Navier - Stokes Equation for incompressible flow (where fluid density is constant):

$$\rho\left(\frac{du}{dt} + (u\cdot\nabla)u\right) = \underbrace{-\nabla p + \mu\nabla^2 u}_{\text{Internal forces}} + \underbrace{F}_{\text{External forces.}}$$

U: velocity field of fluids
t: time
ρ: fluid density
p: pressure fluid
μ: dynamic viscocity
F: body forces e.g gravity

∇p is the gradient of the pressure field

$\nabla\cdot u$ is divergence of the velocity field, which is zero for incompressible flow

$\nabla^2 u$ is laplacian of the velocity field, representing viscous diffusion

---

$$\nabla\cdot U = 0 \quad — ①$$

$$\rho\frac{Du}{Dt} = -\nabla p + \mu\nabla^2 u + \rho F \quad — ②$$

universal laws of physics → can model any fluid, some gases & some solids too.

① → mass is conserved
→ fluid may change shape but mass remains same.
→ Conservation of mass

② → Conservation of Momentum
→ Newton's 2nd law
→ F = ma.

$\rho = m$ in fluid dynamics.

$\nabla \cdot U = 0$ → U is velocity which is a vector → speed, → direction.

So $\underline{U} = (u, v, w)$ [a component in the x direction (u), a component in the y direction (v), a component in the z direction (w)]

→ How fast the water is flowing
How fast is air going around F1 car, etc.

∇ → nabla → gradient, which is a derivative → Telling us to differentiate $\underline{U}$ in a particular way

→ Telling us what to do with our velocity

So we have our 3 components $\underline{U} = (u, v, w)$ →

$\underline{\underline{\frac{\partial u}{\partial x}}}$ (differentiate the first bit w.r.t its coordinate which is x), similarly $\underline{\underline{\frac{\partial v}{\partial y}}}$, $\underline{\underline{\frac{\partial w}{\partial z}}}$

So $\nabla \cdot U$ is the divergence of our velocity which is these 3 derivatives. Basically it is telling us how does the x component of my velocity u changes as I move in the x direction, how v changes w.r.t y & how w changes w.r.t z direction

$$\partial \quad \nabla \cdot \underline{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Change in x Change in y Change in z

This is the conservation of mass equation.

② equation is Newton's 2nd law which is $F = ma$.

U is our velocity & when we take a time derivative of our velocity we get acceleration $\longrightarrow$ so $\frac{\partial U}{\partial t}$ is acceleration.

& ρ which is density which mass here when it comes to fluid dynamics (sort of)

$\therefore$ $\rho \cdot \frac{\partial U}{\partial t} = F$ $\longrightarrow$ Forces here are:

① Internal forces:

(a) • ∇p : Nabla p or grad p.

∇p : pressure gradient is a vector representing the change in pressure $\rightarrow$ water/air moves from a region of high pressure to low pressure.

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right)$$

H.P $\longrightarrow$ L.P

(b) $\mu \nabla^2 \underline{U}$: viscocity $\rightarrow$ think of fluid as made up of layers, viscocity is the friction b/w those layers $\rightarrow$ i.e. who these layers resist to motion. Higher the viscocity higher the resistance to flow (e.g Honey > water)

Further Simplification / explanation.

Two equations → conservation of mass → $\nabla \cdot U = 0$

→ ~~New~~ Conservation of momentum

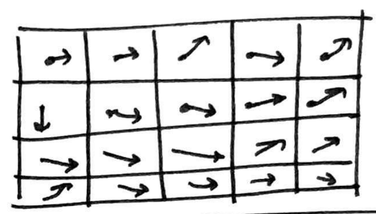
$$\rho \cdot \frac{\partial U}{\partial t} = -\nabla p + \mu \nabla^2 U + F$$

$\nabla \cdot U$ → mass is conserved within a fluid

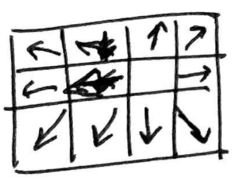
$\nabla \cdot$ → divergence operator. divergence of vector field → so it shows divergence of U

U → vector field. (velocity)

Vector field → we get when we assign every single point in space to a vector field

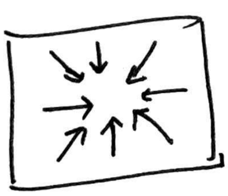


Divergence of a vector field tells us how much a point tends to diverge vectors away from it.



→ ~~appears~~ appears that vectors seem to be diverging from the origin this indicates positive divergence

$$\boxed{\operatorname{div} \vec{F} > 0}$$



→ appears that vectors are flowing into the origin. which indicates negative divergence

$$\boxed{\operatorname{div} \vec{F} < 0}$$

numerically we write divergence as a dot product b/w the gradient vector & its vector field

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \vec{F}$$

In terms of a fluids divergence tells us how much a point acts as a source of a fluid.

In terms of water → imagine ~~in~~ water in some area it's impossible for the water to simply disappear, it could change forms but the mass is never destroyed & hence the divergence across the fluid has to be zero $\boxed{\nabla\cdot U = 0}$

2nd equation is just the rewritten version of Newton's 2nd law → i.e. sum of forces acting on a body can we written as ~~its~~ it's mass times the acceleration $\boxed{ma = \Sigma F}$

Consider this for a single molecule of a fluid let's see how we can derive Navier Stokes equation

Firstly, since we are considering each individual point let's replace mass w/ density → the mathematical reason for this is → to consider each individual point we have to divide by volume & mass/Volume is equal to density (ρ).

Next, let's consider acceleration → we have the velocity of vector field $\underline{u}$ & acceleration is just the derivative of velocity vector field so we can replace a with $\frac{\partial u}{\partial t}$

<u>Side note</u>: $\frac{\partial u}{\partial t} = \boxed{\frac{\partial u}{\partial t} + u \cdot \nabla u}$ → expansion of $\frac{\partial u}{\partial t}$ by <u>Chain rule.</u>

~~Thirf~~ <u>Internal forces</u> → ① Pressure

$-\nabla p$ → pressure gradient.

② Viscocity → $\mu \Delta^2 u$ for newtonian fluid.

③ <u>External forces</u> → ① Gravity (ρg)

↳ <u>F</u>

Equation to be used for Simulation.

movement of liquid in space

$$\frac{\partial u}{\partial t} = \underbrace{-(u\cdot\nabla)u}_{\text{Advection or Convection of fluid velocity field}} - \frac{1}{\rho}\nabla p + \mu\nabla^2 u + F$$

In fluid $\underline{u} = (u, v, w)$ is velocity vector field of the fluid where $u \rightarrow$ velocity component in x-axis

$v \rightarrow$ y axis

$w \rightarrow$ z axis

<u>Advection term</u> $\rightarrow$ $(u\cdot\nabla)u$ describes how the velocity field is transported by the flow itself. It is non linear term that represents the change in the velocity of fluid particles as they move through the velocity field; Detailed explanation is as follows:

① Gradient operator is 3-D $\rightarrow$ $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

② Dot product $\rightarrow (u\cdot\nabla)$ This dot product produces a differential operator that acts on the velocity field u

It can be written as

$$u\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

③ Advection term $\rightarrow (u\cdot\nabla)u$ applying the operator $(u\cdot\nabla)$ to veloci field gives us

$$(u\cdot\nabla)u = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

This term represents how the velocity components u, v, w change due to the movement of the fluid itself.

<u>Physical interpretation</u> ① Advection - $(u\cdot\nabla)u$ describes the transport of the fluid's momentum by the fluid's own flow. Essentially, it accounts for how the velocity of a fluid parcel changes as it moves through velocity field.

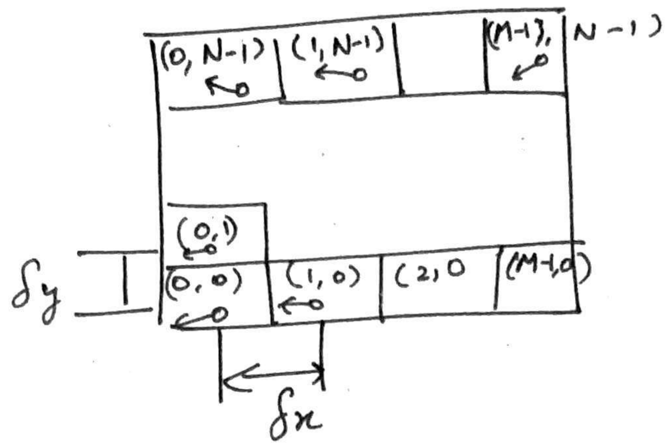
② Non-linearity : Advection term is non-linear because velocity term u is multiplied by its own spatial derivatives

<u>For 2D flow</u> → $(u\cdot\nabla)u = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y},\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)$

Each component of this vector represents the change in the x & y component of the velocity due to advection process

<u>Conclusion</u> → Advection term represents advection of velocity
fluid, describing how fluid parcels carry their
momentum as they move through the flow

Process → ① To represent our liquid in a program, we need to obtain a mathematical representation of the state of each particle of the liquid at any arbitrary point in time.



In each cell of our 2-D array we will store the velocity of the particle at the time instant

$t: \ \vec{U} = \underbrace{U(\vec{x},t)}$, $\underbrace{\vec{x} = (x,y)}_{\text{space dim}}$

velocity at position $\vec{x}$ in space at given time t.

Distance b/w each particle
↓
$\delta x \ \delta y$

operator	Definition.	Discrete analog.
① grad	$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)$	$\frac{p_{i+1,j} - p_{i-1,j}}{2\delta x}, \frac{p_{i,j+1} - p_{i,j-1}}{2\delta y}$
② div	$\nabla \cdot \vec{U} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$	$\frac{\vec{U}_{(x)i+1,j} - \vec{U}_{(x)i-1,j}}{2\delta x} + \frac{\vec{U}_{(y)i,j+1} - \vec{U}_{(y)i,j-1}}{2\delta y}$
③ Δ	$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}$	$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{(\delta x)^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{(\delta y)^2}$

(4) rot $\nabla \times \vec{U} = \frac{\partial \vec{U}}{\partial y} - \frac{\partial u}{\partial x}$ $\frac{\vec{U}_{(y)i,j+1} - \vec{U}_{(y)i,j-1}}{2\delta y} - \frac{\vec{U}_{(x)i+1,j} - \vec{U}_{(x)i-1,j}}{2\delta x}$

Further simplify $\delta x = \delta y = 1$

Moving particle → These statements will only work if we find the nearest particles relative to the one being considered at the movement. For this, we will track not their movement but where the particles come from at the beginning of the iteration by projecting the trajectory of movement back in time, in other words, subtract the velocity vector multiplied by the change in time from current position. This will ensure we will be absolutely sure that any particle will have neighbours

$$q(\vec{x}, t+\delta t) = q(\vec{x} - \vec{u}\,\delta t, t)$$

Bilinear interpolation of the states of the four nearest particle and take it as true value at the point ← Done to avoid loss of accuracy if projection hits cell boundary or if non- ~~linear~~ integer coordinates are obtained

Viscosity → directly affects the acceleration acquired by a fluid & can be expressed by the formula

$$\frac{\partial \vec{u}}{\partial t} = \mu \nabla^2 \vec{u}$$

Momentum Equation $\rightarrow$ disguise of Newton's 2nd law

$\vec{F} = m\vec{a}$

① Fluid is being simulated using a particle system. Each particle might represent a blob of fluid w/ mass (m), a volume (V) & velocity ($\vec{u}$).

② To integrate the system forward in time, we need to figure out what forces are acting on each particle

(a) $\vec{a} = \frac{\partial \vec{u}}{\partial t}$ $\therefore$ $m\frac{\partial \vec{u}}{\partial t} = F$

(b) Forces acting are $\rightarrow$ $m\vec{g}$ (gravity)

→ pressure
- → high P to low P
- → if no pressure diff, no flow
- → gradient is in the direction of "steepest ascent" thus negative gradient points away from H.P. regions towards L.P. regions
  $\therefore -\nabla p$
- → Integrate this over our Volume of blob to get pressure force. For simple approximation, we'll just multiply by Volume. $-V\nabla p$

→ Viscosity
- → resistance to flow
- → differential operator that measures how far a quantity is from the average
  it is Laplacian $\nabla \cdot \nabla$
- → Viscous force will be provided when we integrate it over the volume of blob
- → We'll use dynamic viscocity coefficient, denoted by μ

Putting it all together $m\frac{\partial \vec{u}}{\partial t} = m\vec{g} - V\nabla p + V\mu\nabla \cdot \nabla \vec{u}$

Since actual fluids have huge (but finite) number of ~~blobs~~ blobs we take this limit which approaches to infinity but this poses a problem which makes mass (m) & Volume (V) approach zero. To solve this we divide the equation by Volume & then take the limit.

$$\frac{m}{V}\frac{\partial \vec{U}}{\partial t} = \frac{m}{V}\vec{g} - \frac{V}{V}\nabla p + \frac{V}{V}\mu \nabla\cdot\nabla\vec{U}$$

(remember $\frac{m}{V} = \rho$)

$$\rho\frac{\partial \vec{U}}{\partial t} = \rho\vec{g} - \nabla p + \mu\nabla\cdot\nabla\vec{U}$$

dividing L.H.S & R.H.S w ρ

$$\frac{\partial \vec{U}}{\partial t} + \frac{1}{\rho}\nabla p = \vec{g} + \frac{\mu}{\rho}\nabla\cdot\nabla\vec{U}$$

Two ways of analyzing fluid motion → ① Lagrangian Viewpoint
② Euler's Viewpoint

Lagrange's
→ Track individual fluid particles as they move through space & time
→ Each particle has properties like position, velocity, temp etc.
→ Material Derivative: Describe how these properties change over time for a specific particle.

Mathematically $\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \vec{U}\cdot\nabla q$

q: properties of particle (e.g temp, velocity)

$\frac{Dq}{Dt}$: Total change in property q over time.

$\frac{\partial q}{\partial t}$: Change in q w.r.t time at a fixed position.

$\vec{U}\cdot\nabla q$: Change in q due to particle's motion in space

Euler's : examines specific locations in space as fluid flows through them

: look at properties like velocity, pressure, and temp at these fixed points over time

: Navier - Stokes Equation : Describes how the velocity field evolves at these fixed points

Mathematically → $$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \vec{f}$$

$\vec{u}$: velocity , $\frac{\partial \vec{u}}{\partial t}$: Change in $\vec{u}$ w.r.t time

$(\vec{u} \cdot \nabla)\vec{u}$: Convective acceleration term (how the flows changes due to its own movement)

$-\nabla p$: pressure gradient , $\nu \nabla^2 \vec{u}$: Viscous term

$\vec{f}$: External forces.