Part (2) — Analytic Geometry Proof (Short, copy into ipynb)

Goal: Prove

$$BE \cdot DH = BD \cdot EH.$$

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1) Coordinate setup

Let - $C = (0, 0)$ - $B = (-1, 0)$ - $D = (0, 1)$

Then $\angle C = 90^\circ$ and $BC = CD = 1$. Since $AD \parallel BC$, line AD is horizontal, so $A = (a, 1)$.

Let $E \in CD \Rightarrow E = (0, t)$, where $0 < t < 1$. Given $DE = AD$: - $DE = 1 - t$ - $AD = |a|$

So $|a| = 1 - t$. From the diagram A is left of D, hence $a = -(1 - t) = t - 1$. Therefore

$$A = (t - 1, 1), \qquad E = (0, t).$$

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2) Lines BD and EF

Line BD through $(-1, 0)$ and $(0, 1)$:

$$BD: \ y = x + 1.$$

Slope of AB:

$$m_{AB} = \frac{1 - 0}{(t - 1) - (-1)} = \frac{1}{t}.$$

So $EF \perp AB$ has slope $-t$, and since it passes through $E = (0, t)$:

$$EF: \ y = t - tx.$$

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3) Intersection $H = BD \cap EF$

Solve

$$x + 1 = t - tx \Rightarrow x = \frac{t - 1}{t + 1}, \quad y = x + 1 = \frac{2t}{t + 1}.$$

So

$$H=\left(\frac{t-1}{t+1},\frac{2t}{t+1}\right).$$

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4) Distances via a common scale factor

Compute vectors:

$$H-D=\left(\frac{t-1}{t+1},\frac{t-1}{t+1}\right)=\frac{t-1}{t+1}(1,1),$$
$$H-E=\left(\frac{t-1}{t+1},\frac{2t}{t+1}-t\right)=\left(\frac{t-1}{t+1},-\frac{t(t-1)}{t+1}\right)=\frac{t-1}{t+1}(1,-t).$$

Hence

$$DH=\frac{1-t}{t+1}\,\|(1,1)\|=\frac{1-t}{t+1}\sqrt{2},\qquad EH=\frac{1-t}{t+1}\,\|(1,-t)\|=\frac{1-t}{t+1}\sqrt{1+t^2}.$$

Also

$$BD=\|(1,1)\|=\sqrt{2},\qquad BE=\|(1,t)\|=\sqrt{1+t^2}.$$

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5) Finish

$$BE\cdot DH=\sqrt{1+t^2}\cdot\frac{1-t}{t+1}\sqrt{2}=\sqrt{2}\cdot\frac{1-t}{t+1}\sqrt{1+t^2}=BD\cdot EH.$$

✅Proven.