In [ ]:

16/20

Assignment 3

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Imports

In [3]:

```
#from scipy import stats
import numpy as np
import matplotlib.pyplot as plt
import urllib.request as url
from decimal import *
#from statistics import mean
%matplotlib inline
```

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Problem 1

Proofs

Proof of Three point Stencil

$$\text{To prove : } y^{''}(x)_2 = \frac{[y(x+h) - 2y(x) + y(x-h)]}{(h * h)}$$

Taking left and right derivatives, i.e when h > 0 or when h < 0, we get -

$$\text{Eqn- 1 : } f(x_0 + h) = f(x_0) + f^{'}(x_0)h + \frac{f^{''}(x_0)h^2}{2} + \frac{f^{(3)}(x_0)h^3}{6} + \frac{f^{(4)}(\xi_1)h^4}{24}$$

$$\text{Eqn- 2 : } f(x_0 - h) = f(x_0) - f^{'}(x_0)h + \frac{f^{''}(x_0)h^2}{2} - \frac{f^{(3)}(x_0)h^3}{6} + \frac{f^{(4)}(\xi_2)h^4}{24}$$

Adding Eqn1 and Eqn2

$$f(x_0 + h) + f(x_0 - h) = 2f(x_0) + f^{''}(x_0)h^2 + \frac{f^{(4)}(\xi)h^4}{12}$$

$$f^{''}(x_0) = \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} - \frac{f^{(4)}(\xi)h^2}{12} \text{ QED}$$

Proof of Five point Stencil

$$\text{Eqn- 1 : } f(x_0 + h) = f(x_0) + f^{'}(x_0)h + \frac{f^{''}(x_0)h^2}{2} + \frac{f^{(3)}(x_0)h^3}{6} + \frac{f^{(4)}(x_0)h^4}{24} + \frac{f^{(5)}(x_0)h^5}{120} + \frac{f^{(6)}(\xi_1)h^6}{720}$$

$$\text{Eqn- 2 : } f(x_0 - h) = f(x_0) - f^{'}(x_0)h + \frac{f^{''}(x_0)h^2}{2} - \frac{f^{(3)}(x_0)h^3}{6} + \frac{f^{(4)}(x_0)h^4}{24} - \frac{f^{(5)}(x_0)h^5}{120} + \frac{f^{(6)}(\xi_2)h^6}{720}$$

$$\text{Egn- 3 : } f(x_0 + 2h) = f(x_0) + 2f^{'}(x_0)h + \frac{4}{2!}f^{''}(x_0)h^2 + \frac{8}{3!}f^{(3)}(x_0)h^3 + \frac{16f^{(4)}(x_0)h^4}{24} + \frac{32f^{(5)}(x_0)h^5}{120} + \frac{64f^{(6}}{7}$$

$$\text{Egn- 4 : } f(x_0 - 2h) = f(x_0) - 2f^{'}(x_0)h + \frac{4}{2!}f^{''}(x_0)h^2 - \frac{8}{3!}f^{(3)}(x_0)h^3 + \frac{16f^{(4)}(x_0)h^4}{24} - \frac{32f^{(5)}(x_0)h^5}{120} + \frac{64f^{(6)}}{72}$$

Adding Eqn 1 and Eqn 2 -

$$16f(x_0 + h) + 16f(x_0 - h) = 32f(x_0) + 16f^{''}(x_0)h^2 + \frac{32f^{(4)}(x_0)h^4}{24} + \frac{32f^{(6)}(\xi_5)h^6}{720}$$

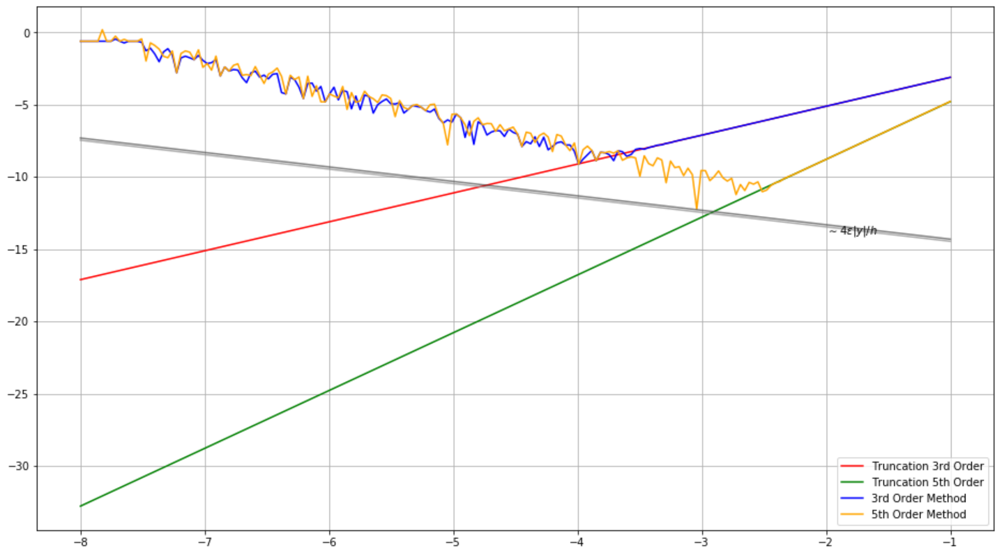
Adding Eqn 3 and Eqn 4 -

$$f(x_0 + 2h) + f(x_0 - 2h) = 2f(x_0) + 4f^{''}(x_0)h^2 + \frac{32f^{(4)}(x_0)h^4}{24} + \frac{128f^{(6)}(\xi_6)h^6}{720}$$

In [23]:
```
# Below are the mathematical functions for f(x), f'(x), df^
4/d^4x(f(x)), df^6/d^6x(f(x))
def f(x):
    return np.sqrt(x)
def f_2(x):
    return -1/4*x**(-3/2)
def f_4(x):
    return -15/(16*x**(7/2))
def f_6(x):
    return -945/(64*x**(11/2))

# main program
# differrent differentiation schemes
logh = -np.linspace(8,1,200)   # -8...0
h = 10.**logh
eps = 2.22e-16
x = 1
analyitical_value = -1/4

# numerical differentiation
# p = 2 diff. formula
d1f_2 = (f(x+h)+f(x-h)-2*f(x))/h**2
# p = 4 diff. formula
d1f_4 = (16*f(x+h)+16*f(x-h)-f(x-2*h)-f(x+2*h) -30*f(x))/(1
2*h*h)
residual_2 =  np.absolute(d1f_2 -f_2(x))
residual_4 = np.absolute(analyitical_value - d1f_4 )
trucation_2 = (abs(f_4(x))*h**2)/12
trucation_4 = np.absolute((96*f_6(x)*h**4)/8640)
plt.figure(figsize=(16,9)) # Setting Larger Graph Size
plt.plot(logh,np.log10(trucation_2),"red")
plt.plot(logh,np.log10(trucation_4),"green")
plt.plot(logh,np.log10(residual_2), "blue")
plt.plot(logh,np.log10(residual_4),"orange")
for p in [2,4]:
    """
    Not sure how this code segment inside for loop works, d
irectly copied from the example text file
    """
    E_round = np.array((5*p)**0.5/2*eps*abs(f(x))/h)
    logE = np.log10(E_round)
    c = 0.8-p/8
    plt.plot(logh,logE,color=(c,c,c),alpha=0.7)
plt.grid()
plt.text(-2.,-13.9,'$\sim 4\epsilon |y|/h$')
axes = plt.gca()
axes.legend(["Truncation 3rd Order","Truncation 5th Orde
r","3rd Order Method","5th Order Method"])
plt.show()
```



Q1 5/6 Need to comment on the result/whether it is what you expect.

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Problem 2

Planet	Semi-major Axis
Mercury	0.387098 AU
Venus	0.723332 AU
Earth	1.000001018 AU
Mars	1.523679 AU
Ceres	2.7691651545 AU
Jupiter	5.2044 AU
Saturn	9.5826 AU
Uranus	19.2184 AU
Neptune	30.11 AU

References

"Ceres (Dwarf Planet)". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Ceres_(dwarf_planet (https://en.wikipedia.org/wiki/Ceres_(dwarf_planet)).

"Earth's Orbit". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Earth%27s_orbit (https://en.wikipedia.org/wiki/Earth%27s_orbit).

"Jupiter". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Jupiter (https://en.wikipedia.org/wiki/Jupiter).

"Mars". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Mars (https://en.wikipedia.org/wiki/Mars).

"Mercury (Planet)". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Mercury_(planet (https://en.wikipedia.org/wiki/Mercury_(planet)).

"Neptune". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Neptune (https://en.wikipedia.org/wiki/Neptune).

"Saturn". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Saturn (https://en.wikipedia.org/wiki/Saturn).

"Uranus". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Uranus (https://en.wikipedia.org/wiki/Uranus).

"Venus". En.Wikipedia.Org, 2019, https://en.wikipedia.org/wiki/Venus (https://en.wikipedia.org/wiki/Venus).

In [93]:
```
obs_axis = [0.387098,0.723332,1.000001018,1.523679,2.769165
1545,5.2044,9.5826,19.2184,30.11]
d_obs = np.array(obs_axis)
def d_unvec(x):
    """
    Calculate all 9 distance values for given X
    """
    n = np.array([-2,-1,0,1,2,3,4,5,6])
    return x**n
d = np.vectorize(d_unvec)
def e(distance):
    """
    Calculate square of the arithmetic average of relative
deivations
    """
    return np.sum( ((distance - d_obs)/d_obs)**2 )/9
e_vec = np.vectorize(e)
def dE_dx(x):
    """
    First Derivative of the function e(d(x)) using first pr
inciples, for arbitarily small number
    1e-10 as h, the smaller this number the more accurate t
he derivative is
    """
    return (e(d(x+1e-10)) -e(d(x)))/1e-10
```

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In [96]:
```
a_n,b_n = 1.5,2.0
m = (a_n + b_n)/2
f_m = dE_dx(m)
list_m, list_f_m = [],[]
n = 0
while not(np.isclose([f_m],[0])):
    """
    Bisection Method Implementation, but instead of restric
ting to N iterations, this method will continue until
    the midoint isclose to 0
    """
    m = (a_n + b_n)/2
    f_m = dE_dx(m)
    f_a = dE_dx(a_n)
    f_b = dE_dx(b_n)
    list_m.append(m)
    list_f_m.append(f_m)
    if f_m*f_a < 0:
        a_n,b_n = a_n,m
    elif f_m*f_b < 0:
        a_n,b_n = m,b_n
    n += 1
print("X Value for Minimun E((d(x))) - ",m,"Value of E(d
(x)) at the X Value - ",f_m)
print("Number of Iterations -", n)
```

```
X Value for Minimun E((d(x))) -  1.7509384118020535 Value o
f E(d(x)) at the X Value -  0.0
Number of Iterations - 27
```

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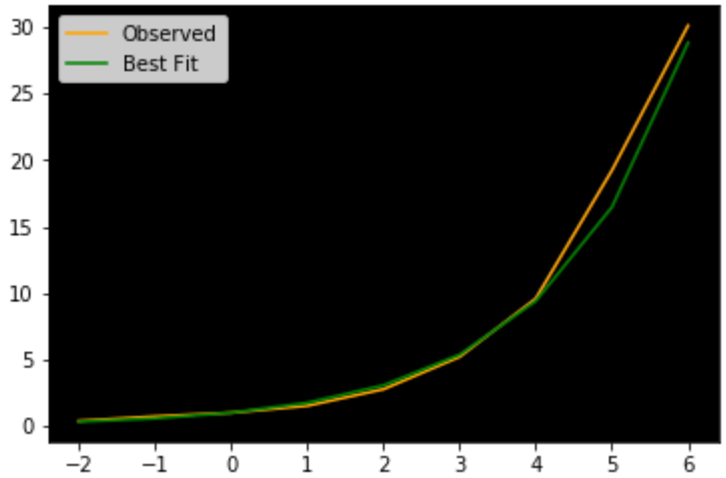
In [99]:
```
n = np.array([-2,-1,0,1,2,3,4,5,6])
plt.plot(n,d_obs,'orange')
plt.plot(n,d(m),'green')
axes = plt.gca()
axes.set_facecolor('xkcd:black') # Setting Background colou
r as black
axes.legend(['Observed','Best Fit'])
plt.show()
residuals = np.array(d(m) - d_obs)
for i in range(9):
    print("Planet Number(N) : ",n[i],"Residual : ",residual
s[i])
```




```
Planet Number(N) :  -2 Residual :  -0.060917300768110 58
Planet Number(N) :  -1 Residual :  -0.15220968452643224
Planet Number(N) :  0 Residual :  -1.0180000000747924e-06
Planet Number(N) :  1 Residual :  0.22725941180205345
Planet Number(N) :  2 Residual :  0.2966201674238973
Planet Number(N) :  3 Residual :  0.16360128249547667
Planet Number(N) :  4 Residual :  -0.18356035987598496
Planet Number(N) :  5 Residual :  -2.761260460056711
Planet Number(N) :  6 Residual :  -1.2945622311269247
```

Q2 3.5/4 Requested a semilog plot to show power law.

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Problem 3

Proofs:

Proving the term in iterations is

$$x_{n+1} = x_n \times (2 - c \times x_n)$$

General form for Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}, \text{where f(x) and f'(x) are as follows}$$

$$f(x) : 1/x - c, f^{'}(x) : -1/x^2$$

$$\implies x_{n+1} = x_n - \frac{(1/x_n - c)}{(-1/x_n^2)}$$

$$\implies x_{n+1} = \frac{\frac{-1}{x_n} - \frac{1}{x_n} + c}{\frac{-1}{x_n^2}}$$

$$\implies x_{n+1} = \frac{2 - c \times x_n}{x_n} \times x_n^2$$

$$\implies x_{n+1} = x_n \times (2 - c \times x_n) \text{ QED}$$

Solving for C = 7, x_1 = 0.09 < 1/7

Need to prove the inequality in general.

In [29]:
```
# Newtons method function from Tutorial 7 starter code
def newton(f, df, x0):
    """
    Find the root of the function f(x) where df(x) is a fun
ction
    which computes the derivative wrt x of f evaluated at
x, x0 is
    and initial guess of the root
    """
    x0 = x0 - f(x0)/df(x0)
    return x0
```

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In [30]:
```
def f(x):
    """
    Function f(x): 1/x -c, solving for f(x) = 0
    """
    return 1/x - 7
def df(x):
    """
    Derivative of f(x), defined above
    """
    return -1/((x)**2)
```

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In [131]:
```
n = 0
lastVal, currentVal = 0, 0.09
approximation = []
while not lastVal == currentVal:
    """
    Newton's Method Implementation, using while loop to sto
p when the previous value is the same as the current value
    instead of using vectorized function for a given n valu
e
    """
    lastVal = currentVal
    currentVal =  newton(f,df,currentVal)
    approximation.append(currentVal)
    n += 1
print("Newton's Method : Value, Iterations-",approximation
[-1],n)
iterations = np.arange(n)
count = 0
a,b = 0.1, 1
m = (a+b)/2
f_m = f(m)
f_a = f(a)
f_b = f(b)
m_values = []
while not np.isclose([f_m],[0]):
    """
    Bisection Algorithm, exactly same as the one used in Pr
oblem 2
    """
    m = (a+b)/2
    if count > 52:
        break
    f_m = f(m)
    f_a = f(a)
    f_b = f(b)
    m_values.append(m)
    if f_m*f_a < 0:
        a,b = a,m
    elif f_m*f_b < 0:
        a,b = m,b
    count += 1
print("Bisection Method : Value, Iterations-", m, count,'\n
')
bisection = np.array(m_values)
approximation = np.array(approximation)
residuals = 1/7 - approximation
residual_bisection = 1/7 - bisection
counts = np.arange(count)
print('Residual Plots','\n')
plt.figure(figsize=(16,9)) # Setting Larger Graph Size
plt.plot(iterations,residuals)
axes = plt.gca()
axes.legend(["Newton's Method "])
plt.show()
```

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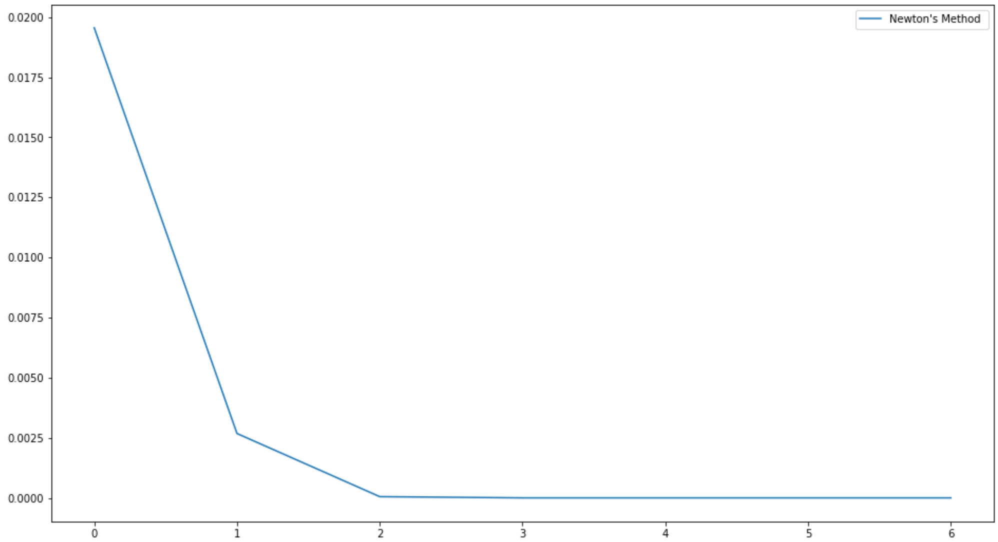
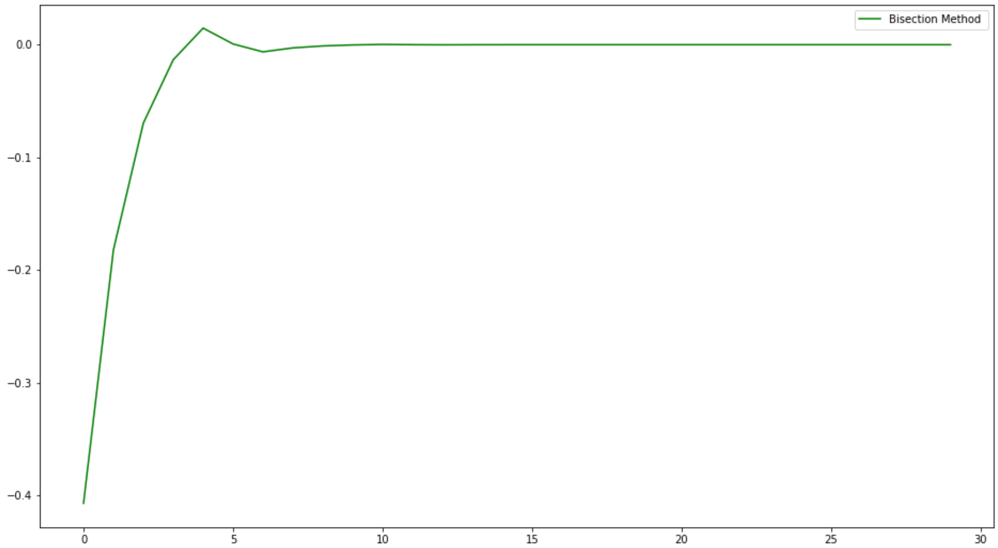
```
Newton's Method : Value, Iterations- 0.14285714285714285 7
Bisection Method : Value, Iterations- 0.14285714281722903 3
0

Residual Plots
```





Q3 3/4 Proofs missing last part + comment on result.

Problem 4

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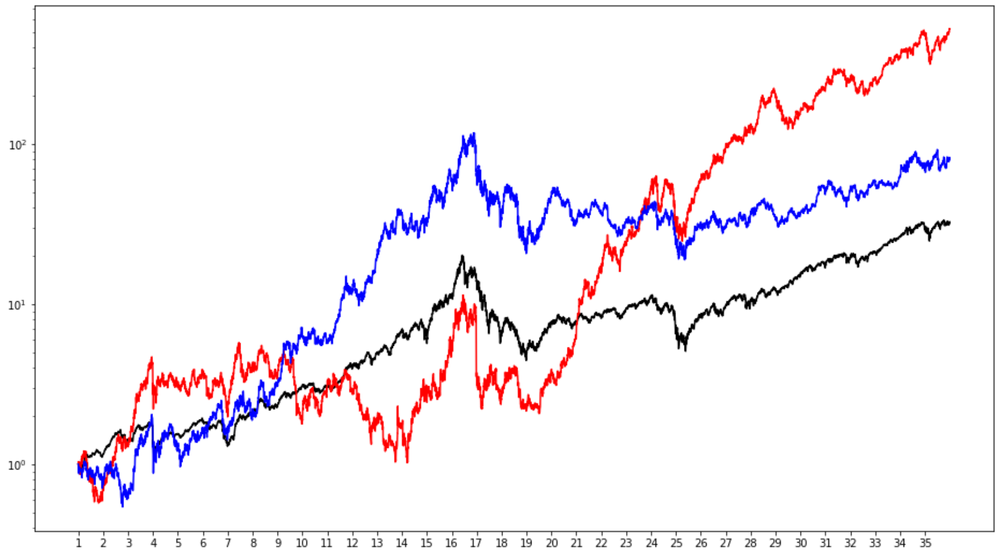
In [61]:
```
url.urlretrieve("http://planets.utsc.utoronto.ca/~pawel/pyth/IXIC.dat", "IXIC.dat")
url.urlretrieve("http://planets.utsc.utoronto.ca/~pawel/pyth/AAPL.dat", "AAPL.dat")
url.urlretrieve("http://planets.utsc.utoronto.ca/~pawel/pyth/INTC.dat", "INTC.dat")
pass
#Retriving Files
```

In [4]:
```
ixic = np.loadtxt('IXIC.dat')
aapl = np.loadtxt('AAPL.dat')
intc = np.loadtxt('INTC.dat')
t = np.arange(0, len(ixic))
plt.figure(figsize=(16,9)) # Setting Larger Graph Size
# Semilog plot
plt.semilogy(t,ixic,color=(0,0,0))
plt.semilogy(t,aapl,color=(1,0,0))
plt.semilogy(t,intc,color=(0,0,1))
#Setting the axis
plt.xticks(range(1,len(ixic),252),range(1,36))
plt.show()
```



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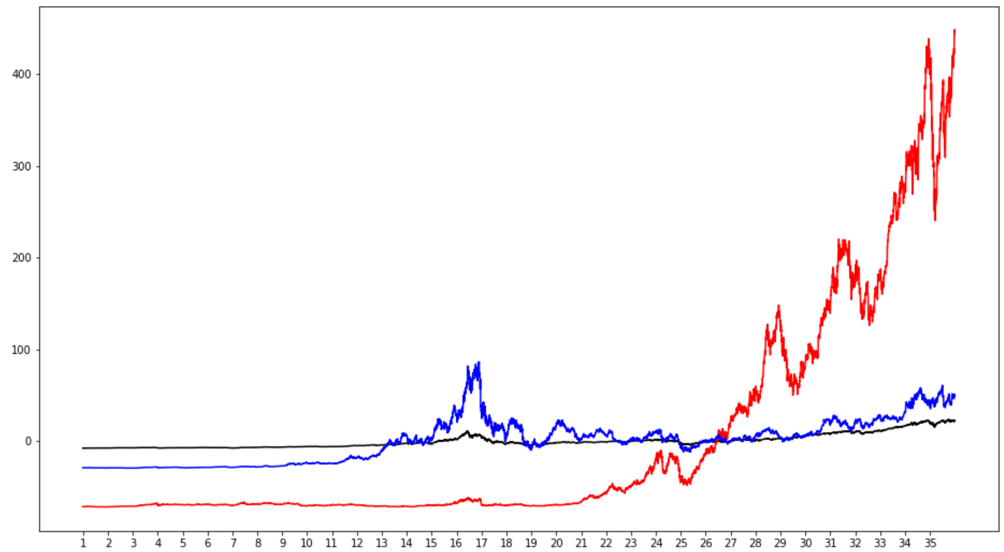
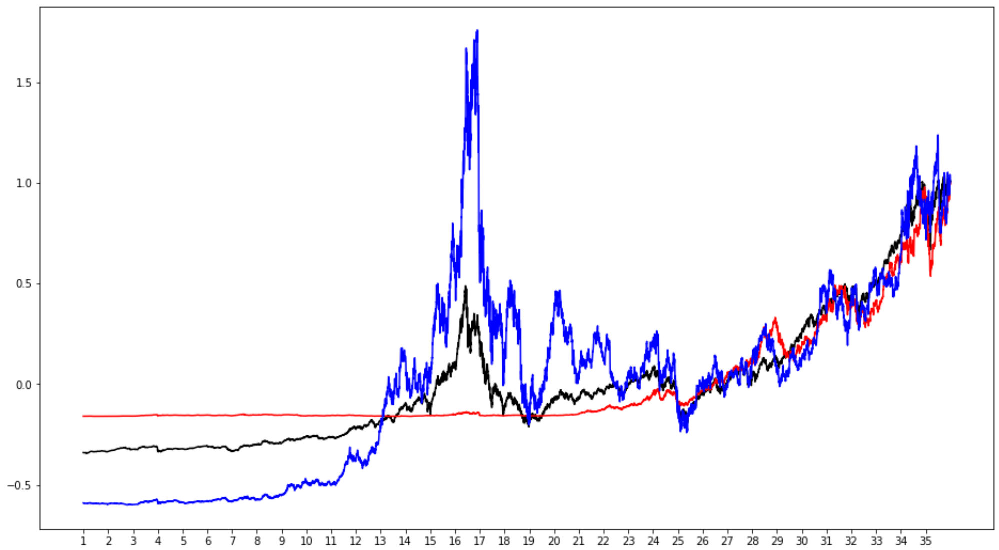
In [5]:
```
# Detrending the inflation trend line of 3%
# Line in the form y = mx+c
def detrend(data):
    """
    Returns detrended list, substracting the inflation rate
    """
    m  = 0.03/252
    # This is the inflation rate per market day, as 252 mar
ket days a year and 0.03 inflation a year
    # c = mean(y) - m*mean(x)
    c = np.mean(data) - m*np.mean(t)
    line = m*t + c
    return data-line
def normalize(data):
    """
    Returns normalized list, where the data ends at value 1
    """
    lastVal = data[-1]
    return data/lastVal
# Creating Detrended Time Series
ixic_detrended = detrend(ixic)
aapl_detrended = detrend(aapl)
intc_detrended = detrend(intc)
plt.figure(figsize=(16,9)) # Setting Larger Graph Size
plt.plot(t,ixic_detrended,color=(0,0,0))
plt.plot(t,aapl_detrended,color=(1,0,0))
plt.plot(t,intc_detrended,color=(0,0,1))
plt.xticks(range(1,len(ixic),252),range(1,36))
plt.show()
print("The above graph is the detrended data graph and belo
w graph is normalized data graph ")
#Creating Renormalized Time Series
ixic_normal = normalize(ixic_detrended)
aapl_normal = normalize(aapl_detrended)
intc_normal = normalize(intc_detrended)
plt.figure(figsize=(16,9)) # Setting Larger Graph Size
plt.plot(t,ixic_normal,color=(0,0,0))
plt.plot(t,aapl_normal,color=(1,0,0))
plt.plot(t,intc_normal,color=(0,0,1))

plt.xticks(range(1,len(ixic),252),range(1,36))
plt.show()
```




```
The above graph is the detrended data graph and below graph
is normalized data graph
```



This is not what we meant by detrending. Just subtract a line with slope 3% per year.

Looking at the normalized data graph, it is clear that the general growth of the market or the IXIC data growth is mostly due to historic inflation graph. While the blue graph, intel stock price, INTC is very slightly above the general historic inflation growth, however about 20 years ago, the INTC stock was growing significantly faster than the inflation rate. Finally, the apple stock or AAPL, was growing negatively relative to the inflation rate for about the first 20 years of this data but in recent years has been growing significantly faster than inflation rate.

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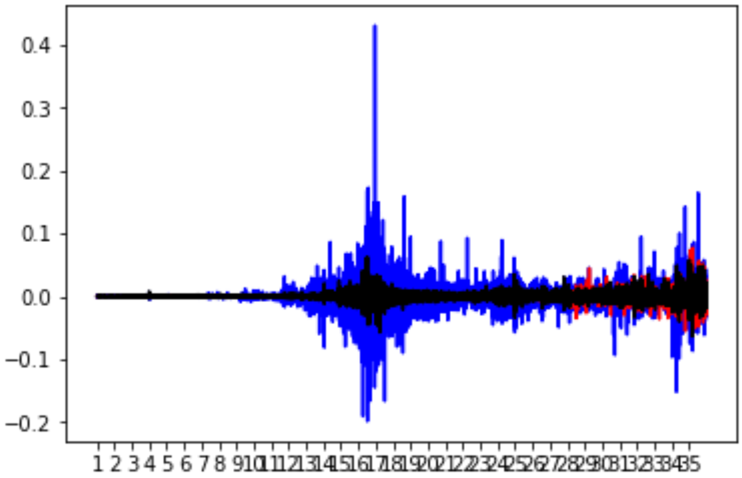
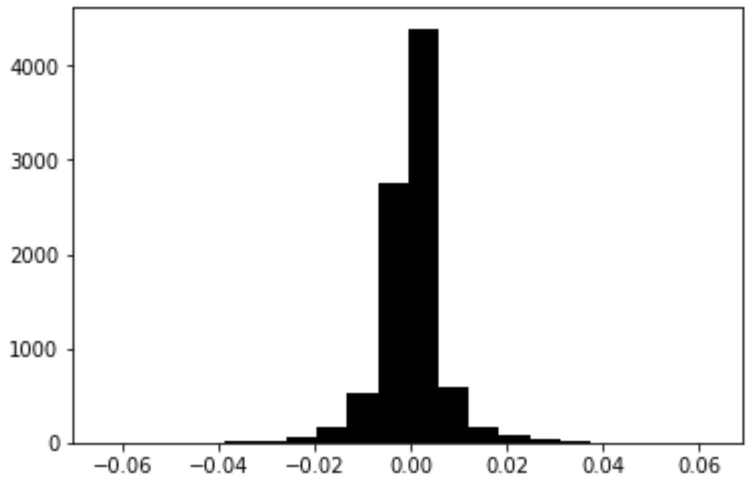
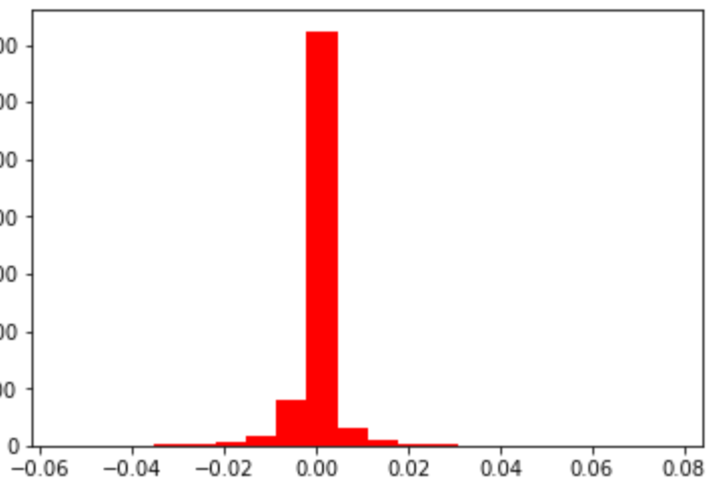
In [6]:
```
def derrivativeP(data):
    """
    Returns rate of change of the time series by comparing
the previous and next value
    """
    firstValue = data[:-1]
    nextValue = data[1:]
    #print(len(firstValue),len(nextValue))
    return firstValue - nextValue
#Creating Rate of change
dIxic = derrivativeP(ixic_normal)
dAapl = derrivativeP(aapl_normal)
dIntc = derrivativeP(intc_normal)
# Plotting rate of change
plt.plot(t[:-1],dIntc,color=(0,0,1))
plt.plot(t[:-1],dAapl,color=(1,0,0))
plt.plot(t[:-1],dIxic,color=(0,0,0))

plt.xticks(range(1,len(ixic),252),range(1,36))
plt.show()
#Plot rate of change as histogram
plt.hist(dIxic,bins=20,color=(0,0,0))
print("IXIC- Mean, Std",dIxic.mean(),dIxic.std())
plt.show()
plt.hist(dAapl,bins=20,color=(1,0,0))
print("AAPL- Mean, Std",dAapl.mean(),dAapl.std())
plt.show()
plt.hist(dIntc,bins=20,color=(0,0,1))
print("INTC- Mean, Std",dIntc.mean(),dIntc.std())
plt.show()
```




```
IXIC- Mean, Std -0.00015185066751938043 0.00681536123856707
3
```




```
AAPL- Mean, Std -0.00013140212236247294 0.00503116722039568
2
```




```
INTC- Mean, Std -0.00018019146294228 0.01947618869723907
```

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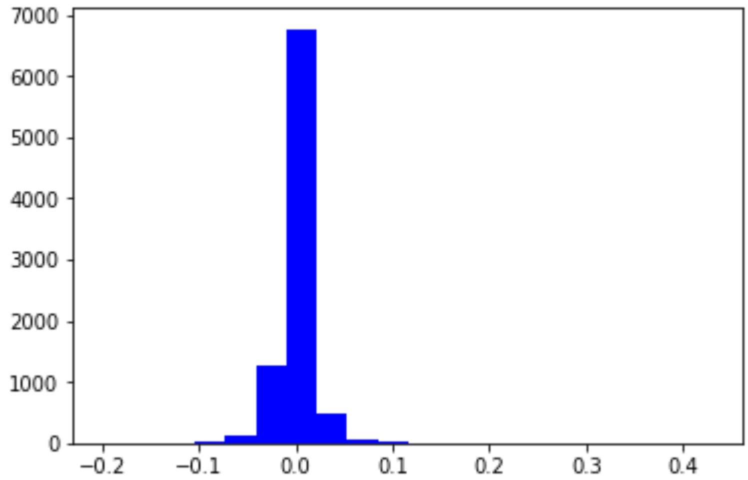

Proof :

Given: Y = c*X + b

To prove: r = +- 1, r = sign(c)

Symbols μ for mean

$$Cov(X, Y) = E(X - \mu_x)(cX - c\mu_x) \implies c \times E(X - \mu_x)^2$$

$$\implies \sigma^2 \times c$$

$$Variance(Y) \implies Var(cX + b) \implies Var(cX) \implies \sigma_x^2 \times c^2$$

$$r = \frac{Cov(X, Y)}{\sigma_x \times \sigma_y} \implies \frac{\sigma^2 \times c}{\sigma_x^2 \times |c|}$$

$$\implies r = \frac{c}{|c|} \text{ QED}$$

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In [22]:
```
def covariance(x,y):
    """
    Returns covariance of X and Y
    """
    return (x- np.mean(x))*(y- np.mean(y))
def r(x,y,cov):
    """
    Returns coeffient of covarience for each year between X
and Y
    """
    relations = []
    for i in range(0,35):
        # Ap
        relations.append(np.sum(cov[i*252:(i+1)*252])/(np.s
um((x[i*252:(i+1)*252] - np.mean(x))**2)*np.sum((y[i*25
2:(i+1)*252] - np.mean(y))**2))**(1/2))
    return relations
def r_full(x,y,cov):
    """
    Returns coeffient of covarience for entire data set of
X and Y
    """
    return np.mean(cov)/(np.std(x)*np.std(y))
r_NA = r(ixic_normal,aapl_normal,covariance(ixic_normal,aap
l_normal)) # NASDAQ-Apple
r_NI = r(ixic_normal,intc_normal,covariance(ixic_normal,int
c_normal)) # NASDAQ-INTEL
r_IA = r(intc_normal,aapl_normal,covariance(intc_normal,aap
l_normal)) # INTEL-Apple
print("Year\t","NA\t\t","NI\t\t","IA\n")
for i in range(0,35):
    print(i+1,r_NA[i],r_NI[i],r_IA[i])
r_NA_full = r_full(ixic_normal,aapl_normal,covariance(ixic_
normal,aapl_normal)) # NASDAQ-Apple
r_NI_full = r_full(ixic_normal,intc_normal,covariance(ixic_
normal,intc_normal)) # NASDAQ-INTEL
r_IA__full = r_full(intc_normal,aapl_normal,covariance(intc
_normal,aapl_normal)) # INTEL-Apple
print("\nAverage",r_NA_full,r_NI_full,r_IA__full)
```

```
Year     NA              NI              IA

1 0.9999351771413449 0.9999465993019827 0.9999981126661226
2 0.999905082123299 0.999832324625482 0.9999743862719062
3 0.9999640653187579 0.9999697748261174 0.9999966381482311
4 0.9999333553223355 0.9999544424140836 0.9999895403868692
5 0.9998767189501863 0.9999145322612504 0.9999915361546918
6 0.999873166105377 0.999852491199096 0.9999591724440073
7 0.9995380892317527 0.9995960505042796 0.9999623630729927
8 0.9999094019725308 0.9999164853115835 0.9998848374936237
9 0.9993745516999291 0.999806148419211 0.9988489016369941
10 0.9998407338287316 0.9999046862522329 0.9999024457134069
11 0.9972993456154381 0.9979161400380376 0.9921729590626722
12 0.9978673821728263 0.9949734796339822 0.991194867370334
13 0.9867094989328324 0.33886599981572085 0.199325364772887
6
14 0.9755458845785038 -0.19774440496976772 -0.2567191774776
1186
15 -0.3037161306581286 0.5920037079923467 -0.90686082929086
83
16 -0.9467384144718061 0.9571242322447178 -0.94440016824970
29
17 -0.018217752344679903 0.4233090812473575 -0.900015258976
9097
18 0.9286455694058353 -0.3704651369611163 -0.68198904650309
51
19 0.9737346179467342 0.039671345184796425 -0.1829107125044
2079
20 0.9747266179213485 -0.8059953678745773 -0.89399664362782
48
21 0.9625098903195777 -0.848420420124127 -0.930942683250846
2
22 0.8060843740663133 -0.15195473403650306 -0.3800132929554
6543
23 -0.7525218929615805 0.9442229310774368 -0.67490513092973
95
24 -0.006608566709371569 0.8067718185106827 -0.517293455083
3561
25 0.9953318664277514 0.9718861722522337 0.9622325903156156
26 0.297160900116172 0.11139065321876443 0.1611310327196377
2
27 0.8907166668965475 0.8854630585480904 0.8456105374808474
28 0.9914566931019279 0.9379864568239813 0.9324813342672365
29 0.9127242448045143 0.9331158570248247 0.795646449864182
30 0.9925445130900916 0.9344619097169077 0.9621439858279555
31 0.9986738038501282 0.970155689879934 0.97093377546489
32 0.9964136613554837 0.9951914948050716 0.9912287153921686
33 0.9982476651741266 0.9923433354528138 0.986104619113764
34 0.9974302750081178 0.9920422881954374 0.9851166905770773
35 0.996329246941892 0.9937047615236932 0.9877805484957529

Average 0.9177502025836916 0.8578807347233194 0.61818290509
85176
```

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Q4 4.5/6 Comment on the last part results.

In [ ]:

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