

# Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)
2. for twice differentiable functions, show  $\nabla^2 f(x) \succeq 0$
3. show that  $f$  is obtained from simple convex functions by operations that preserve convexity
  - nonnegative weighted sum
  - composition with affine function
  - pointwise maximum and supremum
  - composition
  - minimization
  - perspective

# Positive weighted sum & composition with affine function

**nonnegative multiple:**  $\alpha f$  is convex if  $f$  is convex,  $\alpha \geq 0$

**sum:**  $f_1 + f_2$  convex if  $f_1, f_2$  convex (extends to infinite sums, integrals)

**composition with affine function:**  $f(Ax + b)$  is convex if  $f$  is convex

## examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^m \log(b_i - a_i^T x), \quad \text{dom } f = \{x \mid a_i^T x < b_i, i = 1, \dots, m\}$$

- (any) norm of affine function:  $f(x) = \|Ax + b\|$

## Pointwise maximum

if  $f_1, \dots, f_m$  are convex, then  $f(x) = \max\{f_1(x), \dots, f_m(x)\}$  is convex

### examples

- piecewise-linear function:  $f(x) = \max_{i=1,\dots,m}(a_i^T x + b_i)$  is convex
- sum of  $r$  largest components of  $x \in \mathbf{R}^n$ :

$$f(x) = x_{[1]} + x_{[2]} + \dots + x_{[r]}$$

is convex ( $x_{[i]}$  is  $i$ th largest component of  $x$ )

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \dots + x_{i_r} \mid 1 \leq i_1 < i_2 < \dots < i_r \leq n\}$$

## Pointwise supremum

if  $f(x, y)$  is convex in  $x$  for each  $y \in \mathcal{A}$ , then

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

### examples

- support function of a set  $C$ :  $S_C(x) = \sup_{y \in C} y^T x$  is convex
- distance to farthest point in a set  $C$ :

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for  $X \in \mathbf{S}^n$ ,

$$\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y$$

# Composition with scalar functions

composition of  $g : \mathbf{R}^n \rightarrow \mathbf{R}$  and  $h : \mathbf{R} \rightarrow \mathbf{R}$ :

$$f(x) = h(g(x))$$

$f$  is convex if  $\begin{array}{l} g \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing} \end{array}$

- proof (for  $n = 1$ , differentiable  $g, h$ )

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- note: monotonicity must hold for extended-value extension  $\tilde{h}$

## examples

- $\exp g(x)$  is convex if  $g$  is convex
- $1/g(x)$  is convex if  $g$  is concave and positive

# Vector composition

composition of  $g : \mathbf{R}^n \rightarrow \mathbf{R}^k$  and  $h : \mathbf{R}^k \rightarrow \mathbf{R}$ :

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \dots, g_k(x))$$

$f$  is convex if  $\begin{array}{l} g_i \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing in each argument} \\ g_i \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing in each argument} \end{array}$

proof (for  $n = 1$ , differentiable  $g, h$ )

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)$$

## examples

- $\sum_{i=1}^m \log g_i(x)$  is concave if  $g_i$  are concave and positive
- $\log \sum_{i=1}^m \exp g_i(x)$  is convex if  $g_i$  are convex

# Minimization

if  $f(x, y)$  is convex in  $(x, y)$  and  $C$  is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex

## examples

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$  with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \quad C \succ 0$$

minimizing over  $y$  gives  $g(x) = \inf_y f(x, y) = x^T (A - B C^{-1} B^T) x$

$g$  is convex, hence Schur complement  $A - B C^{-1} B^T \succeq 0$

- distance to a set:  $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$  is convex if  $S$  is convex

# Perspective

the **perspective** of a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is the function  $g : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$ ,

$$g(x, t) = tf(x/t), \quad \text{dom } g = \{(x, t) \mid x/t \in \text{dom } f, t > 0\}$$

$g$  is convex if  $f$  is convex

## examples

- $f(x) = x^T x$  is convex; hence  $g(x, t) = x^T x/t$  is convex for  $t > 0$
- negative logarithm  $f(x) = -\log x$  is convex; hence relative entropy  $g(x, t) = t \log t - t \log x$  is convex on  $\mathbf{R}_{++}^2$
- if  $f$  is convex, then

$$g(x) = (c^T x + d)f((Ax + b)/(c^T x + d))$$

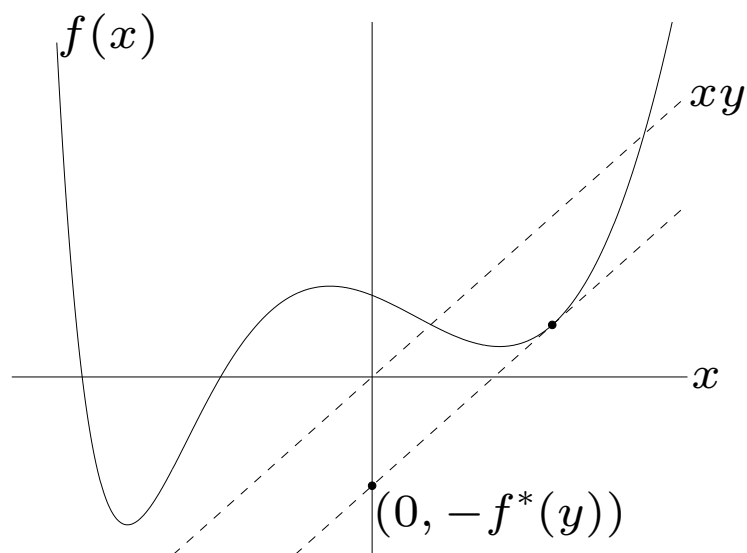
is convex on  $\{x \mid c^T x + d > 0, (Ax + b)/(c^T x + d) \in \text{dom } f\}$



# The conjugate function

the **conjugate** of a function  $f$  is

$$f^*(y) = \sup_{x \in \text{dom } f} (y^T x - f(x))$$



- $f^*$  is convex (even if  $f$  is not)
- will be useful in chapter 5

## examples

- negative logarithm  $f(x) = -\log x$

$$\begin{aligned} f^*(y) &= \sup_{x>0} (xy + \log x) \\ &= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- strictly convex quadratic  $f(x) = (1/2)x^T Qx$  with  $Q \in \mathbf{S}_{++}^n$

$$\begin{aligned} f^*(y) &= \sup_x (y^T x - (1/2)x^T Qx) \\ &= \frac{1}{2} y^T Q^{-1} y \end{aligned}$$