

Polygon Ratio Problem: Many Proofs, One Degree of Freedom

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Introduction

During the Christmas holidays, a basic geometry proof problem showed up in my class alumni chat (Figure 1 is the original screenshot from classmate Lin He). He asked about part (2) because part (1) is a warm-up, and that set the whole discussion in motion. I have always liked Euclidean geometry, and problems like this rarely stop at one proof. Draw one line and a ratio drops out; draw another and a different argument appears. What follows is a cleaned collection: one quick sine-law proof, several synthetic proofs, an analytic-geometry proof (from ChatGPT 5.2, edited for clarity), and a one-parameter generalization that reveals deeper structure. I hope it gives younger students a sense of how much structure a single diagram can hide.

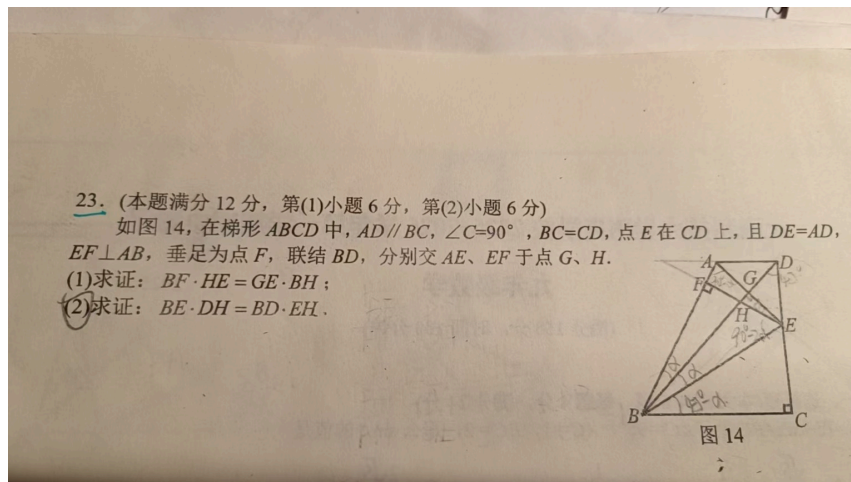


Figure 1: The original problem (photo from the chat).

Let me first restate the problem in words and redraw the diagram more cleanly.

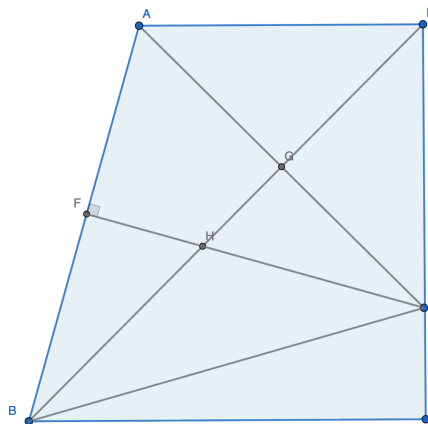


Figure 2: A clean redraw in GeoGebra.

Problem statement (translated). In trapezoid $ABCD$, $AD \parallel BC$, $\angle C = 90^\circ$, and $BC = CD$. Point E lies on CD , and $DE = AD$. Through E , draw $EF \perp AB$ with F as the foot of the perpendicular. Draw BD ; it intersects AE and EF at points G and H , respectively. Prove:

1. $BF \cdot HE = GE \cdot BH$.
2. $BE \cdot DH = BD \cdot EH$.

Part 1. Synthetic geometry (auxiliary lines)

The basic equal angles and lengths are shown in Figure 3.

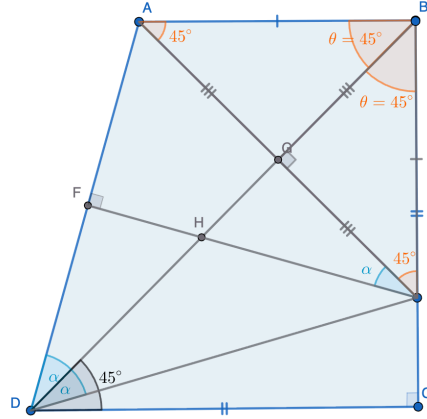


Figure 3: Basic angle and length relations.

Question (1) is a warm-up; it follows from similar triangles visible in Figure 3. For part (2), a quick start is the sine law. For a triangle ABC with side lengths a, b, c opposite vertices A, B, C , we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This gives Proof 0 immediately.

Proof 0 (sine law)

The ratios $DB : BE$ and $DH : HE$ are both $\sin(45^\circ + \alpha) : \sin 45^\circ$, so they are equal. This proof uses no auxiliary lines.

To avoid assuming the sine law, I drew auxiliary lines. Rather than separate diagrams per proof, the next figure collects all extra points. Each added point carries a subscript that matches the proof number (for example, D_1 is used only in Proof 1).

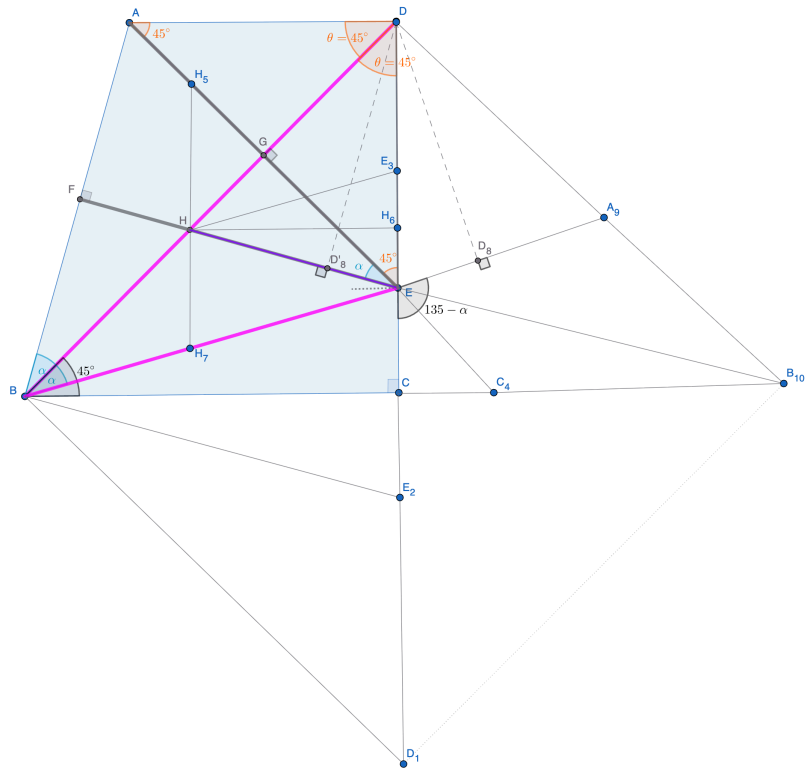


Figure 4: All auxiliary lines used in the synthetic proofs.

Below are brief proof sketches. Each can be completed as a standard middle-school geometry exercise.

Proof 1

Extend DC to D_1 so that D_1 is symmetric to D with respect to BC . Then $\triangle EBD_1 \sim \triangle EHD$, so the desired ratio follows.

Proof 2

Extend DC to E_2 so that E_2 is symmetric to E with respect to BC . Then $\triangle E_2BD \sim \triangle EHD$, so the desired ratio follows.

Proof 3

From H draw a line parallel to BE and intersect DE at E' . Then $HE = HE'$. The rest follows.

Proof 4

Extend AE to meet BC at C_4 . Then $AC_4 = BD$. Also $\triangle HED \sim \triangle ABC_4$, so the ratio follows. (Credit: Prof. Xueheng Lan.)

Proof 4.1

Use the reflection property: the ray from B to E reflects across DE and passes through H . This gives the same angle relation at DE , so the ratio follows.

Proof 5

Draw a line from H parallel to DE and intersect EA at H_5 . Then $DH = EH_5$, and $\triangle EHH_5 \sim \triangle BED$, which leads to the result.

Proof 6

Draw a line from H parallel to BC and intersect DE at H_6 . Then $DH/DB = HH_6/BD = EH/BD$

Proof 7

Draw a line from H parallel to DE and intersect BE at H_7 . Then $HE : BE = H_7E : BE = HD : BD$.

Proof 8

The distances from D to BE and EH are equal (use DD_8 and DD'_8 ; these are non-essential since $\angle BEC = \angle HED = 45^\circ + \alpha$). The distances from E to BD and HD are also equal. So area BDE to area DHE can be expressed as $BE \cdot DD_8/2 : EH \cdot DD'_8/2 = BE : EH$, and also as $BD \cdot GE/2 : DH \cdot GE/2 = BD : DH$. Hence they are equal.

Proof 9

Extend BE to A_9 where $EA_9 = EH$. Then $DH = DA_9$, $EH = EA_9$, and $\angle EDA_9 = \angle EDH = 45^\circ$. Since DE bisects $\angle BDA_9$, we get $BD : DA_9 = BE : EA_9$. The rest is trivial.

Proof 10

Extend EH to B_{10} where B_{10} intersects the extension of line BC . Then B_{10} is the reflection of B across BC (since HE is the reflection path of light BE at mirror DC). One can show $\angle CDB_{10} = \angle BDC = 45^\circ$, so $DH : HE = DB_{10} : EB_{10} = DB : BE$.

Proof 11

This one also needs no auxiliary lines: notice $HD = HG + GE$ and $BD = BG + GE$, and $\triangle HGE \sim \triangle EGB$, so the result follows.

A note on degrees of freedom: the only adjustable angle in the original configuration is $\alpha < 45^\circ$. If $\alpha = 45^\circ$, then H and B overlap and the result is trivial. If $\alpha > 45^\circ$, then H lies on the extension of DB , and E lies on the extension of DC . The roles of B and H swap, and the same style of arguments still works. Similarly, one can show that a ray from B to E reflects across DE and reaches H .

Other proofs

By now we should be done. However, if you followed the solutions so far, you likely noticed two patterns: (1) we have essentially drawn a square $BDB_{10}D_1$ in Figure 4, and many proofs are based on various points on this square; (2) polygon $ABED$ is symmetric about BD , which is why we could leverage $AB = BE$ and $AH = HE$ in several steps. It follows that any proof obtained by extending lines in the original diagram has a mirror proof across BD . These are indicated again by subscripts, and they give Proofs 12, 13, 14, 15, and 16. They form the symmetric rectangle $BDB_{15}D_{12}$ and are shown in Figure 5.

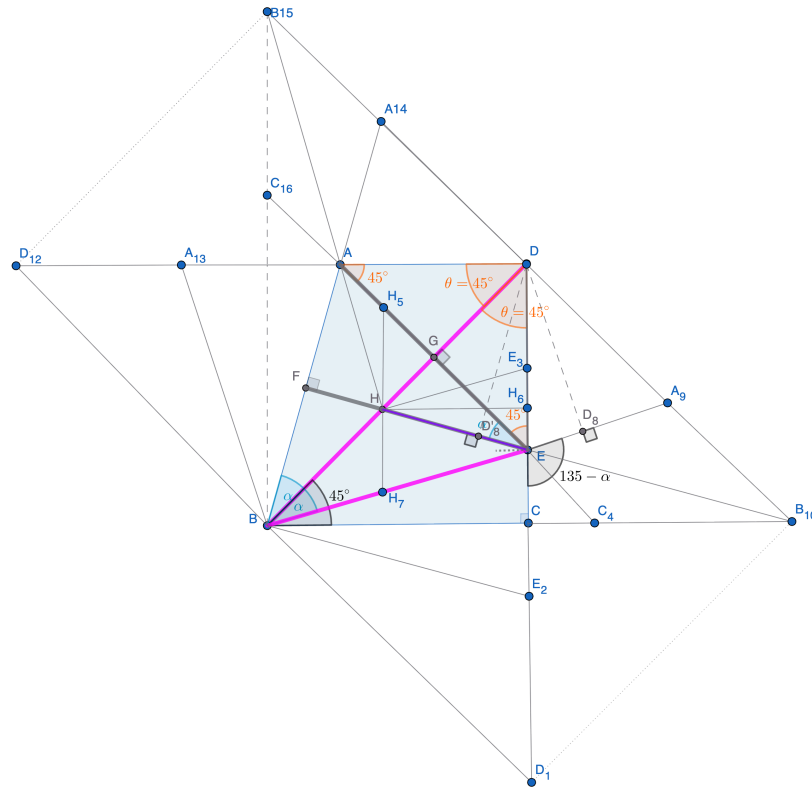


Figure 5: Five additional symmetric proofs.

Finally, several proofs such as Proofs 3, 5, 6, and 7 focus on what happens inside polygon $ABED$. Each has a symmetric counterpart (Proofs 17, 18, 19, 20) reflected across BD . For clarity, these are not displayed.

Part 2. Analytic geometry proof

This is a short coordinate proof.

Goal

$$BE \cdot DH = BD \cdot EH.$$

Proof

Setup. Let $C = (0, 0)$, $B = (-1, 0)$, $D = (0, 1)$. Then $AD \parallel BC$ implies $A = (t - 1, 1)$ when $E = (0, t)$ and $DE = AD$ with $0 < t < 1$.

Lines. $BD : y = x + 1$. The slope of AB is $1/t$, so $EF \perp AB$ has slope $-t$ and passes through E : $EF : y = t - tx$.

Intersection. Solve $x + 1 = t - tx$ to get

$$H = \left(\frac{t-1}{t+1}, \frac{2t}{t+1} \right).$$

Common factor.

$$H - D = \frac{t-1}{t+1}(1, 1), \quad H - E = \frac{t-1}{t+1}(1, -t).$$

Hence

$$DH = \frac{1-t}{t+1}\sqrt{2}, \quad EH = \frac{1-t}{t+1}\sqrt{1+t^2}.$$

Also $BD = \sqrt{2}$ and $BE = \sqrt{1+t^2}$.

Finish.

$$BE \cdot DH = \sqrt{1+t^2} \cdot \frac{1-t}{t+1} \sqrt{2} = BD \cdot EH.$$

Conclusion: $BE \cdot DH = BD \cdot EH$.

Part 3. One more degree of freedom (ellipse view)

The 45° angle is special. If we relax that angle, the clean right-angle structure disappears, but the problem still has a simple geometric core: reflection on an ellipse. The key insight is to generalize the reflection: the ray from B to E reflects across DE and heads to H . This reveals an ellipse that reflects light from one focus to the other at any point E on it. That reflection property is equivalent to the constant-sum-of-distances definition of an ellipse.

Construction

Let O be the ellipse with foci B and H . Take a point A on O , with A closer to H . Draw the tangent line AD at A , and let it intersect BH at D . Take a point E on O , the reflection of A with respect to the major axis. Through E draw the line $D - E - C$, and choose C so that $BC \parallel AD$. Draw EH , and let it intersect AB at F . In general, $\angle BFE$ is not 90° , but when $\theta = 45^\circ$, $\angle BFE = 90^\circ$. The resulting shape is shown in Figure 6.

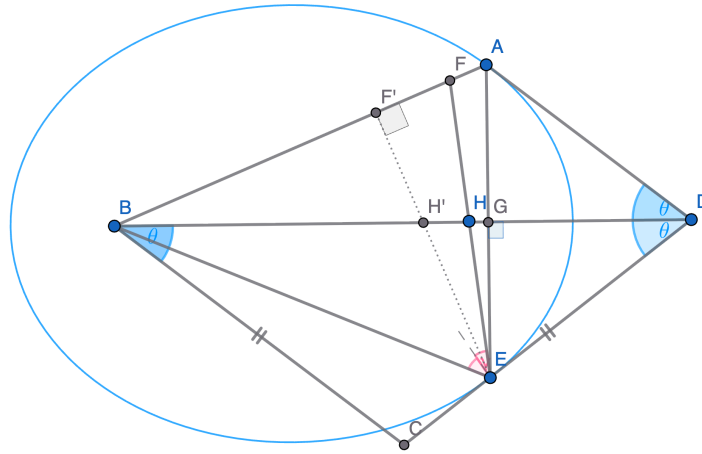


Figure 6: Ellipse construction with foci B and H .

Proof sketch

By the reflection property of the ellipse (a light ray from B to E reflects and then passes through H), we obtain

$$\frac{EH}{HD} = \frac{BE}{BD}.$$

Reflect H across line DE to H' . Then

$$\frac{DB}{BE} = \frac{DH'}{EH'} = \frac{DH}{EH},$$

which is the same proportion.

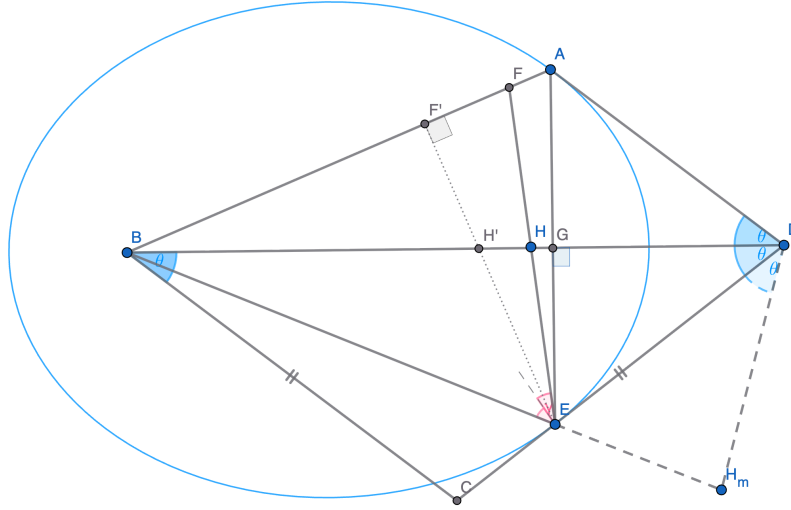


Figure 7: Reflection proof for the ellipse construction.

In this generalized setup, EF is no longer perpendicular to AB . In fact, the perpendicular line intersects AB and DB at F' and H' , as shown in Figure 7. Here H' differs from H and does not satisfy the ratio property. The original problem is special because the 45° angles create a surplus of right angles. The single degree of freedom here is essentially the ellipse eccentricity, with A and C determined by the right angle $\angle ADC$. When the ellipse degenerates to a circle (foci coincide), the identity becomes obvious.

In this setup, points A , F , C , and G , and the lines through them, are not even essential; only the foci B and H , and the tangent line DE , are necessary, with D on the major axis. This shows the true simplicity of the original problem, as shown in Figure 8 with the pure geometric approach.

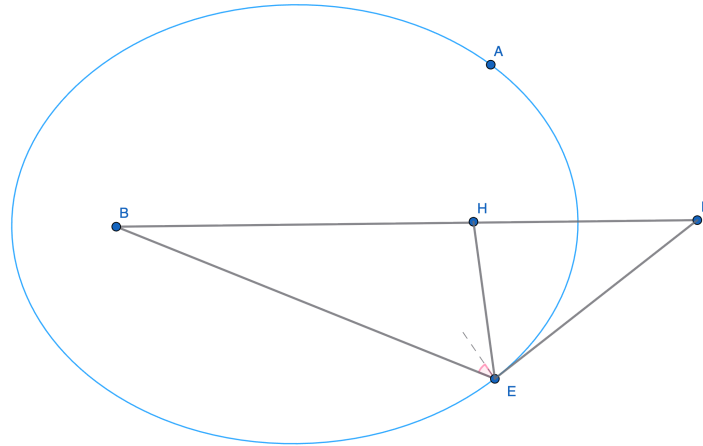


Figure 8: The simplified ellipse setup, which reveals the essence of the original problem.

Part 4. Vector analysis

Once one length is fixed, the trapezoid configuration has a single degree of freedom: the angle α . In the broader ellipse construction, a second parameter θ (the tangent point/slope) appears, and the same ratio follows from the reflection property. The special case $\theta = 45^\circ$ is exactly the original problem; that is why so many proofs exist, and why each auxiliary line exposes the same invariant ratio in a different way. Part 3 frees the θ dimension while keeping the reflection property, and that is what ultimately shows the simplicity behind the original picture.

This arc also mirrors the historical split between geometry and algebra in physics and mechanics. Newton, in the *Principia* [3], favored coordinate-free arguments because they were the clearest language of rigor in 1687, while modern physics often relies on coordinates for universal computation. The conclusions, however, are invariant, and

the geometric viewpoint keeps that symmetry visible. Several later authors also gave elementary, coordinate-free proofs (see [2]). It is good for young students to appreciate both techniques.

For folks familiar with Maxwell's equations, the following table shows how mathematical language for geometry evolved over time: from coordinate-expanded component equations, to vector analysis, to geometric algebra, and to differential forms. The tools change, but the physics does not.

Formulation	Coordinate status (what's really true)	Maxwell equations written in that language	What you gain / lose	Good reference
Maxwell (1865, component-heavy)	Coordinate-invariant laws, but written in components (explicit x, y, z bookkeeping) and bundled with extra relations (constitutive laws, Ohm, forces, potentials).	Not packaged as the modern four; a larger coupled component system (often summarized as ~20 equations/unknowns, plus auxiliaries).	Gain: historically close to derivations. Lose: compactness; structure is obscured by bookkeeping.	Maxwell's 1865 paper [4]; Deschamps [10].
Vector calculus (Heaviside/Gibbs)	Coordinate-free in content; coordinates enter only when writing component formulas.	(SI, vacuum) $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial_t \mathbf{E}$.	Gain: four clean equations; practical in 3D Euclidean space. Caveat: $\nabla \cdot, \nabla \times$ depend on the chosen metric + orientation, though invariantly defined once fixed.	Wilson [5]; cross product [7]; vector analysis [8].
Geometric algebra (Clifford; dot + wedge unified)	Coordinate-free once metric + orientation are fixed. Vectors are bold (e.g., \mathbf{a}, \mathbf{b}); higher-grade objects (bivectors, trivectors) are not vectors in disguise.	Core algebra: $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$. Maxwell can be written compactly (common spacetime GA convention): $\nabla F = \mu_0 J$, with the grade split $\nabla F = \nabla \cdot F + \nabla \wedge F$ giving $\nabla \wedge F = 0$ (homogeneous) and $\nabla \cdot F = \mu_0 J$ (sources).	Gain: cross product becomes derived (dual of \wedge); div/curl unify as one operator split by grade; rotations often cleaner. Lose: conventions vary (3D GA vs spacetime GA; units/signatures); less standard in mainstream EM courses.	Peeter Joot [9].
Differential forms (exterior calculus on spacetime)	Manifestly coordinate-invariant; works naturally on curved manifolds (GR). Metric enters via the Hodge star.	$dF = 0$, $d*F = J$. Here $*$ maps k -forms to $(n - k)$ -forms using the metric + orientation (in 4D, $2 \leftrightarrow 2$).	Gain: topology/relativity are transparent; $d^2 = 0$ exposes structure. Lose: abstraction up front; computations often unpack to components.	Deschamps [10]; Lindell [11]; Epstein [12].

Table 1: Forms of Maxwell's equations and their coordinate status.

Notes:

- Geometric algebra (Clifford; dot + wedge unified): metric + wedge fused immediately via $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$.
- Differential forms (exterior calculus on spacetime): Forms: wedge-first, metric-later via $*$.

The concept of vector analysis has evolved significantly since Newton's time, and that evolution echoes the flavor of Part 1 in these notes. While we did not use the differential side of geometry here, Newton's proofs of planetary motion combined geometric constructions, similarity with auxiliary lines, and infinitesimal limiting arguments (see [3] and [2]). Much later, Maxwell's equations were first written in a coordinate-expanded form, which has the flavor of our solution in Part 2, along with differential operators that were by then well understood. Only later did divergence and curl become standard, giving a more compact, coordinate-free expression of Maxwell's equations using $\nabla \cdot$ and $\nabla \times$. Those operators are coordinate-free in meaning, which parallels the shift from the coordinate solution in Part 2 to the coordinate-free constructions in Part 1, but with greater power. Vector analysis keeps the geometric insight

while still enabling the algebraic computations of the coordinate approach. To explain the power of this toolbox, I include a couple of short proofs below.

It is noteworthy that the mathematical techniques developed further into geometric algebra, which combines vectors and bivectors (no longer purely vectors), and later into differential forms that skip vectors altogether. Our proofs below can use the cross product in vector analysis, or use the geometric-algebra bivector concept equivalently (they carry the same information when the geometry is 2D Euclidean, as in our case). We will focus on the vector-analysis form in the following. Here we want to emphasize how these more advanced tools enable coordinate-free proofs while still allowing more advanced analytic methods than the purely ancient Euclidean geometry used in Part 1.

Proof 4.1 (reflection on DE)

Use the reflection property: a ray from B to E reflects across DE and passes through H . Let $p = |\overrightarrow{EB}|$, $q = |\overrightarrow{EH}|$, and define unit vectors $\hat{u} = \overrightarrow{EB}/p$, $\hat{v} = \overrightarrow{EH}/q$. Then $\overrightarrow{EB} = p\hat{u}$ and $\overrightarrow{EH} = q\hat{v}$, with $|\hat{u}| = |\hat{v}| = 1$. The tangent at E is perpendicular to $\hat{u} + \hat{v}$, so its direction is parallel to $\hat{u} - \hat{v}$. Hence

$$\overrightarrow{ED} = \tau(\hat{u} - \hat{v})$$

for some scalar τ . Because $D \in BH$,

$$\overrightarrow{BH} = \overrightarrow{EH} - \overrightarrow{EB} = q\hat{v} - p\hat{u}, \quad \overrightarrow{BD} = \mu\overrightarrow{BH}.$$

Using $\overrightarrow{ED} = \overrightarrow{EB} + \overrightarrow{BD}$ gives

$$\tau(\hat{u} - \hat{v}) = p(1 - \mu)\hat{u} + \mu q\hat{v}.$$

Matching coefficients yields $\tau = p(1 - \mu)$ and $-\tau = \mu q$, so $\mu = \frac{p}{p-q}$. Finally,

$\overrightarrow{HD} = \overrightarrow{BD} - \overrightarrow{BH} = (\mu - 1)\overrightarrow{BH}$, so

$$\frac{BD}{HD} = \frac{\mu}{\mu - 1} = \frac{p}{q} = \frac{BE}{EH},$$

which is equivalent to $\frac{BE}{BD} = \frac{EH}{HD}$.

Proof 4.2 (using the cross product)

Let the tangent at E meet BH at D , and take any nonzero tangent vector \overrightarrow{ED} . The reflection law says the tangent bisects the angle between \overrightarrow{EB} and \overrightarrow{EH} , so the perpendicular components match:

$$\frac{|\overrightarrow{EH} \times \overrightarrow{ED}|}{EH} = \frac{|\overrightarrow{EB} \times \overrightarrow{ED}|}{EB}.$$

Now $\overrightarrow{EH} = \overrightarrow{ED} + \overrightarrow{DH}$ and $\overrightarrow{EB} = \overrightarrow{ED} + \overrightarrow{DB}$. Cross with \overrightarrow{ED} and drop the $\overrightarrow{ED} \times \overrightarrow{ED}$ term:

$$\overrightarrow{EH} \times \overrightarrow{ED} = \overrightarrow{DH} \times \overrightarrow{ED}, \quad \overrightarrow{EB} \times \overrightarrow{ED} = \overrightarrow{DB} \times \overrightarrow{ED}.$$

So

$$\frac{|\overrightarrow{DH} \times \overrightarrow{ED}|}{EH} = \frac{|\overrightarrow{DB} \times \overrightarrow{ED}|}{BE}.$$

Because B, D, H are collinear, $\overrightarrow{DB} \parallel \overrightarrow{DH}$, hence

$$|\overrightarrow{DB} \times \overrightarrow{ED}| : |\overrightarrow{DH} \times \overrightarrow{ED}| = BD : HD.$$

Substitute to get

$$\frac{BD}{HD} = \frac{BE}{EH},$$

which is equivalent to $\frac{BE}{BD} = \frac{EH}{HD}$.

I explored several alternative constructions. For instance, in the original configuration, the condition $EH \perp AB$ immediately yields the dot-product constraint $ecEH \cdot ecAB = 0$. This orthogonality can be used to reduce the degrees of freedom without appealing to any reflection argument. More generally, one can start from essentially any point in the diagram, parameterize the remaining configuration by a small set of length (or scaling) parameters, and then drive the argument to the target length-ratio identity. There are many such routes, and most do not require introducing any auxiliary lines.

I chose the two proofs above because they are the simplest, they naturally extend to the generalized case discussed in Part 3, and they keep the presentation as coordinate-free as possible. I hope they provide a distinct flavor from the proofs in previous parts of the note, yet reveal powerful mathematical properties of the geometric shape.

In the present 2D Euclidean setting, the geometric-algebra approach does not introduce much additional structure. The wedge product \wedge carries essentially the same geometric information as the (embedded) cross product: both encode the oriented (signed) area of the parallelogram spanned by two vectors. Likewise, the differential-forms formulation is not fundamentally different in this context: a 2-form takes two vectors as input and returns a scalar, so the resulting identities reduce to the same area or determinant computations already used in the cross-product argument (e.g., Proof 4.2).

For this reason, the forms and geometric-algebra reconstructions in our problem are largely transferable, and, at least in this 2D case, do not change the essential mechanism of the proof. If a genuinely distinct use of these frameworks becomes useful later (for example, one that clarifies higher-dimensional generalizations or invariant structure), it can be added as an optional extension.

To emphasize the common backbone: Proof 8 is driven by area ratios. The relevant area can be represented equivalently as (i) the determinant of the two side vectors of a triangle, (ii) the magnitude of their (embedded) cross product, (iii) the magnitude of their wedge product, or (iv) the evaluation of the canonical area 2-form on the same pair of vectors. These are simply different notational realizations of the same underlying invariant. In this sense, Proof 4.2 and Proof 8 are exploiting the same area-ratio principle, expressed through slightly different languages.

By now I hope this part of the note have conveyed an appreciation for coordinate-free methods in our mathematical toolbox.

Postscript

It can be interesting to see how the following software would approach the problem:

1. Medal-level AI geometry solvers (closest to "gold/silver" performance in the International Math Olympiad)
 - AlphaGeometry / AlphaGeometry 2 (research systems that can solve many Olympiad problems)
2. Geometry theorem provers (strong automation, not always Olympiad-style proofs)
 - GCLC / WinGCLC (area method, Wu's method, Grobner bases)
 - JGEX (Wu/Grobner/full-angle methods with visualization)
 - OpenGeoProver (Wu/Grobner style)
 - GeoLogic (interactive Euclidean prover)
3. Proof assistants (formal, but manual)
 - Lean 4 + mathlib
 - LeanGeo

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