LAURA INCERA

EJERCICIO

Tenemos un campo al que se le pueden asignar tres valores:

$$\phi \to \phi_1 \quad \phi_2 \quad \phi_3 \qquad (\text{valores del campo})$$

Hay una "cosa":

$$COSA = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\,\phi_1\phi_2 - \sqrt{2}\,\phi_2\phi_3$$

Para cada configuración del campo (para cada valor de ϕ_i), COSA valdrá un número:

Se pide:

a) Matriz A / $(\phi_1\ \phi_2\ \phi_3)\,(A)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = COSA$

b) Diagonalizar A (Encontrar la matriz M, sus valores y vectores propios)

c) Cambio de variable eliminando "productos cruzados" y mostrar que

$$COSA = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$$

a)

$$(\phi_1\ \phi_2\ \phi_3)\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

Hacemos la primera multiplicación:

$$(\phi_1\ \phi_2\ \phi_3)\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = (*)$$

$$\circledast = \Big(a_{11}\phi_1 + a_{21}\phi_2 + a_{31}\phi_3 \, ,$$

$$a_{12}\phi_1 + a_{22}\phi_2 + a_{32}\phi_3 \, ,$$

$$a_{13}\phi_1 + a_{23}\phi_2 + a_{33}\phi_3 \Big)$$

Ahora multiplico por $\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} =$

$$= a_{11}\phi_1^2 + a_{21}\phi_1\phi_2 + a_{31}\phi_1\phi_3 +$$

$$a_{12}\phi_1\phi_2 + a_{22}\phi_2^2 + a_{32}\phi_2\phi_3 +$$

$$a_{13}\phi_1\phi_3 + a_{23}\phi_2\phi_3 + a_{33}\phi_3^2$$

Comparando con "cosa" sacamos:

$$a_{11} = -6 \qquad a_{21} + a_{12} = -\sqrt{2}$$

$$a_{22} = -6 \qquad a_{31} + a_{13} = 0$$

$$a_{33} = -6 \qquad a_{32} + a_{23} = -\sqrt{2}$$

$$\begin{array}{ccc} a_{21} & & a_{12} \\ \frac{-\sqrt{2}}{2} & + & \frac{-\sqrt{2}}{2} \end{array}$$

$$a_{31} = a_{13} = 0$$

$$\begin{array}{ccc} \frac{-\sqrt{2}}{2} & + & \frac{-\sqrt{2}}{2} \\ a_{32} & & a_{23} \end{array}$$

Para que A sea SIMÉTRICA.

$$A = \begin{pmatrix} -6 & \frac{-\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & -6 & \frac{-\sqrt{2}}{2} \\ 0 & -\sqrt{2}/2 & -6 \end{pmatrix} \equiv$$

$$\equiv \begin{pmatrix} -6 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -6 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -6 \end{pmatrix}$$

b) Tenemos que diagonalizar A. Tenemos que encontrar una base en la que A sea diagonal.

La base será la matriz M de vectores propios que estamos buscando. La matriz A en esa base tendrá los valores propios en la diagonal.

Si $\bar{v}_i$ son los vectores propios y λ_i son los valores propios se cumplirá:

$$A\bar{v}_i = \lambda_i \bar{v}_i$$

Esto implica que $(A - \lambda_i I)v_i = 0$

Para que el sistema tenga infinitas soluciones:

$$|A - \lambda_i I| = 0$$

$$\begin{vmatrix} -6-\lambda & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -6-\lambda & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -6-\lambda \end{vmatrix} = 0$$

$$(-6-\lambda)^3 + \cancel{(-6-\lambda)\cdot\tfrac{1}{2}0} - (-6-\lambda)\cdot\frac{1}{2} - (-6-\lambda)\cdot\frac{1}{2} = 0$$

$$(-6-\lambda)^3 - (-6-\lambda) = 0$$

Resolviendo:

$$\boxed{\begin{aligned} \lambda_1 &= -6 \\ \lambda_2 &= -5 \\ \lambda_3 &= -7 \end{aligned}}$$

Para $\lambda_1 = -6$

$$\begin{pmatrix} 0 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \vec{0}$$

$$\left.\begin{array}{l} \frac{-1}{\sqrt{2}} y_1 = 0 \rightarrow y_1 = 0 \\ \frac{-1}{\sqrt{2}} x_1 + \frac{-1}{\sqrt{2}} z_1 = 0 \quad x_1 = -z_1 \end{array}\right\} \boxed{V_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

$$|V_1| = \sqrt{2}$$

Para $\lambda_2 = -5$

$$\begin{pmatrix} -1 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -x_2 + \frac{-1}{\sqrt{2}} y_2 = 0 \\ -\frac{1}{\sqrt{2}} x_2 - y_2 - \frac{1}{\sqrt{2}} z_2 = 0 \\ \frac{-1}{\sqrt{2}} y_2 - z_2 = 0 \end{array}\right\} \boxed{V_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}} \quad |V_2| = 2$$

Para $\lambda_3 = -7$

$$\begin{pmatrix} 1 & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \vec{0}$$

$$\left.\begin{aligned} & x_3 - \frac{1}{\sqrt{2}} y_3 = 0 \\ & \frac{-1}{\sqrt{2}} x_3 + y_3 - \frac{1}{\sqrt{2}} z_3 = 0 \\ & \frac{-1}{\sqrt{2}} y_3 + z_3 = 0 \end{aligned}\right\} \quad \boxed{V_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}} \qquad |V_3| = 2$$

Entonces $\boxed{M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ -1 & 1 & 1 \end{pmatrix}}$

En esta base de vectores, la matriz A será:

$$A_M = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -7 \end{pmatrix} \quad \text{Diagonal.}$$

Si normalizamos los vectores

$$\boxed{M = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}}$$

c) Tengo que hacer un cambio de variable que haga que "COSA" en las nuevas variables no tenga productos cruzados.

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}}_{M} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$\boxed{\begin{aligned} \phi_1 &= \psi_1/\sqrt{2} + \psi_2/2 + \psi_3/2 \\ \phi_2 &= -\sqrt{2}\,\psi_2/2 + \sqrt{2}\,\psi_3/2 \\ \phi_3 &= -\psi_1/\sqrt{2} + \psi_2/2 + \psi_3/2 \end{aligned}}$$

lo sustituimos en COSA

$$\mathrm{COSA} = -6\left(\frac{\psi_1}{\sqrt{2}} + \frac{\psi_2}{2} + \frac{\psi_3}{2}\right)^2 - 6\left(-\sqrt{2}\,\frac{\psi_2}{2} + \frac{\sqrt{2}}{2}\psi_3\right)^2$$

$$-6\left(\frac{-1}{\sqrt{2}}\psi_1 + \frac{1}{2}\psi_2 + \frac{1}{2}\psi_3\right)^2 - \sqrt{2}\left(\frac{\psi_1}{\sqrt{2}} + \frac{\psi_2}{2} + \frac{\psi_3}{2}\right)\left(\frac{-\sqrt{2}}{2}\psi_2 + \frac{\sqrt{2}}{2}\psi_3\right)$$

$$-\sqrt{2}\left(-\sqrt{2}\,\frac{\psi_2}{2} + \sqrt{2}\,\frac{\psi_3}{2}\right)\left(-\frac{\psi_1}{\sqrt{2}} + \frac{\psi_2}{2} + \frac{\psi_3}{2}\right)$$

$$= \frac{-6}{4}\left(2\psi_1^2 + \psi_2^2 + \psi_3^2 + \cancel{2\sqrt{2}\psi_1\psi_2} + 2\psi_2\psi_3 + \cancel{2\sqrt{2}\psi_1\psi_3}\right)$$

$$-6\left(\frac{1}{2}\psi_2^2 + \frac{1}{2}\psi_3^2 - \psi_2\psi_3\right)$$

$$-\frac{6}{4}\left(2\psi_1^2 + \psi_2^2 + \psi_3^2 - \cancel{2\sqrt{2}\psi_1\psi_2} - \cancel{2\sqrt{2}\psi_1\psi_3} + 2\psi_2\psi_3\right)$$

$$-\frac{\sqrt{2}}{4}\left(-\cancel{2\psi_1\psi_2} + \cancel{2\psi_1\psi_3} - \cancel{\sqrt{2}\psi_2^2} + \cancel{\sqrt{2}\psi_2\psi_3} - \cancel{\sqrt{2}\psi_2\psi_3} + \sqrt{2}\psi_3^2\right)$$

$$-\frac{\sqrt{2}}{4}\left(\cancel{2\psi_1\psi_2} + \cancel{\sqrt{2}\psi_2^2} - \sqrt{2}\psi_2\psi_3 - \cancel{2\psi_1\psi_3} + \cancel{\sqrt{2}\psi_2\psi_3} + \sqrt{2}\psi_3^2\right)$$

$$= \psi_1^2(-3-3) + \psi_2^2\left(\frac{-6}{4} - 3 - \frac{6}{4}\right) + \psi_3^2\left(\frac{-6}{4} - 3 - \frac{6}{4} - \frac{1}{2} - \frac{1}{2}\right)$$

$$+ \psi_2\psi_3(-3+6-3)$$

$$= \boxed{-6\psi_1^2 - 5\psi_2^2 - 7\psi_3^2} \qquad \text{c.q.d.}$$

λ_i