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Discrete Optimization

The exact solutions of several types of container loading problems

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ABSTRACT

In this paper, we address multiple container loading problems, consisting of placing rectangular boxes, orthogonally and without overlapping, inside containers in order to optimize a given objective function, generally maximizing the value of the packed boxes or minimizing the number of containers required to pack all available boxes. Four techniques to enumerate the possible locations of boxes inside a container, some of them not yet tested in the literature, are evaluated. We also propose new techniques to obtain primal and dual bounds for these problems. In addition, we study the constraints related to box orientation, load stability, and separation of boxes. Detailed analysis on well-known benchmark instances shows that our method is very competitive, generating mathematical models containing significantly fewer variables and constraints than the traditional approach existing in the literature. We test our methods on five different benchmark sets. We provide a detailed comparison with different approaches from the CLP literature, proving new optimal solutions and improving the best-known results for several instances.

1. Introduction

In the Container Loading Problem (CLP) the goal is to place a set of boxes with their edges orthogonal to those of a large recipient called a container. Both the boxes and the container are three dimensional, parallelepiped shaped items. Variations of this problem arise in freight distribution when packing boxes inside actual containers for maritime or air transportation, or in trucks for road distribution. Due to the nature of the problem, one must consider all three dimensions of the boxes and the containers; special and simplified cases arise in one and two dimensions as well.

A feasible loading pattern is an arrangement of boxes inside a container where boxes do not overlap and lay entirely inside the container. Better use of the container's space can substantially reduce the cost of freight, yielding significant financial implications for carriers and shippers. Moreover, practical constraints arise when designing a loading pattern, such as the orientations of the boxes, the stability of the load, and the distribution of weight inside the containers. Bischoff and Ratcliff (1995) list 12 practical constraints commonly observed in CLPs, but some of them have not yet been properly studied in the literature, e.g., loading priorities and complete shipment constraints (Bortfeldt & Wäscher, 2013).

CLPs are classified into two groups of problems (Wäscher, Haußner, & Schumann, 2007). The first one is the *input minimization problem*, in which the storage space is sufficient to pack all boxes; here, the number of containers is usually not binding, and the objective function minimizes the number of containers required to load all available boxes. In the second group, called *output maximization problem*, the space of a limited set of containers is not sufficient to store all the boxes. The goal is then to select a subset of boxes maximizing the volume or value associated with the load.

In this paper, we present exact approaches to solve problems dealing with multiple containers, both for input minimization and output maximization. We also enhance several loading patterns to account for the three-dimensional issues inherent to CLPs. To generate an upper bound for input minimization problems, we use a set partitioning approach. To create the columns for this problem, we use the heuristic of George and Robinson (1980) and also present a variation of it, extending the work of Moura and Oliveira (2005). The 0–1 integer linear programming approach presented in this work, based on the formulations for the single container loading problems of Junqueira, Morabito, and Yamashita (2012), allows us to determine the exact solution of problems in different sce-

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narios, combining practical constraints in ways that have not been previously tested in the CLP literature. These practical constraints include the orientation of the boxes, their stability, and the separation of boxes that cannot be loaded into the same container. We highlight that no exact formulation exists to deal with the separation of boxes. Finally, this paper provides a detailed classification and comparison of literature on multiple CLPs, gathering together relevant information about these problems, instances, and best-known results.

The remainder of this paper is organized as follows. In Section 2 we provide a review of works concerning the CLP. Section 3 presents mathematical formulations, along with the practical constraints of box orientation, load stability, and box separation. In Section 4 we derive tight bounds for input minimization problems. Section 5 analyzes the extensive computational results obtained on classical benchmark instances and presents the comparison with many competing algorithms. Finally, Section 6 presents the main conclusions as well as perspectives for future work.

2. Previous work

Following the typology for cutting and packing problems presented by Wäscher et al. (2007), CLPs can be divided in two broad groups: input minimization and output maximization problems. The assortment of containers and boxes available to be loaded allows the division of CLPs into specific problem categories, which we briefly review in this section.

Input minimization problems can be divided in seven categories:

1. Single Stock-Size Cutting Stock Problem (SSSCSP) (Alonso, Alvarez-Valdes, Iori, Parreño, & Tamarit, 2017; Bortfeldt, 2000; Eley, 2002; Grunewald, Volling, Müller, & Spengler, 2018; Ivancic, Mathur, & Mohanty, 1989; Zhu, Huang, & Lim, 2012);
2. Multiple Stock-Size Cutting Stock Problem (MSSCSP) (Correcher, Alonso, Parreño, & Alvarez-Valdes, 2017; Eley, 2003; Toffolo, Esprit, Wauters, & Berghe, 2017);
3. Residual Cutting Stock Problem (RCSP) (Chen, Lee, & Shen, 1995);
4. Single Bin-Size Bin Packing Problem (SBSBPP) (Elhedhli, Gzara, & Yan, 2017; Gonçalves & Resende, 2013; Martello, Pisinger, & Vigo, 2000; Parreño, Alvarez-Valdes, Oliveira, & Tamarit, 2010);
5. Multiple Bin-Size Bin Packing Problem (MBSBPP) (Alvarez-Valdes, Parreño, & Tamarit, 2013; Tian, Zhu, Lim, & Wei, 2016; Wei, Zhu, & Lim, 2015);
6. Residual Bin Packing Problem (RBPP) (de Almeida & Figueiredo, 2010; Jin, Ito, & Ohno, 2003);
7. Open Dimension Problem (ODP) (Allen, Burke, & Kendall, 2011; Bortfeldt & Jungmann, 2012; de Queiroz, Miyazawa, Wakabayashi, & Xavier, 2012).

Our methods for input minimization problems can solve the MSSCSP, in which weakly heterogeneous boxes (i.e., relatively few types) have to be packed in a weakly assortment of containers. Our methods can also be used to solve the SSSCSP, in which a set of identical bins are available to pack a weakly heterogeneous set of boxes. According to the state-of-the-art review of Bortfeldt and Wäscher (2013), out of 163 papers surveyed, only 4.9% (i.e., 8 papers) deal with the MSSCSP, and only 11.7% (19 out 163) address the SSSCSP.

CLPs of output maximization type can also be classified in seven categories based on the assortment of boxes and containers:

1. Identical Item Packing Problem (IIPP) (Junqueira et al., 2012);
2. Single Large Object Placement Problem (SLOPP) (Araya, Guerrero, & Nuñez, 2017; Araya & Riff, 2014; George & Robinson, 1980; Junqueira et al., 2012; Ramos, Silva, & Oliveira, 2018);
3. Multiple Identical Large Object Placement Problem (MILOPP) (Hifi, 2002);
4. Multiple Heterogeneous Large Object Placement Problem (MHLOPP) (Eley, 2003; Mohanty, Mathur, & Ivancic, 1994; Ren, Tian, & Sawaragi, 2011);
5. Single Knapsack Problem (SKP) (Junqueira et al., 2012; Moura & Oliveira, 2005; Ramos et al., 2018; Sheng, Hongxia, Xisong, & Changjian, 2016);
6. Multiple Identical Knapsack Problem (MIKP) (Koloch & Kaminski, 2010);
7. Multiple Heterogeneous Knapsack Problem (MHKP) (Ceschia & Schaerf, 2013).

Our formulations for the output maximization case can be used to solve the MHLOPP, in which weakly heterogeneous boxes have to be packed inside a limited number of (weakly or strongly heterogeneous) containers. The methods presented in this work can also be used to address the MILOPP, in which a set of identical bins are available to pack a weakly heterogeneous set of boxes. Bortfeldt and Wäscher (2013) state that only 4 out of 163 papers (2.45%) address the MHLOPP, while only the paper of Hifi (2002), according to Bortfeldt and Wäscher (2013) and Zhao, Bennell, Bektaş, and Dowsland (2016), presents formulations that can be used to address the MILOPP. The formulations presented in this paper can also be reduced to consider single container problems, such as the IIPP, the SLOPP, and the SKP.

In the CLP, boxes must be placed inside the container respecting some practical constraints. Bischoff and Ratcliff (1995) present 12 common requirements that may arise during the loading procedure. These requirements, usually treated as constraints in the CLP, are the following: orientation of the boxes, load bearing of boxes, handling constraints, load stability, grouping of boxes, multi-drop situations, separation of boxes, complete shipment of boxes, shipment priorities, complexity of the load, container weight limit, and weight distribution in the container. Bortfeldt and Wäscher (2013) divide these requirements in four groups: (i) container-related constraints, (ii) box-related practical constraints, (iii) cargo-related constraints, and (iv) load-related practical constraints. Ramos et al. (2018) separates them into two classes: safety-related and logistics-related constraints.

Some practical constraints are a common feature in the literature, while some constraints have been overlooked so far. Out of the 163 papers studied by Bortfeldt and Wäscher (2013), the orientation of boxes and stability were considered in 115 (70.55%) and 61 (37.42%) papers, respectively; however, the practical constraint of complete shipment of boxes is present in only 1 paper (i.e., 0.61% of the sample) while shipment priorities are addressed in just 3 (1.84%) papers.

In our work, we focus on three of the most practical constraints: the orientation of the boxes, the stability of the load, and the separation of the boxes. While stability and orientation constraints are often taken into account in the CLP literature (Bortfeldt and Wäscher, 2013), the requirement of separation of boxes has been less studied in 3D cutting and packing problems; besides, no mathematical formulation has been presented to encompass this practical consideration.

Although some papers present techniques to handle the practical constraint of separation of items in one- and two-dimensional problems, these approaches cannot be directly extended to three-dimensional problems, because adding a third geometric dimension affects the formulation as a whole (Pollaris, Braekers, Caris, Janssens, & Limbourg, 2015). According to Bortfeldt and Wäscher (2013), only the work of Eley (2003) addresses the constraint of separation of items in three-dimensional packing problems.

Several approaches to solve the problems addressed in this paper have been proposed in the literature. Ivancic et al. (1989) solve

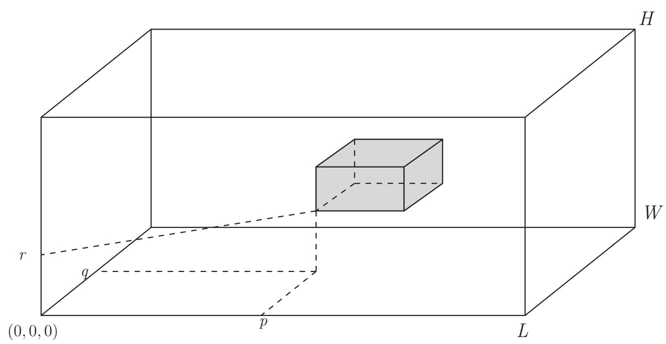

Fig. 1. Box allocated inside a container with its vertex at the point (p, q, r).

the input minimization problem with a hybrid of heuristic and integer programming that fills the containers sequentially. Eley (2002) presents a greedy heuristic that creates blocks of identical boxes to be packed into the container, and a tree-search heuristic to improve the loading patterns. The method of Lim and Zhang (2005) uses a greedy heuristic that fills the containers sequentially, prioritizing large boxes. Both Che, Huang, Lim, and Zhu (2011) and Zhu et al. (2012) use a procedure to generate columns representing packings to solve an extended set covering problem.

Output maximization problems have also attracted the interest of many researchers. Mohanty et al. (1994) present a sequential solution strategy using a column generation procedure. Bortfeldt (2000) uses a sequential strategy to fill the containers, outlining some strategies to select the boxes to be placed in the containers. Modifications were made to deal with input minimization problems as well. Eley (2003) uses a column generation heuristic to obtain a sufficient number of loading patterns and then solves an integer programming problem, considering only this limited number of patterns. Takahara (2008) introduces a multi-start local search procedure to determine the best loading sequence of types of boxes and their orientation to be placed in the container. Ren et al. (2011) solve several single container problems, filling each one with the cuboid arrangement approach and then improve the solution using a tree search algorithm. Junqueira et al. (2012) present a 0–1 integer linear programming, based on the formulations for the two-dimensional cutting problem of Beasley (1985), for an output maximization problem considering a single container, which we use as the base for the formulations presented in our work.

3. Mathematical formulations

We consider a set of m distinct types of boxes available, and each box of type $i \in \{1, \dots, m\}$ has length l_i, width w_i, height h_i, volume (or value) v_i, and availability b_i. We also consider $\mathcal{K}$ types of containers, each container of type $k \in \{1, \dots, \mathcal{K}\}$ being associated with length L_k, width W_k, height H_k and volume V_k. For output maximization problems, every container of type k has availability of C_k; for input minimization problems, the availability is not an issue and limited by a valid upper bound.

Taking the Cartesian coordinates system, let (p, q, r) be the back-bottom-left corner of a box inside a container (see Fig. 1). Boxes can assume different orientations within the container. Although a box can be placed inside a container in up to six different orientations, more restricted situations can be addressed, in which it is not possible to place a box in a given orientation (e.g., boxes with a “this way up!” sign).

In our formulations, we address the practical consideration of box orientation, assuming that a box can be loaded in up to six orientations, by means of orthogonal rotations. To this end, we derive the orientations that a given box of type i can take by decomposing the dimensions (l_i, w_i, h_i) into new items of dimensions (l_{ig}, w_{ig}, h_{ig}), where $g \in \Omega_i \neq \emptyset$, with $\Omega_i \subseteq \{1, 2, 3, 4, 5, 6\}$, such that $(l_{i1}, w_{i1}, h_{i1}) = (l_i, w_i, h_i)$, $(l_{i2}, w_{i2}, h_{i2}) = (l_i, h_i, w_i)$, $(l_{i3}, w_{i3}, h_{i3}) = (w_i, l_i, h_i)$, $(l_{i4}, w_{i4}, h_{i4}) = (w_i, h_i, l_i)$, $(l_{i5}, w_{i5}, h_{i5}) = (h_i, l_i, w_i)$ and $(l_{i6}, w_{i6}, h_{i6}) = (h_i, w_i, l_i)$, as shown in Fig. 2.

Symmetrical rotations can be avoided limiting the possible orientations of a box. For example, let a box of dimensions (l_i, w_i, h_i) with $l_i = w_i$. In this case, the orientation pairs with $g = 1$ and $g = 3$ ($g = 2$ and $g = 4$, and also $g = 5$ and $g = 6$) are equivalents. So, we can forbid these similar orientations, considering only $\Omega_i = \{1, 2, 5\}$, thus reducing the number of variables without loss of generality.

The following sets, which we denote as Unit Discretization (UD), indicate the possible positions that a box can take in relation to the dimensions of the container:

$$X_k = \{p \in \mathbb{Z} \,|\, 0 \le p \le L_k - \min_i(l_{ig}), \quad \forall i \in \{1, \dots, m\}, \quad \forall g \in \Omega_i\}, \quad \forall\, k \in \{1, \dots, \mathcal{K}\} \tag{1}$$

$$Y_k = \{q \in \mathbb{Z} \,|\, 0 \le q \le W_k - \min_i(w_{ig}), \quad \forall i \in \{1, \dots, m\}, \quad \forall g \in \Omega_i\}, \quad \forall\, k \in \{1, \dots, \mathcal{K}\} \tag{2}$$

$$Z_k = \{r \in \mathbb{Z} \,|\, 0 \le r \le H_k - \min_i(h_{ig}), \quad \forall i \in \{1, \dots, m\}, \quad \forall g \in \Omega_i\}, \quad \forall\, k \in \{1, \dots, \mathcal{K}\}. \tag{3}$$

Enumerating the sets that describe all possible positions that a box can take along the axes of a container allows solving only small instances due to the large number of possible positions that must be considered when deciding where to place an item, many of them redundant or symmetric. In this paper, four techniques to generate the possible points that can be occupied by the back-bottom-left corner of a box are adapted for the three-dimensional packing case: Normal Patterns (NP), Reduced Raster Points (RRP), Regular Normal Patterns (RNP), and the Meet in the Middle (MiM) Principle.

The NP, originally described by Herz (1972) and Christofides and Whitlock (1977), takes into account that the boxes can be moved toward the front, bottom and/or left of the container until they are adjacent to other boxes or the container walls, eliminating symmetrical positions that they may occupy. In an attempt to obtain sets with fewer elements, Terno, Lindemann, and Scheithauer (1987) and Scheithauer and Terno (1996) introduced the RRP sets, derived from NP. Although there is no guarantee that no loss of generality occurs in the RRP, in the empirical tests performed by de Queiroz et al. (2012) and also in this work, no optimal solution was missed.

Boschetti, Mingozzi, and Hadjiconstantinou (2002) introduce the RNP, in which the possible positions that a box of type i can take inside a container can be computed by determining the positions of all box types except i. Finally, the MiM Principle, proposed by Côté and Iori (2018), is defined for each box and for a given threshold $\mathbb{T}$ (i.e., half of the length), and consists in the combination of two patterns: those formed by the items aligned to the left of $\mathbb{T}$, and the ones of items arranged to the right of the threshold.

These discretization techniques aim to generate sets of lower cardinality than those presented by formulation (1)–(3), leading to models with fewer variables and constraints. The formulations used to obtain these sets can be found in Appendix A.

3.1. Input minimization problems

In order to formulate input minimization problems, we use the following sets and decision variables. The sets described by (4)–(6)

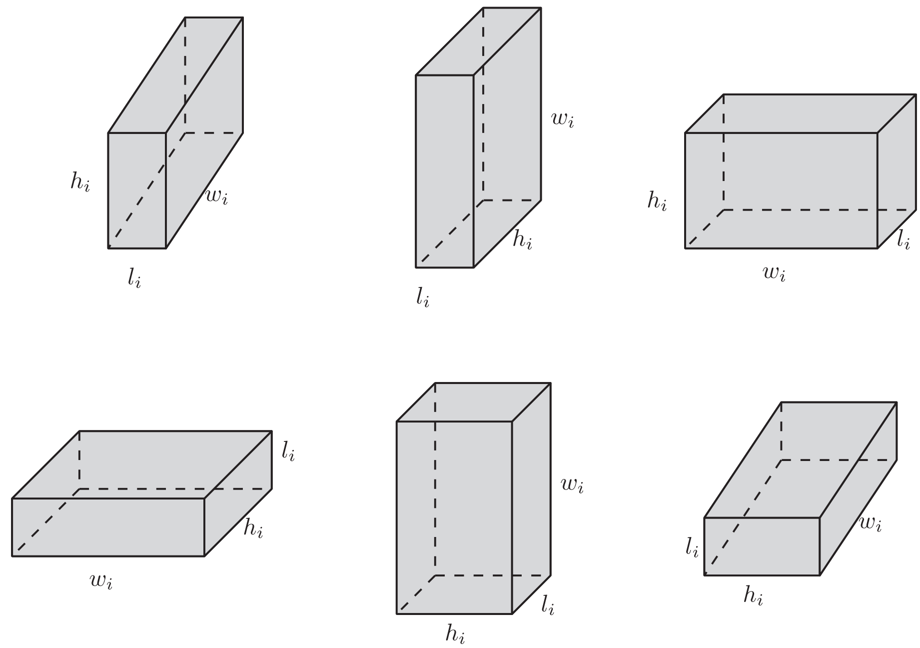

Fig. 2. Six possible orientations of a box of type i.

enumerate the possible coordinates that a box of type i, in its g^{th} orientation, can assume inside a container:

$$X_{igk} = \{p \in X_k \,|\, 0 \le p \le L_k - l_{ig}\}, \quad \forall i \in \{1, \dots, m\}, \quad \forall g \in \Omega_i, \quad \forall k \in \{1, \dots, \mathcal{K}\} \tag{4}$$

$$Y_{igk} = \{q \in Y_k \,|\, 0 \le q \le W_k - w_{ig}\}, \quad \forall i \in \{1, \dots, m\}, \quad \forall g \in \Omega_i, \quad \forall k \in \{1, \dots, \mathcal{K}\} \tag{5}$$

$$Z_{igk} = \{r \in Z_k \,|\, 0 \le r \le H_k - h_{ig}\}, \quad \forall i \in \{1, \dots, m\}, \quad \forall g \in \Omega_i, \quad \forall k \in \{1, \dots, \mathcal{K}\}. \tag{6}$$

The decision variables of the model are defined as follows. Let x_{ig}^{jkpqr} be equal to 1 if a box of type i, in its g^{th} orientation, has its back-bottom-left vertex at point (p, q, r) of the j^{th} container of type k, and 0 otherwise, and e_{jk} be equal to 1 if the j^{th} container of type k is used, and 0 otherwise.

The mathematical formulation for the input minimization problem is given by:

$$\min \sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{C_k} V_k e_{jk} \tag{7}$$

subject to:

$$\sum_{i=1}^{m} \sum_{g \in \Omega_i} \sum_{\{p \in X_{igk} | s - l_{ig} + 1 \le p \le s\}} \sum_{\{q \in Y_{igk} | t - w_{ig} + 1 \le q \le t\}} \sum_{\{r \in Z_{igk} | u - h_{ig} + 1 \le r \le u\}} x_{ig}^{jkpqr} \le e_{jk}, \quad s \in X_k, \quad t \in Y_k, \quad u \in Z_k, \quad k \in \{1, \dots, \mathcal{K}\}, \quad j \in \{1, \dots, C_k\} \tag{8}$$

$$\sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{C_k} \sum_{g \in \Omega_i} \sum_{p \in X_{igk}} \sum_{q \in Y_{igk}} \sum_{r \in Z_{igk}} x_{ig}^{jkpqr} = b_i, \quad i \in \{1, \dots, m\} \tag{9}$$

$$e_{jk} \in \{0, 1\}, \; x_{ig}^{jkpqr} \in \{0, 1\}, \; i \in \{1, \dots, m\}, \; g \in \Omega_i, \; k \in \{1, \dots, \mathcal{K}\}, \; j \in \{1, \dots, C_k\}, \quad p \in X_{igk}, \quad q \in Y_{igk}, \quad r \in Z_{igk}. \tag{10}$$

The objective function (7) minimizes the total volume to pack the boxes. Constraints (8) guarantee that no overlapping of boxes inside the j^{th} container of type k occurs, if it is used. These constraints work by ensuring point (s, t, u) is occupied by at most one item. Constraints (9) ensure that all boxes must be packed, allowing the boxes to be rotated, assuming up to six positions, while constraints (10) define the domain of the decision variables.

Since this formulation is indifferent to the heterogeneity of the boxes and the containers, it can be used to solve all kinds of input minimization problems, except the ODP. However, it may yield poor results when solving problems with strongly heterogeneous characteristics, since the discretizations tend to generate a greater number of possible points, which, in turn, will reflect in a greater number of variables in comparison with weakly heterogeneous problems. Hence, the formulation is better suited to solve the MSSCSP and, if there is only one type of container (i.e., $\mathcal{K} = 1$), the SSSCSP. In the case of the former problem, the objective function then minimizes the number of containers used.

Equivalent solutions may arise by simply interchanging the usage of containers of the same type. To avoid such symmetries, we consider the following constraints, which impose an order on the usage of the available containers:

$$e_{jk} \le e_{(j-1)k}, \quad k \in \{1, \dots, \mathcal{K}\}, \quad j \in \{2, \dots, C_k\}. \tag{11}$$

The modifications made in the original formulation of Junqueira et al. (2012), such as the inclusion of new indices and new summations in the constraints and objective function, as well as the symmetry breaking inequality, were critical to cope with different categories of CLPs in a way that can lead to optimal solutions efficiently for the problems at hand.

3.2. Output maximization problems

Some adjustments in the objective function and constraints in the input minimization model allow the construction of a mathematical formulation for output maximization problems with multiple containers.

The decision variables x_{ig}^{jkpqr} have the same interpretation of the ones in (7)–(10); since all available containers will be used to load the boxes, the variables e_{jk} are not necessary in this context. Hence, the mathematical formulation for the output maximization problem is given by:

$$\max \sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{C_k} \sum_{i=1}^{m} \sum_{g\in\Omega_i} \sum_{p\in X_{igk}} \sum_{q\in Y_{igk}} \sum_{r\in Z_{igk}} v_i x_{ig}^{jkpqr} \tag{12}$$

subject to

$$\sum_{i=1}^{m} \sum_{g\in\Omega_i} \sum_{\{p\in X_{igk}|s-l_{ig}+1\le p\le s\}} \sum_{\{q\in Y_{igk}|t-w_{ig}+1\le q\le t\}} \sum_{\{r\in Z_{igk}|u-h_{ig}+1\le r\le u\}} x_{ig}^{jkpqr} \le 1,$$
$$s\in X_k,\quad t\in Y_k,\quad u\in Z_k,\quad k\in\{1,\dots,\mathcal{K}\},\quad j\in\{1,\dots,C_k\} \tag{13}$$

$$\sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{C_k} \sum_{g\in\Omega_i} \sum_{p\in X_{igk}} \sum_{q\in Y_{igk}} \sum_{r\in Z_{igk}} x_{ig}^{jkpqr} \le b_i,\quad i\in\{1,\dots,m\} \tag{14}$$

$$x_{ig}^{jkpqr}\in\{0,1\},\quad i\in\{1,\dots,m\},\quad g\in\Omega_i,\quad k\in\{1,\dots,\mathcal{K}\},$$
$$j\in\{1,\dots,C_k\},\quad p\in X_{igk},\quad q\in Y_{igk},\quad r\in Z_{igk}. \tag{15}$$

The objective function (12) maximizes the volume (or the associated value) of the items packed in the available containers. Constraints (13) guarantee that no overlapping occurs, i.e., a point (s, t, u) is occupied by at most one item, while constraints (14) ensure that, at most, all available boxes of type i will be packed. Finally, constraints (15) impose the domain of the variables.

The formulation for output maximization problems, like the minimization one, is indifferent to the heterogeneity of the boxes and the containers and can be used to solve all types of output maximization problems with more than one container available. Nevertheless, for the same reasons stated in Section 3.1, the formulation (12)–(15) is better suited to solve the MHLOPP and, if there is only one type of container (i.e., $\mathcal{K}=1$), the MILOPP.

3.3. Practical constraints

In this section, we study some practical constraints not often considered in other approaches. Nevertheless, they are of high importance in practice. These include the separation of boxes in Section 3.3.1 and load stability in Section 3.3.2.

3.3.1. Separation of boxes

According to Bischoff and Ratcliff (1995), requirements of separation of boxes are related to items that cannot be packed side by side, having some space between them when they share the same container. Eley (2003) extends this concept to items that cannot be loaded into the same container. We follow this interpretation of conflicting items, i.e., they cannot be loaded into the same container.

To this end, let B_1 and B_2 be two sets with the types of conflicting items, i.e., each box type in B_1 cannot be loaded in the same container with items of set B_2. A practical situation arises when one must load some containers with a set of different foods and another set of chemicals, or separating frozen and refrigerated items from general cargo. Binary variables $z_{\gamma jk}$, $\gamma\in B_1\cup B_2$, $j\in\{1,\dots,C_k\}$ and $k\in\{1,\dots,\mathcal{K}\}$, take value 1 if boxes of type γ, with availability b_γ, are allocated in the j^{th} container of type k, and 0 otherwise.

To consider the of separation of items, we add the following constraints to the formulations presented in the previous sections, for each $\gamma\in B_1\cup B_2$:

$$\sum_{g\in\Omega_\gamma} \sum_{p\in X_{\gamma gk}} \sum_{q\in Y_{\gamma gk}} \sum_{r\in Z_{\gamma gk}} x_{\gamma g}^{jkpqr} \le b_\gamma z_{\gamma jk},\quad \forall\gamma\in B_1\cup B_2$$
$$B_1\subset\{1,\dots,m\},\quad B_2\subset\{1,\dots,m\},\quad B_1\cap B_2=\emptyset,$$
$$k\in\{1,\dots,\mathcal{K}\},\quad j\in\{1,\dots,C_k\}. \tag{16}$$

To guarantee that boxes of type $\theta\in B_1$ are loaded in different containers than boxes of type $\mu\in B_2$, the following constraints must be added to the formulations:

$$z_{\theta jk}+z_{\mu jk}\le 1,\ \theta\in B_1,\ \mu\in B_2,\ k\in\{1,\dots,\mathcal{K}\},\ j\in\{1,\dots,C_k\} \tag{17}$$

$$z_{\gamma jk}\in\{0,1\},\quad z_{\theta jk}\in\{0,1\},\quad z_{\mu jk}\in\{0,1\},\quad \gamma\in B_1\cup B_2,$$
$$\theta\in B_1,\quad \mu\in B_2. \tag{18}$$

We stress that the constraints (17) can be strengthened for input minimization problems replacing the right-hand side constant for the variable e_{jk} defined in (10), since the constraint will be considered only if the container is used. We also note that formulation (16)–(18) can be extended to any finite number of sets of items that must be packed in different containers.

3.3.2. Stability of boxes

Stability is one of the most important practical constraints that can be incorporated in the CLP, as stable loads prevent cargo damage during transport and ensure the safety of operators, especially during the loading/unloading procedures. According Bortfeldt and Wäscher (2013), the stability of boxes can be divided in two variants: vertical (or static) stability, regarding the ability of the boxes to resist the action of the force of gravity (i.e., “fall” towards the floor of the container); and the horizontal (or dynamic) stability, when the boxes do not shift their positions when the container is moved horizontally (e.g., when a truck starts a trip). The formulations presented in this work deal only with the vertical stability case. Hence, for the sake of simplicity, we will refer to the constraint as stability throughout this paper.

One way to approach the static stability requirement is to determine a minimum percentage of the base of each box that must be supported by the base of the container or by other boxes. The stability coefficient α indicates the minimum portion of the items that must be supported. When $\alpha=1$, 100% of the base of the box must be supported, and when $\alpha=0$, no stability requirement is considered, i.e., the items may be partially supported or “floating” within the containers.

To consider the stability in the CLP, the following constraints must be added to formulations (7)–(10) and (12)–(15), noting that one can assume that $\mathcal{K}=1$ if there is only one type of container available:

$$\sum_{i=1}^{m} \sum_{\{g\in\Omega_i|r'-h_{ig}\ge 0\}} \sum_{\{p\in X_{igk}|p'-l_{ig}+1\le p\le p'+l_{\lambda a}-1\}} \sum_{\{q\in Y_{igk}|q'-w_{ig}+1\le q\le q'+w_{\lambda a}-1\}}$$
$$L_{i\lambda}W_{i\lambda}x_{ig}^{jkpq(r'-h_{ig})} \ge \alpha l_{\lambda a} w_{\lambda a} x_{\lambda a}^{jkp'q'r'}$$
$$\lambda\in\{1,\dots,m\},\quad a\in\Omega_\lambda,\quad k\in\{1,\dots,\mathcal{K}\},\quad j\in\{1,\dots,C_k\},$$
$$p'\in X_{\lambda ak},\quad q'\in Y_{\lambda ak},\quad r'\in Z_{\lambda ak} \tag{19}$$

where

$$L_{i\lambda}=\min(p+l_{ig},\,p'+l_{\lambda a})-\max(p,\,p') \tag{20}$$

and

$$W_{i\lambda}=\min(q+w_{ig},\,q'+w_{\lambda a})-\max(q,\,q'). \tag{21}$$

Expressions (19)–(21), adapted from Junqueira et al. (2012), guarantee a minimum percentage of support for the base for a box of type λ positioned at (p', q', r') of the j^{th} container of type k. This support is provided by a box of type i (including $i = \lambda$) that is located at (p, q, r) within the container, where $r = r' - h_{ig}$.

We note, however, that while the parameter α controls the minimum bearing area of the boxes, it does not guarantee the stability of the load when it is subject to the action of external forces such as speed, acceleration, and vehicle oscillations (Ramos et al., 2018).

4. Bounds for input minimization problems

In this section, we describe how we obtain bounds for input minimization problems. Bounds for output maximization problems are unnecessary, given that these problems already have a limited number of containers available. We describe an upper bound procedure in Section 4.1 and a lower bound method in Section 4.2. Computational results showing their efficiency are presented in Section 5.

4.1. Upper bound

For input minimization problems, one usually assumes an unlimited number of containers available to pack the boxes. A trivial bound is to set the number of each container type as the number of items to be packed. In order to obtain efficient mathematical formulations, it is essential to determine tight bounds, so that fewer variables and constraints are needed.

4.1.1. Upper bound without practical constraints

In order to obtain an upper bound for the problem, we solve a set partitioning problem, minimizing the number of containers needed to pack all the boxes. Inspired by the work of Zhu et al. (2012), we provide packing patterns to the set partitioning problem but, instead of using the prototype column generation presented by them, we use two wall-building techniques to obtain the packings: the heuristic G&R of George and Robinson (1980) and a variation of the heuristic GRMod of Moura and Oliveira (2005).

Originally designed for loading a single container, the G&R heuristic fills the container by partitioning it into layers and filling them with stacks of identical boxes. The depth of each layer is defined by the first box placed in the layer; George and Robinson (1980) present three criteria for choosing this box. These criteria must be applied sequentially, i.e., the next criterion is only used as a tie-breaker: (i) select the box with the largest of the smallest dimensions; (ii) select the item with the largest quantity available; (iii) select the box with the largest dimension.

While filling the container, it is possible that the chosen box type does not have sufficient availability to fill the entire layer or that this type of box cannot be placed in the remaining space. This entails the generation of new spaces inside the container: a depth space in front of the layer, a width space on the side of the layer, and a height space on the top of the layer. These spaces must be filled in the reverse order in which they are created, i.e., one must first try to fill the height space, then the width and, finally, the depth space. However, these spaces may be insufficient to be occupied by any other type of box during the filling of the current layer. Even so, they are stored in a list of temporarily rejected spaces, because they can be combined with other idle spaces in filling a new layer, generating a more dense loading.

To generate feasible packing patterns, we applied the heuristic G&R sequentially, for each type of container, updating the availability of the boxes as each container is filled. One of the disadvantages of this strategy is that, although the first containers tend to be almost full, the succeeding ones may have a low ratio of occupancy. This is mainly due to the number of boxes available for loading, as well as the need to choose the boxes that will open the layers based on the original heuristic criteria.

In order to obtain different, high-quality packings for the set partitioning, we propose a variation of the GRMod heuristic of Moura and Oliveira (2005). Being itself a variation of the G&R heuristic, the GRMod is a two-phase heuristic for problems in which a single container is available: first, in a building phase, a packing pattern is obtained, and in the second phase, the solution is improved using a local search technique. To generate the packing pattern, the criteria for choosing the boxes of the original heuristic are disregarded, and the choice of the type of box that will be used to create a new layer or to fill a space is made based on a restricted candidates list. The boxes that will be part of this list are chosen based on the following expression:

$$\mathcal{T} = Vol_{max} + \beta(Vol_{min} - Vol_{max}) \tag{22}$$

where $\mathcal{T}$ is a volume utilization threshold, Vol_{max} and Vol_{min} are, respectively, the maximum and minimum volume that can be used by the arrangements that can be formed with the boxes available for loading, and β is a parameter that controls the level of randomness of the algorithm: when $\beta = 1$, the boxes are chosen randomly, whereas when $\beta = 0$, the choice of the boxes assumes a greedy behavior.

For the set partitioning formulation used for input minimization problems addressed in this paper, it is desirable to have a large number of packing patterns for the set partitioning formulation in order to increase the chance of obtaining good solutions. Since the GRMod has a random feature, each execution can yield a different loading pattern. A key difference in our approach is the way that we execute the building phase: while Moura and Oliveira (2005) build a solution and perform an improvement phase, we execute the building phase multiple times, for each container type, storing every different packing pattern obtained. The different restricted candidate lists obtained with different values of β yield a great variety of packing patterns.

One of the advantages of these methods to obtain the packings is that, besides the number of each box type loaded in a container, the heuristics also provide an actual packing pattern, with a feasible arrangement of boxes inside a container, providing useful information for the mathematical models.

After running the strategies above, we obtain $\mathcal{P}_k$ different packing patterns for each container of type k. The number of boxes types loaded in each packing becomes a column vector y, whose elements y^i_{jk} indicate the number of boxes of type i in the j^{th} packing using the container type k. This leads to the following set partitioning formulation:

$$\min \sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{\mathcal{P}_k} V_k \delta_{jk} \tag{23}$$

subject to:

$$\sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{\mathcal{P}_k} y^i_{jk} \delta_{jk} = b_i, \qquad i = 1, \dots, m \tag{24}$$

$$y^i_{jk} \in \{0, \dots, b_i\}, \quad \forall i \in \{1, \dots, m\}, \quad \forall j \in \{1, \dots, \mathcal{P}_k\},$$
$$\forall k \in \{1, \dots, \mathcal{K}\} \quad \text{and } \delta_{jk} \in \mathbb{N}. \tag{25}$$

Here, the objective function (23) minimizes the volume needed to pack the whole set of boxes. Variables δ_{jk} indicate the number of times a packing pattern is select; therefore, by solving (23)–(25), an upper bound $\mathcal{U}$ for the input minimization problem without practical constraints is obtained.

Moreover, formulation (23)–(25) can be used to obtain a particular upper bound $\mathcal{U}_k$ of a given container type k if the following modifications are made:

$$\min \sum_{j=1}^{\mathcal{P}_k} \delta_{jk} \tag{26}$$

subject to:

$$\sum_{j=1}^{\mathcal{P}_k} y^i_{jk}\delta_{jk} = b_i, \qquad i = 1, \dots, m \tag{27}$$

$$y^i_{jk} \in \{0, \dots, b_i\}, \quad \forall i \in \{1, \dots, m\}, \quad \forall j \in \{1, \dots, \mathcal{P}_k\} \quad \text{and} \quad \delta_{jk} \in \mathbb{N}. \tag{28}$$

where the objective function (26) represents the maximal number of containers of type k needed to pack the boxes.

4.1.2. Upper bound with load stability

One of the key improvements made by Moura and Oliveira (2005) to the original G&R heuristic is a new scheme to generate spaces on top of the boxes already placed inside a container, in order to obtain stable loadings. If no box has dimensions suitable to fit into a width space, the GRMod heuristic restricts the height space considering only the space above layers already packed. To obtain packing patterns with proved stability, we not only use this concept in our variation of the GRMod but also use the same idea in the sequential application of the G&R to obtain an upper bound for problems with cargo stability.

4.1.3. Upper bound with separation of boxes

To consider the separation of boxes constraint, we execute the G&R and the variation of GRMod grouping the conflicting box types in two sets, B_1 and B_2. Then, we apply the upper bound heuristics in two steps: first, we only consider boxes not belonging to B_1. The process is repeated, now taking into account boxes not belonging to B_2. Let $\mathcal{P}'_k$ and $\mathcal{P}''_k$ be packing patterns obtained in the two steps described. Then, we have $|\mathcal{P}_k| = |\mathcal{P}'_k \cup \mathcal{P}''_k|$ different patterns, and the upper bound can be achieved solving (23)–(25) and (26)–(28).

4.2. Lower bound

To obtain a general lower bound $\mathcal{L}$ for the problem considering only its basic features, we solve a mathematical model without the non-overlapping constraints; thus, we effectively reduced the CLP to a 1D bin packing problem. It is a classic approach to achieve a valid lower bound for the CLP, used, for example, by Zhu et al. (2012). The decision variables $\bar{e}_{jk}$ are equal to 1 if the j^{th} container of type k is used and 0 otherwise, and $\bar{y}_{ijk}$, which indicates the number of boxes of type i are loaded in the j^{th} container of type k. Our formulation is given by:

$$\min \sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{\mathcal{U}_k} V_k \bar{e}_{jk} \tag{29}$$

subject to:

$$\sum_{i=1}^{m} v_i \bar{y}_{ijk} \le V_k \bar{e}_{jk}, \quad k \in \{1, \dots, \mathcal{K}\}, \quad j \in \{1, \dots, \mathcal{U}_k\} \tag{30}$$

$$\sum_{k=1}^{\mathcal{K}} \sum_{j=1}^{\mathcal{U}_k} \bar{y}_{ijk} = b_i, \quad i \in \{1, \dots, m\} \tag{31}$$

$$\bar{y}_{ijk} \in \mathbb{N}, \quad \bar{e}_{ijk} \in \{0, 1\}, \quad \forall i \in \{1, \dots, m\}, \quad \forall j \in \{1, \dots, \mathcal{U}_k\}, \quad \text{and} \quad \forall k \in \{1, \dots, \mathcal{K}\}. \tag{32}$$

In the expressions above, the objective function (29) determines the minimum volume needed to pack the boxes. Constraints (30) guarantee that the volume of the items packed in the j^{th} container of type k does not exceed the volumetric capacity of the container, while expressions (31) ensure that all boxes are loaded. Constraints (32) present the domain of the decision variables. The lower bound is also valid for problems with the practical consideration of stability since it only takes into account the volumetric features of the boxes and containers.

Formulation (29)–(32) can be extended to obtain a valid lower bound considering the separation of boxes. The following constraints must be added:

$$\bar{y}_{\gamma jk} \le b_\gamma \bar{z}_{\gamma jk}, \quad k \in \{1, \dots, \mathcal{K}\}, \quad j \in \{1, \dots, \mathcal{U}_k\}, \quad \gamma \in B_1 \cup B_2 \tag{33}$$

$$\bar{z}_{\theta jk} + \bar{z}_{\mu jk} \le \bar{e}_{jk}, \quad \theta \in B_1, \quad \mu \in B_2, \quad \forall k \in \{1, \dots, \mathcal{K}\}, \quad \forall j \in \{1, \dots, \mathcal{U}_k\} \tag{34}$$

$$\bar{z}_{\theta jk} \in \{0, 1\}, \quad \bar{z}_{\mu jk} \in \{0, 1\}, \quad \bar{z}_{\gamma jk} \in \{0, 1\}, \quad \bar{y}_{\gamma jk} \in \mathbb{N}, \quad k \in \{1, \dots, \mathcal{K}\}, \quad j \in \{1, \dots, \mathcal{U}_k\}. \tag{35}$$

Constraints (33)–(35) can be interpreted like (16)–(18).

The formulations presented in (29)–(32) can be adapted to obtain a particular lower bound $\mathcal{L}_k$ for a given type of container k as follows:

$$\min \sum_{j=1}^{\mathcal{U}_k} \bar{e}_{jk} \tag{36}$$

subject to:

$$\sum_{i=1}^{m} v_i \bar{y}_{ijk} \le V_k \bar{e}_{jk}, \quad j \in \{1, \cdots, \mathcal{U}_k\} \tag{37}$$

$$\sum_{j=1}^{\mathcal{U}_k} \bar{y}_{ijk} = b_i, \quad i \in \{1, \dots, m\} \tag{38}$$

$$\bar{y}_{ijk} \in \mathbb{N}, \quad \bar{e}_{ijk} \in \{0, 1\}, \quad i \in \{1, \dots, m\}, \quad j \in \{1, \dots, \mathcal{U}_k\}, \quad \text{and} \quad k \in \{1, \dots, \mathcal{K}\}. \tag{39}$$

The objective function (36) represents the minimum number of containers of type k needed to pack the boxes.

In problems of type SSSCSP, since $\mathcal{K} = 1$, i.e., there is only one type of container, if $\mathcal{L}_k \le \mathcal{U}_k$, one can assign the value 1 to all variables e_{jk}, $j \in \{1, \dots, \mathcal{L}_k\}$, leaving the mathematical formulation to minimize the use of remaining variables, and to ensure the arrangement of the boxes in the selected containers, considerably reducing the effort to solve the models. Finally, if a loading pattern is known and $\mathcal{L}_k = \mathcal{U}_k$, it constitutes an optimal solution for both types of input minimization problems addressed in this work.

4.3. Overall strategy to solve input minimization problems

In summary, the strategy used to solve the input minimization problems is presented in Algorithm 1. We start by computing the upper and lower bounds for the problem, using the appropriated strategy, according to the practical constraints addressed (lines 1–2). Then, we compare the bounds: if the bounds have the same value, we have an optimal solution for the problem (line 4); otherwise, if the problem addressed is a SSSCSP (line 6), we generate the mathematical model, using $\mathcal{U}_k - 1$ as the available number of containers (line 7), set the variables e_{jk} with index k less than or

Algorithm 1 Strategy to solve input minimization problems.

1: Compute the upper bounds $\mathcal{U}_k$ for each k, and general upper bound $\mathcal{U}$
2: Compute the lower bounds $\mathcal{L}_k$ for each k, and general lower bound $\mathcal{L}$
3: **if** $\mathcal{L} = \mathcal{U}$ **then**
4:   Optimal solution found
5: **else**
6:   **if** $\mathcal{K} = 1$ **then**
7:     $C_k \leftarrow \mathcal{U}_k - 1$
8:     Generate the model
9:     Set $e_{jk} = 1,\ j \in \{1, \dots, \mathcal{L}_k\}$
10:     Add symmetry breaking constraints $e_{jk} \le e_{(j-1)k}$, $\forall j \in \{\mathcal{L}_k + 1, \dots, C_k\}$
11:     Solve the model
12:     **if** model is infeasible **then**
13:       The upper bound $\mathcal{U}_k$ is the optimal solution
14:     **else**
15:       The solution of the model is feasible
16:     **end if**
17:   **else**
18:     **for** each k **do**
19:       $C_k \leftarrow \mathcal{U}_k$
20:     **end for**
21:     Generate the model
22:     Add symmetry breaking constraints $e_{jk} \le e_{(j-1)k}$, $\forall j \in \{1, \dots, C_k\}$
23:     Solve the model
24:   **end if**
25: **end if**

equal to the lower bound $\mathcal{L}_k$ to 1 (line 9), and we add symmetry breaking constraints (line 10); by doing so, we can generate models with fewer constraints and variables, possibly reducing the effort needed to obtain a solution. If this model proves to be infeasible, we then conclude that the upper bound $\mathcal{U}_k$ consists of the optimal solution for the problem (line 13); if not, solving the model provides a solution for the problem at hand (line 15). If the problem addressed is a MSSSCP (line 17), we then generate the mathematical model, using $\mathcal{U}_k$ as the available number of containers (line 19), add symmetry breaking constraints (line 22), and then solve the model.

5. Computational experiments

We now describe the detailed computational experiments used to assess the performance of our methods. In order to evaluate the algorithms proposed in the previous sections, we have used public datasets from the literature for both input minimization and output maximization problems. We have implemented all models and procedures in C++ and executed each instance using computers equipped with Xeon processors running at 2.77 gigahertz and up to 120 gigabytes of RAM. A time limit of 3600 seconds was imposed for each execution. The mathematical models were solved using Gurobi 8.1.0 with default parameters.

To obtain the upper bound, we created the columns for the set partitioning problem applying the G&R heuristic sequentially. Then, more columns were created using the adapted GR-Mod approach presented in Section 4.1, with parameters $\beta = 0$, $\beta = 0.5$ and $\beta = 1$. As a stop criterion, we set a time limit of 15 seconds or a maximum of 100,000 executions for each parameter.

Section 5.1 describes the instances used and the competing methods available in the literature. Section 5.2 presents a discussion about the discretization techniques used to solve the models. In Section 5.3, we detail the results obtained for input minimization problems, followed by those for output maximization in Section 5.4.

5.1. Instances

Five sets of instances were used to test our formulations. First, to analyze the impact of the discretizations, we restricted the number of containers, in order to solve the single container problems of Junqueira et al. (2012). Composed by 320 instances, this set is divided into two groups, A_m and B_m, where $m \in \{1, 5, 10, 20\}$ stands for the number of types of boxes. In both groups, the dimensions of the boxes are randomly chosen as a fraction of the corresponding container's dimensions (between 25% and 75% of the dimensions of the containers for group A_m, and 10% and 50% for group B_m). The containers are always cubic, with dimensions of 10, 20, 30, 50, and 100 for group A_m, and 10, 20, and 30 in group B_m. In this set, boxes have fixed orientations, except if $m = 1$, when they are allowed to rotate. Besides running tests as in Junqueira et al. (2012), we also ran tests in a more general framework, allowing the boxes to assume up to six different orientations to better evaluate our methods.

For input minimization problems, we used the instances of Ivancic et al. (1989). For the MSSCSP, the data set is composed of 17 instances, ranging from two to five types of boxes, and from 47 to 180 boxes. The number of containers available in each instance varies between two and three. The objective is to find the minimum volume necessary to pack the whole set of boxes. Different methods solved these instances, and we compare our results against the following ones:

- IVA: the integer programming-based heuristic of Ivancic et al. (1989);
- BOR: the heuristic box and container selection criteria of Bortfeldt (2000);
- ELY: the bottleneck approach of Eley (2003);
- TAK: the multi-start local search procedure of Takahara (2008);
- REN: the column generation procedure of Ren et al. (2011).

The 17 instances of the MSSCSP were used to generate 47 instances for the SSSCSP decomposing them by container, i.e., an instance with three containers gives origin to three new instances, each one with a single container type and the same set of boxes from the original instance. The objective is to find the lowest number of containers needed to pack all the boxes. Besides the techniques cited above, which also present approaches to the SSSCSP, we compared our algorithm against the following approaches:

- LZG: the greedy heuristic of Lim and Zhang (2005);
- CHE: the column generation technique of Che et al. (2011);
- LIM: the iterated construction approach of Lim, Ma, Xu, and Zhang (2012);
- ZHU: the column generation procedures of Zhu et al. (2012).

We stress that the methods of Zhu et al. (2012) and Eley (2003) also take into account practical constraints – the first considers the stability of the cargo for the SSSCSP, while the latter considers the separation of boxes for the SSSCSP. Additionally, for the SSSCSP, the tree search heuristic with sequential packing and the tree search strategy with parallel packing of Eley (2002), denoted as ELS and ELP, respectively, are also used as benchmarks in cases with load stability.

For output maximization problems with multiple containers, we used the instances of Mohanty et al. (1994). For the MHLOPP,

Table 1
Comparison between the discretization techniques using the instances of Junqueira et al. (2012) with $L = W = H = 10$.

		UD		NP		RRP		RNP		MiM	
		Variables	Constraints	Variables	Constraints	Variables	Constraints	Variables	Constraints	Variables	Constraints
A_1	Min	600.0	344.0	36.0	28.0	**24.0**	**9.0**	36.0	28.0	36.0	28.0
	Avg	1430.4	561.2	326.1	142.5	**313.2**	**127.9**	326.1	142.5	316.2	129.8
	Max	2160.0	730.0	**1344.0**	**513.0**	**1344.0**	**513.0**	**1344.0**	**513.0**	**1344.0**	**513.0**
	Sum	14304.0	5612.0	3261.0	1425.0	**3132.0**	**1279.0**	3261.0	1425.0	3162.0	1298.0
A_5	Min	1294.0	509.0	**293.0**	145.0	**293.0**	**125.0**	**293.0**	145.0	**293.0**	**125.0**
	Avg	1603.4	598.0	570.8	255.0	541.10	228.2	569.0	251.8	**539.3**	**225.0**
	Max	2283.0	734.0	**1236.0**	453.0	**1236.0**	**389.0**	**1236.0**	453.0	**1236.0**	**389.0**
	Sum	16034.0	5980.0	5708.0	2550.0	5411.0	2282.0	5690.0	2518.0	**5393.0**	**2250.0**
A_{10}	Min	3076.0	658.0	1326.0	330.0	**1284.0**	**298.0**	1326.0	330.0	**1284.0**	**298.0**
	Avg	3467.1	730.9	1954.4	472.4	**1930.0**	**454.8**	1954.4	472.4	**1930.0**	**454.8**
	Max	4002.0	739.0	**2397.0**	**522.0**	**2397.0**	**522.0**	**2397.0**	**522.0**	**2397.0**	**522.0**
	Sum	34671.0	7309.0	19544.0	4724.0	**19300.0**	**4548.0**	19544.0	4724.0	**19300.0**	**4548.0**
A_{20}	Min	6366.0	749.0	**3948.0**	**532.0**	**3948.0**	**532.0**	**3948.0**	**532.0**	**3948.0**	**532.0**
	Avg	7063.2	749.0	**4468.9**	**532.0**	**4468.9**	**532.0**	**4468.9**	**532.0**	**4468.9**	**532.0**
	Max	8327.0	749.0	**5422.0**	**532.0**	**5422.0**	**532.0**	**5422.0**	**532.0**	**5422.0**	**532.0**
	Sum	70632.0	7490.0	**44689.0**	**5320.0**	**44689.0**	**5320.0**	**44689.0**	**5320.0**	**44689.0**	**5320.0**
B_1	Min	1470.0	730.0	**1176.0**	**513.0**	**1176.0**	**513.0**	**1176.0**	**513.0**	**1176.0**	**513.0**
	Avg	2569.2	946.8	**2454.0**	**903.4**	**2454.0**	**903.4**	**2454.0**	**903.4**	**2454.0**	**903.4**
	Max	**3780.0**	**1001.0**	**3780.0**	**1001.0**	**3780.0**	**1001.0**	**3780.0**	**1001.0**	**3780.0**	**1001.0**
	Sum	25692.0	9468.0	**24540.0**	**9034.0**	**24540.0**	**9034.0**	**24540.0**	**9034.0**	**24540.0**	**9034.0**
B_5	Min	2482.0	805.0	**1250.0**	**405.0**	**1250.0**	**405.0**	**1250.0**	**405.0**	**1250.0**	**405.0**
	Avg	2953.2	936.0	**2609.7**	**835.0**	**2609.7**	**835.0**	**2609.7**	**835.0**	**2609.7**	**835.0**
	Max	**3411.0**	**1005.0**	**3411.0**	**1005.0**	**3411.0**	**1005.0**	**3411.0**	**1005.0**	**3411.0**	**1005.0**
	Sum	29532.0	9360.0	**26097.0**	**8350.0**	**26097.0**	**8350.0**	**26097.0**	**8350.0**	**26097.0**	**8350.0**
B_{10}	Min	5586.0	910.0	**4888.0**	**810.0**	**4888.0**	**810.0**	**4888.0**	**810.0**	**4888.0**	**810.0**
	Avg	6023.8	990.0	**5880.0**	**970.0**	**5880.0**	**970.0**	**5880.0**	**970.0**	**5880.0**	**970.0**
	Max	**6627.0**	**1010.0**	**6627.0**	**1010.0**	**6627.0**	**1010.0**	**6627.0**	**1010.0**	**6627.0**	**1010.0**
	Sum	60238.0	9900.0	**58800.0**	**9700.0**	**58800.0**	**9700.0**	**58800.0**	**9700.0**	**58800.0**	**9700.0**
B_{20}	Min	**11447.0**	**1020.0**	**11447.0**	**1020.0**	**11447.0**	**1020.0**	**11447.0**	**1020.0**	**11447.0**	**1020.0**
	Avg	**12207.8**	**1020.0**	**12207.8**	**1020.0**	**12207.8**	**1020.0**	**12207.8**	**1020.0**	**12207.8**	**1020.0**
	Max	**12905.0**	**1020.0**	**12905.0**	**1020.0**	**12905.0**	**1020.0**	**12905.0**	**1020.0**	**12905.0**	**1020.0**
	Sum	**122078.0**	**10200.0**	**122078.0**	**10200.0**	**122078.0**	**10200.0**	**122078.0**	**10200.0**	**122078.0**	**10200.0**

Bold indicates smallest value

the data set is composed of 16 instances, which range from two to six types of boxes, with availability varying from 47 to 200 items, and with two and three different container types, with availability ranging from two to 15 containers. The objective is to maximize the value associated with the cargo. Again, different methods have been used to solve these instances, and we compare our results against:

- MOH: the sequential solution strategy using a column generation of Mohanty et al. (1994);
- BOR: the heuristic box and container selection criteria of Bortfeldt (2000);
- ELY: the bottleneck approach of Eley (2003);
- TAK: the multi-start local search procedure of Takahara (2008);
- REN: the column generation procedure of Ren et al. (2011).

The study of Eley (2003) also considers the separation of boxes for the MHLOPP and is used as a benchmark.

Since we were unable to find instances for the MILOPP, we use those of Mohanty et al. (1994) for the MHLOPP to generate 42 instances decomposing them, similarly as the instances of SSSCSP were generated by Ivancic et al. (1989). Since in the MILOPP the number of containers is given, we chose as the number of containers the maximum availability in the original problem, not having any instance with fewer than two containers.

5.2. Comparison of the discretization techniques

In order to assess the discretization techniques presented in Section 3, we first solve the single container problem. We restricted the number of containers, the number of types of containers, and the orientations that the boxes may take in the mathematical formulation given by (12)–(15). We also solved the same instances considering a more general framework, in which the boxes can take up to six orientations. Likewise, we did the same solving the problems considering the stability requirement, given by constraints (19).

Due to space limitations, we only present results for the problem defined by (12)–(15). However, the results for different scenarios, with different discretizations in every problem solved in this paper can be downloaded from https://www.leandro-coelho.com/container-loading-problems/. Table 1 presents the minimum, average, maximum, and total number of variables and constraints for the instances of Junqueira et al. (2012) in all different discretizations presented in this paper, considering a container with $L = W = H = 10$. We also add the information regarding the size of the models using the unit discretization (UD).

Although the NP and the RNP yield similar results in terms of the number of variables and constraints, the RNP presented slightly smaller numbers than the NP. The RRP and the MiM can generate smaller mathematical models than all other discretiza-

Table 2
Comparison between the discretization techniques using the instances of Ivancic et al. (1989) and Mohanty et al. (1994).

Problem		NP		RRP		RNP		MiM	
		Variables	Constraints	Variables	Constraints	Variables	Constraints	Variables	Constraints
MILOPP	Min	**130.00**	**52.00**	**130.00**	**52.00**	**130.00**	**52.00**	**130.00**	**52.00**
	Avg	37523.64	5228.33	**31799.17**	**4000.86**	37523.64	5228.33	32666.74	4121.19
	Max	**337924.00**	38051.00	**337924.00**	**25764.00**	**337924.00**	38051.00	**337924.00**	**25764.00**
	Sum	1575993.00	219590.00	**1335565.00**	**168036.00**	1575993.00	219590.00	1372003.00	173090.00
MHLOPP	Min	**1493.00**	**279.00**	**1493.00**	**279.00**	**1493.00**	**279.00**	**1493.00**	**279.00**
	Avg	59385.06	8468.75	**51612.13**	**6655.00**	59385.06	8468.75	52750.81	6812.94
	Max	**264270.00**	35677.00	**264270.00**	**22636.00**	**264270.00**	35677.00	**264270.00**	**22636.00**
	Sum	950161.00	135500.00	**825794.00**	**106480.00**	950161.00	135500.00	844013.00	109007.00
SSSCSP	Min	**1458.00**	**435.00**	**1458.00**	**435.00**	**1458.00**	**435.00**	**1458.00**	**435.00**
	Avg	108078.23	15255.81	**98276.96**	**12903.66**	108078.23	15255.81	99173.64	13056.98
	Max	884170.00	75604.00	**838042.00**	**62612.00**	884170.00	75604.00	**838042.00**	**62612.00**
	Sum	5079677.00	717023.00	**4619017.00**	**606472.00**	5079677.00	717023.00	4661161.00	613678.00
MSSCSP	Min	**8988.00**	**1755.00**	**8988.00**	**1755.00**	**8988.00**	**1755.00**	**8988.00**	**1755.00**
	Avg	430185.94	56358.53	**390685.71**	**47166.65**	430185.94	56358.53	393991.12	47731.82
	Max	3337194.00	297042.00	**3147992.00**	**243676.00**	3337194.00	297042.00	**3147992.00**	**243676.00**
	Sum	7313161.00	958095.00	**6641657.00**	**801833.00**	7313161.00	958095.00	6697849.00	811441.00

Bold indicates smallest value.

Table 3
Results for instances A_1.

	Stat.	NP with rotation				MiM with rotation			
		Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)
10	Min.	10/10	0.00	0.00	64.00	10/10	0.00	0.00	64.00
	Avg.		0.00	2.15	79.42		0.00	2.24	79.42
	Max.		0.00	21.09	96.00		0.00	22.01	96.00
20	Min.	10/10	0.00	0.00	39.20	10/10	0.00	0.00	39.20
	Avg.		0.00	0.01	67.35		0.00	0.01	67.35
	Max.		0.00	0.06	90.00		0.00	0.04	90.00
30	Min.	10/10	0.00	0.00	37.33	10/10	0.00	0.00	37.33
	Avg.		0.00	0.01	59.43		0.00	0.01	59.43
	Max.		0.00	0.10	80.00		0.00	0.05	80.00
50	Min.	10/10	0.00	0.00	33.18	10/10	0.00	0.00	33.18
	Avg.		0.00	0.03	59.74		0.00	0.02	59.74
	Max.		0.00	0.15	87.09		0.00	0.12	87.09
100	Min.	10/10	0.00	0.00	20.33	10/10	0.00	0.00	20.33
	Avg.		0.00	0.04	54.11		0.00	0.02	54.11
	Max.		0.00	0.20	97.20		0.00	0.12	97.20

Table 4
Results for instances A_5.

	Stat.	NP without rotation				MiM without rotation				NP with rotation				MiM with rotation			
		Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (s)	Vol. (%)	Feas./Opt.	Gap (%)	Time (s)	Vol. (%)
10	Min.	10/10	0.00	0.03	77.20	10/10	0.00	0.02	77.20	10/10	0.00	0.40	88.00	10/10	0.00	0.40	88.00
	Avg.		0.00	0.33	91.34		0.00	0.32	91.34		0.00	196.57	97.32		0.00	208.70	97.32
	Max.		0.00	1.55	100.00		0.00	1.51	100.00		0.00	1752.75	100.00		0.00	1869.38	100.00
20	Min.	10/10	0.00	0.01	60.69	10/10	0.00	0.01	60.69	10/10	0.00	0.83	74.98	10/10	0.00	0.31	74.98
	Avg.		0.00	0.17	78.54		0.00	0.10	78.54		0.00	89.80	91.47		0.00	62.64	91.47
	Max.		0.00	0.79	91.10		0.00	0.25	91.10		0.00	312.40	99.00		0.00	289.17	99.00
30	Min.	10/10	0.00	0.00	42.93	10/10	0.00	0.00	42.93	10/10	0.00	0.14	54.33	10/10	0.00	0.03	54.33
	Avg.		0.00	0.02	60.90		0.00	0.01	60.90		0.00	30.82	77.29		0.00	10.38	77.29
	Max.		0.00	0.06	76.18		0.00	0.05	76.18		0.00	184.14	95.78		0.00	81.97	95.78
50	Min.	10/10	0.00	0.01	51.48	10/10	0.00	0.01	51.48	10/8	0.00	27.71	71.75	10/8	0.00	5.60	71.75
	Avg.		0.00	0.08	70.08		0.00	0.04	70.08		1.06	1121.08	85.31		0.59	838.23	85.66
	Max.		0.00	0.40	84.85		0.00	0.17	84.85		5.47	3600.00	92.95		3.59	3600.00	94.59
100	Min.	10/10	0.00	0.00	48.70	10/10	0.00	0.00	48.70	9/8	0.00	17.03	62.24	10/10	0.00	0.61	62.24
	Avg.		0.00	0.05	68.24		0.00	0.02	68.24		0.51	1172.24	78.96		0.00	118.40	79.58
	Max.		0.00	0.19	85.04		0.00	0.07	85.04		4.57	3600.00	86.88		0.00	477.31	91.22

Table 5
Results for instances A_{10}.

	Stat.	NP without rotation				MiM without rotation				NP with rotation				MiM with rotation			
		Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)
10	Min.	10/10	0.00	0.15	98.40	10/10	0.00	0.15	98.40	10/10	0.00	1.26	100.00	10/10	0.00	1.21	100.00
	Avg.		0.00	7.29	99.54		0.00	6.95	99.54		0.00	6.95	100.00		0.00	5.48	100.00
	Max.		0.00	34.30	100.00		0.00	34.10	100.00		0.00	16.25	100.00		0.00	9.77	100.00
20	Min.	10/10	0.00	0.13	76.45	10/10	0.00	0.06	76.45	10/8	0.00	123.06	92.64	10/8	0.00	123.41	92.64
	Avg.		0.00	5.37	89.95		0.00	3.94	89.95		0.09	1251.33	97.65		0.09	1257.04	97.65
	Max.		0.00	26.48	94.30		0.00	18.96	94.30		0.43	3600.00	99.55		0.43	3600.00	99.55
30	Min.	10/10	0.00	0.09	70.06	10/10	0.00	0.02	70.06	10/9	0.00	43.06	90.48	10/9	0.00	20.01	90.48
	Avg.		0.00	2.80	83.05		0.00	1.35	83.05		0.11	718.04	94.54		0.11	621.73	94.54
	Max.		0.00	14.39	93.56		0.00	5.95	93.56		1.11	3600.00	97.60		1.11	3600.00	97.60
50	Min.	10/10	0.00	0.25	77.36	10/10	0.00	0.12	77.36	8/2	0.00	736.94	65.75	10/4	0.00	323.15	89.18
	Avg.		0.00	10.66	82.62		0.00	5.46	82.62		3.11	3223.30	89.38		2.78	2631.60	92.98
	Max.		0.00	39.44	87.13		0.00	36.55	87.13		8.46	3600.00	95.66		6.47	3600.00	95.66
100	Min.	9/9	0.00	0.63	65.70	9/9	0.00	0.12	65.70	1/1	0.00	1463.48	66.70	9/3	0.00	547.90	63.02
	Avg.		0.00	365.42	83.11		0.00	360.89	83.11		0.00	3386.35	77.82		2.31	2854.54	87.94
	Max.		0.00	3600.00	93.89		0.00	3600.00	93.89		0.00	3600.00	92.91		5.92	3600.00	96.54

Table 6
Results for instances A_{20}.

	Stat.	NP without rotation				MiM without rotation				NP with rotation				MiM with rotation			
		Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)	Feas./opt.	Gap (%)	Time (second)	Vol. (%)
10	Min.	10/10	0.00	0.57	99.60	10/10	0.00	0.57	99.60	10/10	0.00	2.17	99.60	10/10	0.00	2.08	99.60
	Avg.		0.00	2.12	99.96		0.00	2.12	99.96		0.00	11.30	99.96		0.00	10.93	99.96
	Max.		0.00	4.61	100.00		0.00	4.61	100.00		0.00	26.70	100.00		0.00	27.91	100.00
20	Min.	10/10	0.00	10.02	90.71	10/10	0.00	4.76	90.71	10/7	0.00	52.37	98.45	10/7	0.00	49.48	98.45
	Avg.		0.00	46.65	95.19		0.00	35.40	95.19		0.20	2007.97	99.50		0.17	1985.58	99.52
	Max.		0.00	104.19	97.85		0.00	93.55	97.85		0.88	3600.00	100.00		0.88	3600.00	100.00
30	Min.	10/10	0.00	6.54	86.42	10/10	0.00	3.22	86.42	10/3	0.00	329.23	96.37	10/4	0.00	256.27	96.37
	Avg.		0.00	20.05	91.84		0.00	18.76	91.84		1.48	3007.41	97.63		1.34	3047.06	97.72
	Max.		0.00	41.56	97.21		0.00	83.20	97.21		3.28	3600.00	100.00		3.28	3600.00	100.00
50	Min.	10/9	0.00	36.80	85.56	10/10	0.00	15.10	85.56	0/0	-	3600.00	62.75	0/0	-	3600.00	62.75
	Avg.		0.31	993.05	91.49		0.00	284.81	91.49		-	3600.00	77.16		-	3600.00	77.16
	Max.		3.19	3600.00	94.60		0.00	1490.58	94.60		-	3600.00	85.55		-	3600.00	85.55
100	Min.	8/7	0.00	113.31	79.71	9/9	0.00	11.25	79.71	0/0	-	3600.00	80.57	0/0	-	3600.00	63.27
	Avg.		0.64	1796.03	86.99		0.00	503.65	87.42		-	3600.00	80.57		-	3600.00	75.81
	Max.		6.46	3600.00	93.87		0.00	3600.00	93.87		-	3600.00	80.57		-	3600.00	85.23

tions. Although the difference between the statistics presented for the RRP and the MiM is small when compared with the NP, in some cases, these discretizations can obtain models with 66.36% fewer variables and 85.32% fewer constraints than their counterparts generated using the NP. When compared with the UD, the reductions are even more significant – in some cases, resulting in models with 99% fewer variables and constraints. A similar outcome is observed using the discretizations with the instances used for MILOPP, MHLOPP, SSSCSP, and MSSCSP, as can be seen in Table 2.

It is interesting to observe that, although in our tests the models generated using RNP always have more constraints and variables than the ones obtained by the RRP, there is no established dominance relation between the cardinality of the sets generated using the RRP and RNP. Sometimes the RRP can generate fewer points than the RNP, while in other cases, the opposite may occur, and these discretizations can even yield the same points. Côté and Iori (2018) demonstrated that the MiM patterns generate, at most, the same number of points of the RNP. Although no relation holds between the RRP and the MiM, an interesting feature of the MiM pattern is that it ensures that no loss of generality occurs when used in general cutting and packing problems (Côté and Iori, 2018).

The loss of optimal solutions was a concern in tests considering the stability of the cargo. So, two preliminary tests were made: first, all dimensions of the container were discretized using one of the techniques presented in Section 3 (including the UD) and the support factor value $\alpha = 1$; we observed loss of generality in at least one instance in all discretizations, except when using the NP, RNP, and the UD. In the second round of tests, we used RRP, RNP, and MiM to discretize the floor of the container, using the NP for the height axis, in an attempt to obtain more points to place the

Table 7
Bounds for the SSSCSP for the instances of Ivancic et al. (1989).

#	LB (Eley, 2002)	Without practical constraints			With stability			With separation of boxes			With stability and separation of boxes		
		$\mathcal{L}$	$\mathcal{U}$	Time (second)	$\mathcal{L}$	$\mathcal{U}$	Time (second)	$\mathcal{L}$	$\mathcal{U}$	Time (second)	$\mathcal{L}$	$\mathcal{U}$	Time (second)
1	19	19	25	11.03	19	25	2.38	19	27	2.63	19	27	2.87
2	7	7	10	6.87	7	10	5.30	8	10	4.46	8	10	5.89
3	19	**19**	**19**	8.83	**19**	**19**	8.10	19	22	10.64	19	22	10.83
4	26	**26**	**26**	8.97	**26**	**26**	5.15	26	28	6.82	26	28	8.35
5	46	46	51	4.87	46	51	4.88	46	51	7.78	46	51	7.95
6	10	**10**	**10**	4.52	**10**	**10**	4.62	**10**	**10**	5.54	**10**	**10**	5.87
7	16	**16**	**16**	3.28	**16**	**16**	3.27	**16**	**16**	4.20	**16**	**16**	4.07
8	4	**4**	**4**	6.55	**4**	**4**	6.19	**4**	**4**	8.21	**4**	**4**	7.91
9	16	17	19	2.20	17	19	2.24	18	19	2.61	18	19	2.58
10	37	43	55	2.46	43	55	2.32	47	55	2.58	47	55	3.74
11	14	14	16	3.08	14	16	2.76	16	18	3.42	16	18	3.94
12	45	**53**	**53**	3.88	**53**	**53**	3.36	53	56	4.52	53	56	7.72
13	20	21	25	8.19	21	25	4.15	22	25	4.74	22	25	7.06
14	27	**27**	**27**	6.52	**27**	**27**	3.64	28	29	4.17	28	29	6.98
15	11	**11**	**11**	7.28	**11**	**11**	3.81	11	12	4.61	11	12	7.85
16	21	21	26	3.23	21	26	3.36	21	26	4.26	21	26	5.98
17	7	**7**	**7**	6.22	**7**	**7**	5.95	7	9	6.10	7	9	8.12
18	1	**2**	**2**	14.69	**2**	**2**	10.78	**2**	**2**	8.59	**2**	**2**	9.81
19	1	**3**	**3**	10.66	**3**	**3**	8.09	**3**	**3**	7.63	**3**	**3**	10.32
20	2	4	5	4.45	4	5	4.17	4	5	4.75	4	5	6.29
21	17	17	20	7.62	17	20	7.11	17	20	10.74	17	20	15.00
22	8	**8**	**8**	15.00	**8**	**8**	13.52	**8**	**8**	15.00	**8**	**8**	15.00
23	17	17	21	5.50	17	21	4.93	17	21	7.15	17	21	9.26
24	5	**5**	**5**	11.50	**5**	**5**	9.52	**5**	**5**	12.08	**5**	**5**	14.97
25	4	4	5	5.96	4	5	5.17	4	5	6.92	4	5	7.75
26	3	**3**	**3**	15.00	**3**	**3**	15.00	**3**	**3**	15.00	**3**	**3**	15.00
27	4	**4**	**4**	6.71	**4**	**4**	5.77	4	5	6.98	4	5	7.12
28	9	**9**	**9**	5.27	**9**	**9**	4.78	9	10	5.89	9	10	7.33
29	15	15	16	5.48	15	17	4.96	15	16	6.77	15	17	7.33
30	18	18	22	5.34	18	22	4.82	18	23	5.98	18	23	6.12
31	11	11	13	9.51	11	13	6.05	11	13	7.95	11	13	7.90
32	4	**4**	**4**	10.98	**4**	**4**	8.47	**4**	**4**	6.67	**4**	**4**	6.85
33	4	**4**	**4**	8.91	**4**	**4**	6.77	4	5	7.23	4	5	6.60
34	5	7	8	4.67	7	8	4.42	7	8	5.09	7	8	5.17
35	2	**2**	**2**	5.21	**2**	**2**	4.36	**3**	**3**	3.58	**3**	**3**	5.55
36	10	10	14	3.55	10	14	2.81	11	19	2.80	11	19	5.00
37	12	12	23	7.26	12	23	5.90	12	23	6.84	12	23	7.52
38	25	26	45	6.49	26	45	4.95	26	45	6.16	26	45	7.12
39	12	12	15	6.24	12	15	4.29	12	15	4.64	12	15	5.25
40	7	7	8	13.74	7	8	11.15	7	9	12.48	7	9	12.99
41	14	14	15	5.88	14	15	5.00	14	16	7.13	14	16	7.54
42	4	**4**	**4**	12.00	**4**	**4**	7.52	**4**	**4**	6.65	**4**	**4**	6.56
43	3	**3**	**3**	15.00	**3**	**3**	11.38	**3**	**3**	9.53	**3**	**3**	11.46
44	3	3	4	6.64	3	4	5.58	3	4	6.81	3	4	10.17
45	2	2	3	15.00	2	3	15.00	2	3	15.00	2	3	15.00
46	2	**2**	**2**	15.00	**2**	**2**	15.00	**2**	**2**	15.00	**2**	**2**	15.00
47	3	**3**	**3**	15.00	**3**	**3**	15.00	3	4	15.00	3	4	15.00
Sum	572	596	693		596	694		608	723		608	724	
Avg.				7.92			6.46			7.22			8.29

Bold indicates optimal solution

boxes in a stable fashion. Here, again, there was a loss of optimal solution using RRP and MiM. We provide a numerical example in Appendix B.

In a final remark regarding the stability of the load, we note that there is no loss of generality using the NP only if the minimum area of contact is $\alpha = 1$, i.e., the bottom of the boxes must be fully supported (Junqueira et al., 2012). Different procedures to tackle stability, which consider mechanical forces acting in the load, have been proposed (e.g., Silva, Soma, & Maculan, 2003, Ramos, Oliveira, & Lopes, 2016 and Bracht, de Queiroz, Schouery, & Miyazawa, 2016).

Tables 3,4,5,6 show the minimum, average, and maximal values obtained by the NP and the MiM discretizations for the test sets A_m for the single container problem given by (12)–(15) with and without allowing the rotation of boxes. The first column represents the size of the containers, followed by information on the number of feasible and optimal solutions. Note that the number of optimal solutions obtained using MiM increases with the value of m when compared against the NP.

5.3. Results for input minimization problems

In this section we provide results for input minimization problems. In Section 5.3.1 we describe the results obtained for the SSSCSP, and in Section 5.3.2 for the MSSCSP.

Table 8
Results for the instances of Ivancic et al. (1989).

#	IVA	BOR	ELY	LZG	CHE	LIM	ZHU	This paper		
								Containers	Time (second)	Deviation to the BKS
1	26	*25*	*25*	*25*	*25*	*25*	*25*	***25***	0.04	0
2	11	10	10	10	10	10	10	***9***	2.66	−1
3	20	20	20	*19*	*19*	*19*	*19*	***19***	8.83	0
4	27	28	*26*	*26*	*26*	*26*	*26*	***26***	8.97	0
5	65	*51*	*51*	*51*	*51*	*51*	*51*	***51***	82.09	0
6	*10*	*10*	*10*	*10*	*10*	*10*	*10*	***10***	4.52	0
7	*16*	*16*	*16*	*16*	*16*	*16*	*16*	***16***	3.28	0
8	5	*4*	*4*	*4*	*4*	*4*	*4*	***4***	6.55	0
9	*19*	*19*	*19*	*19*	*19*	*19*	*19*	***19***	0.85	0
10	*55*	*55*	*55*	*55*	*55*	*55*	*55*	***55***	0.02	0
11	18	18	17	*16*	*16*	*16*	*16*	***16***	31.82	0
12	55	*53*	*53*	*53*	*53*	*53*	*53*	***53***	3.88	0
13	27	*25*	*25*	*25*	*25*	*25*	*25*	***25***	1.30	0
14	28	28	*27*	*27*	*27*	*27*	*27*	***27***	6.52	0
15	*11*	*11*	*11*	*11*	*11*	*11*	*11*	***11***	7.28	0
16	34	*26*	*26*	*26*	*26*	*26*	*26*	***26***	0.76	0
17	8	*7*	*7*	*7*	*7*	*7*	*7*	***7***	6.22	0
18	3	*2*	*2*	*2*	*2*	*2*	*2*	***2***	14.69	0
19	*3*	*3*	*3*	*3*	*3*	*3*	*3*	***3***	10.66	0
20	*5*	*5*	*5*	*5*	*5*	*5*	*5*	***5***	49.90	0
21	24	21	*20*	*20*	*20*	*20*	*20*	***20***	3217.77	0
22	10	9	*8*	9	*8*	9	*8*	***8***	15.00	0
23	21	20	20	20	*19*	20	*19*	***19***	930.95	0
24	6	6	6	*5*	*5*	*5*	*5*	***5***	11.50	0
25	6	*5*	*5*	*5*	*5*	*5*	*5*	*5*	3600.00	0
26	*3*	*3*	*3*	*3*	*3*	*3*	*3*	***3***	15.00	0
27	5	5	5	5	5	5	*4*	***4***	6.71	0
28	10	10	10	*9*	10	*9*	10	***9***	5.27	0
29	18	17	17	17	17	17	17	***16***	1116.65	−1
30	24	*22*	*22*	*22*	*22*	*22*	*22*	*22*	3600.00	0
31	13	13	13	*12*	*12*	*12*	*12*	13	3600.00	1
32	5	*4*	*4*	*4*	*4*	*4*	*4*	***4***	10.98	0
33	5	5	5	*4*	*4*	*4*	*4*	***4***	8.91	0
34	9	*8*	*8*	*8*	*8*	*8*	*8*	***8***	26.41	0
35	3	2	2	2	2	2	2	***2***	5.21	0
36	18	*14*	*14*	*14*	*14*	*14*	*14*	***14***	0.39	0
37	26	*23*	*23*	*23*	*23*	*23*	*23*	***23***	3.14	0
38	50	*45*	*45*	*45*	*45*	*45*	*45*	***45***	0.77	0
39	16	*15*	*15*	*15*	*15*	*15*	*15*	***15***	2139.96	0
40	9	9	8	9	*8*	9	*8*	***8***	104.49	0
41	16	*15*	*15*	*15*	*15*	*15*	*15*	***15***	27.96	0
42	*4*	*4*	*4*	*4*	*4*	*4*	*4*	***4***	12.00	0
43	*3*	*3*	*3*	*3*	*3*	*3*	*3*	***3***	15.00	0
44	4	*3*	4	*3*	*3*	*3*	*3*	***3***	747.40	0
45	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	3600.00	0
46	*2*	*2*	*2*	*2*	*2*	*2*	*2*	***2***	15.00	0
47	4	*3*	*3*	*3*	*3*	*3*	*3*	***3***	15.00	0
Sum	763	705	699	694	692	694	691	689		
Avg. Time	–	–	30.00	6.43	45.94	6.43	246.20		491.54	

Bold indicates optimal solution
Italics indicates the BKS

5.3.1. SSSCSP

The formulations for the SSSCSP presented in this paper were tested in four different configurations. First, we tested the basic problem, defined by (7)–(10) with $\mathcal{K} = 1$. Then, we added the separation of boxes requirement, given by formulation (16)–(18), followed by the consideration of stability, given by (19). Finally, we considered a scenario ensuring the separation of boxes and requiring stability of the load simultaneously.

After an upper bound is obtained through the procedures of Section 4.1, the set partitioning model is solved. After that, optimal solutions were already obtained by comparing the upper and lower bounds in 23 instances for the general case, 23 for the scenario with stability, 13 when the separation of boxes are considered, and also 13 when the separation of boxes and stability are considered simultaneously, as highlighted in bold in Table 7. Our lower bound is tighter than the one computed by Eley (2002), which we refer as *LB(Eley, 2002)* in the table. The lower bounds, the upper bounds, and the corresponding loading pattern for each instance were computed in an average time of fewer than 10 seconds.

Table 8 compares our results for the basic problem against the competing methods described in Section 5.1, where optimal solutions are highlighted in bold and best-known solutions (BKS) in italics. We improved two solutions and matched the BKS in 46 out of 47 instances of this set. Moreover, we prove optimality for 43 instances, the highest number in literature. We obtained significant results, optimally solving 91.48% of the instances. Comparing Tables 7 and 8, besides the 23 instances in which optimality was

Table 9
Results for the instances of Ivancic et al. (1989) with load stability.

#	ELS	ELP	ZST	This paper		
				Containers	Time (second)	Deviation to the BKS
1	27	26	*25*	***25***	0.06	0
2	11	*10*	*10*	*10*	3600.00	0
3	21	22	*19*	***19***	8.10	0
4	29	30	*26*	***26***	5.15	0
5	55	*51*	*51*	***51***	44.83	0
6	*10*	*10*	*10*	***10***	4.62	0
7	*16*	*16*	*16*	***16***	3.27	0
8	*4*	*4*	*4*	***4***	6.19	0
9	*19*	*19*	*19*	***19***	13.45	0
10	*55*	*55*	*55*	***55***	0.03	0
11	17	18	17	***16***	27.09	−1
12	*53*	*53*	*53*	***53***	3.36	0
13	*25*	*25*	*25*	***25***	8.07	0
14	*27*	*27*	*27*	***27***	3.64	0
15	12	12	*11*	***11***	3.81	0
16	28	*26*	*26*	***26***	3.14	0
17	8	*7*	*7*	***7***	5.95	0
18	*2*	*2*	*2*	***2***	10.78	0
19	*3*	*3*	*3*	***3***	8.09	0
20	*5*	*5*	*5*	***5***	873.75	0
21	24	26	*20*	*20*	3600.00	0
22	9	9	*8*	***8***	13.52	0
23	21	21	*20*	21	3600.00	1
24	6	6	*5*	***5***	9.52	0
25	6	*5*	*5*	*5*	3600.00	0
26	*3*	*3*	*3*	***3***	15.00	0
27	5	5	*4*	***4***	5.77	0
28	11	10	10	***9***	4.78	−1
29	18	18	*17*	*17*	3600.00	0
30	*22*	23	*22*	*22*	3600.00	0
31	13	14	*12*	13	3600.00	1
32	*4*	*4*	*4*	***4***	8.47	0
33	5	5	*4*	***4***	6.77	0
34	*8*	9	*8*	***8***	297.94	0
35	*2*	*2*	*2*	***2***	4.36	0
36	18	*14*	*14*	***14***	0.80	0
37	26	*23*	*23*	***23***	11.27	0
38	46	*45*	*45*	***45***	2.70	0
39	*15*	*15*	*15*	*15*	3600.00	0
40	9	9	*8*	*8*	3600.00	0
41	16	*15*	*15*	***15***	338.58	0
42	*4*	*4*	*4*	***4***	7.52	0
43	*3*	*3*	*3*	***3***	11.38	0
44	4	4	*3*	***3***	3523.13	0
45	*3*	*3*	*3*	*3*	3600.00	0
46	*2*	*2*	*2*	***2***	15.00	0
47	*3*	*3*	*3*	***3***	15.00	0
Sum	733	721	693	693		
Avg. time	–	–	116.7		880.89	

Bold indicates optimal solution
Italics indicates the BKS

proved early in Table 7, the set partitioning approach found the optimal solution in 19 cases, later proved by solving the models. Overall, the sum of containers needed to pack the boxes is 689, two fewer than the approach of Zhu et al. (2012), currently the best in CLP literature.

Table 9 shows the results considering the stability constraint. We compare our results against the tree search heuristic with sequential packing (ELS) and the tree search strategy with parallel packing (ELP) of Eley (2002), and the column generation approach with full base support requirement (ZST) of Zhu et al. (2012). Out of the 47 instances, we proved optimality for 37 (highlighted in bold). Our results were equal to the BKS in 44 instances, and we improved the BKS for two cases, requiring one fewer container in each instance to load the boxes, improving the best results from the literature.

Table 10 presents the results considering the separation of boxes. Out of 47 instances, we proved the optimality in 37, improving the BKS in 11 instances. Overall, our approach needed 10 fewer containers to pack all the boxes than the procedure of Eley (2003).

5.3.2. MSSCSP

The formulations for the MSSCSP presented in this paper were also tested in four different configurations: the basic problem, defined by (7)–(10), considering the separation of boxes ((16)–(17)), followed by the stability requirement, given by (19), and the separation of boxes and stability of the load simultaneously. Since no benchmark exists for problems with practical constraints, we present these results, as well as the lower and upper bounds for the MSSCSP in all addressed cases, in Appendix C.

Table 10
Results for the instances of Ivancic et al. (1989) with separation of boxes.

#	ELY	This paper				
		With separation of boxes			With stability and separation of boxes	
		Containers	Time (second)	Deviation to the BKS	Containers	Time (second)
1	*27*	***27***	1.60	0	**27**	1.95
2	11	***10***	6.50	−1	**10**	178.80
3	*20*	***20***	69.08	0	**20**	2075.38
4	*28*	***28***	19.27	0	**28**	104.18
5	*51*	***51***	20.04	0	**51**	47.40
6	*10*	***10***	5.54	0	**10**	5.87
7	*16*	***16***	4.20	0	**16**	4.07
8	*4*	***4***	8.21	0	**4**	7.91
9	*19*	***19***	0.42	0	**19**	14.64
10	*55*	***55***	0.03	0	**55**	0.05
11	18	***17***	13.82	−1	18	3600.00
12	*56*	***56***	2.05	0	56	3600.00
13	*25*	***25***	1.37	0	**25**	8.21
14	*28*	***28***	0.79	0	**28**	23.93
15	*12*	***12***	45.81	0	**12**	2597.52
16	*26*	***26***	0.79	0	**26**	2.72
17	*9*	***9***	422.91	0	9	3600.00
18	*2*	***2***	8.59	0	**2**	9.81
19	*3*	***3***	7.63	0	**3**	10.32
20	*5*	***5***	105.42	0	**5**	1027.68
21	21	***20***	2712.63	−1	20	3600.00
22	*8*	***8***	15.00	0	**8**	15.00
23	21	*20*	3600.00	−1	21	3600.00
24	6	***5***	12.08	−1	**5**	14.97
25	*5*	*5*	3600.00	0	5	3600.00
26	*3*	***3***	15.00	0	**3**	15.00
27	*5*	*5*	3600.00	0	5	3600.00
28	10	***9***	1176.35	−1	10	3600.00
29	17	***16***	1112.28	−1	17	3600.00
30	23	*22*	3600.00	−1	23	3600.00
31	*13*	*13*	3600.00	0	13	3600.00
32	*4*	***4***	6.67	0	**4**	6.85
33	*5*	*5*	3600.00	0	5	3600.00
34	9	***8***	31.93	−1	**8**	281.09
35	*3*	***3***	3.58	0	**3**	5.55
36	*18*	***18***	4.40	0	**18**	34.33
37	*23*	***23***	3.29	0	**23**	10.98
38	*45*	***45***	0.83	0	**45**	2.74
39	*15*	***15***	1008.68	0	15	3600.00
40	*9*	*9*	3600.00	0	9	3600.00
41	*16*	***16***	155.73	0	16	3600.00
42	5	***4***	7.13	−1	**4**	6.56
43	4	***3***	6.65	−1	**3**	11.46
44	*4*	*4*	3600.00	0	4	3600.00
45	*3*	*3*	3600.00	0	3	3600.00
46	*2*	***2***	15.00	0	**2**	15.00
47	*3*	4	3600.00	1	4	3600.00
Sum	725	715			720	
Avg. Time			915.56			1517.87

Bold indicates optimal solution
Italics indicates the BKS

Table 11 compares our results for the basic problem with the methods described in Section 5.1. We present the average volumetric occupation of the containers as well as the number of containers used in the solution. We highlight the optimal solutions in bold and the BKS in italics. In 16 out of 17 instances, our solution is equal to, if not better than the BKS, and in five cases we found an optimal solution.

5.4. Results for output maximization problems

We now describe the results for maximization problems. In Section 5.4.1 we present our results for the MILOPP, and in Section 5.4.2 those of the MHLOPP.

5.4.1. MILOPP

The formulations for the MILOPP presented in this paper were tested in four different configurations. First, we tested the basic problem, defined by (12)–(15) with $\mathcal{K} = 1$. Then, we considered situations with the separation of the boxes, given by (16)–(18), as well as with the consideration of stability, given by (19). Finally, we solved problems considering the separation of boxes and stability. Again, because we are the first to provide solutions for these problems, no benchmark exists, and we show our results in Appendix D.

5.4.2. MHLOPP

To evaluate the formulation for the MHLOPP, we considered four different situations: first, we only assessed the formulation

Table 11
Results for for MSSCSP for the instances of Ivancic et al. (1989).

#	IVA	BOR	ELY	TAK	REN	This paper		
						Vol. (#)	Time (second)	Gap (%)
1	71.8 (26/0)	74.7 (25/0)	74.7 (25/0)	74.7 (25/0)	74.7 (25/0)	***78.2 (1/8)***	5.91	0.00
2	97.6 (7/13/6)	95.1 (1/19/11)	*99.9 (2/13/17)*	99.7 (7/15/1)	99.1 (2/22/3)	***99.9 (2/18/8)***	118.53	0.00
3	97.6 (4/6/1)	*99.7 (4/1/2)*	*99.7 (7/4/0)*	99.6 (7/1/0)	*99.7 (4/1/2)*	***99.7 (7/4/0)***	1.06	0.00
4	85.8 (10/1/7)	86.8 (16/0/2)	*87.4 (2/0/14)*	87.1 (9/0/8)	*87.4 (2/0/14)*	*87.4 (2/0/14)*	3600.00	0.86
5	95.8 (3/0/26)	97.9 (7/0/23)	*99.4 (3/0/25)*	98.7 (1/1/25)	98.7 (0/5/20)	***99.4 (3/0/25)***	5.60	0.00
6	92.2 (7/6/1)	96.6 (8/5/0)	96.6 (6/9/0)	96.6 (7/7/0)	96.6 (7/7/0)	*98.9 (3/2/4)*	3600.00	0.61
7	90.6 (1/0/2)	90.6 (1/0/2)	90.6 (1/0/2)	90.6 (1/0/2)	90.6 (1/0/2)	*95.5 (1/1/0)*	3600.00	2.28
8	81.2 (3/3/11)	85.9 (0/4/10)	88.4 (9/2/5)	87.7 (1/6/4)	*91.0 (2/7/0)*	*91.0 (2/7/0)*	3600.00	8.36
9	75.0 (5/1/0)	90.2 (2/1/1)	*93.5 (3/0/1)*	92.7 (5/0/0)	92.7 (5/0/0)	*93.5 (3/0/1)*	3600.00	4.80
10	87.3 (2/5)	87.3 (2/5)	88.5 (1/7)	88.5 (1/7)	*95.0 (4/0)*	*95.0 (4/0)*	3600.00	4.41
11	85.3 (9/1/5)	87.8 (14/1/1)	86.3 (13/1/2)	85.0 (5/11/2)	87.2 (7/3/5)	*91.4 (12/1/2)*	3600.00	5.46
12	88.7 (0/2/4)	*94.0 (2/1/1)*	*94.0 (2/1/1)*	*94.0 (2/1/1)*	*94.0 (2/1/1)*	*94.0 (2/1/1)*	3600.00	5.97
13	74.3 (1/8)	*92.2 (2/0)*	*92.2 (2/0)*	*92.2 (2/0)*	*92.2 (2/0)*	*92.2 (2/0)*	3600.00	7.43
14	76.3 (3/2/11)	79.1 (3/3/10)	79.2 (2/3/11)	78.7 (3/1/11)	*81.3 (3/0/11)*	***81.3 (3/0/11)***	986.09	0.00
15	84.1 (1/14)	89.0 (0/15)	89.0 (0/15)	89.5 (2/11)	89.2 (1/13)	*90.2 (5/5)*	3600.00	3.51
16	82.7 (4/0/0)	*91.6 (2/1/0)*	*91.6 (2/1/0)*	82.9 (1/1/1)	*91.6 (2/1/0)*	82.8 (1/1/1)	3600.00	15.28
17	77.1 (1/0/2)	84.7 (0/0/3)	84.7 (0/0/3)	*91.6 (0/1/1)*	*91.6 (0/1/1)*	*91.6 (0/1/1)*	3600.00	6.56
Avg. vol.	84.9	89.6	90.3	90.0	91.3	91.9		
Avg. time	–	–	30.00	–	20.00	2606.89		

Bold indicates optimal solution.
Italics indicates the BKS.

Table 12
Results for the test sets of Mohanty et al. (1994).

#	MOH	BOR	ELY	TAK	REN	This paper		
						Obj. value	Gap (%)	Time (second)
1	8640.00	8640.00	8640.00	8640.00	8640.00	***9216.00***	0.00	20.14
2	83494.40	85120.00	85376.00	84224.00	85376.00	***85555.20***	0.00	8.75
3	*53262.50*	*53262.50*	*53262.50*	52350.00	*53262.50*	***53262.50***	0.00	2.73
4	2333440.00	2333440.00	2307840.00	2333440.00	2333440.00	***1354752.00****	0.00	0.07
5	495500.00	581250.00	*583750.00*	579250.00	579250.00	***583750.00***	0.00	0.43
6	138240.00	139584.00	141216.00	137952.00	139968.00	*142464.00*	0.54	3600.00
7	16668.00	17409.00	17004.00	17262.00	17226.00	*17664.00*	0.93	3600.00
8	65741.00	68645.60	69121.20	69747.20	71236.40	*71972.40*	0.38	3600.00
9	119772.00	128952.00	*133632.00*	128556.00	130860.00	98748.00	–	3600.00
10	*15360.00*	*15360.00*	*15360.00*	*15360.00*	*15360.00*	***15360.00***	0.00	0.58
11	49995.00	53202.80	52873.60	53202.80	53202.80	***54761.00***	0.00	170.85
12	23529.00	*24235.20*	23673.00	23990.40	23990.40	24076.80	2.75	3600.00
13	*36556.80*	*36556.80*	*36556.80*	*36556.80*	*36556.80*	***36556.80***	0.00	71.59
14	56492.80	65316.80	*68723.20*	*68723.20*	*68723.20*	*68723.20*	0.29	3600.00
15	37558.80	39727.20	39382.20	40590.00	40590.00	***40807.80***	0.00	23.46
16	556458.00	595770.00	591535.00	571290.00	603000.00	*632274.00*	2.28	3600.00
Avg. time	–	–	30.00	–	42.00			1593.66

Bold indicates optimal solution.
Italic indicates BKS.
– indicates that the solver did not provide a solution/dual bound.
*See Appendix F.

given by (12)–(15). Then, we considered the constraints for the separation of boxes, considering the stability of the cargo, and finally, with both practical constraints.

Table 12 compares the results obtained by our approach against the ones mentioned in Section 5.1. In 14 out of the 16 instances, our method yielded the BKS, improving the existing one for 8 instances, highlighted in italics. Additionally, 9 instances, highlighted in bold, were solved optimally.

For the separation of boxes, our results improve the solution of Eley (2003) in 10 instances, and we prove optimality for 10 instances. Detailed solutions are reported in Table 13.

The stability coefficient was again defined with $\alpha = 1$, i.e., requiring 100% of support to the bottom of the boxes, the most restricted approach to static stability. Although the load stability requirement significantly increases the complexity of the problem, we obtained an optimal solution for 7 instances. When the practical constraints of load stability and separation of boxes are jointly considered, the objective value is typically much worse due to the more constrained nature of the problem. Nevertheless, the proposed method also obtained an optimal solution for 7 instances. Results for these scenarios are shown in Appendix E.

Fig. 3 shows the loading pattern obtained for instance 12 of Mohanty et al. (1994) in each of the evaluated cases. All boxes have their bases fully supported by other items or by the floor of the container when the load stability is considered, and in cases considering the separation of boxes, items of type 1 and 2 (in gray and blue, respectively) are allocated to different containers.

Table 13
Results for the test sets of Mohanty et al. (1994) with separation of boxes.

#	ELY	This paper		
		Obj. value	Gap (%)	Time (second)
1	5120.00	***7680.00***	0.00	3.72
2	85376.00	***85555.20***	0.00	17.46
3	*53262.50*	***53262.50***	0.00	3.55
4	*1354752.00*	***1354752.00***	0.00	0.09
5	536250.00	***538750.00***	0.00	1.97
6	139968.00	*140448.00*	0.48	3600.00
7	16707.00	***17664.00***	0.00	3565.08
8	69121.20	*71972.40*	0.38	3600.00
9	*128088.00*	95148.00	–	3600.00
10	*15360.00*	***15360.00***	0.00	0.54
11	52873.60	***54761.00***	0.00	180.49
12	22730.40	*23745.60*	4.12	3600.00
13	*34022.40*	33388.80	9.49	3600.00
14	*66995.20*	***66995.20***	0.00	1351.95
15	39382.20	***40807.80***	0.00	28.58
16	568482.00	*612546.00*	5.63	3600.00
Avg. time	30.00			1672.09

Bold indicates optimal solution.
Italic indicates BKS.
– indicates that the solver did not provide a solution/dual bound.

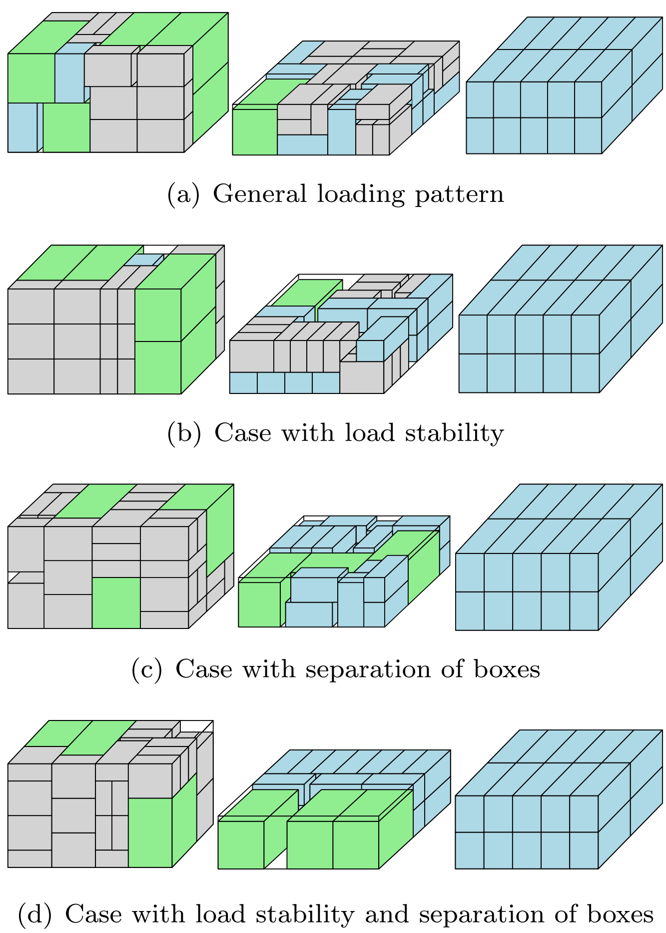

Fig. 3. Loading patterns for instance 12 of Mohanty et al. (1994). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

6. Conclusion

This paper presented mathematical formulations for several container loading problems. We have adapted and tested four discretization techniques to enumerate the possible positions of a box inside a container, and also presented techniques to obtain bounds for input minimization problems. Besides, we have also proposed mathematical formulations to the practical constraints of separation of boxes and stability, allowing the boxes to assume up to six orientations in our approaches. Computational experiments using well-known data sets from the literature were carried out, and in many cases, new optimal or improved BKS were obtained.

The formulations presented in this paper can be used to create new techniques combining exact methods with heuristic or metaheuristics strategies, to achieve good solutions in shorter computational time. Also, new formulations to obtain tighter bounds can be devised to assess the quality of the solutions for the problems.

Mathematical formulations for other practical constraints can be taken into account, in order to address more realistic problems. Finally, the formulations presented in this paper can be combined with routing algorithms, formulating approaches to the capacitated vehicle routing problem with three-dimensional loading requirements.

Acknowledgments

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Appendix A. Formulations for the discretization sets

A1. Normal patterns

$$X_k^{NP} = \{p \in \mathbb{Z} \mid p = \sum_{i=1}^{m} \beta_i \cdot l_{ig}, \quad 0 \le p \le L_k - \min_i(l_{ig}),$$
$$0 \le \beta_i \le b_i, \quad \beta_i \in \mathbb{Z}, \quad \forall g \in \Omega_i\}, \quad \forall k \in \{1, \ldots, \mathcal{K}\} \quad \text{(A.1)}$$

$$Y_k^{NP} = \{q \in \mathbb{Z} \mid q = \sum_{i=1}^{m} \beta_i \cdot w_{ig}, \quad 0 \le q \le W_k - \min_i(w_{ig}),$$
$$0 \le \beta_i \le b_i, \quad \beta_i \in \mathbb{Z}, \quad \forall g \in \Omega_i\}, \quad \forall k \in \{1, \ldots, \mathcal{K}\} \quad \text{(A.2)}$$

$$Z_k^{NP} = \{r \in \mathbb{Z} \mid r = \sum_{i=1}^{m} \beta_i \cdot h_{ig}, \quad 0 \le r \le H_k - \min_i(l_{ig}),$$
$$0 \le \beta_i \le b_i, \quad \beta_i \in \mathbb{Z}, \quad \forall g \in \Omega_i\}, \quad \forall k \in \{1, \ldots, \mathcal{K}\} \quad \text{(A.3)}$$

A2. Reduced raster points

$$X_k^{RRP}=\{\langle L_k-x\rangle | x\in X_k^{NP}\},\ \text{with}\ \langle L_k-x\rangle = max\{p\in X_k^{NP} | p\leq L_k-x\},\quad \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.4}$$

$$Y_k^{RRP}=\{\langle W_k-y\rangle | y\in Y_k^{NP}\},\ \text{with}\ \langle W_k-y\rangle = max\{q\in Y_k^{NP} | q\leq W_k-y\}, \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.5}$$

$$Z_k^{RRP}=\{\langle H_k-z\rangle | z\in Z_k^{NP}\},\ \text{with}\ \langle H_k-z\rangle = max\{r\in Z_k^{NP} | r\leq H_k-z\}, \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.6}$$

A3. Regular normal patterns

$$X_k^{RNP}=\cup_{\forall i}X_k^i,\ \text{with}\ X_k^i=\{p\in\mathbb{Z}\,|\,p=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot l_{\zeta g},\ 0\leq p\leq L-l_{ig},\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta\},\quad \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.7}$$

$$Y_k^{RNP}=\cup_{\forall i}Y_k^i,\ \text{with}\ Y_k^i=\{q\in\mathbb{Z}\,|\,q=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot w_{\zeta g},\ 0\leq q\leq W-w_{ig},\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta\},\quad \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.8}$$

$$Z_k^{RNP}=\cup_{\forall i}Z_k^i,\ \text{with}\ Z_k^i=\{r\in\mathbb{Z}\,|\,r=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot h_{\zeta g},\ 0\leq r\leq H-h_{ig},\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta\},\quad \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.9}$$

A4. Meet in the middle

$$X_k^{MiM}=\cup_{\forall i}X_k^{i\mathbb{T}},$$

with $\cup_{\forall i}X_k^{i\mathbb{T}}=\mathcal{L}_{X_k}^{i\mathbb{T}}\cup\mathcal{R}_{X_k}^{i\mathbb{T}}$

with

$$\mathcal{L}_{X_k}^{i\mathbb{T}}=\{p\in\mathbb{Z}\,|\,p=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot l_{\zeta g},\quad 0\leq p\leq min(\mathbb{T}-1,L_k-l_{ig}),\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta,\quad \forall g\in\Omega_i\},\quad \forall k\in\{1,\ldots,\mathcal{K}\}$$

and

$$\mathcal{R}_{X_k}^{i\mathbb{T}}=\{L_k-l_{ig}-p\in\mathbb{Z}\,|\,p=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot l_{\zeta g},\quad 0\leq p\leq L_k-l_{ig}-\mathbb{T},\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta,\quad \forall g\in\Omega_i\},\quad \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.10}$$

$$Y_k^{MiM}=\cup_{\forall i}Y_k^{i\mathbb{T}},$$

with $\cup_{\forall i}Y_k^{i\mathbb{T}}=\mathcal{L}_{Y_k}^{i\mathbb{T}}\cup\mathcal{R}_{Y_k}^{i\mathbb{T}}$

with

$$\mathcal{L}_{Y_k}^{i\mathbb{T}}=\{q\in\mathbb{Z}\,|\,q=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot l_{\zeta g},\quad 0\leq q\leq min(\mathbb{T}-1,W_k-w_{ig}),\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta,\quad \forall g\in\Omega_i\},\quad \forall k\in\{1,\ldots,\mathcal{K}\}$$

and

$$\mathcal{R}_{Y_k}^{i\mathbb{T}}=\{W_k-w_{ig}-q\in\mathbb{Z}\,|\,q=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot l_{\zeta g},\quad 0\leq q\leq W_k-w_{ig}-\mathbb{T},\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta,\quad \forall g\in\Omega_i\},\quad \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.11}$$

$$Z_k^{MiM}=\cup_{\forall i}Z_k^{i\mathbb{T}},$$

with $\cup_{\forall i}Z_k^{i\mathbb{T}}=\mathcal{L}_{Z_k}^{i\mathbb{T}}\cup\mathcal{R}_{Z_k}^{i\mathbb{T}}$

with

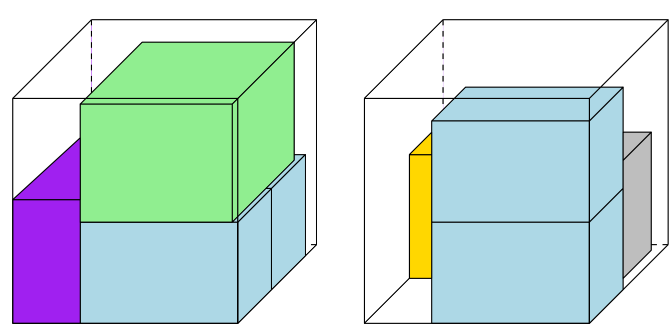
$$\mathcal{L}_{Z_k}^{i\mathbb{T}}=\{r\in\mathbb{Z}\,|\,r=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot l_{\zeta g},\quad 0\leq r\leq min(\mathbb{T}-1,H_k-h_{ig}),\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta,\quad \forall g\in\Omega_i\},\quad \forall k\in\{1,\ldots,\mathcal{K}\}$$

and

$$\mathcal{R}_{Z_k}^{i\mathbb{T}}=\{H_k-h_{ig}-r\in\mathbb{Z}\,|\,r=\sum_{\forall\zeta\neq i}\beta_\zeta\cdot l_{\zeta g},\quad 0\leq r\leq H_k-h_{ig}-\mathbb{T},\quad 0\leq\beta_\zeta\leq b_\zeta,\quad \forall g\in\Omega_\zeta,\quad \forall g\in\Omega_i\},\quad \forall k\in\{1,\ldots,\mathcal{K}\} \tag{A.12}$$

Appendix B. Numerical example on the loss of optimal solution in problems with stability

Situations involving loss of optimality can be observed in the instance depicted in Fig. B.4, where we have a container with dimensions $(L,W,H)=(100,100,100)$ and five types of boxes: $(l_1,w_1,h_1)=(60,40,58)$, $(l_2,w_2,h_2)=(71,39,43)$, $(l_3,w_3,h_3)=(68,69,54)$, $(l_4,w_4,h_4)=(38,53,53)$ and $(l_5,w_5,h_5)=(27,72,54)$. The boxes cannot rotate and each box type has availability of 2, 3, 1, 1 and 2 units, respectively. In the optimal solution for this instance, considering the stability



(a) Solution when the base is discretized using the NP/RNP

(b) Solution when the base is discretized using the RRP/MiM

Fig. B1. Example of loss of optimality using different discretizations approaches.

Table C1
Bounds for the MSSCSP for the instances of Ivancic et al. (1989).

#	Without constraints				With stability				With separation of boxes				With stability and separation of boxes			
	$\mathcal{L}$	$\mathcal{U}$	QTY	Time (second)	$\mathcal{L}$	$\mathcal{U}$	QTY	Time (second)	$\mathcal{L}$	$\mathcal{U}$	QTY	Time (second)	$\mathcal{L}$	$\mathcal{U}$	QTY	Time (second)
1	17,968	24,280	(11/5)	17.90	17,968	24,280	(11/5)	7.67	18,104	24832	(3/8)	7.09	18,104	25,920	(27/0)	8.76
2	116,224	116,224	(2/18/8)	22.67	116,224	116,224	(16/3/0)	18.13	116,224	119,040	(5/10/16)	25.24	116,224	119,040	(5/10/16)	27.12
3	70,500	70,500	(4/9/0)	14.35	70,500	70,500	(1/6/2)	14.07	70,500	70,500	(7/4/0)	17.95	70,500	70,500	(7/4/0)	17.85
4	2793,600	3141,120	(2/0/14)	7.75	2793,600	3141,120	(2/0/14)	7.32	2952,960	3173,760	(11/1/6)	8.61	2952,960	3173,760	(11/1/6)	10.26
5	938,000	938,000	(3/0/25)	18.59	938,000	938,000	(3/0/25)	11.15	952,000	973,000	(1/3/23)	13.43	952,000	973,000	(1/3/23)	21.76
6	117,360	120,960	(8/5/0)	16.72	117,360	120,960	(8/5/0)	13.13	117,360	120,960	(6/9/0)	14.98	117,360	120,960	(6/9/0)	21.95
7	19,584	20,040	(1/1/0)	29.80	19,584	21,132	(1/0/2)	23.04	19,584	21,132	(1/0/2)	20.97	19,584	21,132	(1/0/2)	26.42
8	89,640	97,820	(2/7/0)	28.11	89,180	98,520	(6/4/3)	25.57	89,640	99,360	(0/8/0)	33.94	89,140	99,360	(0/8/0)	46.28
9	108,000	113,440	(3/0/1)	32.46	108,000	113,440	(3/0/1)	29.69	108,000	113,440	(3/0/1)	33.99	108,000	113,440	(3/0/1)	37.72
10	48,750	51,000	(4/0)	11.98	48,750	51,000	(4/0)	10.55	49,500	54,750	(1/7)	12.87	49,500	54,750	(1/7)	14.45
11	129,690	141,114	(12/1/2)	20.33	129,690	146,124	(12/0/3)	15.82	129,690	141,852	(11/4/1)	20.71	129,690	146,124	(12/0/3)	21.35
12	25,200	26,800	(2/1/1)	24.55	25,200	26,800	(2/1/1)	19.66	25,200	26,800	(2/1/1)	19.00	25,200	26,800	(2/1/1)	18.62
13	24,300	26,250	(2/0)	8.76	24,300	26,250	(2/0)	7.17	25,275	34,995	(1/9)	6.39	25,275	34,995	(1/9)	10.55
14	99,040	120,960	(3/0/11)	19.99	99,040	120,960	(3/0/11)	15.14	100,480	120,960	(3/0/11)	17.64	100,480	120,960	(3/0/11)	19.89
15	49,392	53,424	(4/7)	19.62	49,392	53,424	(4/7)	16.15	49,392	53,712	(2/11)	19.61	49,392	53,712	(2/11)	20.53
16	363,430	428,962	(1/1/1)	33.64	363,430	428,962	(1/1/1)	24.48	363,430	430,080	(4/0/0)	22.99	363,430	428,962	(1/1/1)	28.18
17	167,200	178,944	(0/1/1)	45.00	167,200	178,944	(0/1/1)	45.00	167,200	198,044	(1/1/0)	45.00	167,200	198,044	(1/1/0)	45.00

Table C2
Results for MSSCSP for the instances of Ivancic et al. (1989) in different scenarios.

#	With stability			With separation of boxes			With separation of boxes and stability		
	Vol. (#)	Time (second)	Gap (%)	Vol. (#)	Time (second)	Gap (%)	Vol. (#)	Time (second)	Gap (%)
1	74.66 (25/0)	3600.00	0.53	**72.16 (3/8)**	89.16	0.00	**69.13 (27/0)**	3238.17	0.00
2	**99.94 (16/3/0)**	78.08	0.00	97.58 (5/10/16)	3600.00	2.37	97.58 (5/10/16)	3600.00	2.37
3	**99.64 (7/4/0)**	6.13	0.00	**99.64 (7/4/0)**	6.85	0.00	**99.64 (7/4/0)**	25.74	0.00
4	87.4 (2/0/14)	3600.00	1.96	87.08 (9/0/8)	3600.00	0.37	**86.5 (11/1/6)**	2569.13	0.00
5	**99.41 (1/3/22)**	21.61	0.00	95.83 (1/3/23)	3600.00	0.72	95.83 (1/3/23)	3600.00	0.72
6	96.62 (7/7/0)	3600.00	2.98	96.62 (6/9/0)	3600.00	2.98	96.62 (7/7/0)	3600.00	2.98
7	90.57 (1/0/2)	3600.00	7.33	90.57 (1/0/2)	3600.00	7.33	90.57 (1/0/2)	3600.00	7.33
8	90.39 (6/4/3)	3600.00	9.48	89.63 (0/8/0)	3600.00	9.78	89.63 (0/8/0)	3600.00	10.29
9	93.51 (3/0/1)	3600.00	4.80	93.51 (3/0/1)	3600.00	4.80	93.51 (3/0/1)	3600.00	4.80
10	95.04 (4/0)	3600.00	4.41	88.53 (1/7)	3600.00	9.59	88.53 (1/7)	3600.00	9.59
11	88.26 (12/0/3)	3600.00	11.25	90.92 (11/4/1)	3600.00	5.99	88.26 (12/0/3)	3600.00	11.09
12	94.02 (2/1/1)	3600.00	5.97	94.02 (2/1/1)	3600.00	5.97	94.02 (2/1/1)	3600.00	5.97
13	92.16 (2/0)	3600.00	7.43	69.12 (1/9)	3600.00	27.78	69.12 (1/9)	3600.00	27.78
14	**81.34 (3/0/11)**	2780.95	0.00	**81.34 (3/0/11)**	242.09	0.00	**81.34 (3/0/11)**	2699.74	0.00
15	89.95 (4/7)	3600.00	7.55	89.47 (2/11)	3600.00	4.02	89.47 (2/11)	3600.00	8.04
16	82.86 (1/1/1)	3600.00	15.28	82.65 (4/0/0)	3600.00	15.50	82.86 (1/1/1)	3600.00	15.28
17	91.59 (0/1/1)	3600.00	6.56	82.75 (1/1/0)	3600.00	15.57	82.75 (1/1/0)	3600.00	15.57
Avg. vol.	91.02			88.32			87.96		
Avg. time		2922.75			2984.59			3254.87	

Bold indicates optimal solution.

parameter $\alpha = 1$, the container has an occupation of 59.64%. This value is found only when the container is discretized using the NP approach (or, alternatively, the RNP). If one uses the RRP or the MiM, the occupation decreases to 48.40%, thus losing optimality. The loading patterns obtained for this particular problem, using the NP and the RRP/MiM approaches to discretize the container are shown in Fig. B.4. Hence, in all tests performed in this paper considering the stability of the load, we adopted the NP for the discretization of the container.

Appendix C. Bounds and results for the MSSCSP with practical constraints

Table C.14 presents the lower and upper bounds for the MSSCSP for all addressed cases. The columns $\mathcal{L}$ and $\mathcal{U}$ show the general lower and upper bounds, respectively, i.e., the minimum and maximum volume needed to pack the set of boxes. Column *QTY* presents a feasible number of each type of container to allocate the boxes; these values are obtained from the general upper bound procedure.

Table C.15 shows the results with additional constraints. As expected, the volumetric occupation of the containers decreases when the requirements of stability and separation of boxes are considered. In some instances, the stability constraint leads to a far greater number of containers to pack the boxes.

Appendix D. Results for the MILOPP

Table D.16 shows the results for the basic problem, the problem with the practical constraint of stability, with the practical constraint of separation of boxes and, finally, with the separation of boxes and stability. For the basic problem, our approach yielded 31 optimal solutions, i.e., 73.80% of the instances, in an average time of 1100.25 seconds. When the stability constraint is taken into account, the average time to solve this set grows to 1628.03 seconds and the number of optimal solutions decreases to 24. In the scenario with the separation of boxes, we achieved the shortest average time to solve the problems of this test set and the highest number of optimal solutions. Although problems with stability and separation of boxes were the most restrictive scenario assessed, 27 instances were solved to optimality in an average time of 1392.16 seconds.

Table D1
Results for MILOPP for the instances of Mohanty et al. (1994).

#	Without constraint			With stability			With separation of boxes			With separation of boxes and stability		
	Obj. value	Gap (%)	Time (second)	Obj. value	Gap (%)	Time (second)	Obj. value	Gap (%)	Time (second)	Obj. Value	Gap (%)	Time (second)
1	**2304.00**	0.00	0.04	**2304.00**	0.00	0.08	**1920.00**	0.00	0.13	**1920.00**	0.00	0.22
2	**6912.00**	0.00	8.60	6528.00	2.94	3600.00	**5760.00**	0.00	4.24	**4800.00**	0.00	59.65
3	42790.40	0.12	3600.00	**42790.40**	0.00	13.26	**42790.40**	0.00	16.07	**42790.40**	0.00	101.93
4	**32256.00**	0.00	0.08	**32256.00**	0.00	0.18	**32256.00**	0.00	0.09	**32256.00**	0.00	0.23
5	**17920.00**	0.00	0.02	**17920.00**	0.00	0.04	**17920.00**	0.00	0.03	**17920.00**	0.00	0.05
6	**24150.00**	0.00	0.05	**24150.00**	0.00	0.14	**24150.00**	0.00	0.05	**24150.00**	0.00	0.12
7	**14875.00**	0.00	0.02	**14875.00**	0.00	0.20	**14875.00**	0.00	0.11	**14875.00**	0.00	0.12
8	**55375.00**	0.00	3.95	55375.00	0.50	3600.00	**55375.00**	0.00	3.57	55375.00	0.50	3600.00
9	**1016060.00**	0.00	0.07	**1016060.00**	0.00	0.76	**1016060.00**	0.00	0.07	**1016060.00**	0.00	0.99
10	**338688.00**	0.00	0.01	**338688.00**	0.00	0.01	**338688.00**	0.00	0.01	**338688.00**	0.00	0.01
11	**130500.00**	0.00	0.01	**130500.00**	0.00	0.02	**112500.00**	0.00	0.04	**112500.00**	0.00	0.07
12	**308250.00**	0.00	2.88	**308250.00**	0.00	2.43	**281250.00**	0.00	0.67	**281250.00**	0.00	6.38
13	**238500.00**	0.00	0.04	**238500.00**	0.00	0.21	**225000.00**	0.00	0.15	**225000.00**	0.00	0.52
14	**106176.00**	0.00	114.47	106176.00	0.90	3600.00	**103680.00**	0.00	40.15	**103680.00**	0.00	1587.57
15	**57600.00**	0.00	0.12	**57600.00**	0.00	0.43	**57600.00**	0.00	0.09	**57600.00**	0.00	0.46
16	144480.00	0.40	3600.00	141984.00	2.43	3600.00	**133056.00**	0.00	339.59	129024.00	3.94	3600.00
17	22242.00	2.29	3600.00	16740.00	–	3600.00	22269.00	2.22	3600.00	19440.00	–	3600.00
18	**12984.00**	0.00	43.84	12591.00	1.67	3600.00	**12984.00**	0.00	23.49	12585.00	1.26	3600.00
19	**18208.80**	0.00	568.14	11199.20	–	3600.00	**18208.80**	0.00	700.74	18426.44	1.20	3600.00
20	**45409.60**	0.00	2454.58	24363.80	–	3600.00	**45409.60**	0.00	2504.50	21276.40	–	3600.00
21	**18480.00**	0.00	45.99	**18480.00**	0.00	774.82	**18480.00**	0.00	30.74	**18480.00**	0.00	766.16
22	74952.00	2.57	3600.00	64224.00	–	3600.00	73674.00	4.35	3600.00	25452.00	–	3600.00
23	84492.00	6.16	3600.00	73152.00	–	3600.00	74754.00	19.74	3600.00	44856.00	–	3600.00
24	93438.00	–	3600.00	92052.00	–	3600.00	92376.00	–	3600.00	72576.00	–	3600.00
25	**9216.00**	0.00	0.48	**9216.00**	0.00	5.13	**9216.00**	0.00	0.52	**9216.00**	0.00	5.05
26	**12288.00**	0.00	0.44	**12288.00**	0.00	4.07	**12288.00**	0.00	0.52	**12288.00**	0.00	3.89
27	**25539.60**	0.00	20.82	**25539.60**	0.00	269.18	**25539.60**	0.00	18.43	**25539.60**	0.00	276.11
28	**22276.80**	0.00	0.83	**22276.80**	0.00	6.94	**22276.80**	0.00	0.85	**22276.80**	0.00	6.88
29	**36166.00**	0.00	89.80	**36166.00**	0.00	967.94	**36166.00**	0.00	80.69	**36166.00**	0.00	994.51
30	18712.80	–	3600.00	17208.00	–	3600.00	5184.00	–	3600.00	5184.00	–	3600.00
31	17870.40	12.09	3600.00	19281.60	3.88	3600.00	18403.20	7.47	3600.00	10800.00	85.47	3600.00
32	**12960.00**	0.00	4.94	**12960.00**	0.00	50.53	**12960.00**	0.00	6.79	**12960.00**	0.00	42.40
33	34656.00	5.48	3600.00	32755.20	–	3600.00	31488.00	14.09	3600.00	–	–	3600.00
34	**9408.00**	0.00	0.11	**9408.00**	0.00	0.70	**9408.00**	0.00	0.14	**9408.00**	0.00	0.93
35	**32832.00**	0.00	0.65	**32832.00**	0.00	6.03	**32832.00**	0.00	0.75	**32832.00**	0.00	6.16
36	**14112.00**	0.00	8.48	**14112.00**	0.00	221.51	**12384.00**	0.00	3.06	**12384.00**	0.00	9.84
37	**32227.20**	0.00	75.94	32227.20	0.09	3600.00	**30096.00**	0.00	13.61	**30096.00**	0.00	111.28
38	**26769.60**	0.00	19.27	**26769.60**	0.00	1239.66	**26769.60**	0.00	19.32	**26769.60**	0.00	476.90
39	**14256.00**	0.00	1.42	**14256.00**	0.00	13.29	**14256.00**	0.00	1.63	**14256.00**	0.00	12.31
40	**376746.00**	0.00	3144.64	295155.00	–	3600.00	330908.00	0.72	3600.00	268128.00	–	3600.00
41	515418.00	6.10	3600.00	408426.00	–	3600.00	473226.00	15.19	3600.00	243360.00	–	3600.00
42	455167.00	2.10	3600.00	384090.00	–	3600.00	422466.00	3.67	3600.00	194688.00	–	3600.00
Avg. time			1100.25			1628.03			947.87			1392.16

Bold indicates optimal solution
– indicates that the solver did not provide a solution/dual bound.

Appendix E. Results for the MHLOPP with practical constraints

Table E1
Results for the test sets of Mohanty et al. (1994)

#	With load stability			With load stability and separation of boxes		
	Obj. value	Gap (%)	Time (second)	Obj. value	Gap (%)	Time (second)
1	8640.00	1.11	3600.00	**6720.00**	0.00	18.60
2	85376.00	0.18	3600.00	85376.00	0.16	3600.00
3	**53262.50**	0.00	3.75	**53262.50**	0.00	3.31
4	**1354752.00**	0.00	0.73	**1354752.00**	0.00	0.94
5	**583750.00**	0.00	5.12	**538750.00**	0.00	14.33
6	142464.00	0.67	3600.00	139968.00	2.33	3600.00
7	10908.00	–	3600.00	16707.00	7.20	3600.00
8	40556.20	–	3600.00	32633.20	–	3600.00
9	92052.00	–	3600.00	67032.00	–	3600.00
10	**15360.00**	0.00	5.20	**15360.00**	0.00	5.43
11	**54761.00**	0.00	1636.92	**54761.00**	0.00	1544.85
12	18504.00	33.75	3600.00	15336.00	–	3600.00
13	**36556.80**	0.00	446.91	32755.20	11.61	3600.00
14	68723.20	0.62	3600.00	66995.20	2.48	3600.00
15	**40807.80**	0.00	1702.85	**40807.80**	0.00	1770.94
16	469266.00	–	3600.00	304200.00	–	3600.00
Avg. Time			2262.59			2234.90

Bold indicates optimal solution.
– indicates that the solver did not provide a dual bound.

Appendix F. On the results of instance 4 of Mohanty et al. (1994)

Our optimal result for instance 4 in this set is significantly lower than that of competing algorithm. According to the original paper of Mohanty et al. (1994), this particular instance contains 10 containers, five with dimensions $(L_1, W_1, H_1) = (60, 40, 72)$ and five measuring $(L_2, W_2, H_2) = (40, 36, 52)$. This totals 1,238,400.00 units of volume. There are two types of items: boxes of type 1, with dimensions $(l_1, w_1, h_1) = (36, 28, 24)$ and associated with value $v_1 = 1.4$, and availability $b_1 = 50$, and boxes of type 2, with dimensions $(l_2, w_2, h_2) = (40, 32, 20)$, unitary value $v_2 = 1.0$, and availability $b_2 = 60$. Assuming that all volume units of the 10 containers are occupied by boxes of type 1, which has the highest associated value, the maximum value of the load would be 1733760.00. Thus, it is not possible to obtain loading patterns with the values associated to the load presented by the authors we compare our results with.

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