$$\int_{-\infty}^{\infty} x^{2n} e^{-\frac{a}{2}x^2} dx = \qquad e^{-\frac{a}{2}x^2} = u \quad -ax e^{-\frac{a}{2}x^2} dx = du$$

$$x^{2n} dx = dv \qquad v = \frac{x^{2n+1}}{2n+1}$$

Integrando por partes

$$\int_{-\infty}^{\infty} x^{2n} e^{-\frac{a}{2}x^2} dx = \left[\frac{x^{2n+1} e^{-\frac{a}{2}x^2}}{(2n+1)}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{x^{2n+1}}{(2n+1)} (-a) x e^{-\frac{a}{2}x^2} dx$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-\frac{a}{2}x^2} dx = 0 + \frac{a}{(2n+1)} \int_{-\infty}^{\infty} x^{2n+2} e^{-\frac{a}{2}x^2} dx$$

Despejando la integral del segundo miembro

$$\int_{-\infty}^{\infty} x^{2n+2} e^{-\frac{a}{2}x^2} dx = \frac{(2n+1)}{a} \int_{-\infty}^{\infty} x^{2n} e^{-\frac{a}{2}x^2} dx$$

Si $2n+2 = 2k \Rightarrow 2n+1 = 2k-1$ y $2n = 2k-2$ queda

$$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{a}{2}x^2} dx = \frac{(2k-1)}{a} \int_{-\infty}^{\infty} x^{2k-2} e^{-\frac{a}{2}x^2} dx$$

Y esta relación implica que $\displaystyle\int_{-\infty}^{\infty} x^{2k-2} e^{-\frac{a}{2}x^2} dx = \frac{2k-3}{a} \int_{-\infty}^{\infty} x^{2k-4} e^{-\frac{a}{2}x^2} dx$

O sea $$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{a}{2}x^2} dx = \frac{(2k-1)}{a} \frac{(2k-3)}{a} \frac{(2k-5)}{a} \cdots \frac{1}{a} \int_{-\infty}^{\infty} x^0 e^{-\frac{a}{2}x^2} dx$$

$$= \frac{(2k-1)}{a} \frac{(2k-3)}{a} \frac{(2k-5)}{a} \cdots \frac{1}{a} \cdot \sqrt{\frac{2\pi}{a}}$$

Por tanto

$$\langle x^{2k} \rangle = \frac{\int_{-\infty}^{\infty} x^{2k} e^{-\frac{a}{2}x^2} dx}{\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2} dx} = \frac{\frac{(2k-1)}{a} \frac{(2k-3)}{a} \cdot \frac{(2k-5)}{a} \cdots \frac{1}{a} \cancel{\sqrt{\frac{2\pi}{a}}}}{\cancel{\sqrt{\frac{2\pi}{a}}}}$$

$$= \frac{1}{a^k} (2k-1)(2k-3)(2k-5) \cdots 5 \cdot 3 \cdot 1$$

NOTA

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2}dx = \sqrt{\frac{\pi}{\frac{a}{2}}} = \sqrt{\frac{2\pi}{a}} \quad \text{usando} \quad \int_{-\infty}^{\infty} e^{-Ax^2}dx = \sqrt{\frac{\pi}{A}}$$

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x e^{-\frac{a}{2}x^2}dx}{\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2}dx} = \frac{\left(-\frac{1}{a}\right)\cdot(-a)\int_{-\infty}^{\infty} x e^{-\frac{a}{2}x^2}dx}{\sqrt{\frac{2\pi}{a}}} = \frac{-\frac{1}{a}\int_{-\infty}^{\infty} -a x e^{-\frac{a}{2}x^2}dx}{\sqrt{\frac{2\pi}{a}}}$$

$$= \frac{-\frac{1}{a}\int_{-\infty}^{\infty} d\left(e^{-\frac{a}{2}x^2}\right)}{\sqrt{\frac{2\pi}{a}}} = \frac{-\frac{1}{a}\left[e^{-\frac{a}{2}x^2}\right]_{-\infty}^{\infty}}{\sqrt{\frac{2\pi}{a}}} = \frac{-\frac{1}{a}\left[\frac{1}{e^{\frac{a}{2}x^2}}\right]_{-\infty}^{\infty}}{\sqrt{\frac{2\pi}{a}}} = \frac{0}{\sqrt{\frac{2\pi}{a}}} = 0$$

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2 e^{-\frac{a}{2}x^2}dx}{\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2}dx} = \frac{\int_{-\infty}^{\infty} x^2 e^{-\frac{a}{2}x^2}dx}{\sqrt{\frac{2\pi}{a}}}$$

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2}dx = \sqrt{\frac{2\pi}{a}} = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2}dx = a^{-\frac{1}{2}}\cdot\sqrt{2\pi}$$ Derivando ambos términos respecto de "a"

$$\int_{-\infty}^{\infty} -\frac{x^2}{2} e^{-\frac{a}{2}x^2}dx = \left(-\frac{1}{2}\right)a^{-\frac{1}{2}-1}\sqrt{2\pi}$$

$$\cancel{\left(-\frac{1}{2}\right)}\int_{-\infty}^{\infty} x^2 e^{-\frac{a}{2}x^2}dx = \cancel{\left(-\frac{1}{2}\right)}a^{-\frac{3}{2}}\sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{a}{2}x^2}dx = a^{-\frac{3}{2}}\sqrt{2\pi}$$

$$\langle x^2 \rangle = \frac{a^{-\frac{3}{2}}\sqrt{2\pi}}{\sqrt{\frac{2\pi}{a}}} = \frac{a^{-\frac{3}{2}}\cdot\cancel{\sqrt{2\pi}}}{a^{-\frac{1}{2}}\cancel{\sqrt{2\pi}}} = a^{-\frac{3}{2}+\frac{1}{2}} = a^{-\frac{2}{2}} = a^{-1} = \frac{1}{a}$$