# Consistency: modal logic

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Here we first briefly review the structure that is most commonly used for describing the multi-agent settings where agents reason about their knowledge, and explain how the consistency rules for inference tables follow from its assumptions.

**Epistemic logic** The classical modal logic is the framework which is usually used to describe and resolve multi-agent scenarios (such as Three Hats Puzzle or Muddy Children Paradox [1], [2]). We review its most important elements in Appendix **??**. In [3], we have shown that the system of axioms that underlines it leads to a contradiction in quantum settings. Axioms relevant for multi-agent settings like one in Frauchiger-Renner scenario, are axioms which govern how knowledge is operated by an agent and how it is combined with other agents' knowledge. In [3], we have shown that the current system of axioms that underlines modal logic leads to a contradiction in quantum settings; this result yields motivation to reconsider and weaken our assumptions. For completeness, here we list Distribution Axiom [2] and Trust Axiom [3].

[Distribution axiom.] If an agent is aware of a fact $\phi$ and that a fact $\psi$ follows from $\phi$, then the agent can conclude that $\psi$ holds:

$$(M, s) \models (K_i\phi \wedge K_i(\phi \Rightarrow \psi)) \Rightarrow (M, s) \models K_i\psi.$$

[Trust] We say that an agent $i$ trusts an agent $j$ (and denote it by $j \rightsquigarrow i$ ) if and only if

$$(M, s) \models K_i\ K_j\ \phi \implies K_i\ \phi,$$

for all $\phi, s$.

To summarize, the minimal logical rules for a multi-agent setting have to state how the information is combined for the inferences, and when the passing of information between agents is allowed.

An application of logical reasoning operations is a common feature for classical computation; making predictions and conclusions based on obtained results stored in the memory is how information circulates in networks. With scaling of quantum computational systems, it is inevitable that similar type of interaction between quantum memories has to be introduced, which has to be a unitary. While the Frauchiger-Renner setup, as an experiment where the memory entries depend and draw conclusions based on each other, provides us with a good playground to test such protocols, they bear significance for quantum networks which should not be overlooked.

To give an intuition about how the logical reasoning and subsequent memory update can be described as a quantum circuit, let us start with a simple example: Alice has to infer what Bob wears today based on her account of the weather. She also knows how Bob's preferred choice of clothing correlates with the weather, that is, she deduced a following inference table: if she sees rain or hail, then Bob wears a jacket, and if she observes sun outside, he wears a shirt (rain ⇒ jacket, hail ⇒ jacket, sun ⇒ shirt); the truth values for all other possible conclusions, for example, sun ⇒ jacket are not determined. When she reasons about Bob's clothing choice outcome, she essentially checks the validity of statements "Bob wears X", where X = {jacket, shirt}. Whether these statements are true or their truth value is not determined, depends on Alice's observation of the weather W and inference table entries corresponding to sun ⇒ X, rain ⇒ X and hail ⇒ X.

Let us consider the memory structure of a single agent on this simple example before we proceed to a more general one. We can model the weather as a qutrit W with the basis states $|s\rangle$ (sun), $|r\rangle$ (rain) and $|h\rangle$ (hail). Then we introduce a qutrit $T[\text{sun} : X]$, which is initialized prior to the experiment in a state $|X\rangle$, where $X = \{\text{don't know, jacket, shirt}\}$ in accordance with the inference table entries. The computation process which yields the inference table is considered classical, and thus is not required to appear in the quantum circuit. Similarly, one can introduce similar qutrits $T[\text{rain} : X]$ and $T[\text{hail} : X]$ for storing the corresponding inferences. To accommodate Alice's final conclusion about Bob's clothing, we add a qutrit $A : B$, basis state of which are $|X\rangle$, where $X = \{\text{don't know, jacket, shirt}\}$. It is initialized in the state with X = "don't know", which we will label as $|0\rangle$.

First, Alice measures W and coherently copies the result to her memory A, which is also modelled as qutrit; this is shown on the Figure 1 as a controlled gate $U_{copy}$. Then, for all outcomes $w \in \{\text{sun, rain, hail}\}$ (the memory entry in A), she consults the corresponding inference table qutrit $T[w : X]$, and updates the state of the qutrit $A : B$; this is shown on the Figure 1 as controlled Toffoli gates $U_{sun}, U_{rain}$ and $U_{hail}$. Thus, if Alice has measured "rain" and knows that "rain ⇒ jacket" (T[rain:X] is initialized in $|jacket\rangle$), then she concludes "jacket".

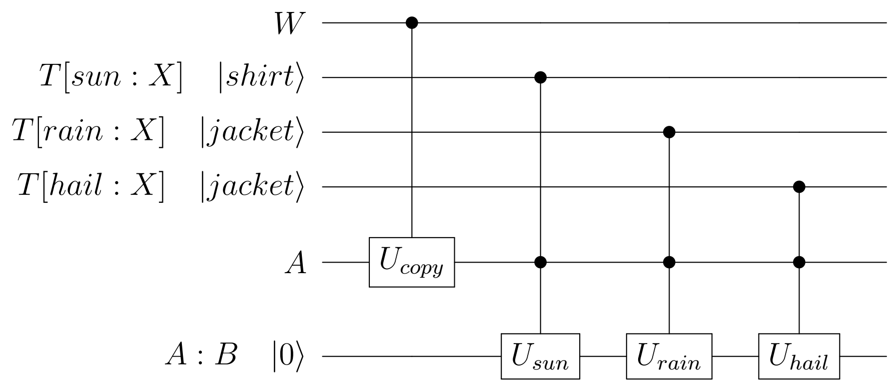


Figure 1: **The circuit for Alice reasoning about Bob's choice of clothing.** Alice performs a coherent observation of weather, and then, based on the entries corresponding to the inferences connecting the weather to Bob's choice of clothing (T[sun:X], T[rain:X] and T[hail:X]), updates the her conclusion about Bob's clothing A:B.

In a general case Alice measures d-dimensional system R (possible outcomes $a_1, a_2, ..., a_d$)and reasons about Bob's outcome which can take k different values $(b_1, ..., b_k)$; thus, her memory is modelled as a d-dimensional system, whereas the systems $T[a_n : b]$ are $(k+1)$-dimensional (as here we also account for possible conclusion "don't know", which we label as $b_0$), and the system $A : B$ is $(k+1)$-dimensional as well, first initialized in $|b_0\rangle$. First, Alice measures the system R and writes down the result to her memory A, initially in a state $|a_0\rangle_A$; this corresponds to a unitary transformation $|a_n\rangle_R|a_0\rangle_A \mapsto |a_n\rangle_R|a_n\rangle_A \quad \forall n \in \{1, ..., d\}$. Then, for each $a_n$ the state of $A : B$ is updated, controlled by states of $T[a_n : b]$ and A; that is, a unitary transformation is performed: $|a_n\rangle_A |b_j\rangle_{T[a_n:b]} |b_0\rangle_{A:B} \mapsto |a_n\rangle_A |b_j\rangle_{T[a_n:b]} |b_j\rangle_{A:B}$.

Though the most general reasoning procedure described above is most efficient with qudits acting as memory entries, due to the technical limitations of the programming languages available today, instead of a single qudit we have to allocate several qubits.

**Inference table** The inference table represents the entries $T[Y : X]$ and prior to the experiment is initialized in the state “I don’t know”.

[numbers=left,xleftmargin=15pt,framexleftmargin=17pt] def set$_inference_table(self, inference_table, n$
$int = 0)$

Then the inference is made, that is, the corresponding prediction qubits are flipped.

## References


[1] The Puzzle of the 3 Hats - The New York Times. https://tierneylab.blogs.nytimes.com/2009/03/16/the-puzzle-of-the-3-hats/.

[2] Fagin, R., Halpern, J. Y., Moses, Y. & Vardi, M. *Reasoning about knowledge* (MIT press, 2004).

[3] Nurgalieva, N. & del Rio, L. Inadequacy of modal logic in quantum settings. *EPCTS* **287**, 267–297 (2019). arXiv:1804.01106.