Ejercicios Teoría Cuántica de Campos. Capítulo 71.

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Ejercicios resueltos por Miguel A. Montañez

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Ejercicio 71.1

Obtener los símbolos de Christoffel (conexión) para la métrica conformemente minkowskiana $ds^2 = e^{2X}(dT^2 - dX^2)$.

Tomamos un sistema (1+1), donde $T \to 0$ y $X \to 1$.
El número de símbolos de Christoffel es $2^3 = 8$, y los coeficientes de la métrica son:

$$g_{00} = e^{2X} \qquad g_{11} = -e^{2X} \qquad g_{01} = g_{10} = 0$$

Entonces:

$$e_0 \cdot e_0 = e^{2X} \qquad 2 e_0 \cdot \partial_0 e_0 = 0 \qquad \Gamma^0_{00} g_{00} = 0 \Rightarrow \boxed{\Gamma^0_{00} = 0}$$

$$\partial_1 (e_0 \cdot e_0) = 2e^{2X} \qquad 2 e_0 \cdot \partial_1 e_0 = 2e^{2X} \qquad \boxed{\Gamma^0_{10} = \Gamma^0_{01} = 1}$$

$$e_1 \cdot e_1 = -e^{2X} \qquad e_1 \cdot \partial_0 e_1 = 0 \qquad \Gamma^1_{01} g_{11} = 0 \Rightarrow \boxed{\Gamma^1_{01} = \Gamma^1_{10} = 0}$$

$$\partial_1 (e_1 \cdot e_1) = -2e^{2X} \qquad 2 e_1 \cdot \partial_1 e_1 = -2e^{2X} \qquad 2\Gamma^1_{11} g_{11} = -2e^{2X} \Rightarrow \boxed{\Gamma^1_{11} = 1}$$

$$\partial_0 (e_0 \cdot e_1) = 0 \qquad \Gamma^1_{00} g_{11} + \Gamma^0_{01} g_{00} = 0 \quad \text{si } \Gamma^0_{01} = 1 \Rightarrow \boxed{\Gamma^1_{00} = 1}$$

$$\partial_1 (e_0 \cdot e_1) = 0 \qquad \Gamma^1_{10} g_{11} + \Gamma^0_{11} g_{00} = 0 \quad \text{si } \Gamma^1_{10} = 0 \Rightarrow \boxed{\Gamma^0_{11} = 0}$$

Luego:

$$\Gamma^0_{10} = \Gamma^0_{01} = \Gamma^1_{11} = \Gamma^1_{00} = 1$$

$$\Gamma^0_{00} = \Gamma^1_{01} = \Gamma^1_{10} = \Gamma^0_{11} = 0$$

Ejercicio 71.2

Demostrar $\partial_t^2 - \partial_x^2 = e^{-2X}(\partial_T^2 - \partial_X^2)$

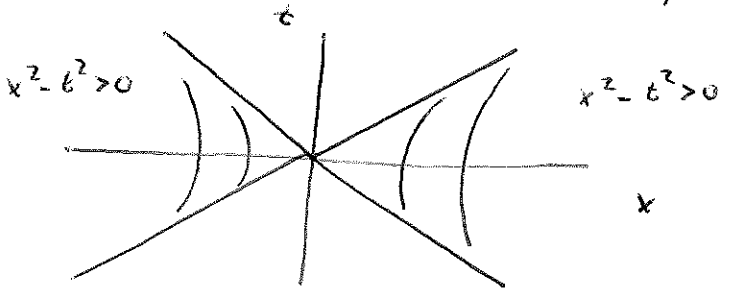
$t, x \to$ coordenadas de Minkowski

$X, T \to$ coordenadas de Rindler

Relación de coordenadas: $t = e^{X} shT \quad x = e^{X} chT$

calculamos $x^2 - t^2 = e^{2X} ch^2T - e^{2X} sh^2T = e^{2X}$

Para todo X, $e^{X} > 0 \Rightarrow x^2 > t^2$ en este cambio de coordenadas, lo cual tiene que ver con el hecho de que las coordenadas de Rindler se definen en esta región:



Entonces: $2X = Ln|x^2 - t^2| = Ln(x^2 - t^2)$ $\boxed{X = Ln\sqrt{x^2 - t^2}}$

Por otra parte:

$\frac{t}{x} = thT$ $\boxed{T = argth\frac{t}{x}}$

Ahora hacemos las derivadas parciales:

$$\frac{\partial T}{\partial t} = \frac{1}{1 - \frac{t^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 - t^2} = \frac{e^{X} chT}{e^{2X}} = e^{-X} chT$$

$$\frac{\partial T}{\partial x} = \frac{1}{1 - \frac{t^2}{x^2}} \left(-\frac{y}{x^2}\right) = -\frac{e^{X} shT}{e^{2X}} = -e^{-X} shT$$

$$\frac{\partial X}{\partial x} = \frac{1}{\sqrt{x^2 - t^2}} \frac{1}{2\sqrt{x^2 - t^2}} 2x = \frac{x}{x^2 - t^2} = e^{-X} chT$$

$$\frac{\partial X}{\partial t} = \frac{1}{\sqrt{x^2 - t^2}} \frac{1}{2\sqrt{x^2 - t^2}} (-2t) = -\frac{t}{x^2 - t^2} = -e^{-X} shT$$

Calculamos:

$$\partial_t = \frac{\partial T}{\partial t}\partial_T + \frac{\partial X}{\partial t}\partial_X = e^{-X} ch T\, \partial_T - e^{-X} sh T\, \partial_X$$

$$\partial_x = \frac{\partial T}{\partial x}\partial_T + \frac{\partial X}{\partial x}\partial_X = - e^{-X} sh T\, \partial_T + e^{-X} ch T\, \partial_X$$

$$\partial_t^2 = \partial_t\left(e^{-X} ch T\, \partial_T - e^{-X} sh T\, \partial_X\right) =$$

$$\frac{\partial T}{\partial t}\frac{\partial}{\partial T}\left(e^{-X} ch T\, \partial_T - e^{-X} sh T\, \partial_X\right) + \frac{\partial X}{\partial t}\frac{\partial}{\partial X}\left(e^{-X} ch T\, \partial_T - e^{-X} sh T\, \partial_X\right)$$

$$\partial_t^2 = e^{-2X}\Big[2 sh T ch T\, \partial_T + ch^2 T\, \partial_T^2 - ch^2 T\, \partial_X - 2 ch T sh T\, \partial_{TX}^2$$
$$- sh^2 T\, \partial_X + sh^2 T\, \partial_X^2\Big]$$

Por otra parte:

$$\partial_x^2 = \partial_x\left(- e^{-X} sh T\, \partial_T + e^{-X} ch T\, \partial_X\right) =$$

$$\frac{\partial T}{\partial x}\frac{\partial}{\partial T}\left(- e^{-X} sh T\, \partial_T + e^{-X} ch T\, \partial_X\right) + \frac{\partial X}{\partial x}\frac{\partial}{\partial X}\left(- e^{-X} sh T\, \partial_T + e^{-X} ch T\, \partial_X\right)$$

$$\partial_x^2 = e^{-2X}\Big[2 sh T ch T\, \partial_T + sh^2 T\, \partial_T^2 - sh^2 T\, \partial_X - 2 sh T ch T\, \partial_{TX}^2$$
$$- ch^2 T\, \partial_X + ch^2 T\, \partial_X^2\Big]$$

Si hacemos la diferencia:

$$\partial_t^2 - \partial_x^2 = e^{-2X}\left[\left(ch^2 T - sh^2 T\right)\partial_T^2 + \left(sh^2 T - ch^2 T\right)\partial_X^2\right]$$

Con lo que demostramos:

$$\boxed{\partial_t^2 - \partial_X^2 = e^{-2X}\left(\partial_T^2 - \partial_X^2\right)}$$

Ejercicio 71.3

Demostrar: $(\partial_t \phi)^2 - (\partial_x \phi)^2 = e^{-2X}\left[(\partial_T \phi)^2 - (\partial_X \phi)^2\right]$

Sabemos del ejercicio anterior que:

$$\partial_t = e^{-X} \operatorname{ch} T\, \partial_T - e^{-X} \operatorname{sh} T\, \partial_X$$

$$\partial_x = -e^{-X} \operatorname{sh} T\, \partial_T + e^{-X} \operatorname{ch} T\, \partial_X$$

Entonces:

$$(\partial_t \phi)^2 = \left(e^{-X} \operatorname{ch} T\, \partial_T \phi - e^{-X} \operatorname{sh} T\, \partial_X \phi\right)^2 =$$

$$e^{-2X} \operatorname{ch}^2 T (\partial_T \phi)^2 + e^{-2X} \operatorname{sh}^2 T (\partial_X \phi)^2 - 2 \operatorname{ch} T \operatorname{sh} T\, \partial_T \phi\, \partial_X \phi$$

$$(\partial_x \phi)^2 = \left(-e^{-X} \operatorname{sh} T\, \partial_T \phi + e^{-X} \operatorname{ch} T\, \partial_X \phi\right)^2 =$$

$$e^{-2X} \operatorname{sh}^2 T (\partial_T \phi)^2 + e^{-2X} \operatorname{ch}^2 T (\partial_X \phi)^2 - 2 \operatorname{ch} T \operatorname{sh} T\, \partial_T \phi\, \partial_X \phi$$

Haciendo la diferencia:

$$(\partial_t \phi)^2 - (\partial_x \phi)^2 = e^{-2X}\left[(\operatorname{ch}^2 T - \operatorname{sh}^2 T)(\partial_T \phi)^2 + (\operatorname{sh}^2 T - \operatorname{ch}^2 T)(\partial_X \phi)^2\right] =$$

$$e^{-2X}\left[(\partial_T \phi)^2 - (\partial_X \phi)^2\right]$$

Así pues:

$$\boxed{(\partial_t \phi)^2 - (\partial_x \phi)^2 = e^{-2X}\left[(\partial_T \phi)^2 - (\partial_X \phi)^2\right]}$$