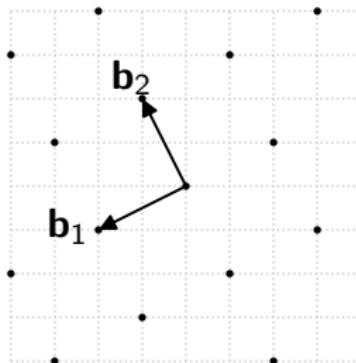
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Definition: LLL– reduced Basis

A basis is called LLL–reduced if:

- It is size-reduced;
- It satisfies the Lovász condition.

Recap: The γ – SVP Problem

Definitions of $\lambda_1, \lambda_2, \dots$ are detailed in (Boudgoust 2023).

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Given a basis B of a lattice $\mathcal{L} \subset \mathbb{R}^n$ and an approximation factor $\gamma > 0$,
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$\gamma = 1$ exact SVP — NP-hard

$\gamma = \text{poly}(n)$ relevant for **lattice-based cryptography**

$\gamma = 2^{\mathcal{O}(n)}$ solvable in **polynomial time** via LLL

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Hence

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$$\mathbf{b} = \lambda_k \mathbf{b}_k^* + \sum_{i < k} \nu_i \mathbf{b}_i^*, \quad \nu_i \in \mathbb{R}$$

Hence

$$\|\mathbf{b}\|^2 = \lambda_k^2 \|\mathbf{b}_k^*\|^2 + \sum_{i < k} \nu_i^2 \|\mathbf{b}_i^*\|^2$$

$$\geq \lambda_k^2 \|\mathbf{b}_k^*\|^2 \geq \|\mathbf{b}_k^*\|^2 \geq \min_{1 \leq i \leq n} \|\mathbf{b}_i\|$$

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Proof.

$$\|\mathbf{b}_1\|^2 = \|\mathbf{b}_1^*\|^2 \leq 2\|\mathbf{b}_2^*\|^2 \leq 2^2\|\mathbf{b}_3^*\|^2 \leq \dots \leq 2^{n-1}\|\mathbf{b}_n^*\|^2.$$

Thus,

$$\|\mathbf{b}\| \geq \min\{\|\mathbf{b}_1^*\|, \dots, \|\mathbf{b}_n^*\|\} \geq 2^{-(n-1)/2} \|\mathbf{b}_1\|$$



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How can we compute a reduced basis in practice?

→ Use the LLL (Lenstra 1982)(Lenstra, Lenstra, Lovasz) algorithm!

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Let's compute a LLL reduced basis of $\mathcal{L}(B)$ with

$$B := \begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

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$$\overbrace{\begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}}^B = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{9} & 1 & 0 \\ 0 & \frac{54}{101} & 1 \end{pmatrix}}^U \times \overbrace{\begin{pmatrix} -2 & 2 & 1 \\ \frac{19}{9} & \frac{8}{9} & \frac{22}{9} \\ \frac{88}{101} & \frac{154}{101} & -\frac{132}{101} \end{pmatrix}}^{B^*}$$

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The vector $(2, 2, 0)$ is a shortest nonzero vector in the lattice, hence:

$$\lambda_1(\mathcal{L}) = 2\sqrt{2}.$$

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3 while  $i \leq n$  do How much?
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Correctness

Key idea: Clearly, if the algorithm LLL **terminates**, the returned basis is by construction LLL-reduced.

*Therefore, it remains **to prove that LLL always terminates.***

How can we prove the termination of the algorithm?

$$\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{i-1} \\ \mathbf{b}_i \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \mu_{2,1} & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mu_{i-1,1} & \cdots & \mu_{i-1,i-2} & \cdots & \cdots & \vdots \\ \mu_{i,1} & \cdots & \mu_{i,i-2} & \mu_{i,i-1} & \cdots & \vdots \\ \vdots & & & & \ddots & \vdots \\ \mu_{n,1} & \cdots & \cdots & \cdots & \cdots & \mu_{n,n-1} 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \vdots \\ \mathbf{b}_{i-1}^* \\ \mathbf{b}_i^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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The matrix is an $n \times n$ upper triangular matrix. The diagonal elements are 1, 0, ..., 0. The sub-diagonal elements are labeled $\mu_{i,j}$ for $i > j$. A red box highlights the sub-diagonal elements from $\mu_{i-1,i}$ to $\mu_{n,i}$. Red marks indicate non-zero values at positions $(i-1, i)$ and (n, i) , while other entries in the highlighted box are marked with \neq .

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If we swap \mathbf{g}_i and \mathbf{g}_{i-1} :

$\|\mathbf{d}_{i-1}^*\|$ decrease by a $\frac{3}{4}$ factor, so \mathbf{d}_{i-1} decrease by a $\frac{3}{4}$ factor.

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$$1 \leq \underbrace{\dots}_{\mathcal{O}\left(\log\left(\max_{1 \leq i \leq n} \|\mathbf{g}_i\|\right)\right) \text{ steps}} \leq \frac{4}{3}D_1 \leq D_0 \leq \left(\max_{1 \leq i \leq n} \|\mathbf{g}_i\| \right)^{n(n-1)}$$

LLL Complexity

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Algorithm 0: LLL

Input: A basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$

Output: An LLL-reduced basis $G = (\mathbf{g}_1, \dots, \mathbf{g}_n)$

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1  $G \leftarrow \text{copy}(B)$ 
2  $(G^*, U) \leftarrow \text{Gram-Schmidt } G$ 
3 while  $i \leq n$  do
4     for  $j = i - 1, i - 2, \dots, 1$  do
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- \Rightarrow LLL uses $\tilde{\mathcal{O}}\left(n^5 \log^2\left(\max_{1 \leq i \leq n} \|\mathbf{b}_i\|\right)\right)$ bit operations.

Theorem.

→ LLL **compute** a reduced basis in **polynomial time**.

Theorem: Complexity of Basis Reduction

Theorem.

- LLL uses $\mathcal{O}\left(n^2 \log\left(\max_{1 \leq i \leq n} \|\mathbf{b}_i\|\right)\right)$ loop iterations.
 - LLL uses $\mathcal{O}(n^2)$ arithmetic operations over rationals per iteration.
 - U represented with rationals of bit-lengths $\mathcal{O}\left(n \log\left(\max_{1 \leq i \leq n} \|\mathbf{b}_i\|\right)\right)$
- \Rightarrow LLL uses $\tilde{\mathcal{O}}\left(n^5 \log^2\left(\max_{1 \leq i \leq n} \|\mathbf{b}_i\|\right)\right)$ bit operations.

Theorem.

- LLL **compute** a reduced basis in **polynomial time**.
- LLL **solve** $2^{\mathcal{O}(n)}$ – SVP in **polynomial time**.

Thank you for your attention!

Questions?