1.a

Q_1: $f(g(2-1)) - g(f(1)) = 0$

So use L'Hôpital's rule

Derivative of numerator: $[f(g(2-x)) - g(f(x^2))]'$

$= (2-x)' g'(2-x) \cdot f'(g(2-x)) - (x^2)' f'(x^2) g'(f(x^2))$

$= -g'(2-x) \cdot f'(g(2-x)) - 2x \, f'(x^2) g'(f(x^2))$

$x = 1$:

$= -g'(1) f'(g(1)) - 2f'(1) g'(f(1))$

$= -g'(1) f'(-2) - 2f'(1) g'(1)$

$= -1 \cdot 3 - 2(-3) \cdot 1 = -3 + 6 = 3$

Derivative of denumerator: $(x-1)' = 1$

$$\lim_{x \to 1} \frac{(\ldots)}{x-1} = \lim_{x \to 1} \frac{(\ldots)'}{(x-1)'} = \frac{3}{1} = 3$$

1b. Find $\lim_{x \to 0} \frac{x}{\sqrt{x^2 - x}} = \lim_{x \to 0^-} \frac{x}{|x|\sqrt{1 - \cancel{x}\frac{1}{x}}} = \lim_{x \to 0^-} \frac{x}{-x\sqrt{1 - \frac{1}{x}}}$

$$= \lim_{x \to 0^-} \frac{1}{-\underbrace{\sqrt{1 - \frac{1}{x}}}_{1+\infty}} = \frac{1}{-\infty} = 0$$

1c. Calculate $\int_0^1 \underbrace{x}_{v'} \underbrace{\ln(x)}_{u} dx$ $\quad \int uv' = uv - \int vu'$

$\longrightarrow u = \ln(x) \quad v = \frac{x^2}{2}$

$u' = \frac{1}{x} \quad v' = x$

$$= \ln(x) \cdot \frac{x^2}{2}\Big|_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2}{4}\Big|_0^1$$

$$= \left(\underbrace{\ln(1)}_{=0} \cdot \frac{1}{2} - \ln(0) \cdot 0\right) - \left(\frac{1}{4} - 0\right)$$

$$= 0 - \frac{1}{4} = \frac{-1}{4}$$

2.

$$f(x) = \frac{1}{\sqrt{x^2+3x}+x} = \frac{\sqrt{x^2+3x}-x}{(\sqrt{x^2+3x}+x)(\sqrt{x^2+3x}-x)} = \frac{\sqrt{x^2+3x}-x}{x^2+3x-x^2} = \frac{\sqrt{x^2+3x}-x}{3x}$$

$$= \frac{|x|\cdot\sqrt{1+\frac{3}{x}}-x}{3x}$$

Break points: $\cancel{\sqrt{x^2+3x}+x=0}$ $\quad x=0$

Vertical Asymptotes:

$$\lim_{x\to 0}\frac{\sqrt{x^2+3x}-x}{3x} = \frac{|x|\sqrt{x^2+3x}-x}{3x}$$

$$\sqrt{x^2+3x} = |x|\cdot\sqrt{1+\frac{3}{x}}$$

$$\lim_{x\to 0^+}\frac{|x|\sqrt{1+\frac{3}{x}}-x}{3x} = \lim_{x\to 0^+}\frac{x\sqrt{1+\frac{3}{x}}-x}{3x} = \lim_{x\to 0^+}\frac{\sqrt{1+\frac{3}{x}}-1}{3} = \infty$$

$$\lim_{x\to 0^-}\frac{-x\sqrt{1+\frac{3}{x}}-x}{3x} = \frac{-\sqrt{1+\frac{3}{x}}<0-1}{3}$$

doesn't exist

Only for right side:

a vertical asymptote $x=0$

Horizontal Asymptotes:

$x>0$: $\quad \frac{3}{x}\to 0$

$$\lim_{x\to\infty}\frac{|x|\sqrt{1+\frac{3}{x}}-x}{3x} = \lim_{x\to\infty}\frac{x\sqrt{1+\frac{3}{x}}-x}{3x} = \frac{\sqrt{1+\frac{3}{x}}-1}{3} = 0$$

$y=0$: one-sided

$$\lim_{x\to-\infty}\frac{-x\sqrt{1+\frac{3}{x}}-x}{3x} = \lim_{x\to-\infty}\frac{-\sqrt{1+\frac{3}{x}}-1}{3} = \frac{-2}{3}$$

$y=\frac{-2}{3}$ one-sided

Oblique Asymptotes:

$$\lim_{x\to\infty}\frac{f(x)}{x} = \cancel{\lim_{x\to\infty}\frac{x\sqrt{1+\frac{3}{x}}-x}{3x\cdot x}} = \cancel{\lim_{x\to\infty}\sqrt{1+\frac{3}{x}}} -$$

$$= \lim_{x\to\infty}\frac{1}{x\cdot(\sqrt{x^2+3x}+x)_{\to\infty}} = 0 \to$$

oblique Asymptotes doesn't exist

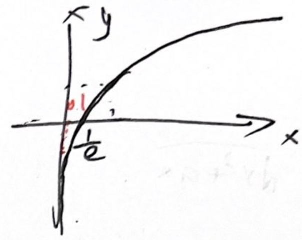
Q3: $f(x) = x\ln(x)$ find global min/max

$f'(x) = 0$

$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1 = 0$

$\ln(x) = -1$

$x = e^{-1} = \frac{1}{e}$



→ has global minimum. because $f'(x) < 0$ when $x < \frac{1}{e}$

↳ $f(x)$ decrease

$f''(x) = 0$.

↳ global min/max

$\sin^2(x) + \cos^2(x) = 1$

↳ $= 1 - \cos^2(x)$

Q4: $\int_0^{\frac{\pi}{2}} (\sin(x))^3 \, dx = \int_0^{\frac{\pi}{2}} \overset{(1-u^2)}{(1-\cos^2(x))} \cdot \sin(x) dx$

$= \sin^2 x) \sin(x) =$

$u = \cos(x) \quad \frac{du}{dx} = -\sin(x) \quad du = -\sin(x)dx \quad dx = \frac{du}{-\sin(x)}$

$= \cancel{\int_0^{\frac{\pi}{2}} u^2 - 1 \, du} = \int_{\cancel{\frac{\pi}{2}}\,1}^{\cancel{\pi}\,0} (1-u^2) \frac{\sin(x)}{-\sin(x)} du$

$= \int_{\cancel{0}\,1}^{\cancel{\frac{\pi}{2}}\,0} \overset{1-u^2}{(\cancel{1-u^2})}(-1)\, du = \int_{\cancel{0}\,1}^{\cancel{\frac{\pi}{2}}\,0} u^2 - 1 \, du = \frac{u^3}{3} - u \Big|_{\cancel{0}\,1}^{\cancel{\frac{\pi}{2}}\,0} = \frac{2}{3}$

I put x back to replace u to get the final result, so I kept 0 to pi/2 and use u only to find the expression of integral result. Here the upper and lower bounds of integral with u should be 1 to 0, which can also be put in expression directly.

$= \left[\frac{\cos^3 x}{3} - \cos(x)\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{\cos^3(\frac{\pi}{2})}{3} - \cos(\frac{\pi}{2})\right) - \left(\frac{\cos^3 0}{3} - \cos(0)\right)$

$= (0 - 0) - (\frac{1}{3} - 1) = \frac{2}{3}$

Q_5:

5a. False.

Because Comparison Test:

When $a_n \geq b_n$. a_n ~~con~~ diverges ~~when~~ if b_n diverges.

5b: False.

① check. if b_n converges → Yes

② check if $a_n \leq b_n$ → No. $a_n = \frac{e^n+n}{e^{2n}-n^2} = \frac{e^n+n}{(e^n+n)(e^n-n)} = \frac{1}{e^n-n}$

$e^n - n \leq e^n$

$\frac{1}{e^n-n} \geq \frac{1}{e^n}$.

$a_n \geq b_n$

5c. True.

check if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

5d. False. not sufficient

6. $\sum_{n=1}^{\infty} n\left(\frac{2}{x}\right)^n \; : x \neq 0$

Ratio test: $\lim_{n\to\infty}\left|\frac{(n+1)\cdot\left(\frac{2}{x}\right)^{n+1}}{n\left(\frac{2}{x}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{2}{x}\right|$

$$= \lim_{n\to\infty}\left|\frac{1+\overset{=0}{\frac{1}{n}}}{1}\cdot\frac{2}{x}\right| \quad \frac{1}{1}$$

$$= \left|\frac{2}{x}\right|$$

It converges if $\left|\frac{2}{x}\right| < 1$: when: $x > 2$ ~~or~~ ~~or~~ , $x < -2$.

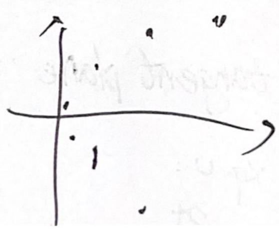
diverges if $\left|\frac{2}{x}\right| > 1$: when: $0 < x < 2$, ~~or~~ $-2 < x < 0$

~~Not~~ Can't conclude when $x = 2$ and -2

$\hookrightarrow \left|\frac{2}{x}\right| = 1$

when $x = 2$: $\sum_{n=1}^{\infty} n \cdot 1^n = \sum_{n=1}^{\infty} n \rightarrow$ diverges

$x = -2$. $\sum_{n=1}^{\infty} n(-1)^n$: Alternating Series test $\rightarrow$ diverges



7. Find solution $y = y(x)$:

$$\begin{cases} y' + y = x \\ y(0) = 3 \end{cases}$$

$y' + p(x)y = f(x)$

$p(x) = 1 \quad f(x) = x$

① $U(x) = e^{\int p(x)\,dx}$

$u(x)y' + u(x)\cdot p(x)y = u(x)\cdot f(x)$

$u(x) = e^{\int 1\,dx} = e^x$

$p(x)\cdot y' + q(x)y = f(x)$

② $(u(x)y)' = u(x)\cdot f(x)$

$y' + \frac{q(x)}{p(x)}y = \frac{f(x)}{p(x)}$

$(e^x y)' = e^x \cdot x$

$u = x \quad v = e^x$

$u' = 1 \quad v' = e^x$

③ $\int (e^x y)'\,dx = \int e^x \cdot x\,dx$

$e^x y = x\cdot e^x - \int e^x\,dx$

$= xe^x - e^x + C$

$y = x - 1 + \frac{C}{e^x}$

$y(0) = 3$

$= 0 - 1 + \frac{C}{e^0 = 1}$

$= -1 + C = 3$

$C = 4$

$\Rightarrow y = x - 1 + \frac{4}{e^x}$

8. ~~find~~ $f(x,y) = \sin(2x - y)$, find tangent plane at $(0, 0, f(0))$

$(2x - y)' = 2$

$\frac{\partial f}{\partial x} = 2\cdot\cos(2x - y)$

$x, y = 0$: $\frac{\partial f}{\partial x} = 2\cos(0 - 0) = 2$

$\frac{\partial f}{\partial y} = -\cos(2x - y)$

$x, y = 0$: $\frac{\partial f}{\partial y} = 0 - \cos(0 - 0) = -1$

$f(0,0) = \sin(0 - 0) = 0$

Tangent plane: $z = \frac{\partial f}{\partial x}(x - a) + \frac{\partial f}{\partial y}(y - b) + f(a,b)$

$z = 2\cdot(x - 0) - 1(y - 0) + 0$

$= 2x - y$

9. Find double integral over a triangle T with vertices (0,0), (1,0), (1,1)

$$\iint_T e^{2x+y}\,dT$$

$0 \le x \le 1$

$0 \le y \le 1$

should rely on x

$\int_0^x \int_0^1 e^{2x+y}\,dx\,dy$

$$\int_0^1 \left[\int_0^1 e^{2x+y}\,dx \right] dy$$

$$= \int_0^1 \left(\frac{1}{2} e^{2x+y} \Big|_0^1 \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} e^{2+y} - \frac{1}{2} e^{0+y} \right) dy$$

$$= \frac{1}{2} \left[\int_0^1 e^{2+y}\,dy - \int_0^1 e^{y}\,dy \right]$$

$$= \frac{1}{2} \left(e^{2+y} \Big|_0^1 - e^{y} \Big|_0^1 \right)$$

$$= \frac{1}{2} \left(e^3 - e^2 - e^1 + e^0 \right)$$

$$= \frac{e^3 - e^2 - e + 1}{2}$$

$\int_0^1 \int_0^1$