Calcular el valor esperado: $\langle \phi_a \phi_b \phi_c \phi_d \rangle$

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle = \frac{1}{Z[0]} \left. \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \frac{\partial}{\partial J_c} \frac{\partial}{\partial J_d} \cdot Z[J] \right|_{J=0}$$

Tenemos que $Z[J] = \exp\left[\frac{1}{2m^2} J^T A^{-1} J\right] \cdot \frac{(\sqrt{2\pi})^n}{m^n \sqrt{|A|}}$

$$\text{y} \quad Z[0] = \frac{(\sqrt{2\pi})^n}{m^n \sqrt{|A|}}$$

Igual que ocurría al calcular el valor esperado $\langle \phi_a \phi_b \rangle$, cada vez que derivemos $Z[J]$ nos saldrá multiplicando como constante un término que coincide con $Z[0]$. Al salir de la derivada como constante se anula con el denominador.

El cálculo se va a reducir a:

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle = \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \frac{\partial}{\partial J_c} \frac{\partial}{\partial J_d} \exp\left[\frac{1}{2m^2} J^T A^{-1} J\right]$$

Recordemos:

$$\frac{1}{2m^2} J^T A^{-1} J = \frac{1}{2m^2} \left(A_{11}^{-1} J_1 J_1 + A_{12}^{-1} J_1 J_2 + \dots \right)$$

$$= \underbrace{a_{11}}_{\frac{A_{11}^{-1}}{2m^2}} J_1 J_1 + \underbrace{a_{12}}_{\frac{A_{12}^{-1}}{2m^2}} J_1 J_2 + \dots$$

En definitiva $\frac{1}{2m^2} J^T A^{-1} J = \underline{a_{ij} J^i J^j}$

Sabemos que es un sumatorio porque tenemos arriba y abajo índices iguales. Según el criterio de Einstein, sabemos que es un sumatorio.

Nuestro cálculo se va a reducir a:

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle = \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \frac{\partial}{\partial J_c} \frac{\partial}{\partial J_d} \cdot \exp\left(a_{ij} J^i J^j\right)$$

Notación $\rightarrow \equiv \partial_a \partial_b \partial_c \partial_d \exp\left(a_{ij} x^i x^j\right)$

$$\boxed{\partial_d \exp\left(a_{ij} x^i x^j\right)} = \exp\left(a_{ij} x^i x^j\right) \cdot \partial_d \left(a_{ij} x^i x^j\right)$$

$\exp() \equiv \exp(a_{ij} x^i x^j)$ $= \exp() \cdot \left(a_{ij} (\partial_d x^i) \cdot x^j + a_{ij} (\partial_d x^j) x^i\right)$

Con la deltra de kronecker sólo sobreviven los términos en los que $i = d$, $j = d$

$$= \exp() \cdot \left(a_{ij} \delta_d^i x^j + a_{ij} \delta_d^j x^i\right)$$

$$= \boxed{\exp() \cdot \left(a_{dj} x^j + a_{id} x^i\right)}$$

$$\boxed{\partial_c \left[\exp() \cdot \left(a_{dj} x^j + a_{id} x^i\right)\right]} =$$

$$= \exp() \cdot \partial_c \left(a_{dj} x^j + a_{id} x^i\right) + \partial_c \exp() \cdot \left(a_{dj} x^j + a_{id} x^i\right)$$

$$= \underbrace{\exp() \cdot 2a_{dc}}_{\equiv F} + \underbrace{\left(a_{cj} x^j + a_{ic} x^i\right) \exp() \cdot \left(a_{dj} x^j + a_{id} x^i\right)}_{\equiv G}$$

Derivemos los dos sumandos.

$$\rightarrow \quad \partial_b \dot{F} = 2a_{dc}\left(a_{bj}x^j + a_{ib}x^i\right)\exp()$$

$$\rightarrow \quad \partial_b G = \partial_b\left[\underline{\left(a_{cj}x^j + a_{ic}x^i\right)\exp()}\cdot \underline{\left(a_{dj}x^j + a_{id}x^i\right)}\right]$$

$$= \left[2a_{cb}\exp() + \left(a_{bj}x^j + a_{ib}x^i\right)\exp()\right]\left(a_{dj}x^j + a_{id}x^i\right)$$

$$+ \; 2a_{bd}\cdot\left(a_{cj}x^j + a_{ic}x^i\right)\cdot\exp()$$

$$\partial_a\Big[\underbrace{2a_{dc}\left(a_{bj}x^j + a_{ib}x^i\right)\cdot\exp()}_{=H} + \underbrace{\left(2a_{cb}\exp() + \left(a_{bj}x^j + a_{ib}x^i\right)\cdot\exp()\cdot\left(a_{dj}x^j + a_{id}x^i\right)\right)}_{=K}$$

$$+ \underbrace{2a_{bd}\left(a_{cj}x^j + a_{ic}x^i\right)\exp()}_{=T}\Big]$$

$$\rightarrow \quad \partial_a H = 2a_{cd}\cdot 2a_{ab}\cdot\exp() + \partial_a\exp()\ldots$$

En $J=0$ $(x=0)$ se va a ir el último término

$$\rightarrow \quad \partial_a K = 2a_{cb}\,\partial_a\exp()\cdot\left(a_{dj}x^j + a_{id}x^i\right) + 2a_{ad}\cdot 2a_{bc}\cdot\exp()$$

este término se va a ir en $J=0$

$$\rightarrow \quad \partial_a T = 2a_{bd}\cdot 2a_{ac}\cdot\exp() + \partial_a\exp()\ldots$$

El último término se va en $J=0$

Sumando las 3 partes de la derivada ($\partial_a H + \partial_a K + \partial_a T$) ya tengo derivado cuatro veces $\exp()$. $\rightarrow$ $\partial_a\partial_b\partial_c\partial_d \exp\left(a_{ij}x^ix^j\right)$

Evaluando en $x=0$ obtengo:

$$\partial_a \partial_b \partial_c \partial_d \exp(a_{ij} x^i x^j) = 4 a_{cd} a_{ab} + 4 a_{ad} a_{bc} + 4 a_{bd} a_{ac}$$

Deshaciendo el cambio

$$a_{ij} = \frac{A_{ij}^{-1}}{2m^2}$$

$$= 4 \frac{A_{cd}^{-1}}{2m^2} \cdot \frac{A_{ab}^{-1}}{2m^2} + 4 \frac{A_{ad}^{-1}}{2m^2} \frac{A_{bc}^{-1}}{2m^2} + 4 \frac{A_{bd}^{-1}}{2m^2} \frac{A_{ac}^{-1}}{2m^2}$$

$$= \boxed{\frac{1}{m^4} \left(A_{cd}^{-1} A_{ab}^{-1} + A_{ad}^{-1} A_{bc}^{-1} + A_{bd}^{-1} A_{ac}^{-1} \right)} =$$

$$= \boxed{\langle \phi_a \phi_b \phi_c \phi_d \rangle}$$