# Combinatorics 25Fall Mid (Recall)

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本文件不保证准确，题面及思路仅供参考，期中考满分为50分，总时间为110分钟

## 1 Problem 1

Given positive integers $n$ and $k$, we choose $k^2$ subsets from $[n]$, arranged in a $k \times k$ matrix:

$$\begin{matrix} S_{1,1} & S_{1,2} & \dots & S_{1,k} \\ S_{2,1} & S_{2,2} & \dots & S_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ S_{k,1} & S_{k,2} & \dots & S_{k,k} \end{matrix}$$

Each set $S_{i,j}$ must satisfy $S_{i,j} \subseteq S_{i-1,j}$ whenever $i > 1$, and $S_{i,j} \subseteq S_{i,j-1}$ whenever $j > 1$. Determine the number of possible choices of such subsets.

## 2 Problem 2

Compute the closed form of

$$\sum_{k=0}^{m} (-1)^k \cdot \binom{n}{k}\binom{n}{m-k}$$

## 3 Problem 3

Given the definition of the Bell number:

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

### 3.1 Task 1 (Dobinski Formula)

Prove that

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

### 3.2 Task 2

Prove that

$$B_n = \sum_{k=1}^{n-1} \binom{n-1}{k} B_k$$

### 3.3 Task 3

Compute the closed form of the following function

$$B(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

# 4 Problem 4

$k_{n,n}$ bipartite, remove a perfect matching

### 4.1 Task 1

compute the number of the remaining distinct perfect matching, where distinct implies that at least one element of the matching are different.

### 4.2 Task 2

compute the closed form of your answer in Task 1, Gaussian symbols($\lfloor\rfloor$, $\lceil\rceil$) may be used here.

# 5 Problem 5

- Is it possible for a simple graph to have a degree sequence 3, 3, 3, 3, 3, 4, 6, 8, 8?
- Is it possible for a **bipartite** graph to have a degree sequence of 3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6?
- Is it possible for a simple graph to have a degree sequence 1, 1, 3, 3, 3, 3, 5, 6, 8, 9?

Note: the sequence may not be exactly accurate but it's adequate to show the point.

# 6 Problem 6

prove or disprove that all the longest paths of a **tree** share a common vertex.

# 7 Problem 7

given a $n$-vertex graph $G$ with maximum degree $\Delta$, prove that:

- There exists a independent set which has at least $\dfrac{n}{\Delta + 1}$ vertices (no rounding)
- There exists a matching which has at least $\dfrac{n}{\Delta + 1}$ edges