

The LLL Algorithm: Lattice Basis Reduction and applications to Approximate Shortest Vector Problem

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Recap: Euclidean Space and Inner Product

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We consider a real finite-dimensional vector space \mathbb{R}^n equipped with the standard **Euclidean inner product**:

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This inner product induces the **Euclidean norm**:

$$\|\mathbf{u}\|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\sum_{i=1}^n u_i^2}$$

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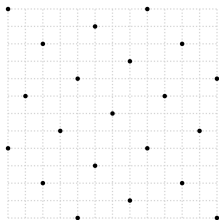


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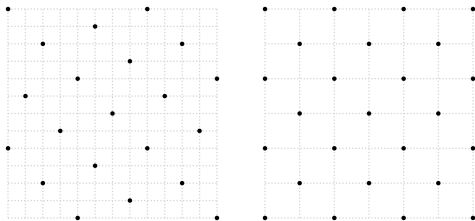


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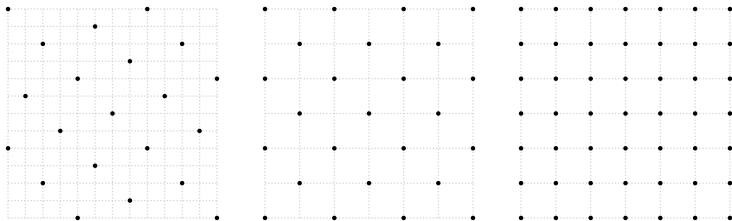


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Any lattice $\mathcal{L} \subseteq \mathbb{R}^n$ admits a maximal \mathbb{Z} -linearly independent family $(\mathbf{b}_i)_{1 \leq i \leq m}$, with $m \leq n$ such that:

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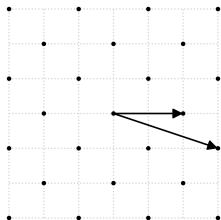


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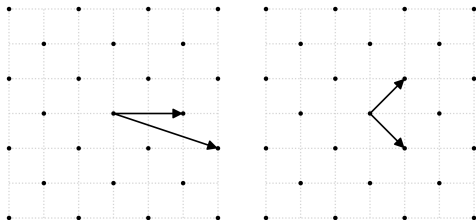


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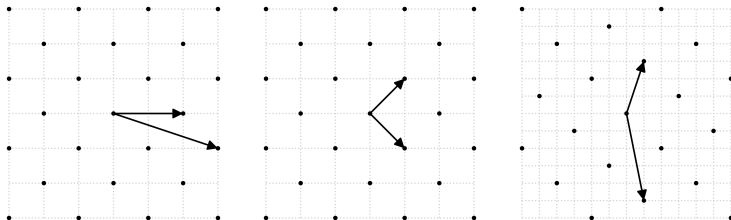
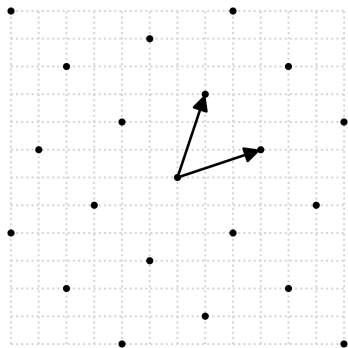


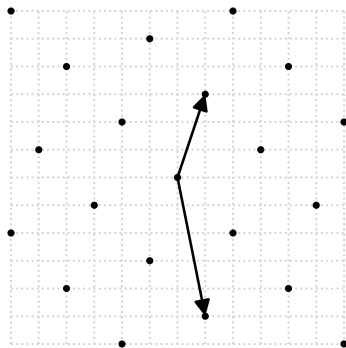
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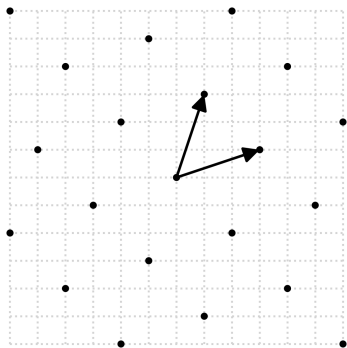


short, nearly orthogonal vectors
looks good

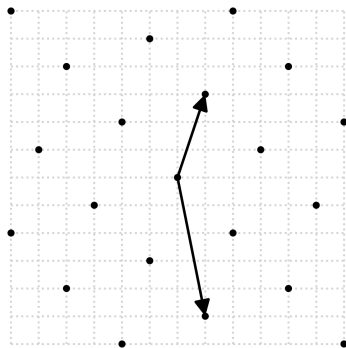


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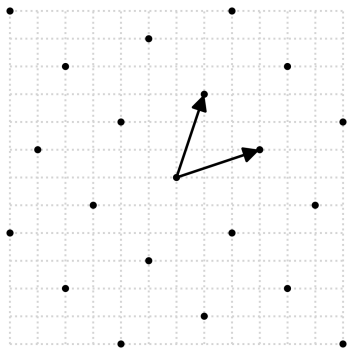
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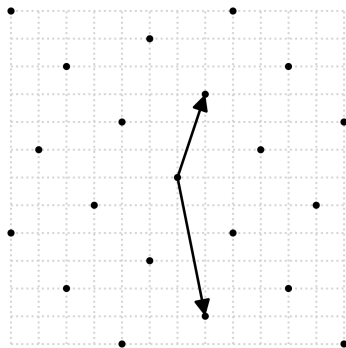
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Can we formalize this?

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→ notion of **quasi-orthogonal** (or **reduced**) bases.

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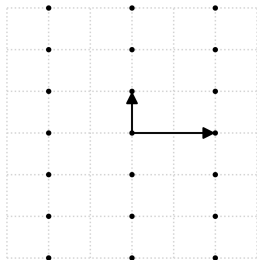


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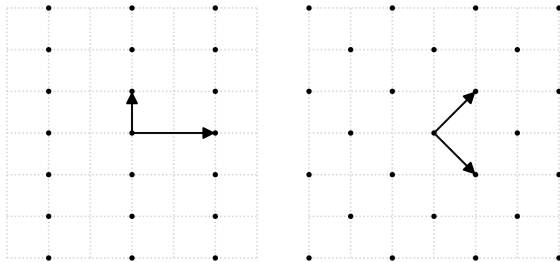


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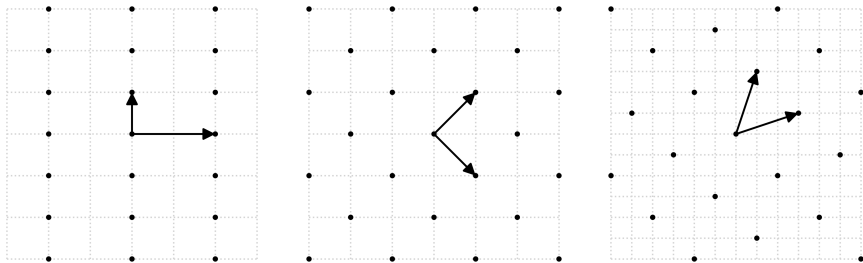


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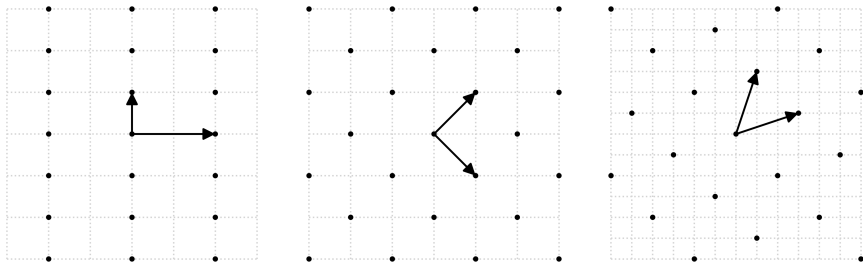


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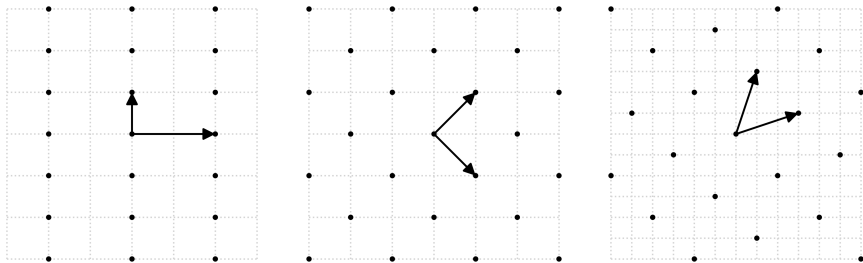


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→ **Gram-Schmidt orthogonalization process**

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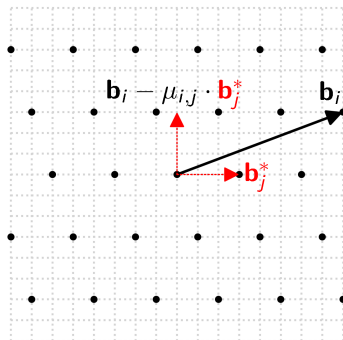
Let $(\mathbf{b}_i)_{1 \leq i \leq n}$ be a basis of \mathbb{R}^n . The associated orthogonal basis $(\mathbf{b}_i^*)_{1 \leq i \leq n}$ is constructed via the **Gram–Schmidt orthogonalization process**:

$$\mathbf{b}_1^* := \mathbf{b}_1, \quad \mathbf{b}_i^* := \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{i,j} \mathbf{b}_j^*, \quad \mu_{i,j} := \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\|\mathbf{b}_j^*\|^2}.$$

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The coefficients $\mu_{i,j}$ are called **Gram–Schmidt coefficients**.

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mu_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mu_{n,1} & \cdots & \mu_{n,n-1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \mathbf{b}_2^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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The resulting family $(\mathbf{b}_i^*)_{1 \leq i \leq n}$ is orthogonal.

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Example: Gram–Schmidt Orthogonalization

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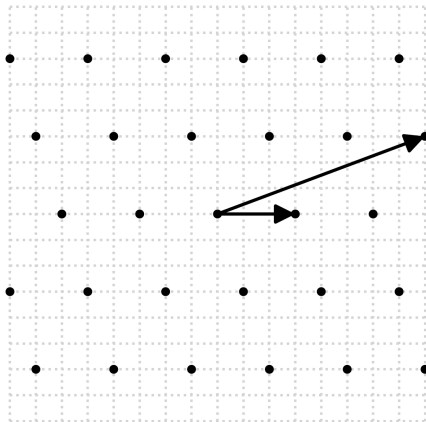
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Problem: The Gram–Schmidt orthogonal basis of B is generally not a basis of the lattice $\mathcal{L}(B)$.

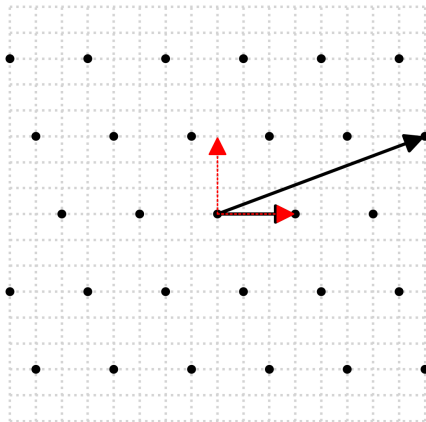
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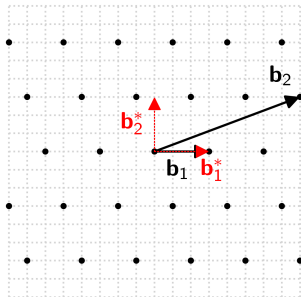
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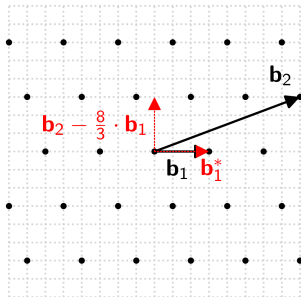
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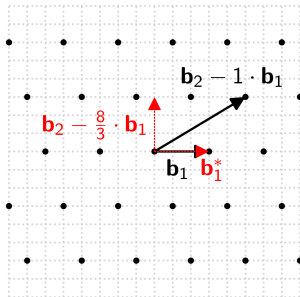
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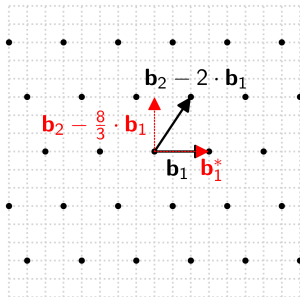
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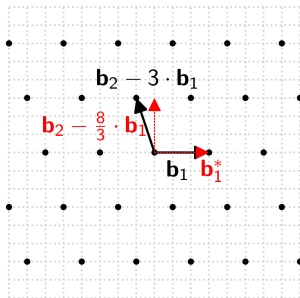
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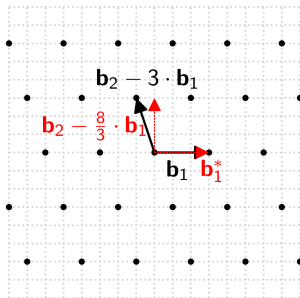
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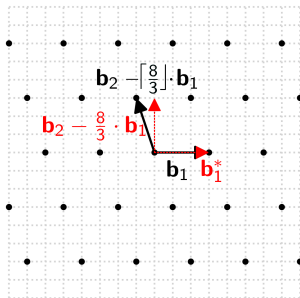
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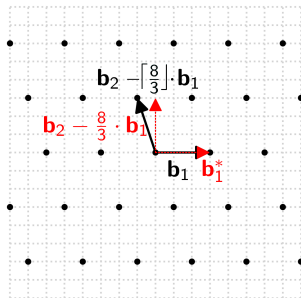
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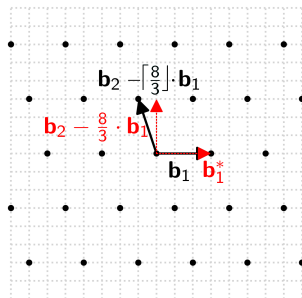


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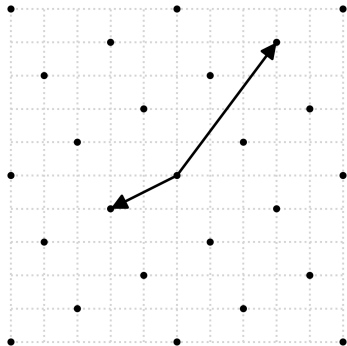
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Definition: A basis is said to be **size-reduced** if:

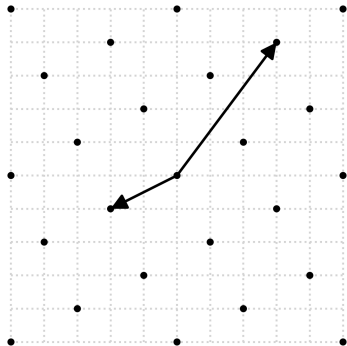
$$\max_{1 \leq j < i \leq n} |\mu_{i,j}| \leq \frac{1}{2}$$

Why Size Reduction is Not Enough



A size-reduced basis.

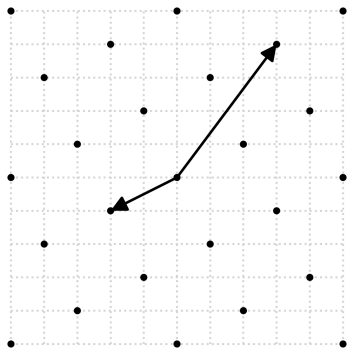
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Length reduction alone **does not imply** almost-orthogonality!

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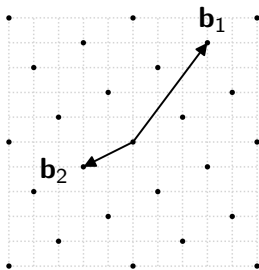
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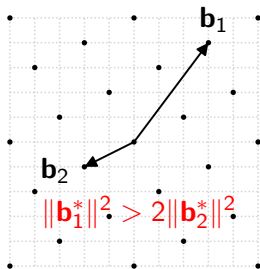
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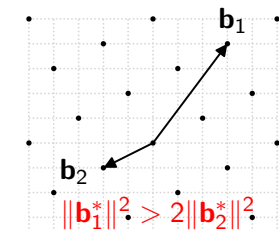
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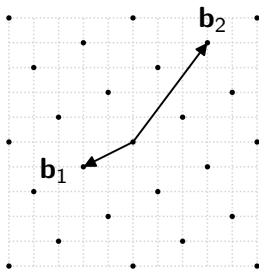
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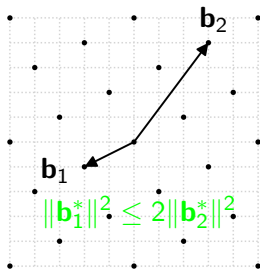
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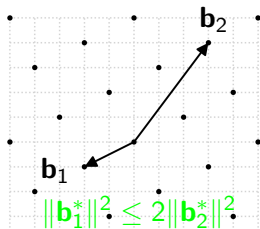
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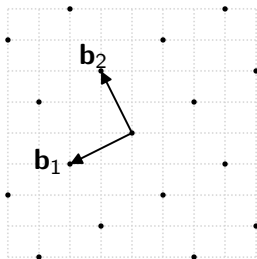
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Definition: LLL– reduced Basis

A basis is called LLL–reduced if:

- It is size-reduced;
- It satisfies the Lovász condition.

Recap: The γ – SVP Problem

Definitions of $\lambda_1, \lambda_2, \dots$ are detailed in (Boudgoust 2023).

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Given a basis B of a lattice $\mathcal{L} \subset \mathbb{R}^n$ and an approximation factor $\gamma > 0$,
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$\gamma = 1$	exact SVP — NP-hard
$\gamma = \text{poly}(n)$	relevant for lattice-based cryptography
$\gamma = 2^{\mathcal{O}(n)}$	solvable in polynomial time via LLL

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Theorem. Let $(\mathbf{b}_i)_{1 \leq i \leq n}$ be a reduced basis of a lattice $\mathcal{L} \subseteq \mathbb{R}^n$, and let $\mathbf{b} \in \mathcal{L} \setminus \{0\}$. Then:

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Proof.

$$\|\mathbf{b}_1\|^2 = \|\mathbf{b}_1^*\|^2 \leq 2\|\mathbf{b}_2^*\|^2 \leq 2^2\|\mathbf{b}_3^*\|^2 \leq \dots \leq 2^{n-1}\|\mathbf{b}_n^*\|^2.$$

Thus,

$$\|\mathbf{b}\| \geq \min\{\|\mathbf{b}_1^*\|, \dots, \|\mathbf{b}_n^*\|\} \geq 2^{-(n-1)/2} \|\mathbf{b}_1\|$$



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→ Use the LLL (Lenstra 1982)(Lenstra, Lenstra, Lovasz) algorithm!

LLL Algorithm

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Let's compute a LLL reduced basis of $\mathcal{L}(B)$ with

$$B := \begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

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$$\overbrace{\begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}}^B = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{9} & 1 & 0 \\ 0 & \frac{54}{101} & 1 \end{pmatrix}}^U \times \overbrace{\begin{pmatrix} -2 & 2 & 1 \\ \frac{19}{9} & \frac{8}{9} & \frac{22}{9} \\ \frac{88}{101} & \frac{154}{101} & -\frac{132}{101} \end{pmatrix}}^{B^*}$$

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The vector $(2, 2, 0)$ is a shortest nonzero vector in the lattice, hence:

$$\lambda_1(\mathcal{L}) = 2\sqrt{2}.$$

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$\mathcal{O}(n^2)$

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3 while  $i \leq n$  do How much?
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10    $\mathcal{O}(n)$ 
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```

Key idea: Clearly, if the algorithm LLL **terminates**, the returned basis is by construction LLL-reduced.

*Therefore, it remains **to prove** that LLL **always terminates**.*

How can we prove the termination of the algorithm?

$$\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{i-1} \\ \mathbf{b}_i \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ \mu_{2,1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{i-1,1} & \dots & \mu_{i-1,i-2} & \cdot & \cdot & \cdot \\ \mu_{i,1} & \dots & \mu_{i,i-2} & \mu_{i,i-1} & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{n,1} & \dots & \dots & \dots & \mu_{n,n-1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \vdots \\ \mathbf{b}_{i-1}^* \\ \mathbf{b}_i^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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[illegible]

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$$\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_j \\ \mathbf{b}_{i-1} \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ & \ddots & & & & & \vdots \\ & \mu_{2,1} & \ddots & & & & \vdots \\ & \vdots & \ddots & \ddots & & & \vdots \\ & \vdots & & \ddots & \ddots & & \vdots \\ & \vdots & & & \ddots & \ddots & \vdots \\ & \vdots & & & & \ddots & \vdots \\ i-1 & \neq & \dots & \dots & \neq & 1 & \vdots \\ i & \neq & \dots & \dots & \neq & \neq & \vdots \\ & \vdots & & & \vdots & \vdots & \vdots \\ & \vdots & & & \vdots & \vdots & 0 \\ \mu_{n,1} & \dots & \dots & \dots & \neq & \neq & \dots & \mu_{n,n-1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \vdots \\ \neq \\ \vdots \\ \neq \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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$$\begin{pmatrix} \|\mathbf{b}_1\|^2 \\ \vdots \\ \|\mathbf{b}_i\|^2 \\ \|\mathbf{b}_{i-1}\|^2 \\ \vdots \\ \|\mathbf{b}_n\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ \mu_{2,1}^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \neq^2 & \dots & \dots & \neq^2 & 1 & \cdot & \cdot \\ \neq^2 & \dots & \dots & \neq^2 & \neq^2 & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{n,1}^2 & \dots & \dots & \neq^2 & \neq^2 & \dots & \mu_{n,n-1}^2 & 1 \end{pmatrix} \times \begin{pmatrix} \|\mathbf{b}_1^*\|^2 \\ \vdots \\ \|\neq\|^2 \\ \|\neq\|^2 \\ \vdots \\ \|\mathbf{b}_n^*\|^2 \end{pmatrix}$$

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If we **swap** \mathbf{g}_i and \mathbf{g}_{i-1} :

$\|\mathbf{d}_{i-1}^*\|$ decrease by a $\frac{3}{4}$ factor, so \mathbf{d}_{i-1} decrease by a $\frac{3}{4}$ factor.

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Termination proof

$$1 \leq \underbrace{\quad \cdots \quad}_{\mathcal{O}\left(\log\left(\max_{1 \leq i \leq n} \|\mathbf{g}_i\|\right)\right)} \leq \frac{4}{3} D_1 \leq D_0 \leq \left(\max_{1 \leq i \leq n} \|\mathbf{g}_i\| \right)^{n(n-1)}$$

$\mathcal{O}\left(\log\left(\max_{1 \leq i \leq n} \|\mathbf{g}_i\|\right)\right)$ steps

LLL Complexity

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Algorithm 0: LLL

Input: A basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$

Output: An LLL-reduced basis $G = (\mathbf{g}_1, \dots, \mathbf{g}_n)$

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1  $G \leftarrow \text{copy}(B)$ 
2  $(G^*, U) \leftarrow \text{Gram-Schmidt } G$ 
3 while  $i \leq n$  do
4   for  $j = i - 1, i - 2, \dots, 1$  do
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Theorem.

- \rightarrow LLL **compute** a reduced basis in **polynomial time**.
- \rightarrow LLL **solve** $2^{\mathcal{O}(n)}$ – SVP in **polynomial time**.

Thank you for your attention!

Questions?