Cálculo del valor esperado

Teoría Cuántica de Campos

Ejercicio Cap. 8

Calcular $\langle \phi_a \phi_b \phi_c \phi_d \rangle$

Siendo $\boxed{\langle \phi_a \phi_b \phi_c \phi_d \rangle = \frac{1}{Z[0]} \left[\frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \frac{\partial}{\partial J_c} \frac{\partial}{\partial J_d} Z[J] \right]_{J=0}}$

y $Z[J] = \frac{(\sqrt{2\pi})^n}{m^n \sqrt{\det A}} \exp\left(\frac{1}{2m^2} J^T A^{-1} J\right) = Z[0] \exp\left(\frac{1}{2m^2} J^T A^{-1} J\right)$

Siendo $A^{-1} \equiv \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} = \{a_{ij}\}_{n \times n}$; $J \equiv \begin{pmatrix} J^1 \\ J^2 \\ \vdots \\ J^n \end{pmatrix} = \{J^i\}_{n \times 1}$

$\therefore A^{-1} J = \begin{pmatrix} a_{11} J^1 + a_{12} J^2 + \cdots + a_{1n} J^n \\ \vdots \\ a_{n1} J^1 + a_{n2} J^2 + \cdots + a_{nn} J^n \end{pmatrix} = \begin{pmatrix} a_{1j} J^j \\ \vdots \\ a_{nj} J^j \end{pmatrix}$ usando criterio de sumatoria de Einstein

$\wedge \; J^T A^{-1} J = (J^1 J^2 \cdots J^n) \begin{pmatrix} a_{1j} J^j \\ a_{2j} J^j \\ \vdots \\ a_{nj} J^n \end{pmatrix} = a_{ij} J^i J^j$

Quedando $\boxed{Z[J] = Z[0] \exp\left(\frac{1}{2m^2} a_{ij} J^i J^j\right) = Z[0] \exp\left(\bar{a}_{ij} J^i J^j\right)}$

donde $\boxed{\bar{a}_{ij} \equiv \frac{1}{2m^2} a_{ij}}$

Denominemos a $\boxed{\bar{a}_{ij} J^i J^j \equiv f(J) = f}$ y $\boxed{\partial_\alpha \equiv \frac{\partial}{\partial J_\alpha}}$ $\boxed{\partial_{\alpha\beta} \equiv \frac{\partial^2}{\partial J_\alpha \partial J_\beta}}$

Derivaré primero con respecto a J_a y por último a J_d ya que el orden de derivación no altera el resultado

$\underline{\partial_a} e^f = e^f \partial_a f$

$\underline{\partial_b}(\partial_a e^f) = e^f \partial_b f \, \partial_a f + e^f \partial_{ab} f$

$\underline{\partial_c}(\partial_b \partial_a e^f) = e^f \partial_c f \, \partial_b f \, \partial_a f + e^f \partial_{bc} f \, \partial_b f \, \partial_a f +$
$+ e^f \partial_c f \, \partial_{ca} f + e^f \partial_c f \, \partial_{ab} f + e^f \partial_{abc} f$

$$\partial_d\left(\partial_c\partial_b\partial_a e^f\right) = \partial_{dcba} e^f =$$

$$= \underbrace{e^f \partial_d f\, \partial_c f\, \partial_b f\, \partial_a f} + \underbrace{e^f \partial_{dc} f\, \partial_b f\, \partial_a f} + \underbrace{e^f \partial_c f\, \partial_{db} f\, \partial_a f} +$$

$$+ \underbrace{e^f \partial_c f\, \partial_b f\, \partial_{ad} f} + \underbrace{e^f \partial_d f\, \partial_{cb} f\, \partial_a f} + \underbrace{e^f \partial_{dcb} f\, \partial_a f} +$$

$$+ \underbrace{e^f \partial_{cb} f\, \partial_{ad} f} + \underbrace{e^f \partial_d f\, \partial_b f\, \partial_{ac} f} + \underbrace{e^f \partial_{db} f\, \partial_{ac} f} +$$

$$+ \underbrace{e^f \partial_b f\, \partial_{acd} f} + \underbrace{e^f \partial_d f\, \partial_c f\, \partial_{ab} f} + \underbrace{e^f \partial_{cd} f\, \partial_{ab} f} +$$

$$+ \underbrace{e^f \partial_c f\, \partial_{abd} f} + \underbrace{e^f \partial_d f\, \partial_{abc} f} + \underbrace{e^f \partial_{abcd} f}$$

Pasando en claro, se observan las siguientes combinaciones:

- Solo derivadas simples: A, B, C, D
- Derivadas simples y dobles: A, B, CD ; A, BD, C ; A, BC, D ; AB, C, D ; AC, B, D
- Derivadas simples y triples: A, BCD ; ABC, D ; ABD, C ; ACD, B
- Solo derivadas dobles: AB, CD ; AC, DB ; AD, BC
- Solo cuádruple derivada: ABCD

Veamos lo que dá una derivada simple

$$\partial_\alpha f(0) = \partial_\alpha\left(\bar{a}_{ij} J^i J^j\right)_{J=0} = \left[\bar{a}_{ij}\, \delta^i_\alpha J^j + \bar{a}_{ij}\, \delta^j_\alpha J^i\right]_{J=0} =$$

$$= \left[\bar{a}_{\alpha j} J^j + \bar{a}_{i\alpha} J^i\right]_{J=0} = \left[2\bar{a}_{\alpha i} J^i\right]_{J=0} \text{ por ser } A \text{ simétrica}$$

Evaluando en $J=0$ $\quad \partial_\alpha f(0) = 0 \quad$ Por lo que:

Los términos ■ ■ ■ son "nulos"

Derivemos nuevamente para obtener una derivada doble

$$\partial_\beta \partial_\alpha f(0) = \partial_{\alpha\beta} f(0) = \partial_\beta\left(2\bar{a}_{\alpha i} J^i\right)_{J=0} = 2\bar{a}_{\alpha i}\delta^i_\beta = 2\bar{a}_{\alpha\beta}$$

Los términos ■ quedan como

$$(2\bar{a}_{cb})(2\bar{a}_{ad}) + (2\bar{a}_{db})(\bar{a}_{ac}\cdot 2) + (2\bar{a}_{cd})(2\bar{a}_{ab})$$

Tanto la derivada tercera como la cuarta son "cero" debido a que $\partial_{\alpha\beta} f = cte$ $\quad \partial_\gamma(\partial_{\alpha\beta} f) = \partial_{\alpha\beta\gamma} f = 0$

y $\partial_\delta \partial_{\alpha\beta\gamma} f = \partial_{\alpha\beta\gamma\delta} f = 0$

Por lo que los términos ■ ■ son nulos

Resumiendo

$$\partial_d \partial_c \partial_b \partial_a e^F = 4\left(\bar{a}_{cb}\bar{a}_{ad} + \bar{a}_{db}\bar{a}_{ac} + \bar{a}_{cd}\bar{a}_{ab}\right)$$

Como $\bar{a}_{\alpha\beta} = \frac{1}{2m^2} a_{\alpha\beta} = \frac{1}{2m^2} A^{-1}_{\alpha\beta}$

Entonces

$$\boxed{\langle \phi_a \phi_b \phi_c \phi_d \rangle = \frac{1}{\cancel{Z[0]}} \cancel{Z[0]} \left(A^{-1}_{cb} A^{-1}_{ad} + A^{-1}_{db} A^{-1}_{ac} + A^{-1}_{cd} A^{-1}_{ab}\right)}$$