Linear Algebra 25Fall Final (Recall)

Crykkkk

本文件不保证准确，题面及思路仅供参考，考试满分为 100 分，总时间为 120 分钟

Problem 1:

Prove or disprove:

1.

$$C(A_1) \subseteq C(A_2) \Longrightarrow C(A_1B) \subseteq C(A_2B)$$

2.

$$N(A_1) \subseteq N(A_2) \Longrightarrow N(A_1B) \subseteq N(A_2B)$$

Problem 2:

Prove that

$$\dim(W_1) + \dim(W_2) > \dim(W_1 + W_2)$$

implies

$$W_1 \cap W_2 \neq \{0\}$$

where

$$W_1 + W_2 \overset{\text{def}}{=} \{v + w | v \in W_1, w \in W_2\}$$

Problem 3:

Given that

$$W = I_n - 2uu^T$$

where $\|u\| = 1$

Prove that:

1. W is both symmetric and orthogonal
2. $Wv = v$ if and only if $u \perp v$

Problem 4:

For $L \in \boldsymbol{T}(V, V)$, define L^k iteratively by

$$L^k(v) = L(L^{k-1}(v))$$

Prove that if for some non-zero L and some $k \in \mathbb{Z}^+$, $L^k = \mathbf{0}$(i.e. $\forall v \in V, L^k(v) \equiv 0$), then

$$\dim(\mathrm{Im}(L^2)) < \dim(\mathrm{Im}(L))$$

Problem 5:

Diagonalize a 3×3 matrix A (in other words, find an orthogonal matrix Q s.t. $Q^T AQ$ is a diagonal matrix)

Problem 6:

Prove that if a $n \times n$ matrix ($n \geq 2$)

$$\begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & b \\ b & b & b & a \end{bmatrix}$$

is positive definite, then $a > b$

Problem 7:

Prove whether each of the following sets is a vector space over $\mathbb{R}$, if so, write its dimension , if not, state which axiom (公理) of vector space it violates and provide an explicit counterexample.

1. Fibonacci-like sequences $\{a_i\}$ s.t. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$
   - $\{a_i\} + \{b_i\} = \{a_i + b_i\}$
   - $c \cdot \{a_i\} = \{ca_i\}$
2. $V = [0, 1)$

- $v_1 \oplus v_2 = v_1 + v_2 - \lfloor v_1 + v_2 \rfloor$
- $c \otimes v_1 = c \cdot v_1 - \lfloor c \cdot v_1 \rfloor$

3. $A = \{(x-\epsilon, x+\epsilon) | x \in \mathbb{R}, \epsilon > 0\} \cup \{\varnothing, \mathbb{R}\}$, in other words, A contains all the open intervals
   - $a_1 \oplus a_2 = a_1 \cap a_2$
   - $a_1 = (x-\epsilon, x+\epsilon) \Rightarrow c \otimes a_1 = \begin{cases} (x - c\epsilon, x + c\epsilon) & c > 0 \\ \varnothing & \text{otherwise} \end{cases}$

Problem 8:

Consider the linear space of 2×2 real matrices $M_{2\times 2}(\mathbb{R})$ with inner product defined as

$$\langle A, B\rangle = \frac{1}{2}\text{trace}(A^T B)$$

and

$$V = \{A \in M | A^T = A\} \qquad U = \{A \in M | A^T = -A\}$$

Prove that:

1. V and U are subspaces of M and $V = U^{\perp}$
2. Consider $W = \{A \in V | \text{trace}(A) = 0\}$ (here you don't need to prove that W is a subspace of V), prove that

$$W^{\perp} \cap V = \text{span}(I_2)$$