Week 11 Tutorial 01 — Big Steps a)

```
let f = fun a -> [(a+1,a-1)] in f 7 ⇒ [(8,6)]
```

Week 11 Tutorial 01 — Big Steps a)

$$\text{LD}\;\dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \qquad \dfrac{}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps a)

$$\pi_0 = \frac{}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps a)

$$\pi_0 = \text{LI}\ \dfrac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7+1 \Rightarrow 8}{\texttt{7+1} \Rightarrow 8} \quad \text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7-1 \Rightarrow 6}{\texttt{7-1} \Rightarrow 6}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)}} \quad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad 7 \Rightarrow 7 \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\texttt{f [3;6]} \Rightarrow 9$$

Week 11 Tutorial 01 — Big Steps b)

$$\pi_f = \mathrm{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}\quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \mathrm{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}\quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

Global Definitions

Week 11 Tutorial 01 — Big Steps b)

$$\pi_f = \text{GD}\ \frac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \text{GD}\ \frac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

T_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\pi_f = \frac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

$$\pi_g = \frac{g = T_g \qquad T_g \Rightarrow T_g}{g \Rightarrow T_g}$$

Global Definitions

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \qquad \dfrac{}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \qquad \dfrac{}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$
\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow 6}}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}
$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \frac{}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \qquad \dfrac{}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{[6] \Rightarrow [6] \quad \text{OP}\ \dfrac{6 \Rightarrow 6 \qquad \dfrac{}{\texttt{f []} \Rightarrow 1}}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{[6] \Rightarrow [6] \quad \text{OP}\ \dfrac{6 \Rightarrow 6 \quad \text{APP'}\ \dfrac{\pi_f \quad [\,] \Rightarrow [\,] \quad \dfrac{}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1}}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{6 \Rightarrow 6 \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad 1 \Rightarrow 1}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1} \qquad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

Week 11 Tutorial 01 — Big Steps b)

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

T_f = fun l -> match l with [] -> 1 | x::xs -> x + g xs

T_g = fun l -> match l with [] -> 0 | x::xs -> x * f xs

$$\pi_f = \frac{f = T_f \qquad T_f \Rightarrow T_f}{f \Rightarrow T_f}$$

$$\pi_g = \frac{g = T_g \qquad T_g \Rightarrow T_g}{g \Rightarrow T_g}$$

$$\pi_0 = \mathrm{PM}\ \frac{[6] \Rightarrow [6] \ \ \mathrm{OP}\ \dfrac{6 \Rightarrow 6 \ \ \mathrm{APP'}\ \dfrac{\pi_f \ \ [\,] \Rightarrow [\,] \ \ \mathrm{PM}\ \dfrac{[\,] \Rightarrow [\,] \quad 1 \Rightarrow 1}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 1}}{\texttt{f []} \Rightarrow 1} \qquad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow 6}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow 6}$$

$$\mathrm{APP'}\ \frac{\pi_f \ \ [3;6] \Rightarrow [3;6] \ \ \mathrm{PM}\ \dfrac{[3;6] \Rightarrow [3;6] \ \ \mathrm{OP}\ \dfrac{3 \Rightarrow 3 \ \ \mathrm{APP'}\ \dfrac{\pi_g \ \ [6] \Rightarrow [6] \ \ \pi_0}{\texttt{g [6]} \Rightarrow 6} \qquad 3 + 6 \Rightarrow 9}{\texttt{3+g [6]} \Rightarrow 9}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow 9}}{\texttt{f [3;6]} \Rightarrow 9}$$

Week 11 Tutorial 02 — Multiplication

Prove that the function

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

terminates for all inputs $a, b \geq 0$.

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Base: a = 0

$$\frac{}{\texttt{mul 0 b} \Rightarrow 0}$$

Week 11 Tutorial 02 — Multiplication

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

Proof by Induction on a

Hypothesis:

$\forall a \geq 0$: mul a b = a*b

Step:

$$\text{APP}\ \frac{}{\texttt{mul (a+1) b} \Rightarrow (a+1)*b}$$