Ejercicio: Dada la siguiente acción:

$$S = \frac{1}{2}\int d^4x \left[\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2\right]$$

a) Demostrar que $\mathcal{L} = \frac{1}{2}\left[\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2\right]$

INVARIANTE BAJO LA TRANSFORMACIÓN:

$$x^{0'} = \gamma x^1 - \gamma\beta x^1$$
$$x^{1'} = -\gamma\beta x^1 + \gamma x^1$$
$$x^{2'} = x^2$$
$$x^{3'} = x^3$$

T. LORENTZ eje x

$$\mathcal{L} = \frac{1}{2}\left[\partial_0\phi\,\partial^0\phi + \partial_1\phi\,\partial^1\phi + \partial_2\phi\,\partial^2\phi + \partial_3\phi\,\partial^3\phi - m^2\phi^2\right]$$

$$\mathcal{L} = \frac{1}{2}\Big[\underbrace{(\partial_0\phi)^2}_{\mathrm{I}} - \underbrace{(\partial_1\phi)^2}_{\mathrm{II}} - \underbrace{(\partial_2\phi)^2}_{\mathrm{III}} - \underbrace{(\partial_3\phi)^2}_{\mathrm{IV}} - m^2\phi^2\Big]$$

I) $$(\partial_0\phi)^2 = \left(\frac{\partial\phi}{\partial x^{0'}}\cdot\cancelto{\gamma}{\frac{\partial x^{0'}}{\partial x^0}} + \frac{\partial\phi}{\partial x^{1'}}\cancelto{-\gamma\beta}{\frac{\partial x^{1'}}{\partial x^0}} + \frac{\partial\phi}{\partial x^{2'}}\cancelto{0}{\frac{\partial x^{2'}}{\partial x^0}} + \frac{\partial\phi}{\partial x^{3'}}\cancelto{0}{\frac{\partial x^{3'}}{\partial x^0}}\right)^2$$

$$= \left(\gamma\,\partial_{0'}\phi - \gamma\beta\,\partial_{1'}\phi\right)^2$$

$$= \gamma^2(\partial_{0'}\phi)^2 - 2\gamma^2\beta\,\partial_{0'}\phi\,\partial_{1'}\phi + \gamma^2\beta^2(\partial_{1'}\phi)^2$$

II) $$(\partial_1\phi)^2 = \left(\frac{\partial\phi}{\partial x^{0'}}\cdot\cancelto{-\gamma\beta}{\frac{\partial x^{0'}}{\partial x^1}} + \frac{\partial\phi}{\partial x^{1'}}\cdot\cancelto{\gamma}{\frac{\partial x^{1'}}{\partial x^1}} + \frac{\partial\phi}{\partial x^{2'}}\cdot\cancelto{0}{\frac{\partial x^{2'}}{\partial x^1}} + \frac{\partial\phi}{\partial x^{3'}}\cdot\cancelto{0}{\frac{\partial x^{3'}}{\partial x^1}}\right)^2$$

$$= \left(-\gamma\beta\,\partial_{0'}\phi + \gamma\,\partial_{1'}\phi\right)^2$$

$$= \gamma^2\beta^2(\partial_{0'}\phi)^2 - 2\gamma^2\beta\,\partial_{0'}\phi\,\partial_{1'}\phi + \gamma^2(\partial_{1'}\phi)^2$$

III) $$(\partial_2\phi)^2 = \left(\frac{\partial\phi}{\partial x^{0'}}\cancelto{0}{\frac{\partial x^{0'}}{\partial x^2}} + \frac{\partial\phi}{\partial x^{1'}}\cdot\cancelto{0}{\frac{\partial x^{1'}}{\partial x^2}} + \frac{\partial\phi}{\partial x^{2'}}\cancelto{1}{\frac{\partial x^{2'}}{\partial x^2}} + \frac{\partial\phi}{\partial x^{3'}}\cdot\cancelto{0}{\frac{\partial x^{3'}}{\partial x^2}}\right)^2$$

$$= (\partial_{2'}\phi)^2$$

IV) $(\partial_3\phi)^2 = \left(\frac{\partial\phi}{\partial x^{0'}}\cancelto{0}{\frac{\partial x^{0'}}{\partial x^3}} + \frac{\partial\phi}{\partial x^{1'}}\cdot\cancelto{0}{\frac{\partial x^{1'}}{\partial x^3}} + \frac{\partial\phi}{\partial x^{2'}}\cancelto{0}{\frac{\partial x^{2'}}{\partial x^3}} + \frac{\partial\phi}{\partial x^{3'}}\cdot\cancelto{1}{\frac{\partial x^{3'}}{\partial x^3}}\right)^2$

$= (\partial_{3'}\phi)^2$

Entonces juntando todo al $\mathcal{L}$.

$$\mathcal{L} = \frac{1}{2}\Big[\gamma^2(\partial_{0'}\phi)^2 - \cancel{2\gamma^2\beta\,\partial_{0'}\phi\,\partial_{1'}\phi} + \gamma^2\beta^2(\partial_{1'}\phi)^2 - \gamma^2\beta^2(\partial_{0'}\phi)^2 + \cancel{2\gamma^2\beta\,\partial_{0'}\phi\,\partial_{1'}\phi}$$
$$-\gamma^2(\partial_{1'}\phi)^2 - (\partial_{2'}\phi)^2 - (\partial_{3'}\phi)^2 - m^2\phi^2\Big]$$

$$\mathcal{L} = \frac{1}{2}\left[\gamma^2(1-\beta^2)(\partial_{0'}\phi)^2 + \gamma^2(\beta^2-1)(\partial_{1'}\phi)^2 - (\partial_{2'}\phi)^2 - (\partial_{3'}\phi)^2 - m^2\phi^2\right]$$

$$\mathcal{L} = \frac{1}{2}\left[\gamma^2(1-\beta^2)(\partial_{0'}\phi)^2 - \gamma^2(1-\beta^2)(\partial_{1'}\phi)^2 - (\partial_{2'}\phi)^2 - (\partial_{3'}\phi)^2 - m^2\phi^2\right]$$

$$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \gamma^2(1-\beta^2) = 1$$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_{0'}\phi)^2 - (\partial_{1'}\phi)^2 - (\partial_{2'}\phi)^2 - (\partial_{3'}\phi) - m^2\phi^2\right]$$

b) Calcular $\frac{\delta S}{\delta\phi}$

$$\frac{\delta S}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$$
$$= \frac{\partial\mathcal{L}}{\partial\phi} - \left[\partial_0\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\right) + \partial_1\left(\frac{\partial\mathcal{L}}{\partial(\partial_1\phi)}\right) + \partial_2\left(\frac{\partial\mathcal{L}}{\partial(\partial_2\phi)}\right) + \partial_3\left(\frac{\partial\mathcal{L}}{\partial(\partial_3\phi)}\right)\right]$$

$$\frac{\delta S}{\delta\phi} = \frac{-1}{2}m^2\cdot 2\phi - \left[\partial_0\left(\tfrac{1}{2}\cdot 2\cdot\partial_0\phi\right) + \partial_1\left(\tfrac{1}{2}\cdot 2\cdot -\partial_1\phi\right) + \partial_2\left(\tfrac{1}{2}\cdot 2\cdot -\partial_2\phi\right) + \partial_3\left(\tfrac{1}{2}\cdot 2\cdot -\partial_3\phi\right)\right]$$

$$\frac{\delta S}{\delta\phi} = -m^2\phi - \left[\partial_0^2\phi - \partial_1^2\phi - \partial_2^2\phi - \partial_3^2\phi\right]$$