Andreas Karrenbauer **Summer 2024**

Assignments for Optimization

Assignment Sheet 1 Due: Wednesday, 24 April 2024

The following students have contributed to the solution of this sheet:

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The cut-off time for submitting this homework 13:00 CEST (using Overleaf).

Exercise	1	2	3	4	$\sum$
Points (self-assessment)					
Check (at random)					

Definition 1 (Convex set) *Let S be a set of points in an Euclidean space. S is convex if for any two points $x, y \in S$ and any $t \in (0,1)$ it holds that $tx + (1-t)y \in S$.*

Definition 2 (Convex hull) *Let $S := \{x_1, \dots, x_n\}$ be a set of points in an Euclidean space. We define conv$(S) := \{\sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0 \ \forall i \in [n]\}$ as the convex hull of S.*

Exercise 1 (*10 points*) Prove the following statements:

a) A polyhedron is convex. *(2 points)*

b) Let $x_1, \dots, x_n \in \mathbb{R}^n$. Show that $conv(\{x_1, \dots, x_n\})$ is convex. *(2 points)*

c) The intersection of a finite number of convex sets is convex. *(3 points)*

d) Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. We define the polyhedra

$$P_L := P \cap \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\},$$
$$P_R := P \cap \{x \in \mathbb{R}^n : \pi^T x \geq \pi_0 + 1\}$$

Show that $P \cap \mathbb{Z}^n = (P_L \cup P_R) \cap \mathbb{Z}^n$.*(3 points)*

Exercise 2 (*10 points*) Let $A' \in \mathbb{Z}^{m' \times n}, b' \in \mathbb{Z}^{m'}$, $P := \{x \in \mathbb{R}^n : A'x \leq b'\}$. The Integer Feasibility problem is to decide if $P \cap \mathbb{Z}^n \neq \emptyset$. Show that Integer Feasibility is NP-hard using the following scheme.

a) Think of a reduction from any NP-hard problem that you prefer (e.g., 3-SAT, Vertex Cover, Independent Set). *(6 points)*

b) Show that this reduction is polynomial. *(4 points)*

Exercise 3 (*10 points*) Consider the following problems and model them as ILPs.

a) Given a set of elements $U := [m] := \{1, \dots, m\}$, a set $S = \{S_1, \dots, S_n\}$ of n subsets of U (i.e. $S_i \subseteq U$ for $i \in [n]$), and a positive integer k. Find a subset $\{S_{i_1}, \dots, S_{i_j}\} \subset S$ with at most k members (that is, $j \leq k$) such that the union of the selected subsets has maximum size (that is, $\left|\bigcup_{k \in [j]} S_{i_k}\right|$ is maximized). *(5 points)*

b) Given an undirected graph $G = (V, E)$, find the largest subset $Q \subseteq V$ such that $\{u, v\} \in E$ for all distinct $u, v \in Q$. *(5 points)*

Exercise 4 (*10 points*) Consider the following optimization problem in $\mathbb{R}^2$.

$$
\begin{array}{lrl}
\text{maximize} & x_1 + x_2 & \\
\text{subject to} & -x_1 + 2x_2 & \leq 3 \\
& 3x_1 + 4x_2 & \leq 22 \\
& -x_1 - 3x_2 & \leq -3 \\
& x_1, x_2 & \geq 0 \\
& x_1, x_2 & \text{integer}
\end{array}
$$

a) Draw the polyhedron of the integer linear program. *(4 points)*

b) Solve the problem with the minimum number of calls to an integer feasibility oracle. *(6 points)*