a) Habiendo expresado u_1 en función de $\vec{p}$, hacer lo mismo con u_2; v_1; v_2

$$u_1 = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_+ \\ \frac{c(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\chi_+ \end{pmatrix}$$

① $\operatorname{ch}\eta/_2 = \sqrt{\frac{1+\operatorname{ch}\eta}{2}}$ $\quad \operatorname{sh}\eta/_2 \sqrt{\frac{-1+\operatorname{ch}\eta}{2}}$

② $\operatorname{ch}\eta = \frac{p^0}{mc} = \frac{E}{mc^2}$

③ $E^2 = |\vec{p}|^2c^2 + (mc^2)^2$

④ $(\vec{\sigma}\cdot\hat{n})\chi_+ = \chi_+ \qquad (\vec{\sigma}\cdot\hat{n})\chi_- = -\chi_-$

$$u_2 = \begin{pmatrix} \operatorname{ch}\eta/2 \\ -\operatorname{sh}\eta/_2 \end{pmatrix} \otimes \chi_- \overset{①}{=} \begin{pmatrix} \sqrt{\frac{1+\operatorname{ch}\eta}{2}} \\ -\sqrt{\frac{-1+\operatorname{ch}\eta}{2}} \end{pmatrix} \otimes \chi_- = \sqrt{\frac{1+\operatorname{ch}\eta}{2}} \begin{pmatrix} \chi_- \\ -\sqrt{\frac{-1+\operatorname{ch}\eta}{+1+\operatorname{ch}\eta}}\chi_- \end{pmatrix}$$

$$\overset{②}{=} \sqrt{\frac{1+E/mc^2}{2}} \begin{pmatrix} \chi_- \\ -\frac{\sqrt{-1+E/mc^2}}{\sqrt{1+E/mc^2}}\chi_- \end{pmatrix} = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_- \\ -\sqrt{\frac{E-mc^2}{E+mc^2}}\chi_- \end{pmatrix}$$

$$\overset{③}{=} \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_- \\ -\frac{\sqrt{E^2-(mc^2)^2}}{E+mc^2}\chi_- \end{pmatrix} = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_- \\ -\frac{|\vec{p}|c}{E+mc^2}\chi_- \end{pmatrix}$$

$$\vec{p} = |\vec{p}|\hat{n} \longrightarrow \hat{n} = \frac{\vec{p}}{|\vec{p}|}$$

$$\vec{\sigma}\cdot\hat{n} = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \overset{④}{\Rightarrow} \left(\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right)\chi_- = -\chi_-$$

$$-|\vec{p}|\chi_- = (\vec{\sigma}\cdot\vec{p})\chi_-$$

$$\boxed{u_2 = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_- \\ \frac{(\vec{\sigma}\cdot\vec{p})c}{E+mc^2}\chi_- \end{pmatrix}}$$

$$v_1 = \begin{pmatrix} \operatorname{sh}\eta/2 \\ \operatorname{ch}\eta/2 \end{pmatrix} \otimes \chi_+ \overset{1}{=} \begin{pmatrix} \sqrt{\frac{-1+\operatorname{ch}\eta}{2}} \\ \sqrt{\frac{1+\operatorname{ch}\eta}{2}} \end{pmatrix} \otimes \chi_+ = \sqrt{\frac{1+\operatorname{ch}\eta}{2}} \begin{pmatrix} \sqrt{\frac{-1+\operatorname{ch}\eta}{1+\operatorname{ch}\eta}}\chi_+ \\ \chi_+ \end{pmatrix}$$

$$\overset{2}{=} \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \sqrt{\frac{-mc^2+E}{mc^2+E}}\chi_+ \\ \chi_+ \end{pmatrix} \overset{3}{=} \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \frac{\sqrt{E^2-(mc^2)^2}}{E+mc^2}\chi_+ \\ \chi_+ \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \frac{|\vec{p}|c}{E+mc^2}\chi_+ \\ \chi_+ \end{pmatrix}$$

4 $\quad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi_+ = \chi_+$

$|\vec{p}|\chi_+ = (\vec{\sigma}\cdot\vec{p})\chi_+$

$$\boxed{v_1 = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \frac{c(\vec{\sigma}\cdot\vec{p})\chi_+}{E+mc^2} \\ \chi_+ \end{pmatrix}}$$

$$v_2 = \begin{pmatrix} -\operatorname{sh}\eta \\ \operatorname{ch}\eta \end{pmatrix} \otimes \chi_- = \sqrt{\frac{1+\operatorname{ch}\eta}{2}} \begin{pmatrix} -\sqrt{\frac{-1+\operatorname{ch}\eta}{1+\operatorname{ch}\eta}}\chi_- \\ \chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} -\frac{\sqrt{E-mc^2}}{\sqrt{E+mc^2}}\chi_- \\ \chi_- \end{pmatrix} = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} -\frac{\sqrt{E^2-(mc^2)^2}}{E+mc^2}\chi_- \\ \chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \frac{-|\vec{p}|c}{E+mc^2}\chi_- \\ \chi_- \end{pmatrix}$$

$-|\vec{p}|\chi_- = (\vec{\sigma}\cdot\vec{p})\chi_-$

$$\boxed{v_2 = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \frac{c(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\chi_- \\ \chi_- \end{pmatrix}}$$

b) Expresar χ_- en función de $\hat{n} = (n_1 ; n_2 ; n_3)$

$$\chi_- = \begin{pmatrix} -e^{-i\phi} \operatorname{sen} \theta/2 \\ \cos \theta/2 \end{pmatrix}$$

$\cancel{\operatorname{sen} \theta/2} = \operatorname{sen}\theta = 2 \operatorname{sen}\frac{\theta}{2} \cos\frac{\theta}{2}$

$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

$$\chi_- = \frac{1}{\cos \theta/2} \begin{pmatrix} -e^{-i\phi} \operatorname{sen}\frac{\theta}{2} \cos\frac{\theta}{2} \\ \cos^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{\sqrt{\frac{1+\cos\theta}{2}}} \begin{pmatrix} -e^{-i\phi} \frac{1}{2} \operatorname{sen}\theta \\ \frac{1+\cos\theta}{2} \end{pmatrix}$$

$$e^{-i\phi} = \cos\phi - i \operatorname{sen}\phi$$

$$\chi_- = \frac{1}{\sqrt{2(1+\cos\theta)}} \begin{pmatrix} -(\cos\phi - i\operatorname{sen}\phi) \operatorname{sen}\theta \\ 1+\cos\theta \end{pmatrix}$$

$$\hat{n} = (\underbrace{\operatorname{sen}\theta \cos\phi}_{n^1}, \underbrace{\operatorname{sen}\theta \operatorname{sen}\phi}_{n^2}, \underbrace{\cos\theta}_{n^3})$$

$$\chi_- = \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -\cos\phi \operatorname{sen}\theta + i \operatorname{sen}\phi \operatorname{sen}\theta \\ 1+n^3 \end{pmatrix}$$

$$\boxed{\chi_- = \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1 + i n^2 \\ n^3 \end{pmatrix}}$$

c) Dadas las matrices

$$M_1 = \begin{pmatrix} p_0 - mc & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 - mc \end{pmatrix} \quad y \quad M_2 = \begin{pmatrix} p_0 + mc & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 + mc \end{pmatrix}$$

Verificar que

$M_1 \cdot u_2 = 0$

$M_2 \cdot v_1 = 0$

$M_2 \cdot v_2 = 0$

$$M_1 \cdot u_2 = \begin{pmatrix} p_0 - mc & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 - mc \end{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_- \\ \frac{(\vec{\sigma}\cdot\vec{p})\,c}{E+mc^2}\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} (p_0 - mc)\chi_- - \frac{(\vec{\sigma}\cdot\vec{p})\,c\,(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\chi_- \\ (\vec{\sigma}\cdot\vec{p})\chi_- + (-p_0 - mc)\frac{(\vec{\sigma}\cdot\vec{p})\,c}{E+mc^2}\chi_- \end{pmatrix}$$

$$(\vec{\sigma}\cdot\vec{p})\chi_- = -|\vec{p}|\chi_-$$

$$M_1 u_2 = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} (p_0 - mc)\chi_- - \frac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})\underbrace{(\vec{\sigma}\cdot\vec{p})\chi_-}_{= -|\vec{p}|\chi_-} \\ -|\vec{p}|\chi_- - \frac{(p_0+mc)\,c}{E+mc^2}(-|\vec{p}|)\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[(p_0 - mc) - \frac{c}{E+mc^2}(-|\vec{p}|)(-|\vec{p}|)\right]\chi_- \\ \left[-|\vec{p}| + \frac{(p_0+mc)\,c}{E+mc^2}|\vec{p}|\right]\chi_- \end{pmatrix}$$

(2) $p_0 = E/c$ y (3) $E^2 - (mc^2)^2 = |\vec{p}|^2 c^2$

$$M_1 u_2 = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[E/c - mc - \frac{c}{(E+mc^2)}\frac{1}{c^2}(E^2 - (mc^2)^2)\right]\chi_- \\ \left[-1 + \frac{(E/c + mc)\,c}{E+mc^2}\right]|\vec{p}|\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[\frac{E-mc^2}{c} - \frac{(E^2 - (mc^2)^2)}{c(E+mc^2)}\right]\chi_- \\ \left[\frac{-E-mc^2 + E+mc^2}{E+mc^2}\right]\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[E^2 - (mc^2)^2 - (E^2 - (mc^2)^2)\right]\frac{\chi_-}{c(E+mc^2)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{M_1 u_2 = 0}$$

$$M_2 . v_1 = \begin{pmatrix} p_0 + mc & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 + mc \end{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \frac{c(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\chi_+ \\ \chi_+ \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} (p_0+mc)\frac{c(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\chi_+ - (\vec{\sigma}\cdot\vec{p})\chi_+ \\ (\vec{\sigma}\cdot\vec{p})\frac{c(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\chi_+ + (-p_0+mc)\chi_+ \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[\frac{E/c+mc}{E+mc^2}\cdot c - 1\right](\vec{\sigma}\cdot\vec{p})\chi_+ \\ \frac{c|\vec{p}|}{E+mc^2}(\vec{\sigma}\cdot\vec{p})\chi_+ + (-p_0+mc)\chi_+ \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[\frac{E+mc^2}{E+mc^2} - 1\right](\vec{\sigma}\cdot\vec{p})\chi_+ \\ \left[\frac{c|\vec{p}|^2}{E+mc^2} - \left(\frac{E}{c} - mc\right)\right]\chi_+ \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} 0 \\ \left[c\,\frac{E^2-(mc^2)^2}{c^2}\frac{1}{E+mc^2} - \frac{1}{c}(E-mc^2)\right]\chi_+ \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} 0 \\ \frac{1}{c}\left(\frac{(E^2-(mc^2)^2) - (E-mc^2)(E+mc^2)}{E+mc^2}\right)\chi_+ \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{M_2 \cdot v_p = 0}$$

$$M_2 \cdot v_2 = \begin{pmatrix} p_0 + mc & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p_0 + mc \end{pmatrix} \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \dfrac{c\,(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\,\chi_- \\ \chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} (p_0+mc)\dfrac{c\,(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\,\chi_- - \vec{\sigma}\cdot\vec{p}\,\chi_- \\ (\vec{\sigma}\cdot\vec{p})\dfrac{c\,(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\,\chi_- + (-p_0+mc)\,\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[\left(E/c+mc\right)\dfrac{c}{E+mc^2} - 1\right](\vec{\sigma}\cdot\vec{p})\,\chi_- \\ \dfrac{c}{E+mc^2}(\vec{\sigma}\cdot\vec{p})(-|\vec{p}|)\,\chi_- + \left(-\dfrac{E}{c}+mc\right)\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \left[\dfrac{E+mc^2}{E+mc^2} - 1\right](\vec{\sigma}\cdot\vec{p})\,\chi_- \\ \left[\dfrac{c\,|\vec{p}|^2}{E+mc^2} + \left(-\dfrac{E}{c}+mc\right)\right]\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} 0 \\ \left[\dfrac{c\,\left(E^2-(mc^2)^2\right)/c^2}{E+mc^2} - \dfrac{1}{c}\left(E-mc^2\right)\right]\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} 0 \\ \dfrac{1}{c}\,\dfrac{\left(E^2-(mc^2)^2\right)-\left(E^2-(mc^2)^2\right)}{E+mc^2}\,\chi_- \end{pmatrix}$$

$$= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{M_2\, v_2 = 0}$$