Ejercicio:

Dada la acción: $S = \frac{1}{2}\int d^4x \left[\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi\right]$

a) Demostrar que

$$L = \frac{1}{2}\left[\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2\right]$$

es invariante bajo la transformación:

$$x^{0\prime} = \gamma x^0 - \gamma \beta x^1$$
$$x^{1\prime} = -\gamma\beta x^0 + \gamma x^1$$
$$x^{2\prime} = x^2$$
$$x^{3\prime} = x^3$$

b) Calcular $\frac{\delta S}{\delta \phi}$.

a) Si L es invariante bajo la transformación, obtendremos que:

$$\frac{1}{2}\left[\partial_\mu \phi \cdot \partial^\mu \phi - m^2 \phi^2\right] = \frac{1}{2}\left[\partial_{\mu'} \phi\, \partial^{\mu'} \phi - m^2 \phi^2\right]$$

Por ello nos bastará demostrar que

$$\partial_\mu \phi\, \partial^\mu \phi = \partial_{\mu'} \phi\, \partial^{\mu'} \phi$$

$$\partial_\mu \phi\, \partial^\mu \phi \equiv \left(\partial_0 \partial^0 + \partial_1 \partial^1 + \partial_2 \partial^2 + \partial_3 \partial^3\right)\phi$$

Además sabemos que:

$$\partial_0 = \partial^0$$
$$\partial_1 = -\partial^1$$
$$\partial_2 = -\partial^2$$
$$\partial_3 = -\partial^3$$

Entonces:

$$(\partial_\mu \partial^\mu)\phi \equiv \left(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2\right)\phi$$

* $$\partial_0 \phi = \frac{\partial \phi}{\partial x^{0'}} \frac{\partial x^{0'}}{\partial x^0} + \frac{\partial \phi}{\partial x^{1'}} \frac{\partial x^{1'}}{\partial x^0} + \frac{\partial \phi}{\partial x^{2'}} \frac{\partial x^{2'}}{\partial x^0} + \frac{\partial \phi}{\partial x^{3'}} \frac{\partial x^{3'}}{\partial x^0}$$

$$= \partial_{0'}\phi\,(\gamma) + \partial_{1'}\phi\,(-\gamma\beta) + \cancel{\partial_{2'}\phi \cdot 0} + \cancel{\partial_{3'}\phi \cdot 0}$$

$$\boxed{\partial_0^2 \phi} = \left[\gamma\left(\partial_{0'} - \beta\,\partial_{1'}\right)\right]^2 \phi$$

$$= \boxed{\gamma^2\left(\partial_{0'}^2 + \beta^2 \partial_{1'}^2 - 2\beta\,\partial_{0'}\partial_{1'}\right)\phi}$$

* $$\partial_1 \phi = \frac{\partial \phi}{\partial x^{0'}} \frac{\partial x^{0'}}{\partial x^1} + \frac{\partial \phi}{\partial x^{1'}} \frac{\partial x^{1'}}{\partial x^1} + \frac{\partial \phi}{\partial x^{2'}} \frac{\partial x^{2'}}{\partial x^1} + \frac{\partial \phi}{\partial x^{3'}} \frac{\partial x^{3'}}{\partial x^1}$$

$$= \partial_{0'}\phi\,(-\gamma\beta) + \partial_{1'}\phi(\gamma)$$

$$\boxed{\partial_1^2 \phi} = \left[\gamma\left(-\beta\,\partial_{0'} + \partial_{1'}\right)\right]^2 \phi =$$

$$= \boxed{\gamma^2\left(\beta^2 \partial_{0'}^2 + \partial_{1'}^2 - 2\beta\,\partial_{0'}\partial_{1'}\right)\phi}$$

* $$\boxed{\partial_2^2 \phi} = \left(\frac{\partial \phi}{\partial x^{0'}} \cancel{\frac{\partial x^{0'}}{\partial x^2}} + \frac{\partial \phi}{\partial x^{1'}} \cancel{\frac{\partial x^{1'}}{\partial x^2}} + \frac{\partial \phi}{\partial x^{2'}} \frac{\partial x^{2'}}{\partial x^2} + \frac{\partial \phi}{\partial x^{3'}} \cancel{\frac{\partial x^{3'}}{\partial x^2}}\right)^2$$

$$= \boxed{\partial_{2'}^2 \phi}$$

$$* \quad \boxed{\partial_3^2 \phi = \partial_{3'}^2 \phi}$$

lo sustituimos:

$$\boxed{\left(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2\right)\phi =}$$

$$= \Big[\gamma^2\left(\partial_{0'}^2 + \beta^2 \partial_{1'}^2 - \cancel{2\beta\, \partial_{0'}\partial_{1'}}\right) - \gamma^2\left(\beta^2 \partial_{0'}^2 + \partial_{1'}^2 - \cancel{2\beta \partial_{0'}\partial_{1'}}\right) - \partial_{2'}^2 - \partial_{3'}^2\Big]\phi$$

$$= \left(\gamma^2 \partial_{0'}^2 + \gamma^2\beta^2 \partial_{1'}^2 - \gamma^2\beta^2\partial_{0'}^2 - \gamma^2\partial_{1'}^2 - \partial_{2'}^2 - \partial_{3'}^2\right)\phi$$

$$= \left[\left(\gamma^2 - \gamma^2\beta^2\right)\partial_{0'}^2 + \left(\gamma^2\beta^2 - \gamma^2\right)\partial_{1'}^2 - \partial_{2'}^2 - \partial_{3'}^2\right]\phi$$

Recordemos que: $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ $\quad \gamma^2 = \frac{1}{1-\beta^2}$

$$\boxed{\gamma^2 - \gamma^2 \cdot \beta^2 = \frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = \frac{1-\beta^2}{1-\beta^2} = 1}$$

$$\boxed{= \left[\partial_{0'}^2 - \partial_{1'}^2 - \partial_{2'}^2 - \partial_{3'}^2\right]\phi}$$

b)

$$\boxed{\frac{\delta S}{\delta \phi}} \equiv \frac{\partial L}{\partial \phi} - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \right) =$$

"Deshago" la notación compacta $= \dfrac{\partial L}{\partial \phi} - \left[\partial_0 \left(\dfrac{\partial L}{\partial(\partial_0 \phi)} \right) + \partial_1 \left(\dfrac{\partial L}{\partial(\partial_1 \phi)} \right) + \partial_2 \left(\dfrac{\partial L}{\partial(\partial_2 \phi)} \right) + \partial_3 \left(\dfrac{\partial L}{\partial(\partial_3 \phi)} \right) \right]$

$$L = \frac{1}{2} \left[\partial_0^2 \phi - \partial_1^2 \phi - \partial_2^2 \phi - \partial_3^2 \phi - m^2 \phi^2 \right]$$

$$= \frac{-1}{\cancel{2}} \cdot \cancel{2} m^2 \phi - \left(\partial_0 \left(\frac{1}{\cancel{2}} \cancel{2} \partial_0 \phi \right) + \partial_1 \left(\frac{-1}{\cancel{2}} \cdot \cancel{2}\, \partial_1 \phi \right) \right.$$

$$\left. + \partial_2 \left(- \partial_2 \phi \right) + \partial_3 \left(- \partial_3 \phi \right) \right]$$

$$= \boxed{- m^2 \phi - \left(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 \right) \phi}$$