Gumbel distribution

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Definition and properties

Definition

Let $\xi \sim Gumbel(\mu, \beta)$, then

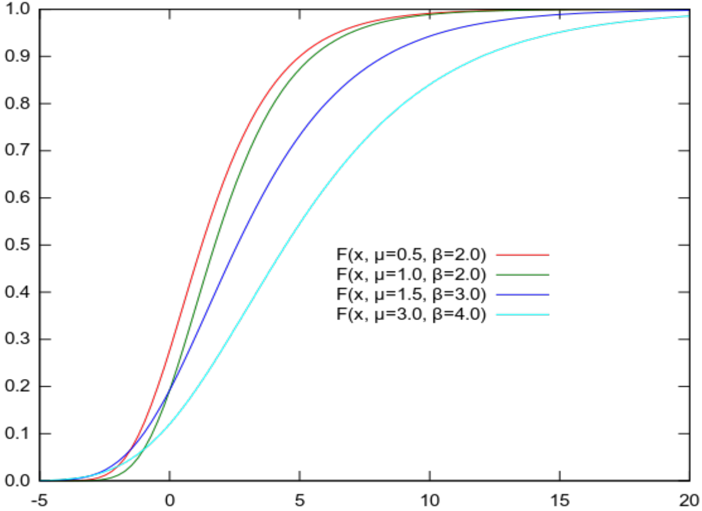
$$f_\xi(x) = \frac{1}{\beta} \exp\left(-\frac{x-\mu}{\beta} + \exp\left[\frac{x-\mu}{\beta}\right]\right), \quad x, \mu \in \mathbb{R}, \beta \in \mathbb{R}_+ \tag{1}$$

Also cumulative distribution function of the Gumbel distribution is

$$F_\xi(x) = \int\limits_{-\infty}^{x} \frac{1}{\beta} \exp\left(-\frac{t-\mu}{\beta} + \exp\left[\frac{t-\mu}{\beta}\right]\right) dt = \exp\left\{-\exp\left(-\frac{x-\mu}{\beta}\right)\right\} \tag{2}$$

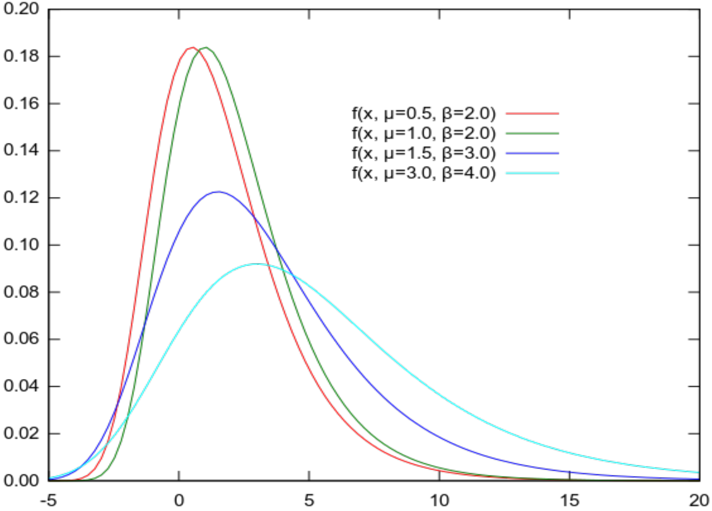
Definition and properties

Figure: Cumulative distribution function



Definition and properties

Figure: Probability density function



Definition and properties

Properties

Let $\xi \sim Gumbel(\mu, \beta)$, then

- $$\mathbb{E}[\xi] = \mu + \beta\gamma$$

  where $\gamma = \lim\limits_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\dfrac{1}{k}\right)$ is Euler–Mascheroni constant

- $$\mathbb{D}[\xi] = \frac{\pi^2}{6}\beta$$

- $$Q(p) = \mu - \beta\log(-\log p), \quad 0 < p < 1$$

Applications

Extreme value theory

Gumbel distribution is I-type of generalized extreme value distribution, which is used to model the distribution of the maximum (or the minimum) of a number of samples of various distributions.
Let $X_1, \dots X_n$ be a sequence of i.i.d random variables with CDF F and let $M_n = \max(X_1, \dots X_n)$. Note, that

$$\Pr(M_n \le z) = \Pr(X_1 \le z, \dots, X_n \le z)$$

$$= \Pr(X_1 \le z) \cdots \Pr(X_n \le z) = (F(z))^n$$

The associated indicator function $I_n = I(M_n > z)$ is a Bernoulli process with a success probability $p(z) = 1 - (F(z))^n$

Applications

Extreme value theory

In practice, we might not have the distribution function F.

Fisher–Tippett–Gnedenko theorem: If there exist sequences of constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\mathbb{P}\left[\frac{M_n - b_n}{a_n} \leq z\right] \xrightarrow[n\to\infty]{} G(z)$$

then

$$G(z) \propto \exp\left[-(1+\zeta z)^{-1/\zeta}\right]$$

In case when M_n has an exponential tail

$$G(z) = \exp\left\{-\exp\left(-\frac{z-b}{a}\right)\right\}$$

Is a gumbel CDF.

Applications

Let's consider application to extreme rainfall data[1]

Application to extreme rainfall data

Let $I(t)$ be the rainfall intensity in the time t, then the quantity $Y_k(d) = \int\limits_k^{k+d} I(t)dt$ is observed. $Y_{ijk}(60)$ the k-th quantity in a year i in day j for the period $d = 60$ minutes. Then, for the i-th year it is reported that $M_i(d) = \max_{j,k}\{Y_{ijk}(d)\}$ would be the annual maximum rainfall for duration d.

The assumption then $Y_{ijk}(d)$ are i.i.d. random variables of a distribution with tails that fall exponentially gives by Fisher–Tippett–Gnedenko theorem that $M_1, \ldots M_n \overset{i.i.d}{\sim} Gumbel(\mu, \beta)$. So, the prior distribution

$$f_\xi(x) = \frac{1}{\beta}\exp\left(-\frac{x-\mu}{\beta} + \exp\left[\frac{x-\mu}{\beta}\right]\right)$$

[1] **A Bayesian analysis of the Gumbel distribution: an application to extreme rainfall data**; https://link.springer.com/article/10.1007/s00477-013-0773-3

Gumbel Trick

Gumbel Trick

Consider "log-sum-exp" quantities

$$\log\left(\sum_{i=1}^{n}\exp x_i\right)$$

They often used in optimization, they are a standard way of performing a soft maximum. In this context, the function defined as

$$f_\varepsilon(x) = \varepsilon\log\left(\sum_{i=1}^{n}\exp\frac{x_i}{\varepsilon}\right) \Rightarrow \lim_{\varepsilon\to 0^+} f_\varepsilon(x) = \max\{x_1, \ldots x_n\}$$

Is it possible to go from the maximum back to the softmax $f_\varepsilon(x)$?

Gumbel Trick

Gumbel Trick

The Gumbel trick provides a surprising randomized solution. Let $\gamma_1, \dots \gamma_n$ be the i.i.d. random variables. Reparametrize vector $x_i \in \mathbb{R}^n$

$$y = x + \gamma, \quad z = \max\{y_1, \dots y_n\} = \max\{x_1 + \gamma_1, \dots x_n + \gamma_n\}$$

The trick is that by choosing $\gamma_1, \dots, \gamma_n \sim \mathit{Gumbel}(\mu, \beta)$ we get

$$\mathbb{E}z = \mathbb{E}[\max\{x_1 + \gamma_1, \dots x_n + \gamma_n\}] = \log\left(\sum_{i=1}^{n} \exp x_i\right)$$

Application in machine learning

The Gumbel trick can be used in mainly two ways: (a) the main goal is to approximate the expectation using a finite average or using stochastic gradients (then only the value of the maximum is used), (b) the minimizer is used for sampling purposes.