1 Analysis by S-function

In this section, we analyze the BCT of an addition by the S-function technique from [MVDP11]. We also compare this analysis with ours in the submission to explain the differences explicitly.

1.1 Defining the BCT Entry

The original definition of a BCT entry is

$$\begin{aligned}\mathrm{BCT}_n(\Delta_l, \Delta_r, \nabla_l, \nabla_r) = \#\Big\{ (L, R) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \;\Big|\; & \big((L \boxplus_n R) \oplus \nabla_l \boxminus_n (R \oplus \nabla_r)\big) \\ & \oplus \Big(\big((L \oplus \Delta_l) \boxplus_n (R \oplus \Delta_r)\big) \oplus \nabla_l \boxminus_n (R \oplus \Delta_r \oplus \nabla_r)\Big) = \Delta_l \Big\}.\end{aligned} \tag{1}$$

To use the S-function technique, this formula should be rewritten. Given n-bit words x_1, y_1, Δx, Δy, ∇x, and ∇y, we calculate Δz using

$$x_2 \leftarrow x_1 \oplus \Delta x, \tag{2}$$
$$y_2 \leftarrow y_1 \oplus \Delta y, \tag{3}$$
$$t_1 \leftarrow x_1 + y_1, \tag{4}$$
$$t_2 \leftarrow x_2 + y_2, \tag{5}$$
$$u_1 \leftarrow t_1 \oplus \nabla x, \tag{6}$$
$$v_1 \leftarrow y_1 \oplus \nabla y, \tag{7}$$
$$u_2 \leftarrow t_2 \oplus \nabla x, \tag{8}$$
$$v_2 \leftarrow y_2 \oplus \nabla y, \tag{9}$$
$$z_1 \leftarrow u_1 - v_1, \tag{10}$$
$$z_2 \leftarrow u_2 - v_2, \tag{11}$$
$$\Delta z \leftarrow z_1 \oplus z_2. \tag{12}$$

Then, the BCT entry is redefined as

$$\mathrm{BCT}_n(\Delta_l, \Delta_r, \nabla_l, \nabla_r) = \# \left\{ (x_1, y_1) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \mid \Delta x = \Delta_l, \Delta y = \Delta_r, \nabla x = \nabla_l, \nabla y = \nabla_r, \Delta z = \Delta_l \right\}.$$

1.2 Constructing the S-Function

Equations (2)-(12) are rewritten on a bit level:

$$x_2[i] \leftarrow x_1[i] \oplus \Delta x[i], \tag{13}$$
$$y_2[i] \leftarrow y_1[i] \oplus \Delta y[i], \tag{14}$$
$$t_1[i] \leftarrow x_1[i] \oplus y_1[i] \oplus c_1[i], \tag{15}$$
$$t_2[i] \leftarrow x_2[i] \oplus y_2[i] \oplus c_2[i], \tag{16}$$
$$c_1[i+1] \leftarrow (x_1[i] + y_1[i] + c_1[i]) \gg 1, \tag{17}$$
$$c_2[i+1] \leftarrow (x_2[i] + y_2[i] + c_2[i]) \gg 1, \tag{18}$$
$$u_1[i] \leftarrow t_1[i] \oplus \nabla x[i], \tag{19}$$
$$v_1[i] \leftarrow y_1[i] \oplus \nabla y[i], \tag{20}$$
$$u_2[i] \leftarrow t_2[i] \oplus \nabla x[i], \tag{21}$$
$$v_2[i] \leftarrow y_2[i] \oplus \nabla y[i], \tag{22}$$
$$z_1[i] \leftarrow u_1[i] \oplus v_1[i] \oplus b_1[i], \tag{23}$$
$$z_2[i] \leftarrow u_2[i] \oplus v_2[i] \oplus b_2[i], \tag{24}$$
$$b_1[i+1] \leftarrow \big((2 + u_1[i] - v_1[i] - b_1[i]) \gg 1\big) \oplus 1, \tag{25}$$
$$b_2[i+1] \leftarrow \big((2 + u_2[i] - v_2[i] - b_2[i]) \gg 1\big) \oplus 1, \tag{26}$$
$$\Delta z[i] \leftarrow z_1[i] \oplus z_2[i], \tag{27}$$

where carries $c_1[0] = c_2[0] = 0$ and borrows $b_1[0] = b_2[0] = 0$. Let the states be

$$\begin{aligned} S[i] &\leftarrow (c_1[i], b_1[i], c_2[i], b_2[i]), \\ S[i+1] &\leftarrow (c_1[i+1], b_1[i+1], c_2[i+1], b_2[i+1]). \end{aligned}$$

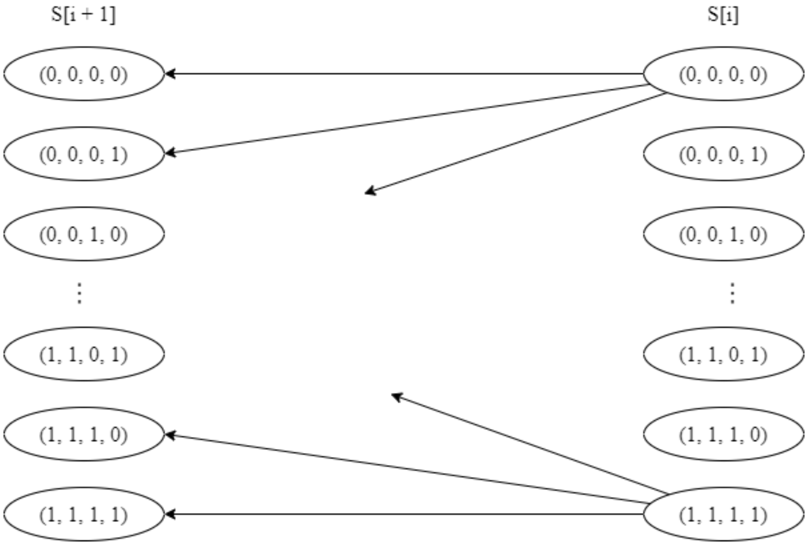


Figure 1: A bipartite graph from the S-function.

Then, Equations (13)-(27) corresponds to the S-function for the BCT of an addition, as the following.

$$(\Delta z[i], S[i+1]) = f(x_1[i], y_1[i], \Delta x[i], \Delta y[i], \nabla x[i], \nabla y[i], S[i]), 0 \le i < n \tag{28}$$

1.3 Computing the Entry

In order to derive the formula for computing the BCT entry, a graph is generated from the S-function (Equation (28)). For $0 \le i \le n$, every state $S[i]$ is represented as a vertex in the graph. Each edge is drawn according to Equation (28). Specifically, for every pair of states, $S[i]$ and $S[i+1]$, if there are values of $x_1[i]$, $y_1[i]$, $\Delta x[i]$, $\Delta y[i]$, $\nabla x[i]$, and $\nabla y[i]$ such that $\Delta z[i]$ is equal to $\Delta x[i]$, then an edge is drawn between vertices $S[i]$ and $S[i+1]$. The resulting graph consists of bipartite graphs each of which contains the vertices $S[i]$ and $S[i+1]$. Figure 1 shows an example of a bipartite graph. Note that, for any bit position i, there are 16 values for $S[i]$, which generates 16 different vertices.

According to [MVDP11], the BCT entry is equal to the number of the paths from $(c_1[0], b_1[0], c_2[0], b_2[0]) = (0, 0, 0, 0)$ to any of the 16 vertices $(c_1[n], b_1[n], c_2[n], b_2[n])$. Let $w[i]$ be $\Delta_l[i] \| \Delta_r[i] \| \nabla_l[i] \| \nabla_r[i]$. Thanks to the nice properties of a bipartite graph, we can use a matrix $A_{w[i]} = [x_{kj}]$ to describe it, where x_{kj} is the number of edges that connect vertices $j = S[i]$ and $k = S[i+1]$. Define a 1×16 matrix $L = [1\ 1\ \ldots\ 1]$ and a 16×1 matrix $C = [1\ 0\ \ldots\ 0]^T$. Then, the formula of the BCT entry is

$$\mathrm{BCT}_n(\Delta_l, \Delta_r, \nabla_l, \nabla_r) = L A_{w[n-1]} A_{w[n-2]} \cdots A_{w[0]} C. \tag{29}$$

Finally, the FSM reduction algorithm in [MVDP11] helps to reduce the number of states.

1.4 Discussion

In this subsection, we explain the differences between our technique and the technique in [MVDP11] as well as the arxtools from [Leu12].

First, our technique focuses on the recursive form of Equation 1 instead of the S-function and the bipartite graphs, which is more elegant. As shown in the previous subsections, the previous technique requires writing down many equations and rewriting them. In Section 1.3, we also have to rely on the bipartite graphs to derive the matrices and the final formula. However, our technique starts by transforming Equation (1) and turns it into a recursive form. This transformation does not require any complicated functions and graphs. Essentially, our idea is from the perspective of dynamic programming. After obtaining the recursive form, the algorithm comes out naturally.

Second, to reduce the number of states, our technique digs out the mathematical property behind the recursive form. The previous technique uses an FSM reduction algorithm to achieve this goal. Indeed, this is a nice idea and can keep the number of states to be the fewest. On the other hand, this algorithm does not tell us what mathematical properties are being used while ours does. Moreover, our computation allows us to find out more properties explained in Section 3.3 of our paper, which lead to the SAT models in Section 4.

Furthermore, the aim of arxtools and that of our technique are completely different. To the best of our knowledge, the arxtools is used to compute the probability of a differential (or boomerang) characteristic and detect incompatibility. It cannot find out a boomerang characteristic with a high probability. Nonetheless, our technique aims to search for such a characteristic. Except the arxtools, we have not found out any other automatic search technique targeting this problem. Therefore, we think that our technique is the first attempt.

In conclusion, our technique is different from the previous ones. It is just similar to the existing techniques, but not identical. In fact, in this scenario, ours has advantages over the others.

[MVDP11] Nicky Mouha, Vesselin Velichkov, Christophe De Cannière, and Bart Preneel. The differential analysis of S-functions. In Alex Biryukov, Guang Gong, and Douglas R. Stinson, editors, SAC 2010, volume 6544 of LNCS, pages 36–56. Springer, Heidelberg, August 2011.
[Leu12] Analysis of differential attacks in ARX constructions. Gaëtan Leurent, Asiacrypt 2012.

2 Results from ARXTools

In this section, we present the results from the arxtools. The first graph is generated from the command:

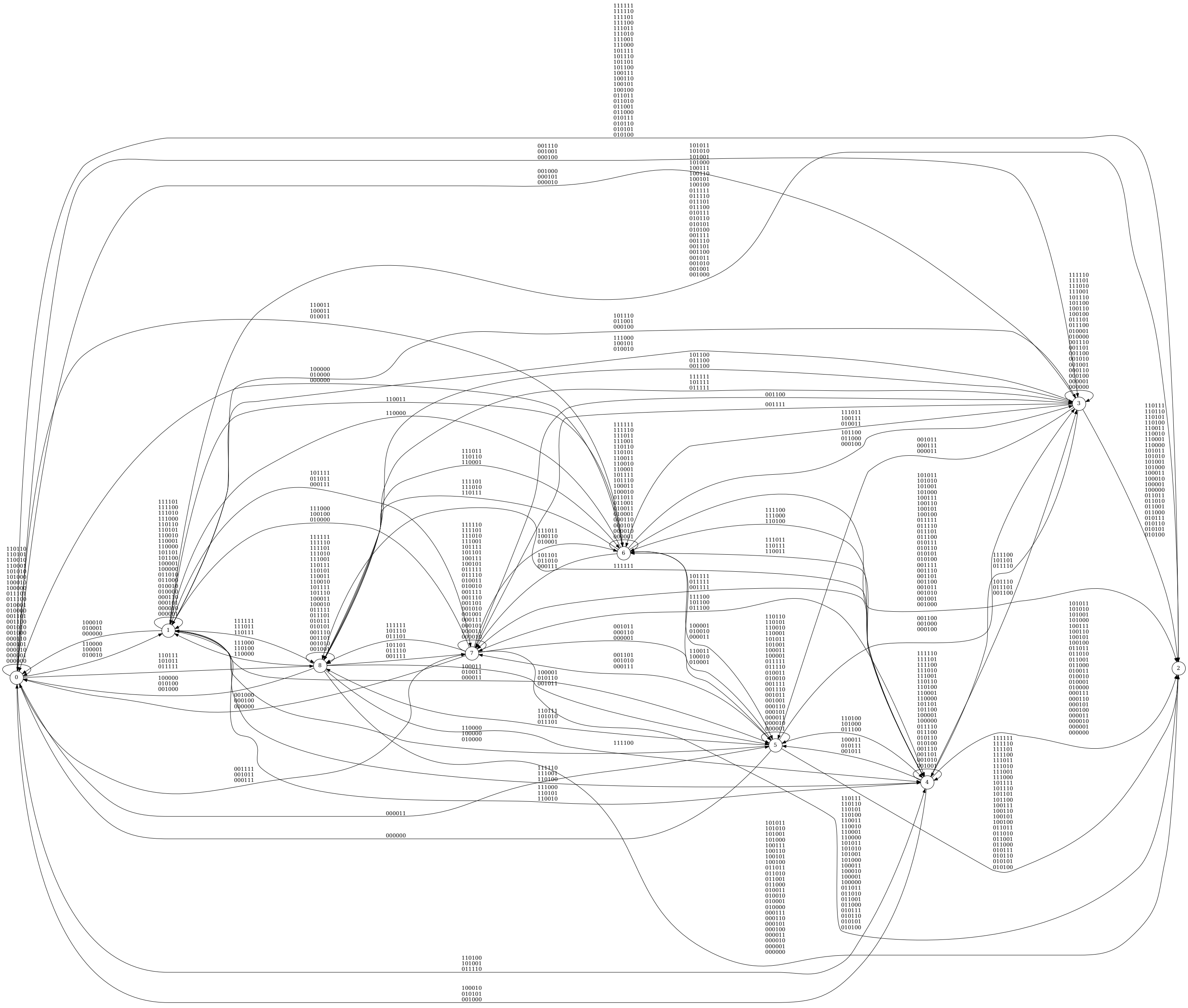
```
build_fsm -e '(((V0+V1)^P2)-(V1^P3))^((((V0^P0)+(V1^P1))^P2)-(V1^P1^P3))==P0' -t -g.
```

The second graph is generated from the command:

```
build_fsm -e '(((V0+V1)^P2)-(V1^P3))^((((V0^P0)+(V1^P1))^P2)-(V1^P1^P3))==P0' -d -g.
```

We have tried them, but we currently do not find out any possibility to simplify our technique based on these results. It seems that our current computation is easier to derive the SAT models. Thanks for the ideas from Review-#6D.

(Please see the next two pages for the results.)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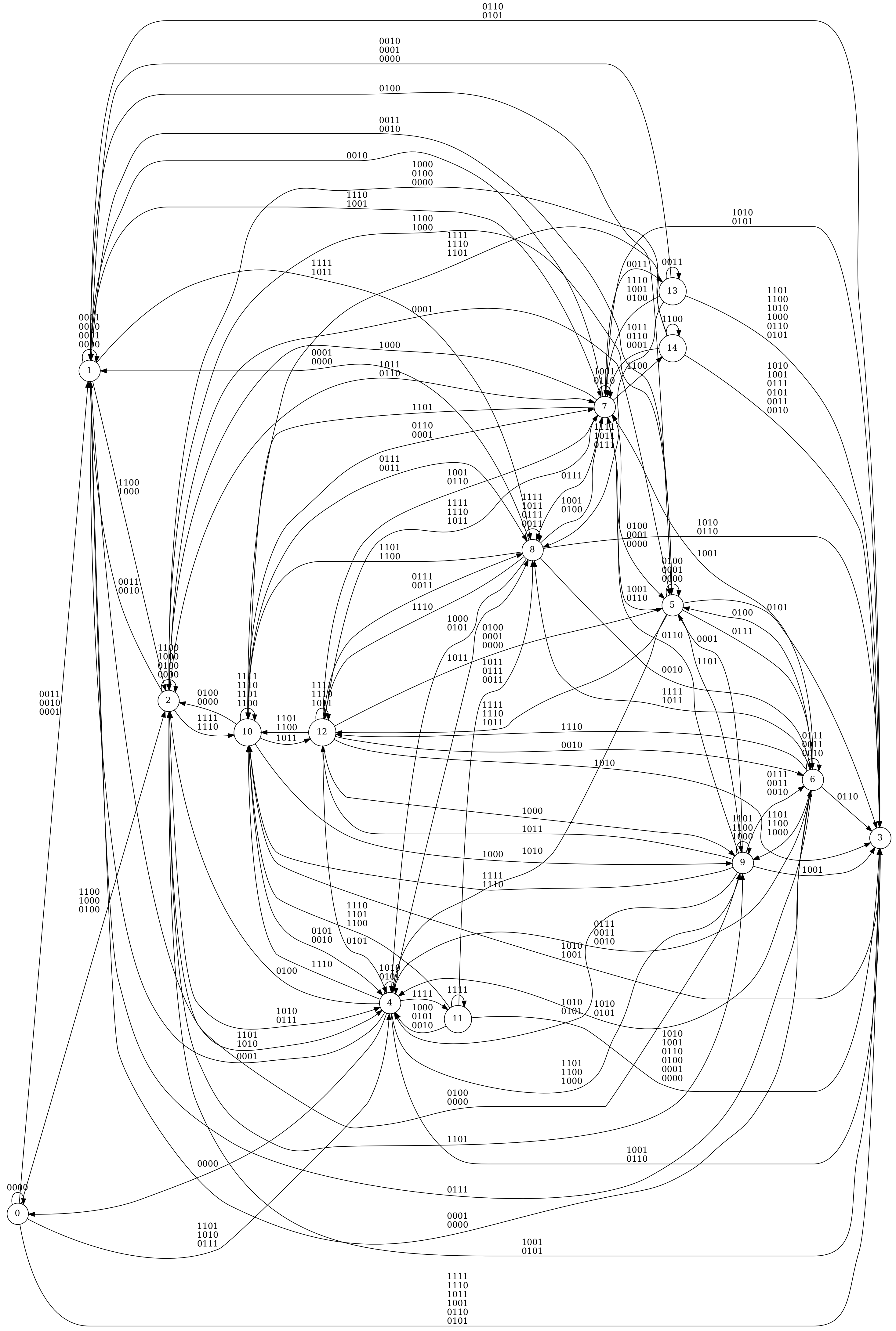
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