Ejercicios Teoría Cuántica de Campos. Capítulo 53

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1. Calcular $\left[a_p^\dagger a_n, a_n^\dagger a_p\right]$.

Conociendo las relaciones

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}, \qquad a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0, \qquad a_i a_j + a_j a_i = 0 \tag{1}$$

Podemos usarlas en la forma $a_n a_n^\dagger = 1 - a_n^\dagger a_n$, $a_p a_p^\dagger = 1 - a_p^\dagger a_p$ y $a_p^\dagger a_n^\dagger = -a_n^\dagger a_p^\dagger$, $a_n a_p = -a_p a_n$.

$$\begin{aligned}
\left[a_p^\dagger a_n, a_n^\dagger a_p\right] &= a_p^\dagger a_n a_n^\dagger a_p - a_n^\dagger a_p a_p^\dagger a_n = a_p^\dagger(1 - a_n^\dagger a_n)a_p - a_n^\dagger(1 - a_p^\dagger a_p)a_n \\
&= a_p^\dagger a_p - a_p^\dagger a_n^\dagger a_n a_p - a_n^\dagger a_n + a_n^\dagger a_p^\dagger a_p a_n = a_p^\dagger a_p - \cancel{a_n^\dagger a_p^\dagger a_p a_n} - a_n^\dagger a_n + \cancel{a_n^\dagger a_p^\dagger a_p a_n} \\
&= a_p^\dagger a_p - a_n^\dagger a_n
\end{aligned}$$

2. Invertir las siguientes relaciones

$$\begin{aligned}
\left|\Delta^{++}\right\rangle &= \left|\pi^+, p\right\rangle \\
\left|\Delta^{+}\right\rangle &= \frac{1}{\sqrt{3}}\left|\pi^+, n\right\rangle + \sqrt{\frac{2}{3}}\left|\pi^0, p\right\rangle \\
\left|\Delta^{0}\right\rangle &= \sqrt{\frac{2}{3}}\left|\pi^0, n\right\rangle + \frac{1}{\sqrt{3}}\left|\pi^-, p\right\rangle \\
\left|\Delta^{-}\right\rangle &= \left|\pi^-, n\right\rangle \\
\left|N^{+}\right\rangle &= \frac{1}{\sqrt{3}}\left|\pi^0, p\right\rangle - \sqrt{\frac{2}{3}}\left|\pi^+, n\right\rangle \\
\left|N^{0}\right\rangle &= \sqrt{\frac{2}{3}}\left|\pi^-, p\right\rangle - \frac{1}{\sqrt{3}}\left|\pi^0, n\right\rangle
\end{aligned}$$

Estas relaciones, escogiendo las siguientes bases:

$$\mathscr{B}_1 = \{\left|\Delta^{++}\right\rangle, \left|\Delta^{+}\right\rangle, \left|\Delta^{0}\right\rangle, \left|\Delta^{-}\right\rangle, \left|N^{+}\right\rangle, \left|N^{0}\right\rangle\}$$
$$\mathscr{B}_2 = \{\left|\pi^+, p\right\rangle, \left|\pi^+, n\right\rangle, \left|\pi^0, p\right\rangle, \left|\pi^0, n\right\rangle, \left|\pi^-, p\right\rangle, \left|\pi^-, n\right\rangle\}$$

Se pueden resumir con la matriz de cambio de base

$$M(\mathscr{B}_1 \leftarrow \mathscr{B}_2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0
\end{pmatrix} \tag{2}$$

Esta matriz, al relacionar dos bases ortonormales, debe ser una matriz ortogonal. Es decir que $M^{-1} = M^t$. Esto implica que la matriz de cambio de base inversa será:

$$M(\mathscr{B}_2 \leftarrow \mathscr{B}_1) = M(\mathscr{B}_1 \leftarrow \mathscr{B}_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & \frac{-1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{3}$$

O, equivalentemente:

$$\begin{aligned} \left|\pi^+, p\right\rangle &= \left|\Delta^{++}\right\rangle \\ \left|\pi^+, n\right\rangle &= \frac{1}{\sqrt{3}}\left|\Delta^+\right\rangle - \sqrt{\frac{2}{3}}\left|N^+\right\rangle \\ \left|\pi^0, p\right\rangle &= \sqrt{\frac{2}{3}}\left|\Delta^+\right\rangle + \frac{1}{\sqrt{3}}\left|N^+\right\rangle \\ \left|\pi^0, n\right\rangle &= \sqrt{\frac{2}{3}}\left|\Delta^0\right\rangle - \frac{1}{\sqrt{3}}\left|N^0\right\rangle \\ \left|\pi^-, p\right\rangle &= \frac{1}{\sqrt{3}}\left|\Delta^0\right\rangle + \sqrt{\frac{2}{3}}\left|N^0\right\rangle \\ \left|\pi^-, n\right\rangle &= \left|\Delta^-\right\rangle \end{aligned}$$

3. Calcular $\sigma(\pi^- p \to \pi^- p)$

$$\begin{aligned} \sigma(\pi^- p \to \pi^- p) \sim \left|\left\langle \pi^- p \middle| S \middle| \pi^- p \right\rangle\right|^2 &= \left| \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix} \right|^2 \\ &= \left| \frac{\alpha}{3} + \frac{2\beta}{3} \right|^2 = \frac{1}{9}\left|\alpha + 2\beta\right|^2 \end{aligned}$$