Andreas Karrenbauer **Summer 2024**

Assignments for Optimization

Assignment Sheet 1 Due: Wednesday, 24 April 2024

The following students have contributed to the solution of this sheet:

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The cut-off time for submitting this homework 13:00 CEST (using Overleaf).

Exercise	1	2	3	4	$\sum$
Points (self-assessment)					
Check (at random)					

Definition 1 (Convex set) *Let S be a set of points in an Euclidean space. S is convex if for any two points $x, y \in S$ and any $t \in (0,1)$ it holds that $tx + (1-t)y \in S$.*

Definition 2 (Convex hull) *Let $S := \{x_1, \ldots, x_n\}$ be a set of points in an Euclidean space. We define $conv(S) := \{\sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0 \ \forall i \in [n]\}$ as the convex hull of S.*

Exercise 1 (*10 points*) Prove the following statements:

a) A polyhedron is convex. *(2 points)*

Soluion : $Ax \leq B; Ay \leq B$

We know by definition of convexity that $tx + (1-t)y \in S$

Therefore, by linearity and handling inequalities we get
$A(tx + (1-t)y) \leq tb + (1-t)b$
$\implies Atx + (1-t)Ay \leq b$
$\implies tAx + (1-t)Ay \leq b$
since $t \in [0,1] \therefore x + (1-t)y \in P$

Ref - https://people.orie.cornell.edu/dpw/orie6300/fall2008/Lectures/lec05.pdf

b) Let $x_1, \ldots, x_n \in \mathbb{R}^n$. Show that $conv(\{x_1, \ldots, x_n\})$ is convex. *(2 points)*
Solution : Let $S = x_1, x_2, ..., x_n$

WKT $conv(S) := \{\sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0 \ \forall i \in [n]\}$

Now,

We take two points a and b $\in conv(s)$

Then there exist
$a_1, a_2, a_3, ...a_n \in S$
$b_1, b_2, b_3, ...b_n \in S$

$$a = \sum_{i=i}^{n} \lambda_i a_i where \lambda_i \in R > 0; \sum_{i=i}^{n} \lambda_i = 1 \tag{1}$$

$$b = \sum_{j=i}^{m} \alpha_j a_j where \alpha_j \in R > 0; \sum_{j=i}^{m} \lambda_j = 1 \tag{2}$$

We have-

$ta + (1-t)b \in S$

Substituting 1 and 2

$$\sum_{i=i}^{n} \lambda_i a_i + \sum_{j=i}^{m} \lambda_j a_j \in S$$

Which Implies

$$\sum_{i=i}^{n} \lambda_i . t + \sum_{j=i}^{m} \alpha_j (1-t) = 1$$

Since
$a_i \in S \& b_j \in S$

Therefore;

$ta + (1-t)b \in conv(S)$

Therefore,

conv(s) is Convex

c) The intersection of a finite number of convex sets is convex. *(3 points)*
Solution: Let $x_1 \& x_2$ be 2 convex sets

Let $x_3 = x_1 \cap x_2$
Lets $a_1 and a_2$ be any point in x_3

Which means $a_1 \& a_2 \in x_3$

$$\implies a_1, a_2 \in x_1 \cap x_2$$

$$a_1, a_2 \in x_1$$

$$a_1, a_2 \in x_2$$

Condition for convexity

$$\lambda a_1 + (1-\lambda)a_2 \in x_1$$

$$\lambda a_1 + (1-\lambda)a_2 \in x_2$$

Where $\lambda \in [0,1]$

Since $x_1 and x_2 are convex sets$

$$\implies \lambda a_1 + (1-\lambda)a_2 \in x_1 \cap x_2; \lambda \in [0,1]$$

$$\implies \lambda a_1 + (1-\lambda)a_2 \in x_3; \lambda \in [0,1]$$

$\therefore x_3$ is a convex set.

d) Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. We define the polyhedra

$$P_L := P \cap \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\},$$
$$P_R := P \cap \{x \in \mathbb{R}^n : \pi^T x \geq \pi_0 + 1\}$$

Show that $P \cap \mathbb{Z}^n = (P_L \cup P_R) \cap \mathbb{Z}^n$.*(3 points)*

Exercise 2 (*10 points*) Let $A' \in \mathbb{Z}^{m' \times n}, b' \in \mathbb{Z}^{m'}$, $P := \{x \in \mathbb{R}^n : A'x \leq b'\}$. The Integer Feasibility problem is to decide if $P \cap \mathbb{Z}^n \neq \emptyset$. Show that Integer Feasibility is NP-hard using the following scheme.

a) Think of a reduction from any NP-hard problem that you prefer (e.g., 3-SAT, Vertex Cover, Independent Set). *(6 points)*

Solution: Choosing a 3SAT problem,

3Sat problem will have variables $x = x_1, x_2, ...x_n x \in 0, 1$

Usually in 3 SAT problem we have 3 term expressions involving variables in x such as $a_1 = x_1 \wedge x_3' \wedge x_4$

This can be converted to ILP by assuming that for each x_i in 3SAT problem there is a variable z_i in ILP and each z have constraints

$$0 \leq z_i \leq 1$$

$\therefore$ the expression $x_1 \wedge x_3' \vee x_4$ can be transformed to $z_1 + (1 - z_3) + z_4 > 0$

The two problems are now similar and NP-Hard since 3SAT problem is NP-Hard.

Ref : https://math.stackexchange.com/questions/2564236/how-can-i-formulate-the-3-sat-problem-as-a-0-1-linear-integer-program

b) Show that this reduction is polynomial. *(4 points)*
To prove that the reduction is polynomial, we have to show that its runtime is polynomial in the size of the input 3SAT instance. The size of an instance is number of variables in it (n) + number of clauses in it (m).
We take a look at the runtime of the operations we do.
For each variable in the 3SAT instance, we create an arithmetic constraint (expression). This is doable in constant time for each variable, so the time these operations take is polynomial in the number of variables in the instance (poly(n)).
Additionally, for each clause, we also create an arithmetic constraint (expression), which is also doable in constant time for each clause, so the time these operations take is polynomial in the number of clauses in the instance (poly(m)).
In both cases, it's not "just" n and m, but poly(n) and poly(m) because we assume some representation is needed for the variables and clauses. For an efficient one, such operations/transformations (like transforming into an arithmetic constraint) would have at most polynomial slowdown.
In the algorithm (reduction) we do n+m constant (or at worst polynomial, but the polynomial is smaller than n or m else the representation is not efficient) operations, so the runtime is at most polynomial in n+m (the size of the 3SAT instance).

Exercise 3 (*10 points*) Consider the following problems and model them as ILPs.

a) Given a set of elements $U := [m] := \{1, \ldots, m\}$, a set $S = \{S_1, \ldots, S_n\}$ of n subsets of U (i.e. $S_i \subseteq U$ for $i \in [n]$), and a positive integer k. Find a subset $\{S_{i_1}, \ldots, S_{i_j}\} \subset S$ with at most k members (that is, $j \leq k$) such that the union of the selected subsets has maximum size (that is, $\left|\bigcup_{k\in[j]} S_{i_k}\right|$ is maximized). *(5 points)*

b) Given an undirected graph $G = (V, E)$, find the largest subset $Q \subseteq V$ such that $\{u, v\} \in E$ for all distinct $u, v \in Q$. *(5 points)*

Exercise 3
Solution

a) Maximum Coverage problem with Subsets:
This problem can be expressed as:

$$
\begin{aligned}
&max \sum_{m\in U} y_m \\
\text{s.t.} \quad & y_m \leq 1 && \forall m \in U \\
& x_j \leq 1 && \forall j \in S_i \\
& \sum_{j\in S_i} x_j \leq k
\end{aligned}
$$

So we have that the decision variables y_m and x_j are binary and defined as: y_m is 1 if the element m is contained in the subset S_i otherwise 0, and x_j is 1 if S_j is in the selected subset cover otherwise 0.

b) Maximum Clique:
We can state the problem as an ILP as follows:

$$
\begin{aligned}
&max \sum_{x_v\in V} \\
\text{s.t.} \quad & x_u + x_v \leq 1 && \forall \{u, v\} \notin E \\
& x_v \in \{0, 1\} && \forall x_v \in V
\end{aligned}
$$

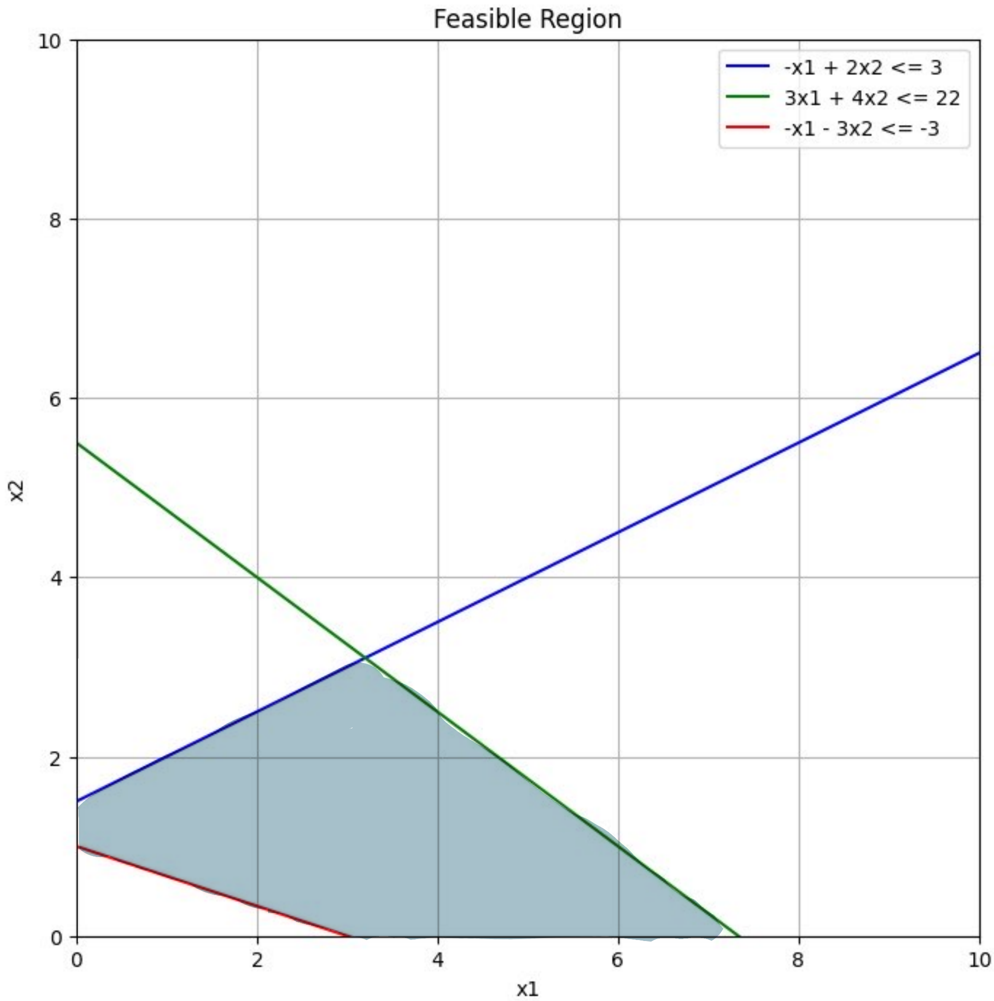
The decision variables x_v are binary and are defined in the following way: x_v is 1, if the vertex $v \in V$ is in $Q \subseteq V$ (the clique set) and 0 otherwise. With the given definition of the decision variables, the constraint $x_u + x_v \leq 1 \quad \forall \{u, v\} \notin E$ guarantees that for any two vertices x_u and x_v that are not adjacent, only one is contained in the clique set Q. Essentially, this guarantees us that the sum (and correspondingly the clique set Q) contain only adjacent vertices. Thus, maximizing the sum of the decision variables while satisfying the given constraint gives us the vertices in the maximum clique as they correspond to the decision variables contained in that sum.

Exercise 4 (*10 points*) Consider the following optimization problem in $\mathbb{R}^2$.

$$\begin{array}{lrl}
\text{maximize} & x_1 + x_2 & \\
\text{subject to} & -x_1 + 2x_2 & \leq 3 \\
 & 3x_1 + 4x_2 & \leq 22 \\
 & -x_1 - 3x_2 & \leq -3 \\
 & x_1, x_2 & \geq 0 \\
 & x_1, x_2 & \text{integer}
\end{array}$$

a) Draw the polyhedron of the integer linear program. *(4 points)*

Solution :



b) Solve the problem with the minimum number of calls to an integer feasibility oracle. *(6 points)*

b) We have that P $\cap$ $\mathbb{Z} \neq \emptyset$

Query P_0
Result: P $\cap$ $\mathbb{Z} \neq \emptyset$

Query P_1
Result: $\mathrm{P} \cap \mathbb{Z} \neq \emptyset$

Query P_2
Result: $\mathrm{P} \cap \mathbb{Z} \neq \emptyset$

Query P_4
Result: $\mathrm{P} \cap \mathbb{Z} \neq \emptyset$

Query P_8
Result: $\mathrm{P} \cap \mathbb{Z} = \emptyset$

Query P_6
Result: $\mathrm{P} \cap \mathbb{Z} \neq \emptyset$

Query P_7
Result: $\mathrm{P} \cap \mathbb{Z} \neq \emptyset$

Query P_0^7
Result: $P_0^7 \cap \mathbb{Z} \neq \emptyset$

Query P_1^7
Result: $P_1^7 \cap \mathbb{Z} \neq \emptyset$

Query P_2^7
Result: $P_2^7 \cap \mathbb{Z} \neq \emptyset$

Query P_4^7
Result: $P_4^7 \cap \mathbb{Z} \neq \emptyset$

Query P_8^7
Result: $P_8^7 \cap \mathbb{Z} = \emptyset$

Query P_6^7
Result: $P_6^7 \cap \mathbb{Z} \neq \emptyset$

Query P_7^7
Result: $P_7^7 \cap \mathbb{Z} \neq \emptyset$

Query $\mathrm{P}_7^{7,0}$
Result: $P_7^{7,0} \cap \mathbb{Z} \neq \emptyset$

Query $\mathrm{P}_7^{7,1}$
Result: $P_7^{7,1} \cap \mathbb{Z} = \emptyset$

Hence, the maximum is obtained in (7,0) with value 7