Parte principal de una integral — Teoría Cuántica de Campos — ejercicios capítulo 17

- Calcular la parte principal de:

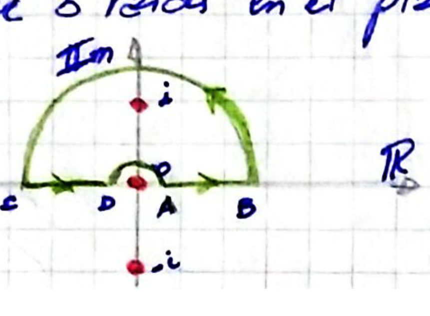
$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2+1)}\, dx$$

- $$\int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2+1)}\, dx = \int_{-\infty}^{\infty} \frac{\sin(x)}{x(x+i)(x-i)}$$

Plantearemos su homólogo en el plano complejo

- $$F(z) = \frac{e^{iz}}{z(z^2+1)}$$

La cual posee 3 raíces en el plano $\mathbb{C}$: $i, 0, -i$.



Planteemos la integral de línea de

$$\boxed{\oint_{ABCD} \frac{e^{iz}}{z(z+i)(z-i)}\,dz \overset{*}{=} \frac{2\pi i}{0!}\left[F^{(0)}(z)(z-i)\right]_{z=i} = \frac{2\pi i}{1}\cdot\frac{e^{ii}}{i(i+i)} = -\pi i\, e^{-1}}$$

Así mismo

$$\oint_{ABCD} F(z)\,dz = \underbrace{\int_A^B F(z)dz}_{1} + \underbrace{\int_B^C F(z)dz}_{2} + \underbrace{\int_C^D F(z)dz}_{3} + \underbrace{\int_D^A F(z)dz}_{4}$$

$$1 + 3 \left(\text{cuando } DA \to 0 \text{ y } B \to \infty\right) = P.P. \int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x\,(x^2+1)}\ \Big]$$

$$4 = \int_B^C F(z)\,dz$$ Si hacemos que $\begin{cases} z = Re^{i\theta} \\ dz = iRe^{i\theta}d\theta \end{cases}$ Luego $R \to \infty$

Así 4 (con $R\to\infty$) $= \lim\limits_{R\to\infty} \displaystyle\int_0^{\pi} \frac{e^{iR(\cos\theta + i\,\text{sen}\,\theta)}}{\cancel{Re^{i\theta}}(R^2e^{i2\theta}+1)}\, i\cancel{R}\cancel{e^{i\theta}}\, d\theta =$

$$= \int_0^{\pi} i \lim_{R\to\infty} \frac{e^{iR\cos\theta}\, e^{-R\,\text{sen}\,\theta}}{(R^2e^{i2\theta}+1)}\,d\theta \quad = 0\ \Big]$$

② Si $\begin{cases} z = Re^{i\theta} \\ dz = iRe^{i\theta} d\theta \end{cases}$ Además $R \to 0$

$$\Rightarrow ② = \lim_{R \to 0} \int_{\pi}^{0} d\theta \frac{e^{iR\cos\theta} e^{-R\operatorname{sen}\theta}}{(R^2 e^{i2\theta} + 1)} i = i[\theta]_{\pi}^{0} = -i\pi$$

Finalmente

$$-i\pi e^{-1} = PP\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x(x^2+1)} - i\pi$$

$$PP\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)} dx = i(1-e^{-1})\pi$$

$$\underbrace{PP\int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x(x^2+1)}}_{} + \underbrace{PP\, i\int_{-\infty}^{\infty} \frac{\operatorname{sen}(x)}{x(x^2+1)} dx}_{} = \underbrace{0}_{} + \underbrace{i(1-e^{-1})\pi}_{}$$

$$\boxed{PP\int_{-\infty}^{\infty} \frac{\operatorname{sen}(x)}{x(x^2+1)} dx = (1-e^{-1})\pi \cong 1{,}986}$$