Teoría Cuántica de Campos

Ejercicio Capítulo 19

$$S = \frac{1}{2}\int d^4x\,[\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2]$$

a) Demostrar que $\mathcal{L} = \frac{1}{2}[\partial_\mu\phi\,\partial^\mu\phi - m^2\phi]$ es invariante bajo la transformación de Lorenz

$$\begin{cases} x^{0'} = \gamma x^0 - \gamma\beta x^1 \\ x^{1'} = -\gamma\beta x^0 + \gamma x^1 \\ x^{2'} = x^2 \\ x^{3'} = x^3 \end{cases}$$

donde

$$\begin{cases} \gamma = \dfrac{1}{\sqrt{1-\beta^2}} \\ \beta = \dfrac{v}{c} \end{cases}$$

Inversamente:

$$\begin{cases} x^0 = \gamma x^{0'} + \gamma\beta x^{1'} \\ x^1 = \gamma\beta x^{0'} + \gamma x^{1'} \\ x^2 = x^{2'} \\ x^3 = x^{3'} \end{cases}$$

Empezaremos calculando las $\partial_\mu\phi$ en funciones de $\partial_{\mu'}\phi$

(*) $\partial_0\phi = \dfrac{\partial\phi}{\partial x^0} = \dfrac{\partial\phi}{\partial x^{0'}}\dfrac{\partial x^{0'}}{\partial x^0} + \dfrac{\partial\phi}{\partial x^{1'}}\dfrac{\partial x^{1'}}{\partial x^0} = \partial_{0'}\phi(\gamma) + \partial_{1'}\phi(-\gamma\beta)$

(*) $\partial_1\phi = \dfrac{\partial\phi}{\partial x^1} = \dfrac{\partial\phi}{\partial x^{0'}}\dfrac{\partial x^{0'}}{\partial x^1} + \dfrac{\partial\phi}{\partial x^{1'}}\dfrac{\partial x^{1'}}{\partial x^1} = \partial_{0'}\phi(-\gamma\beta) + \partial_{1'}\phi(\gamma)$

$\partial_2\phi = \partial_{2'}\phi$

$\partial_3\phi = \partial_{3'}\phi$

(*) nota: los términos que incluyen $\dfrac{\partial x^{2'}}{\partial x^0}$, $\dfrac{\partial x^{3'}}{\partial x^0}$, $\dfrac{\partial x^{2'}}{\partial x^1}$ y $\dfrac{\partial x^{3'}}{\partial x^1}$ son excluídos al ser dichas derivadas iguales a cero.

Asimismo recordemos que bajo la métrica de Minkowsky

$\partial^0\phi = \partial_0\phi$; $\partial^1\phi = -\partial_1\phi$; $\partial^2\phi = -\partial_2\phi$; $\partial^3\phi = -\partial_3\phi$

y $\partial^{0'}\phi = -\partial_{0'}\phi$; $\partial^{1'}\phi = -\partial_{1'}\phi$; $\partial^{2'}\phi = -\partial_{2'}\phi$; $\partial^{3'}\phi = -\partial_{3'}\phi$

Es decir que tendremos:

$\partial^0\phi = \partial_0\phi = \partial_{0'}\phi(\gamma) + \partial_{1'}\phi(-\gamma\beta) = \partial^{0'}\phi(\gamma) - \partial^{1'}\phi(-\gamma\beta)$

$\partial^1\phi = -\partial_1\phi = -\partial_{0'}\phi(-\gamma\beta) - \partial_{1'}\phi(\gamma) = -\partial^{0'}\phi(-\gamma\beta) + \partial^{1'}\phi(\gamma)$

$\partial^2\phi = -\partial_2\phi = -\partial_{2'}\phi = \partial^{2'}\phi$

$\partial^3\phi = -\partial_3\phi = -\partial_{3'}\phi = \partial^{3'}\phi$

Realizemos las multiplicaciones

① $\partial_0\phi\,\partial^0\phi = [\gamma\partial_{0'}\phi - \gamma\beta\,\partial_{1'}\phi][\gamma\partial^{0'}\phi + \gamma\beta\,\partial^{1'}\phi] =$

$= \gamma^2\partial_{0'}\phi\,\partial^{0'}\phi - \gamma^2\beta\,\partial_{1'}\phi\,\partial^{0'}\phi + \gamma^2\beta\,\partial_{0'}\phi\,\partial^{1'}\phi - \gamma^2\beta^2\,\partial^{1'}\phi\,\partial_{1'}\phi =$

$= \gamma^2\partial_{0'}\phi\,\partial^{0'}\phi + 2\gamma^2\beta\,\partial_{0'}\phi\,\partial^{1'}\phi - \gamma^2\beta^2\,\partial_{1'}\phi\,\partial^{1'}\phi$

② $\partial_1\phi\,\partial^1\phi = [\gamma\partial_{1'}\phi - \gamma\beta\,\partial_{0'}\phi][\gamma\partial^{1'}\phi + \gamma\beta\,\partial^{0'}\phi] =$

$= \gamma^2\partial_{1'}\phi\,\partial^{1'}\phi + \gamma^2\beta\,\partial_{1'}\phi\,\partial^{0'}\phi - \gamma^2\beta\,\partial_{0'}\phi\,\partial^{1'}\phi - \gamma^2\beta^2\,\partial_{0'}\phi\,\partial^{0'}\phi =$

$= \gamma^2\partial_{1'}\phi\,\partial^{1'}\phi - 2\gamma^2\beta\,\partial_{0'}\phi\,\partial^{1'}\phi - \gamma^2\beta^2\,\partial_{0'}\phi\,\partial^{0'}\phi$

③ $\partial_2\phi\,\partial^2\phi = \partial_{2'}\phi\,\partial^{2'}\phi$

④ $\partial_3\phi\,\partial^3\phi = \partial_{3'}\phi\,\partial^{3'}\phi$

Quedando:

$$\boxed{\mathcal{L}} = \frac{1}{2}\left[① + ② + ③ + ④ - m^2\phi^2\right]|J(x^\mu, x^{\mu'})| =$$

$$= \frac{1}{2}\Big[\gamma^2\partial_{0'}\phi\,\partial^{0'}\phi + \cancel{2\gamma^2\beta\,\partial_{0'}\phi\,\partial^{1'}\phi} - \gamma^2\beta^2\,\partial_{1'}\phi\,\partial^{1'}\phi + \gamma^2\partial_{1'}\phi\,\partial^{1'}\phi - \cancel{2\gamma^2\beta\,\partial_{0'}\phi\,\partial^{1'}\phi} - \gamma^2\beta^2\,\partial_{0'}\phi\,\partial^{0'}\phi + \partial_{2'}\phi\,\partial^{2'}\phi + \partial_{3'}\phi\,\partial^{3'}\phi - m^2\phi^2\Big]|J|$$

$$= \frac{1}{2}\Big[\underbrace{\gamma^2(1-\beta^2)}_{=1}\partial_{0'}\phi\,\partial^{0'}\phi + \underbrace{\gamma^2(1-\beta^2)}_{=1}\partial_{1'}\phi\,\partial^{1'}\phi + \partial_{2'}\phi\,\partial^{2'}\phi + \partial_{3'}\phi\,\partial^{3'}\phi - m^2\phi^2\Big]|J(x^\mu, x^{\mu'})| = \frac{1}{2}\left[\partial_{\mu'}\phi\,\partial^{\mu'}\phi - m^2\phi^2\right]|J(x^\mu, x^{\mu'})|$$

Calculemos el Jacobiano

$$J(x^\mu, x^{\mu'}) = \begin{bmatrix} \frac{\partial x^0}{\partial x^{0'}} & \frac{\partial x^0}{\partial x^{1'}} & \frac{\partial x^0}{\partial x^{2'}} & \frac{\partial x^0}{\partial x^{3'}} \\ \frac{\partial x^1}{\partial x^{0'}} & \frac{\partial x^1}{\partial x^{1'}} & \frac{\partial x^1}{\partial x^{2'}} & \frac{\partial x^1}{\partial x^{3'}} \\ \frac{\partial x^2}{\partial x^{0'}} & \frac{\partial x^2}{\partial x^{1'}} & \frac{\partial x^2}{\partial x^{2'}} & \frac{\partial x^2}{\partial x^{3'}} \\ \frac{\partial x^3}{\partial x^{0'}} & \frac{\partial x^3}{\partial x^{1'}} & \frac{\partial x^3}{\partial x^{2'}} & \frac{\partial x^3}{\partial x^{3'}} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Por lo cual

$$\det\left[J(x^{\mu} x^{\mu'})\right] = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1$$

Finalmente

$$\boxed{\mathcal{L}(x') = \frac{1}{2}\left[\partial_{\mu'}\phi\,\partial^{\mu'}\phi - m^2\phi^2\right]}$$

Por lo cual se nota la invarianza del Lagrangiano dado bajo una transformación de Lorentz.

b) Calcular $\frac{\delta S}{\delta \phi}$

Según fórmula 19.12 (derivada de un funcional

$$\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)$$

$$\mathcal{L} = \frac{1}{2}\left[\partial_0\phi\,\partial^0\phi + \partial_1\phi\,\partial^1\phi + \partial_2\phi\,\partial^2\phi + \partial_3\phi\,\partial^3\phi - m^2\phi^2\right] =$$

$$= \frac{1}{2}\left[(\partial_0\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 - m^2\phi^2\right]$$

Así que

$$\partial_0\left(\frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\right) = \partial_0\left(\frac{1}{\cancel{2}}\cdot \cancel{2}\,\partial_0\phi\right) = \partial_0^{(2)}\phi$$

$$\partial_1\left(\frac{\partial \mathcal{L}}{\partial(\partial_1\phi)}\right) = \partial_1\left(-\frac{1}{\cancel{2}}\,\cancel{2}\,\partial_1\phi\right) = -\partial_1^{(2)}\phi$$

$$\partial_2\left(\frac{\partial \mathcal{L}}{\partial(\partial_2\phi)}\right) = \partial_2\left(-\frac{1}{\cancel{2}}\,\cancel{2}\,\partial_2\phi\right) = -\partial_2^{(2)}\phi$$

$$\partial_3\left(\frac{\partial \mathcal{L}}{\partial(\partial_3\phi)}\right) = \partial_3\left(-\frac{1}{\cancel{2}}\,\cancel{2}\,\partial_3\phi\right) = -\partial_3^{(2)}\phi$$

y $\quad \dfrac{\partial L}{\partial \phi} = -\dfrac{1}{\cancel{2}} \cdot \cancel{2} \cdot m^2\phi = -m^2\phi$

Por lo cual queda

$$\frac{\delta S}{\delta \phi} = -m^2\phi - \partial_0^{(2)}\phi + \partial_1^{(2)}\phi + \partial_3^{(2)}\phi + \partial_4^{(2)}\phi$$

Simplificadamente

$$\boxed{\frac{\delta S}{\delta \phi} = -m^2\phi - \eta^{\mu\mu}\partial_\mu^2\phi}$$

$$\boxed{\eta^{\mu\mu} = \begin{cases} 1 & : \mu = 0 \\ -1 & : \mu = 1, 2, 3 \end{cases}}$$