Teoría Cuántica de Campos

$$\text{COSA} = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\,\phi_1\phi_2 - \sqrt{2}\,\phi_2\phi_3$$

A) Halla **A** tal que $(\phi_1\ \phi_2\ \phi_3)\, A \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \text{COSA}$

Como $\vec{\phi} \in 3D \Rightarrow A \in 3\times3 D$.

Es decir A es de la forma $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$

Primeramente $A\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} a_{11}\phi_1 + a_{12}\phi_2 + a_{13}\phi_3 \\ a_{21}\phi_1 + a_{22}\phi_2 + a_{23}\phi_3 \\ a_{31}\phi_1 + a_{32}\phi_2 + a_{33}\phi_3 \end{pmatrix} = \sum_{i=1}^{3}\begin{pmatrix} a_{1i}\phi_i \\ a_{2i}\phi_i \\ a_{3i}\phi_i\end{pmatrix} = A'$

Luego $(\phi_1\ \phi_2\ \phi_3)\, A' = (\phi_1\ \phi_2\ \phi_3)\sum_{i=1}^{3}\begin{pmatrix} a_{1i}\phi_i \\ a_{2i}\phi_i \\ a_{3i}\phi_i\end{pmatrix} =$

$$= \phi_1\sum_{i=1}^{3} a_{1i}\phi_i + \phi_2\sum_{i=1}^{3} a_{2i}\phi_i + \phi_3\sum_{i=1}^{3} a_{3i}\phi_i =$$

$$= \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ji}\phi_j\phi_i = a_{ji}\phi_j\phi_i = a_{ij}\phi_i\phi_j$$

Criterio de sumatoria de Einstein.

Finalmente $\boxed{\text{COSA} = \vec{\phi}^{T} A \phi = a_{ij}\phi_i\phi_j}$

Así es que los coeficientes a_{ij} acompañan a ϕ_i y a ϕ_j y son intercambiables por lo cual A es simétrica

En este caso particular

$$A = \begin{bmatrix} -6 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -6 & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & -6 \end{bmatrix}$$

B) Diagonalizar A ; Hallar M, λ_1, λ_2 y λ_3 Tal que

$(\varphi_1\ \varphi_2\ \phi_3) \longrightarrow (\psi_1\ \psi_2\ \psi_3)$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \phi_3 \end{pmatrix} = M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \quad \text{Recordando que} \quad A = \begin{bmatrix} -6 & -\sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -6 & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & -6 \end{bmatrix}$$

Se tiene que dar que

$$A\vec{V}_1 = \lambda_1 \vec{V}_1$$
$$A\vec{V}_2 = \lambda_2 \vec{V}_2$$
$$A\vec{V}_3 = \lambda_3 \vec{V}_3$$

Así que $A\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ ó $A\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} - \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = 0$

Quedando
$$\begin{bmatrix} -6-\lambda & -\sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -6-\lambda & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & -6-\lambda \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Ya que
$A\phi - \lambda\phi = (A - I\lambda)\phi$

Para hallar la S.C.I. debe ser $\det(A - I\lambda) = 0$

$$-(6+\lambda)\cdot\left[(6+\lambda)^2 - \left(\frac{\sqrt{2}}{2}\right)^2\right] - \left(-\frac{\sqrt{2}}{2}\right)\left[-\frac{\sqrt{2}}{2}\cdot(-(6+\lambda)) - 0\right] + 0\cdot(\dots) =$$

$$-(6+\lambda)^3 + \frac{1}{2}(6+\lambda) + \frac{1}{2}(6+\lambda) = -(6+\lambda)^3 + (6+\lambda) = 0$$

Si llamamos a $6+\lambda = \mu \Rightarrow -\mu^3 + \mu = 0$

Soluciones $\mu_1 = -1$ $\mu_2 = 0$ $\mu_3 = 1$; siendo $\lambda = \mu - 6$

Entonces $\boxed{\lambda_A = -7 \quad \lambda_B = -6 \quad \lambda_C = -5}$

Tomando $\lambda = -7$

$$\begin{cases} (-6+7)\phi_1 - \frac{\sqrt{2}}{2}\phi_2 = 0 & (1) \\ -\frac{\sqrt{2}}{2}\phi_1 + (-6+7)\phi_2 - \frac{\sqrt{2}}{2}\phi_3 = 0 & (2) \\ -\frac{\sqrt{2}}{2}\phi_2 + (-6+7)\phi_3 = 0 & (3) \end{cases}$$

$$\left.\begin{array}{ll} \text{de } (1) & \phi_1 = \sqrt{2}/2\,\phi_2 \\ \text{de } (3) & \phi_3 = \frac{\sqrt{2}}{2}\phi_2 \end{array}\right\} \text{Tomemos } \phi_2 = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\phi_A = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix};\ |\phi_A| = 1}$$

Reemplazando en (2) $-\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + (-6+7)\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2}\frac{1}{2} =$

$$= -\frac{\sqrt{2}}{4} + \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{4} = \frac{-2+4-2}{4\sqrt{2}} = 0 \ ✓$$

Tomando $\lambda = -6$ $\quad (-6-\lambda) = 0$

$$\begin{cases} -\frac{\sqrt{2}}{2}\phi_2 = 0 & (4) \\ -\frac{\sqrt{2}}{2}\phi_1 - \frac{\sqrt{2}}{2}\phi_3 = 0 & (5) \end{cases}$$

$$\left.\begin{array}{l} \text{de } (5) \quad \phi_1 = -\phi_3 \\ \text{y } \phi_2 \text{ puede tomar cualquier valor} \end{array}\right\} \text{Tomemos } \phi_3 = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\phi_B = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix};\ |\phi_B| = 1}$$

Tomando $\lambda = -5$ $\quad (-6-\lambda) = -1$

$$\begin{cases} -\phi_1 - \frac{\sqrt{2}}{2}\phi_2 = 0 & (6) \\ -\frac{\sqrt{2}}{2}\phi_1 - \phi_2 - \frac{\sqrt{2}}{2}\phi_3 = 0 & (7) \\ -\frac{\sqrt{2}}{2}\phi_2 - \phi_3 = 0 & (8) \end{cases}$$

$$\left.\begin{array}{ll} \text{de } (6) & \phi_1 = -\sqrt{2}/2\,\phi_2 \\ \text{de } (8) & \phi_3 = -\frac{\sqrt{2}}{2}\phi_2 \end{array}\right\} \text{Tomemos } \phi_2 = \frac{-1}{\sqrt{2}} \Rightarrow \boxed{\phi_C = \begin{pmatrix} +1/2 \\ -1/\sqrt{2} \\ +1/2 \end{pmatrix};\ |\phi_C| = 1.}$$

Reemplazando en (1)

$$-\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)-\frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)=\frac{\sqrt{2}}{4}-\frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{4}=\frac{2-4+2}{4\sqrt{2}}=0 \quad \checkmark$$

Así
$$\boxed{M=\begin{bmatrix}\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2}\end{bmatrix}}$$

Tendremos de esa forma que: $\det M = 1$ y $M^T = M^{-1}$

Además se comprueba que $\boxed{M^T A M = D = \begin{bmatrix}-7 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & -5\end{bmatrix}}$

c) Tenemos que:

$$\begin{cases}\phi^T A \phi = \cos a\\ \phi = M\psi \quad \leftarrow \text{impuesto por cambio de variables}\end{cases}$$

Como $\phi^T = (M\psi)^T = \psi^T M^T$

entonces $\phi^T A \phi = (\psi^T M^T)\, A\, (M\psi)$

por prop. asociativa de matrices $\phi^T A \phi = \psi^T (M^T A M)\psi$

Siendo que $M^T A M = D$ $\quad \phi^T A \phi = \psi^T D \psi$

Como D es diagonal $D : \{ d_{ij} \,/\, d_{ij} = \lambda_i \Leftrightarrow i=j \wedge d_{ij} = 0 \text{ si } i \neq j \}$

Así que $\psi^T D \psi = d_{ij}\psi_i\psi_j = \lambda_i \psi_i^2$

Finalmente se demuestra que

$$\boxed{a_{ij}\phi_i\phi_j = \lambda_k \psi_k^2}$$

En nuestro caso:

$$\boxed{-6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\,\phi_1\phi_2 - \sqrt{2}\,\phi_2\phi_3 = -7\psi_1^2 - 6\psi_2^2 - 5\psi_3^2}$$