Ejercicios Teoría Cuántica de Campos
Capítulo 43

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Partimos de

$$u_1 = S[\Lambda]\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ch\frac{\eta}{2} \\ sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_+ \qquad u_2 = S[\Lambda]\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ch\frac{\eta}{2} \\ -sh\frac{\eta}{2} \end{pmatrix} \otimes \chi$$

$$v_1 = S[\Lambda]\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} sh\frac{\eta}{2} \\ ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_+ \qquad v_2 = S[\Lambda]\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -sh\frac{\eta}{2} \\ ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_-$$

con

$$\chi_+ = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

y

$$\chi_+^\dagger = \begin{pmatrix} \cos\theta/2 & e^{-i\phi}\sin\theta/2 \end{pmatrix} \qquad \chi_-^\dagger = \begin{pmatrix} -e^{i\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

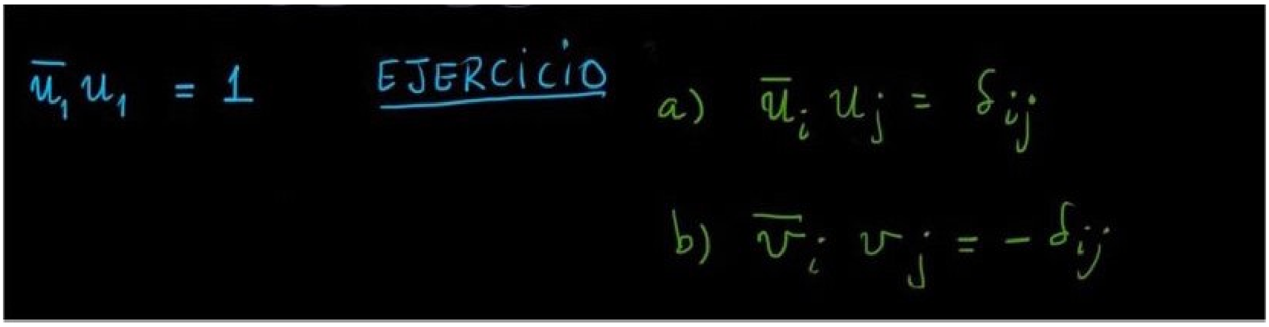
y consecuentemente

$$u_i^\dagger = \begin{pmatrix} ch\frac{\eta}{2} & \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger \qquad v_i^\dagger = \begin{pmatrix} \pm sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger$$

con el signo superior si $i = 1$ y el inferior si $i = 2$
Obtendremos $\overline{u_1}$ $\overline{u_2}$ $\overline{v_1}$ y $\overline{v_2}$ multiplicando $u_1^\dagger$ $u_2^\dagger$ $v_1^\dagger$ $v_2^\dagger$ por $\gamma^0 = \sigma^3 \otimes I$

Se han planteado 4 ejercicios que pasamos a intentar resolver.

EJERCICIO 1



Si $i = j$

$\overline{u_i}u_i = \left[\begin{pmatrix} ch\frac{\eta}{2} & \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger\right] \left[\sigma^3 \otimes I\right] \left[\begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm\right] \qquad \text{con } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

y para $i \neq j$

$\overline{u_i}u_j = \left[\begin{pmatrix} ch\frac{\eta}{2} & \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger\right] \left[\sigma^3 \otimes I\right] \left[\begin{pmatrix} ch\frac{\eta}{2} \\ \mp sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\mp\right]$

Ya que $(C \otimes D)(F \otimes G) = (CF) \otimes (DG)$ también $(A \otimes B)(C \otimes D)(F \otimes G) = [A(CF)] \otimes [B(DG)]$

Calculemos los productos de los segundos factores de los productos directos

$$(\chi_+^\dagger I \chi_+) = \begin{pmatrix} \cos\theta/2 & e^{-i\phi}\sin\theta/2 \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} = (\cos^2\theta/2 + \sin^2\theta/2) = 1$$

$$(\chi_-^\dagger I \chi_-) = \begin{pmatrix} -e^{i\phi}\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} = (\sin^2\theta/2 + \cos^2\theta/2) = 1$$

$$\chi_+^\dagger I \chi_- = \begin{pmatrix} \cos\theta/2 & e^{-i\phi}\sin\theta/2 \end{pmatrix} \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} = (-e^{-i\phi}\sin\theta/2\cos\theta/2 + e^{-i\phi}\sin\theta/2\cos\theta/2) = 0$$

$$\chi_-^\dagger I \chi_+ = \begin{pmatrix} \cos\theta/2 & -e^{i\phi}\sin\theta/2 \end{pmatrix} \begin{pmatrix} e^{i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} = (e^{-i\phi}\sin\theta/2\cos\theta/2 - e^{-i\phi}\sin\theta/2\cos\theta/2) = 0$$

lo que corrobora que $\chi_\pm^\dagger \chi_\pm =< \pm \mid \pm >= 1$ y que $\chi_\pm^\dagger \chi_\mp =< \pm \mid \mp >= 0$

Vemos así que si $i \neq j \rightarrow \overline{u_i}u_j = 0$ independientemente del producto de los primeros factores.

Para el caso $i = j$ calculamos los productos de los 3 primeros factores

Empezamos por los dos ultimos

$\sigma^3 \begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix} = \begin{pmatrix} ch\frac{\eta}{2} \\ \mp sh\frac{\eta}{2} \end{pmatrix}$

y ahora multiplicando por el primero

$\begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix}^\dagger \begin{pmatrix} ch\frac{\eta}{2} \\ \mp sh\frac{\eta}{2} \end{pmatrix} = \begin{pmatrix} ch\frac{\eta}{2} & \pm sh\frac{\eta}{2} \end{pmatrix} \begin{pmatrix} ch\frac{\eta}{2} \\ \mp sh\frac{\eta}{2} \end{pmatrix} = (ch^2\frac{\eta}{2} - sh^2\frac{\eta}{2}) = 1$

O sea para $i = j \rightarrow \overline{u_i}u_i = 1 \otimes 1 = 1$ (estos 1 ya son simples números)

Reuniendo ambos casos

$$\overline{u_i}u_j = \delta_{ij}$$

Para las v solo se cambia el sh por ch y viceversa por lo que al calcular los ultimos productos en lugar de $(ch^2\frac{\eta}{2} - sh^2\frac{\eta}{2}) = 1$ saldrá $(sh^2\frac{\eta}{2} - ch^2\frac{\eta}{2}) = -1$

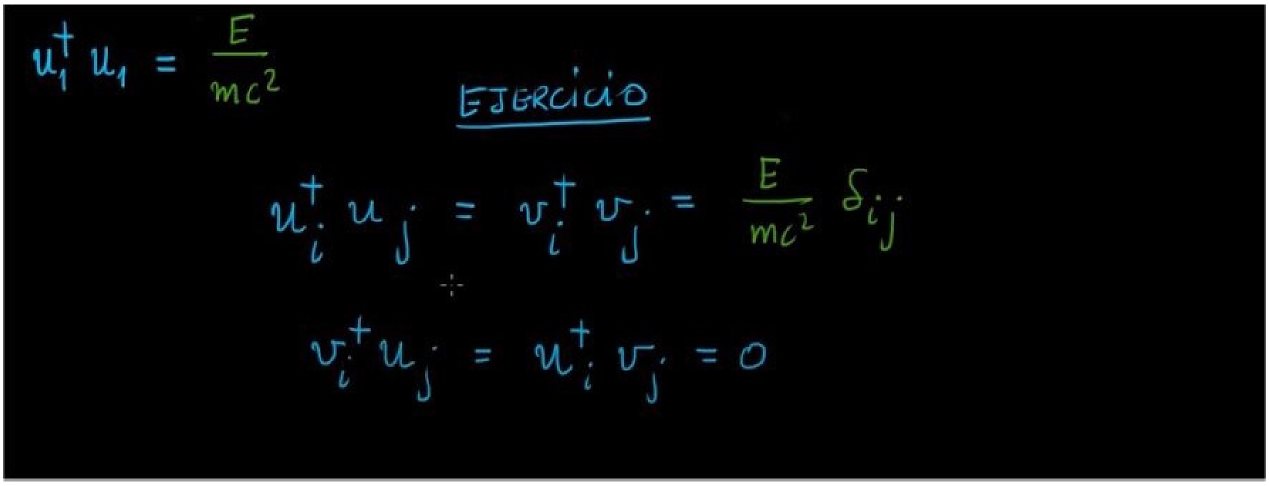
$\sigma^3 \begin{pmatrix} sh\frac{\eta}{2} \\ \pm ch\frac{\eta}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} sh\frac{\eta}{2} \\ \pm ch\frac{\eta}{2} \end{pmatrix} = \begin{pmatrix} sh\frac{\eta}{2} \\ \mp ch\frac{\eta}{2} \end{pmatrix}$

$\begin{pmatrix} sh\frac{\eta}{2} \\ \pm ch\frac{\eta}{2} \end{pmatrix}^\dagger \begin{pmatrix} sh\frac{\eta}{2} \\ \mp ch\frac{\eta}{2} \end{pmatrix} = (sh^2\frac{\eta}{2} - ch^2\frac{\eta}{2}) = -1$

y por tanto

$$\overline{v_i} v_j = -\delta_{ij}$$

EJERCICIO 2



Si $i = j$

$u_i^\dagger u_i = \left[\begin{pmatrix} ch\frac{\eta}{2} & \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger \right] \left[\begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm \right] = (ch^2\frac{\eta}{2} + sh^2\frac{\eta}{2}) \otimes (\chi_\pm^\dagger \chi_\pm)$

y para $i \neq j$

$u_i^\dagger u_j = \left[\begin{pmatrix} ch\frac{\eta}{2} & \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger \right] \left[\begin{pmatrix} ch\frac{\eta}{2} \\ \mp sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\mp \right] = (ch^2\frac{\eta}{2} - sh^2\frac{\eta}{2}) \otimes (\chi_\pm^\dagger \chi_\mp) = 1 \otimes (\chi_\pm^\dagger \chi_\mp)$

Calculemos los productos $(\chi_\pm^\dagger \chi_\pm)$ y $(\chi_\pm^\dagger \chi_\mp)$

$$(\chi_+^\dagger \chi_+) =<+|+>= 1 \; ; \; (\chi_-^\dagger \chi_-) =<-|->= 1$$
$$(\chi_+^\dagger \chi_-) =<+|->= 0 \; ; \; (\chi_-^\dagger \chi_+) =<-|+>= 0$$

ya solo necesitamos calcular $(ch^2\frac{\eta}{2} + sh^2\frac{\eta}{2})$. Aplicaremos $ch^2\frac{\eta}{2} = \frac{1+\cosh\eta}{2}$; $sh^2\frac{\eta}{2} = \frac{-1+\cosh\eta}{2}$

$$(ch^2\frac{\eta}{2} + sh^2\frac{\eta}{2}) = \frac{1+\cosh\eta}{2} + \frac{-1+\cosh\eta}{2} = \cosh\eta$$

y recordando (gracias a Javier) que $p^o = mc{\cdot}\cosh\eta$ y $p^o = E/c$ obtenemos finalmente $\cosh\eta = E/mc^2$

Reuniendo resultados

si $i=j \rightarrow \;\; u_i^\dagger u_i = E/mc^2 \otimes 1 = E/mc^2$ y si $i \neq j \rightarrow \;\; u_i^\dagger u_j = 1 \otimes 0 = 0$
Resumiendo

$$\boxed{\mathrm{u}_i^\dagger \mathrm{u}_j = \delta_{ij} \mathrm{E/mc}^2}$$

Para los v_i vale el mismo razonamiento que hicimos en el ejercicio 1: Ya que solo cambian sinh por cosh y viceversa en los productos de los primeros factores apareceran

si $i=j \;\; (sh^2\frac{\eta}{2} + ch^2\frac{\eta}{2})$ que sigue siendo igual a E/mc^2 y si $i \neq j \;\; (sh^2\frac{\eta}{2} - ch^2\frac{\eta}{2}) = -1$
por lo que

si $i=j \rightarrow \;\; v_i^\dagger v_i = E/mc^2 \otimes 1 = E/mc^2$ y si $i \neq j \rightarrow \;\; v_i^\dagger v_j = -1 \otimes 0 = 0$
O sea

$$\boxed{\mathrm{v}_i^\dagger \mathrm{v}_j = \delta_{ij} \mathrm{E/mc}^2}$$

Para los productos cruzados $u_i^\dagger(-\overrightarrow{p})v_j(\overrightarrow{p})$ tendremos, puesto que cambiar de $\overrightarrow{p}$ a $-\overrightarrow{p}$ se puede hacer

cambiando η por $-\eta$ y eso solo nos cambia el signo del sinh
para $i=j$ resulta cero el producto de los primeros factores

en $u_i^\dagger v_i \rightarrow \begin{pmatrix} ch\frac{\eta}{2} & \mp sh\frac{\eta}{2} \end{pmatrix} \begin{pmatrix} \pm sh\frac{\eta}{2} \\ ch\frac{\eta}{2} \end{pmatrix} = (\mp ch\frac{\eta}{2} sh\frac{\eta}{2} \pm sh\frac{\eta}{2} ch\frac{\eta}{2}) = 0$

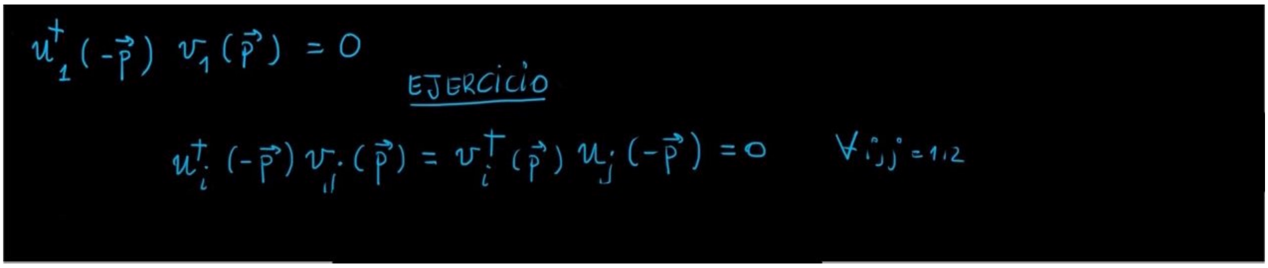
y para $i \neq j$ lo que es cero es el producto de los segundos
en $u_i^\dagger v_j \rightarrow \left[\chi_\pm^\dagger\right] \left[\chi_\mp\right] = <\pm \mid \mp> = 0$

Si ahora hacemos los $v_i^\dagger u_j$ se repiten los razonamientos cambiando sinh por cosh y viceversa. por lo que da los mismos resultados
para $i=j$ en $v_i^\dagger u_i \rightarrow \begin{pmatrix} \mp sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix} \begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix} = (\mp sh\frac{\eta}{2} ch\frac{\eta}{2} \pm ch\frac{\eta}{2} sh\frac{\eta}{2}) = 0$
y para $i \neq j$ en $u_i^\dagger v_j \rightarrow \left[\chi_\pm^\dagger\right] \left[\chi_\mp\right] = <\pm \mid \mp> = 0$

En resumen

$$\boxed{\mathrm{u}_i^\dagger(-\overrightarrow{p})\mathrm{v}_j(\overrightarrow{p}) = \mathrm{v}_i^\dagger(-\overrightarrow{p})\mathrm{u}_j(\overrightarrow{p}) = 0}$$

EJERCICIO 3



La primera parte ya esta demostrada en el ejercicio 2

Pasemos ahora a $v_i^\dagger(\overrightarrow{p})u_j(-\overrightarrow{p})$

Si $i=j$ y ahora el que cambia η por $-\eta$ es el segundo factor
$v_i^\dagger(\overrightarrow{p})u_i(-\overrightarrow{p}) = \left[\begin{pmatrix} \pm sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger\right] \left[\begin{pmatrix} ch\frac{\eta}{2} \\ \mp sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm\right] =$

$$v_i^\dagger(\overrightarrow{p})u_i(-\overrightarrow{p}) = \left[\begin{pmatrix} \pm sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix}\begin{pmatrix} ch\frac{\eta}{2} \\ \mp sh\frac{\eta}{2} \end{pmatrix}\right] \otimes \left[\chi_\pm^\dagger \chi_\pm\right] = (\pm sh\tfrac{\eta}{2}ch\tfrac{\eta}{2} \mp ch\tfrac{\eta}{2} \mp sh\tfrac{\eta}{2}) \otimes (\chi_\pm^\dagger \chi_\pm)$$

resultando

$$v_i^\dagger(\overrightarrow{p})u_i(-\overrightarrow{p}) = 0 \otimes <\pm \mid \pm>= 0$$

Si $i \neq j$

$$v_i^\dagger(\overrightarrow{p})u_j(-\overrightarrow{p}) = \left[\begin{pmatrix} \pm sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger\right]\left[\begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm\right]$$

$$v_i^\dagger(\overrightarrow{p})u_j(-\overrightarrow{p}) = \left[\begin{pmatrix} \pm sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_\pm^\dagger\right]\left[\begin{pmatrix} ch\frac{\eta}{2} \\ \pm sh\frac{\eta}{2} \end{pmatrix} \otimes \chi_\mp\right] = (\pm sh\tfrac{\eta}{2}ch\tfrac{\eta}{2} \pm ch\tfrac{\eta}{2}sh\tfrac{\eta}{2}) \otimes (\chi_\pm^\dagger \chi_\mp)$$

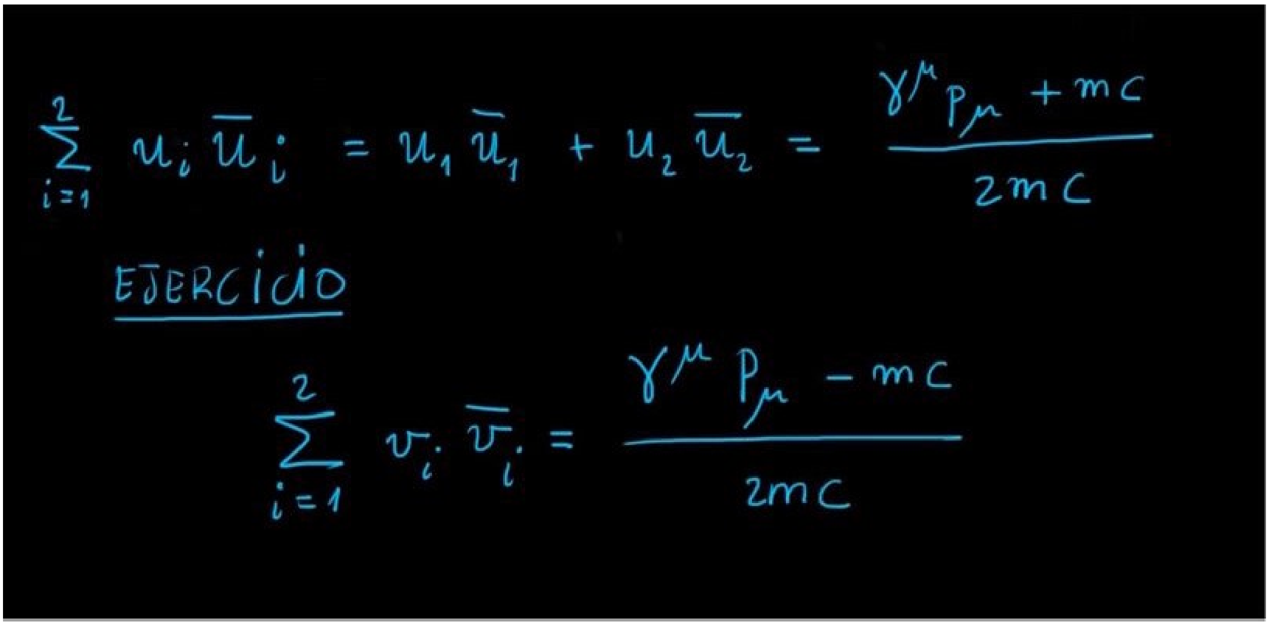
es decir

$$v_i^\dagger(\overrightarrow{p})u_j(-\overrightarrow{p}) = (\pm 2sh\frac{\eta}{2}ch\frac{\eta}{2}) \otimes 0 = 0$$

Es decir en ambos casos da cero

$$\boxed{\mathrm{u}_i^\dagger(\overrightarrow{p})\mathrm{v}_j(\text{-}\overrightarrow{p})=\mathrm{v}_i^\dagger(\overrightarrow{p})\mathrm{u}_j(\text{-}\overrightarrow{p})=0}$$

EJERCICIO 4



Recordando que obtendremos $\overline{v_1}$ y $\overline{v_2}$ multiplicando $v_1^\dagger$ $v_2^\dagger$ por $\gamma^0 = \sigma^3 \otimes I_2$

$$\sum_{i=1}^{2} v_i\overline{v_i} = \left[\begin{pmatrix} sh\frac{\eta}{2} \\ ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_+\right]\left[\begin{pmatrix} sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_+^\dagger\right]\gamma^0 + \left[\begin{pmatrix} -sh\frac{\eta}{2} \\ ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_-\right]\left[\begin{pmatrix} -sh\frac{\eta}{2} & ch\frac{\eta}{2} \end{pmatrix} \otimes \chi_-^\dagger\right]\gamma^0$$

$$\sum_{i=1}^{2} v_i\overline{v_i} = \left[\begin{pmatrix} sh^2\frac{\eta}{2} & sh\frac{\eta}{2}ch\frac{\eta}{2} \\ sh\frac{\eta}{2}ch\frac{\eta}{2} & ch^2\frac{\eta}{2} \end{pmatrix} \otimes \left[\chi_+\chi_+^\dagger\right] + \begin{pmatrix} sh^2\frac{\eta}{2} & -sh\frac{\eta}{2}ch\frac{\eta}{2} \\ -sh\frac{\eta}{2}ch\frac{\eta}{2} & ch^2\frac{\eta}{2} \end{pmatrix} \otimes \left[\chi_-\chi_-^\dagger\right]\right]\gamma^0$$

$$\sum_{i=1}^{2} v_i\overline{v_i} = \begin{pmatrix} sh^2\frac{\eta}{2}(\chi_+\chi_+^\dagger + \chi_-\chi_-^\dagger) & sh\frac{\eta}{2}ch\frac{\eta}{2}(\chi_+\chi_+^\dagger - \chi_-\chi_-^\dagger) \\ sh\frac{\eta}{2}ch\frac{\eta}{2}(\chi_+\chi_+^\dagger - \chi_-\chi_-^\dagger) & ch^2\frac{\eta}{2}(\chi_+\chi_+^\dagger + \chi_-\chi_-^\dagger) \end{pmatrix}\gamma^0$$

$$\sum_{i=1}^{2} v_i\overline{v_i} = \begin{pmatrix} sh^2\frac{\eta}{2}I_2 & sh\frac{\eta}{2}ch\frac{\eta}{2}(\chi_+\chi_+^\dagger - \chi_-\chi_-^\dagger) \\ sh\frac{\eta}{2}ch\frac{\eta}{2}(\chi_+\chi_+^\dagger - \chi_-\chi_-^\dagger) & ch^2\frac{\eta}{2}I_2 \end{pmatrix}\gamma^0$$

Como se recordó en el vídeo $(\chi_+\chi_+^\dagger - \chi_-\chi_-^\dagger)$ es una matriz cuyos vectores propios son χ_+ y χ_- con valores propios respectivos 1 y -1 por lo que resulta ser

$$(\chi_+\chi_+^\dagger - \chi_-\chi_-^\dagger) = \overrightarrow{\sigma}\cdot\overrightarrow{n}$$

Queda entonces

$$\sum_{i=1}^{2} v_i\overline{v_i} = \begin{pmatrix} sh^2\frac{\eta}{2}I_2 & sh\frac{\eta}{2}ch\frac{\eta}{2}\overrightarrow{\sigma}\cdot\overrightarrow{n} \\ sh\frac{\eta}{2}ch\frac{\eta}{2}\overrightarrow{\sigma}\cdot\overrightarrow{n} & ch^2\frac{\eta}{2}I_2 \end{pmatrix}\gamma^0$$

puesto que $\sinh^2\frac{\eta}{2} = \frac{-1+\cosh\eta}{2}$; $\cosh^2 = \frac{1+\cosh\eta}{2}$ y $sh\frac{\eta}{2}ch\frac{\eta}{2} = \frac{\sinh\eta}{2}$

$$\sum_{i=1}^{2} v_i\overline{v_i} = \begin{pmatrix} \frac{-1+\cosh\eta}{2}I_2 & \frac{\sinh\eta}{2}\overrightarrow{\sigma}\cdot\overrightarrow{n} \\ \frac{\sinh\eta}{2}\cdot\overrightarrow{n} & \frac{1+\cosh\eta}{2}I_2 \end{pmatrix}\gamma^0 = \frac{1}{2}\begin{pmatrix} (-1+\cosh\eta)I_2 & (\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} \\ (\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} & (1+\cosh\eta)I_2 \end{pmatrix}\gamma^0$$

$$\sum_{i=1}^{2} v_i\overline{v_i} = \frac{1}{2}\begin{pmatrix} (-1+\cosh\eta)I_2 & (\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} \\ (\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} & (1+\cosh\eta)I_2 \end{pmatrix}\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} =$$

$$\sum_{i=1}^{2} v_i\overline{v_i} = \begin{pmatrix} (-1+\cosh\eta)I_2 & -(\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} \\ (\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} & (-1-\cosh\eta)I_2 \end{pmatrix}$$

Comparando lo anterior con el resultado intermedio del cálculo de $\sum_{i=1}^{2} u_i\overline{u_i}$

$$\sum_{i=1}^{2} u_i\overline{u_i} = \begin{pmatrix} (1+\cosh\eta)I_2 & -(\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} \\ (\sinh\eta)\overrightarrow{\sigma}\cdot\overrightarrow{n} & (1-\cosh\eta)I_2 \end{pmatrix}$$

vemos que la diferencia esta en los -1 de los términos diagonales, lo que nos lleva a que en vez de $\sum_{i=1}^{2} u_i\overline{u_i} = \frac{1}{2}I_4 + \frac{1}{2}\gamma^0 p^0/mc - ...$ aparezca $\sum_{i=1}^{2} v_i\overline{v_i} = -\frac{1}{2}I_4 + ...$

como ese primer término de la suma es el que acababa dando lugar al $+mc$ en el resultado final

Tendremos $\sum_{i=1}^{2} v_i\overline{v_i} = \frac{1}{mc}(-mc + \gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3) = \frac{1}{mc}(-mc + \gamma^\mu p_\mu)$

o sea

$$\sum_{i=1}^{2} v_i\overline{v}_i = \frac{\gamma^\mu p_\mu - mc}{mc}$$