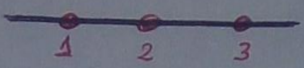
Teoria Cuantica de Campos

Ejercicio



$$\phi \rightarrow \phi_1 \ \phi_2 \ \phi_3$$

$$Cosa = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

Matriz Simétrica

a) $A = ?$

$$(\phi_1 \ \phi_2 \ \phi_3)\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & g \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = cosa.$$

$$(\phi_1 \ \phi_2 \ \phi_3)\begin{pmatrix} a\phi_1 + b\phi_2 + c\phi_3 \\ b\phi_1 + d\phi_2 + e\phi_3 \\ c\phi_1 + e\phi_2 + g\phi_3 \end{pmatrix} = \phi_1(a\phi_1 + b\phi_2 + c\phi_3) + \phi_2(b\phi_1 + d\phi_2 + e\phi_3) + \phi_3(c\phi_1 + e\phi_2 + g\phi_3)$$

$$= a\phi_1^2 + b\phi_1\phi_2 + c\phi_1\phi_3 + b\phi_1\phi_2 + d\phi_2^2 + e\phi_2\phi_3 + c\phi_1\phi_3 + e\phi_2\phi_3 + g\phi_3^2 = cosa.$$

$$= a\phi_1^2 + d\phi_2^2 + g\phi_3^2 + 2b\phi_1\phi_2 + 2c\phi_1\phi_3 + 2e\phi_2\phi_3 = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

$$a = -6$$
$$d = -6$$
$$g = -6$$
$$2b = -\sqrt{2} \Rightarrow b = -\sqrt{2}/2$$
$$2c = 0 \Rightarrow c = 0$$
$$2e = -\sqrt{2} \Rightarrow e = -\sqrt{2}/2$$

Por lo tanto la matriz A es

$$A = \begin{pmatrix} -6 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -6 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -6 \end{pmatrix}$$

$$A = \begin{pmatrix} -6 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -6 \end{pmatrix}$$

b) $M = ?$

$$\vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \qquad A\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$\begin{pmatrix} -6 & \frac{-1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & -6 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

$$\begin{pmatrix} -6\psi_1 - \frac{1}{\sqrt{2}}\psi_2 + 0\psi_3 \\ -\frac{1}{\sqrt{2}}\psi_1 - 6\psi_2 - \frac{1}{\sqrt{2}}\psi_3 \\ 0\psi_1 - \frac{1}{\sqrt{2}}\psi_2 - 6\psi_3 \end{pmatrix} = \begin{pmatrix} \lambda\psi_1 \\ \lambda\psi_2 \\ \lambda\psi_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -6\psi_1 - \frac{1}{\sqrt{2}}\psi_2 - \lambda\psi_1 \\ -\frac{1}{\sqrt{2}}\psi_1 - 6\psi_2 - \frac{1}{\sqrt{2}}\psi_3 - \lambda\psi_2 \\ 0\psi_1 - \frac{1}{\sqrt{2}}\psi_2 - 6\psi_3 - \lambda\psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (-6-\lambda)\psi_1 - \frac{1}{\sqrt{2}}\psi_2 \\ -\frac{1}{\sqrt{2}}\psi_1 + (-6-\lambda)\psi_2 - \frac{1}{\sqrt{2}}\psi_3 \\ -\frac{1}{\sqrt{2}}\psi_2 + (-6-\lambda)\psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Por lo tanto.

$$\left.\begin{aligned} (-6-\lambda)\psi_1 - \tfrac{1}{\sqrt{2}}\psi_2 &= 0 \\ -\tfrac{1}{\sqrt{2}}\psi_1 + (-6-\lambda)\psi_2 - \tfrac{1}{\sqrt{2}}\psi_3 &= 0 \\ -\tfrac{1}{\sqrt{2}}\psi_2 + (-6-\lambda)\psi_3 &= 0 \end{aligned}\right\} \dots (i)$$

El determinante.

$$\begin{vmatrix} -6-\lambda & \frac{-1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6-\lambda & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -6-\lambda \end{vmatrix} = (-6-\lambda)\begin{vmatrix} -6-\lambda & \frac{-1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -6-\lambda \end{vmatrix} + \frac{1}{\sqrt{2}}\begin{vmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -6-\lambda \end{vmatrix} + \cancel{0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}}$$

$$= (-6-\lambda)\left[(-6-\lambda)^2 - \tfrac{1}{2}\right] + \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}(6+\lambda)\right)$$

$\Rightarrow -(6-\lambda)^3 + 6 + \lambda = 0$

$\Rightarrow \quad -(6-\lambda)^3 = -(6+\lambda)$

$-(\lambda^3 + 18\lambda^2 + 108\lambda + 216) + 6 + \lambda = 0$

$-\lambda^3 - 18\lambda^2 - 107\lambda - 210 = 0$

$\lambda_1 = -$

$\lambda_2 = -$

$\lambda_3 =$

$\lambda_1 = -5$

$\lambda_2 = -6$

$\lambda_3 = -7$

Para $\lambda_1 = -5$, la ecuación (i) se tiene.

$$-\psi_1 - \frac{1}{\sqrt{2}}\psi_2 = 0 \quad \Rightarrow \psi_1 = -\frac{1}{\sqrt{2}}\psi_2$$

$$-\frac{1}{\sqrt{2}}\psi_1 - \psi_2 - \frac{1}{\sqrt{2}}\psi_3 = 0$$

$$-\frac{1}{\sqrt{2}}\psi_2 - \psi_3 = 0 \quad \Rightarrow \psi_2 = -\sqrt{2}\,\psi_3$$

$\rightarrow \psi_1 = -\frac{1}{\sqrt{2}}\left(-\sqrt{2}\,\psi_3\right)$

$\psi_1 = \psi_3 = \mathbf{1}$

$\psi_2 = -\sqrt{2}$

$\sqrt{1^2 + (-\sqrt{2})^2 + 1} = \sqrt{4} = 2$

$$\vec{\psi}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Para $\lambda_2 = -6$, la ecuación (i) se tiene:

$$-\frac{1}{\sqrt{2}}\psi_2 = 0 \quad \Rightarrow \psi_2 = 0$$

$$-\frac{1}{\sqrt{2}}\psi_1 - \frac{1}{\sqrt{2}}\psi_3 = 0 \quad \Rightarrow \psi_1 = -\psi_3$$

$$-\frac{1}{\sqrt{2}}\psi_2 = 0 \qquad \psi_1 = 1, \ \psi_3 = -1 \qquad \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\vec{\psi}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Para $\lambda = -7$, la ecuación (i) se tiene:

$$\psi_1 - \frac{1}{\sqrt{2}}\psi_2 = 0 \quad \Rightarrow \psi_1 = \frac{1}{\sqrt{2}}\psi_2$$

$$-\frac{1}{\sqrt{2}}\psi_1 + \psi_2 - \frac{1}{\sqrt{2}}\psi_3 = 0$$

$$-\frac{1}{\sqrt{2}}\psi_2 + \psi_3 = 0 \quad \Rightarrow \psi_3 = \frac{1}{\sqrt{2}}\psi_2$$

$\rightarrow \psi_1 = \frac{1}{\sqrt{2}}\left(\sqrt{2}\,\psi_3\right)$

$\psi_1 = \psi_3$

$\psi_1 = 1, \ \psi_3 = 1, \ \psi_2 = \sqrt{2}$

$\sqrt{1^2 + (\sqrt{2})^2 + (1)^2} = \sqrt{4} = 2$

$$\vec{\psi}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\vec{\psi}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \quad \vec{\psi}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{\psi}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{vmatrix} - \frac{1}{\sqrt{2}}\begin{vmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= \frac{1}{2}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{2\sqrt{2}}{4} + \frac{1}{2}\frac{1}{2} = \frac{1}{4} + \frac{1\cdot 2}{2\cdot 2} + \frac{1}{4} = \frac{1+2+1}{4} = \frac{4}{4} = 1 > 1$$:)

$$\vec{v} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \phi_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \phi_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \phi_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \psi_1\begin{pmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{pmatrix} + \psi_2\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} + \psi_3\begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1/2\,\psi_1 + 1/\sqrt{2}\,\psi_2 + 1/2\,\psi_3 \\ -\sqrt{2}/2\,\psi_1 + 0.\psi_2 + \sqrt{2}/2\,\psi_3 \\ 1/2\,\psi_1 - 1/\sqrt{2}\,\psi_2 + 1/2\,\psi_3 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \quad ; \quad M = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}$$

c) Mostar que: $Cosa = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$

- $\phi_1^2 = \left(\frac{1}{2}\psi_1 + \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3\right)^2 = \frac{1}{4}\psi_1^2 + \frac{1}{2}\psi_2^2 + \frac{1}{4}\psi_3^2 + \frac{1}{\sqrt{2}}\psi_1\psi_2 + \frac{1}{2}\psi_1\psi_3 + \frac{1}{\sqrt{2}}\psi_2\psi_3$
- $\phi_2^2 = \left(-\frac{\sqrt{2}}{2}\psi_1 + \frac{\sqrt{2}}{2}\psi_3\right)^2 = \frac{1}{2}\psi_1^2 - \frac{1}{2}\psi_1\psi_3 + \frac{1}{2}\psi_3^2$
- $\phi_3^2 = \left(\frac{1}{2}\psi_1 - \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3\right)^2 = \frac{1}{4}\psi_1^2 + \frac{1}{2}\psi_2^2 + \frac{1}{4}\psi_3^2 - \frac{1}{\sqrt{2}}\psi_1\psi_2 + \frac{1}{2}\psi_1\psi_3 - \frac{1}{\sqrt{2}}\psi_2\psi_3$
- $\phi_1\phi_2 = \left(\frac{1}{2}\psi_1 + \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3\right)\left(-\frac{\sqrt{2}}{2}\psi_1 + \frac{\sqrt{2}}{2}\psi_3\right) = -\frac{\sqrt{2}}{4}\psi_1^2 + \frac{\sqrt{2}}{4}\psi_3^2 - \frac{1}{2}\psi_1\psi_2 + \frac{1}{2}\psi_2\psi_3$
- $\phi_2\phi_3 = \left(-\frac{\sqrt{2}}{2}\psi_1 + \frac{\sqrt{2}}{2}\psi_3\right)\left(\frac{1}{2}\psi_1 - \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3\right) = -\frac{\sqrt{2}}{4}\psi_1^2 + \frac{\sqrt{2}}{4}\psi_3^2 + \frac{1}{2}\psi_1\psi_2 - \frac{1}{2}\psi_2\psi_3$

$$
\begin{aligned}
Cos\,a = {} & -6\left(\frac{1}{4}\psi_1^2+\frac{1}{2}\psi_2^2+\frac{1}{4}\psi_2^3+\frac{1}{\sqrt{2}}\psi_1\psi_2+\frac{1}{2}\psi_1\psi_3+\frac{1}{\sqrt{2}}\psi_2\psi_3\right)-6\left(\frac{1}{2}\psi_1^2-\frac{1}{2}\psi_1\psi_3+\frac{1}{2}\psi_3^2\right) \\
& -6\left(\frac{1}{4}\psi_1^2+\frac{1}{2}\psi_2^2+\frac{1}{4}\psi_3^2-\frac{1}{\sqrt{2}}\psi_1\psi_2+\frac{1}{2}\psi_1\psi_3-\frac{1}{\sqrt{2}}\psi_2\psi_3\right)-\sqrt{2}\left(-\frac{\sqrt{2}}{4}\psi_1^2+\frac{\sqrt{2}}{4}\psi_3^2-\frac{1}{2}\psi_1\psi_2+\frac{1}{2}\psi_2\psi_3\right) \\
& -\sqrt{2}\left(\frac{-\sqrt{2}}{4}\psi_1^2+\frac{\sqrt{2}}{4}\psi_3^2+\frac{1}{2}\psi_1\psi_2-\frac{1}{2}\psi_2\psi_3\right)
\end{aligned}
$$

$$
\begin{aligned}
Cos\,a = {} & \psi_1^2\left(-\frac{6}{4}-\frac{6}{2}-\frac{6}{4}+\frac{2}{4}+\frac{2}{4}\right)+\psi_2^2\left(-\frac{6}{2}-\frac{6}{2}\right)+\psi_3^2\left(-\frac{6}{4}-\frac{6}{4}-\frac{6}{4}-\frac{2}{4}-\frac{2}{4}\right) \\
& +\psi_1\psi_2\left(\frac{-6}{\sqrt{2}}+\frac{6}{\sqrt{2}}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)+\psi_1\psi_3\left(-\frac{6}{2}+6-\frac{6}{2}\right)+\psi_2\psi_3\left(\frac{-6}{\sqrt{2}}+\frac{6}{\sqrt{2}}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)
\end{aligned}
$$

$$
Cos\,a = -5\psi_1^2-6\psi_2^2-7\psi_3^2
$$