Curso, Teoria Cuantica de Campos ,by:Javier Garcia, ejercicio realizado por A.M.V

Ejercicio 1.

$$Cosa = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

Apartado “(a)”

Hallar (A) tal que,

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} ...A... \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = Cosa = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$

Para hallar A, definimos primero su dimensionalidad, que sera 3x3 por propiedades del producto de matrices.

Por tanto

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Hay que tener en cuenta que trabajaremos con matrices simetricas,

Entonces

$$A = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

Realizamos el producto

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} =$$

$$= \begin{pmatrix} (\phi_1 a + \phi_2 d + \phi_3 e) & (\phi_1 d + \phi_2 b + \phi_3 f) & (\phi_1 e + \phi_2 f + \phi_3 c) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} =$$

$$= (\phi_1^2 a + \phi_1\phi_2 d + \phi_3\phi_1 e) + (\phi_1\phi_2 d + \phi_2^2 b + \phi_2\phi_3 f) + (\phi_1\phi_3 e + \phi_2\phi_3 f + \phi_3^2 c)$$

$$= \phi_1^2 a + \phi_2^2 b + \phi_3^2 c + 2\phi_2\phi_3 f + 2\phi_1\phi_2 d + 2\phi_1\phi_3 e$$

Que ha de ser igual a Cosa

$$Cosa = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3 = \phi_1^2 a + \phi_2^2 b + \phi_3^2 c + 2f\phi_2\phi_3 + 2d\phi_1\phi_2 + 2e\phi_1\phi_3$$

Por tanto,

$$a = b = c = -6$$

$$e = 0$$

$$2f = 2d = -\sqrt{2} \implies f = d = \frac{-\sqrt{2}}{2}$$

$$A = \begin{pmatrix} -6 & \frac{-\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & -6 & \frac{-\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & -6 \end{pmatrix}$$

Apartado "(b)"

Diagonalizar A.

$$Ax = \begin{pmatrix} -6 & \frac{-\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & -6 & \frac{-\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & -6 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = \lambda \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}$$

$$\begin{pmatrix} -6 & \frac{-\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & -6 & \frac{-\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & -6 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} - \lambda \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = 0 = \begin{pmatrix} (-6x1 - \lambda x1)+ & \frac{-\sqrt{2}}{2}x2 & +0 \\ \frac{-\sqrt{2}}{2}x1+ & (-6x2 - \lambda x2) & +\frac{-\sqrt{2}}{2}x3 \\ 0+ & \frac{-\sqrt{2}}{2}x2 & +(-6x3 - \lambda x3) \end{pmatrix}$$

$$(A - I\lambda)x = \begin{pmatrix} -6-\lambda & \frac{-\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & -6-\lambda & \frac{-\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & -6-\lambda \end{pmatrix} x$$

El determinante ha de ser cero,

$$\begin{vmatrix} -6 & \frac{-\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & -6 & \frac{-\sqrt{2}}{2} \\ 0 & \frac{-\sqrt{2}}{2} & -6 \end{vmatrix} = (-6-\lambda)((-6-\lambda)^2 - \frac{1}{2}) - \frac{1}{2}(-6-\lambda) = -\lambda^3 - 18\lambda^2 - 35\lambda - 210 = 0$$

Hallamos los ceros del polinomio caracteristico.

$$\lambda_1 = -6$$

$$\lambda_2 = -5$$

$$\lambda_3 = -7$$

Hallamos los autovectores.
Solucionamos con el metodo de gauss. , La matriz 3x4 es la matriz extendida.
V1

$$(A - I\lambda_1)x = 0 \implies \begin{pmatrix} 0 & \frac{-\sqrt{2}}{2} & 0 & 0 \\ \frac{-\sqrt{2}}{2} & 0 & \frac{-\sqrt{2}}{2} & 0 \\ 0 & \frac{-\sqrt{2}}{2} & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies$$

$$X_{1General} = \begin{pmatrix} -x3 \\ 0 \\ x3 \end{pmatrix} = \delta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \implies X_{1Particular} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

V2

$$(A - I\lambda_2)x = 0 \implies \begin{pmatrix} -1 & \frac{-\sqrt{2}}{2} & 0 & 0 \\ \frac{-\sqrt{2}}{2} & -1 & \frac{-\sqrt{2}}{2} & 0 \\ 0 & \frac{-\sqrt{2}}{2} & -1 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies$$

$$X_{2General} = \begin{pmatrix} x3 \\ -\sqrt{2}x3 \\ x3 \end{pmatrix} = \delta \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \implies X_{2Particular} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

V3

$$(A - I\lambda_3)x = 0 \implies \begin{pmatrix} -2 & \frac{-\sqrt{2}}{2} & 0 & 0 \\ \frac{-\sqrt{2}}{2} & -2 & \frac{-\sqrt{2}}{2} & 0 \\ 0 & \frac{-\sqrt{2}}{2} & -2 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies$$

$$X_{3General} = \begin{pmatrix} x3 \\ \sqrt{2}x3 \\ x3 \end{pmatrix} = \delta \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \implies X_{3Particular} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Por tanto
Nuestra matriz diagonal

$$D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -7 \end{pmatrix}$$

La Matriz M(de autovectores)

$$M = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \neq M^T$$

Hay que ortogonalizarla.
Seguimos el proceso de GRAM-SCHMIDT.
Entonces

$$M \perp= \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Y ahora.

$$M^{-1=}M^T = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}$$

Apartado "(c)"

Mostrar que

$$Cosa = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$$

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \Longrightarrow \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \end{pmatrix}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} & \\ ..M.. \\ & \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

Podemos ver lo siguiente.

$$Recordamos \rightarrow (Mx)^T = x^T M^T$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} & \\ ..M.. \\ & \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \Longrightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}^T = (\begin{pmatrix} & \\ ..M.. \\ & \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix})^T = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \end{pmatrix} \begin{pmatrix} & \\ ..M.. \\ & \end{pmatrix}^T$$

Por tanto

$$\begin{pmatrix}\psi_1 & \psi_2 & \psi_3\end{pmatrix}\begin{pmatrix} & & \\ & ..D.. & \\ & & \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}\psi_1 & \psi_2 & \psi_3\end{pmatrix}\begin{pmatrix} & & \\ & ..M.. & \\ & & \end{pmatrix}^T\begin{pmatrix} & & \\ & ..A.. & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & ..M.. & \\ & & \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} =$$

$$= \begin{pmatrix}\phi_1 & \phi_2 & \phi_3\end{pmatrix}\begin{pmatrix} & & \\ & ..A.. & \\ & & \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

Explicitamente:

$$\begin{pmatrix}\psi_1 & \psi_2 & \psi_3\end{pmatrix}\begin{pmatrix}-6 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -7\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = Cosa = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2 =$$

$$= \begin{pmatrix}\phi_1 & \phi_2 & \phi_3\end{pmatrix}\begin{pmatrix}-\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2}\\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-6 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -7\end{pmatrix}\begin{pmatrix}-\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2}\\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} =$$

$$= \begin{pmatrix}\phi_1 & \phi_2 & \phi_3\end{pmatrix}\begin{pmatrix}-6 & \frac{-\sqrt{2}}{2} & 0\\ \frac{-\sqrt{2}}{2} & -6 & \frac{-\sqrt{2}}{2}\\ 0 & \frac{-\sqrt{2}}{2} & -6\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} =$$

$$= Cosa = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$$