Polygon Ratio Problem: Many Proofs, One Degree of Freedom

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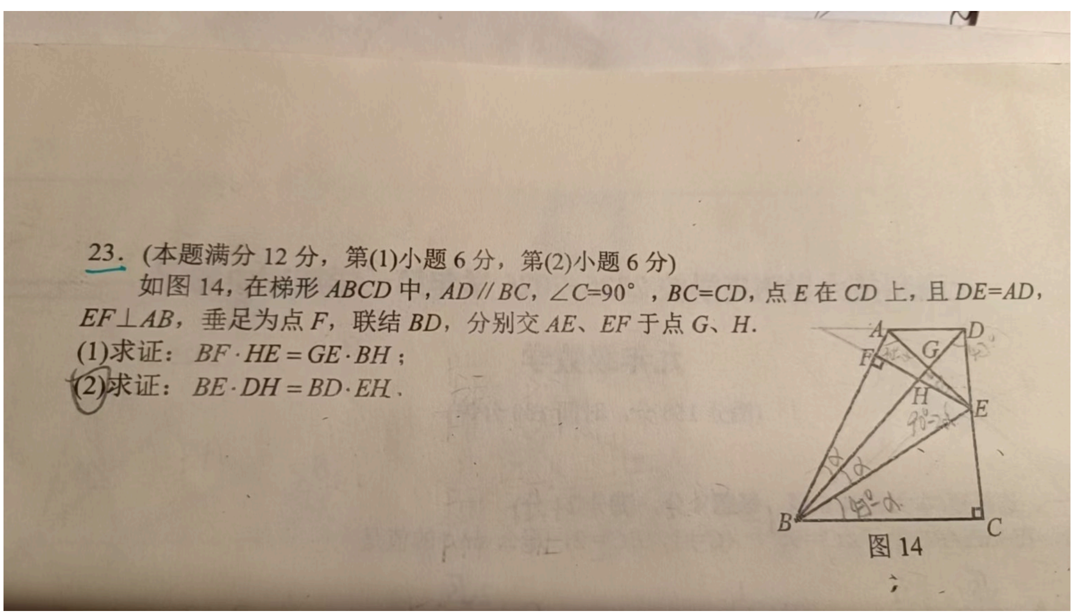
Introduction

During the Christmas holidays, a basic geometry proof problem showed up in my class alumni chat (Figure 1 is the original screenshot from classmate Lin He). He asked about the second question because the first is almost trivial, and that set the discussion in motion. I have always liked Euclidean geometry and excelled in my teenage competitions (I don't recall failing to prove any geometry problems in that period). Problems like this rarely stop at one proof. Draw one line and a ratio drops out; draw another and a different argument appears. What follows is a cleaned collection: one quick sine-law proof, several synthetic proofs, an analytic-geometry proof (with help from ChatGPT 5.2-thinking, edited for clarity), and a one-parameter generalization that reveals deeper structure. I hope it gives younger students a sense of how much structure a single diagram can hide. I also plan to use this pilot problem to introduce state-of-the-art AI approaches for solving such problems in separate notes.



23.（本题满分 12 分，第(1)小题 6 分，第(2)小题 6 分）
如图 14，在梯形 $ABCD$ 中，$AD /\!/ BC$，$\angle C=90°$，$BC=CD$，点 E 在 CD 上，且 $DE=AD$，$EF \perp AB$，垂足为点 F，联结 BD，分别交 AE、EF 于点 G、H.
(1)求证：$BF \cdot HE = GE \cdot BH$；
(2)求证：$BE \cdot DH = BD \cdot EH$.
图 14

Figure 1: The original problem (photo from the chat).

Let me first restate the problem in words and redraw the diagram more cleanly.

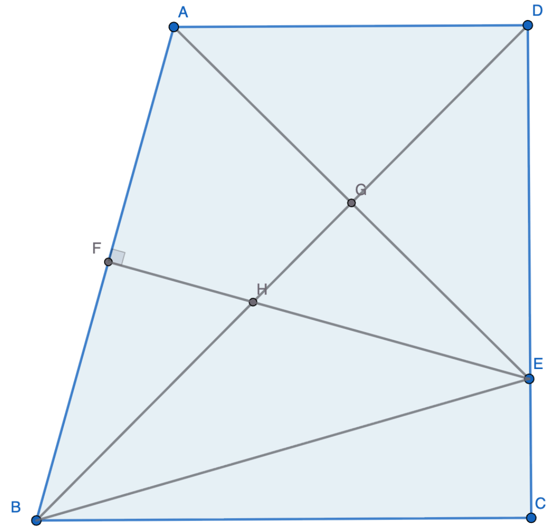


Figure 2: A clean redraw in GeoGebra.

Problem statement (translated). In trapezoid $ABCD$, $AD \parallel BC$, $\angle C = 90^\circ$, and $BC = CD$. Point E lies on CD, and $DE = AD$. Through E, draw $EF \perp AB$ with F as the foot of the perpendicular. Draw BD; it intersects AE and EF at points G and H, respectively. Prove:

1. $BF \cdot HE = GE \cdot BH$.
2. $BE \cdot DH = BD \cdot EH$.

Part 1. Synthetic geometry (auxiliary lines)

The basic equal angles and lengths are shown in Figure 3.

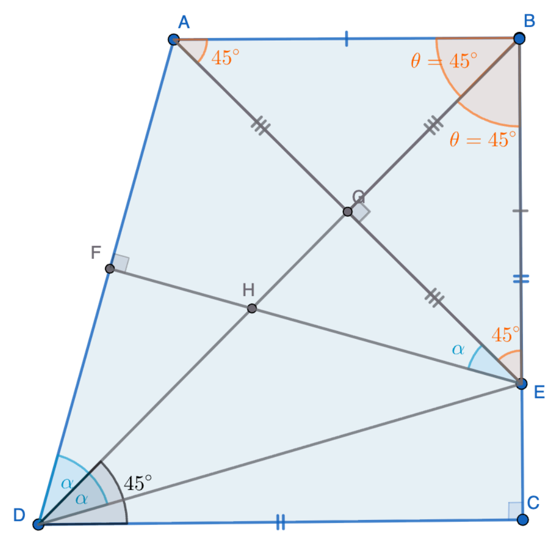


Figure 3: Basic angle and length relations.

Question (1) is a warm-up; it follows from similar triangles visible in Figure 3. For part (2), a quick start is the sine law. For a triangle ABC with side lengths a, b, c opposite vertices A, B, C, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This gives Proof 0 immediately.

Proof 0 (sine law)

> The ratios $DB : BE$ and $DH : HE$ are both $\sin(45^\circ + \alpha) : \sin 45^\circ$, so they are equal. This proof uses no auxiliary lines.

To avoid assuming the sine law, I drew auxiliary lines. Rather than separate diagrams per proof, the next figure collects all extra points. Each added point carries a subscript that matches the proof number (for example, D_1 is used only in Proof 1).

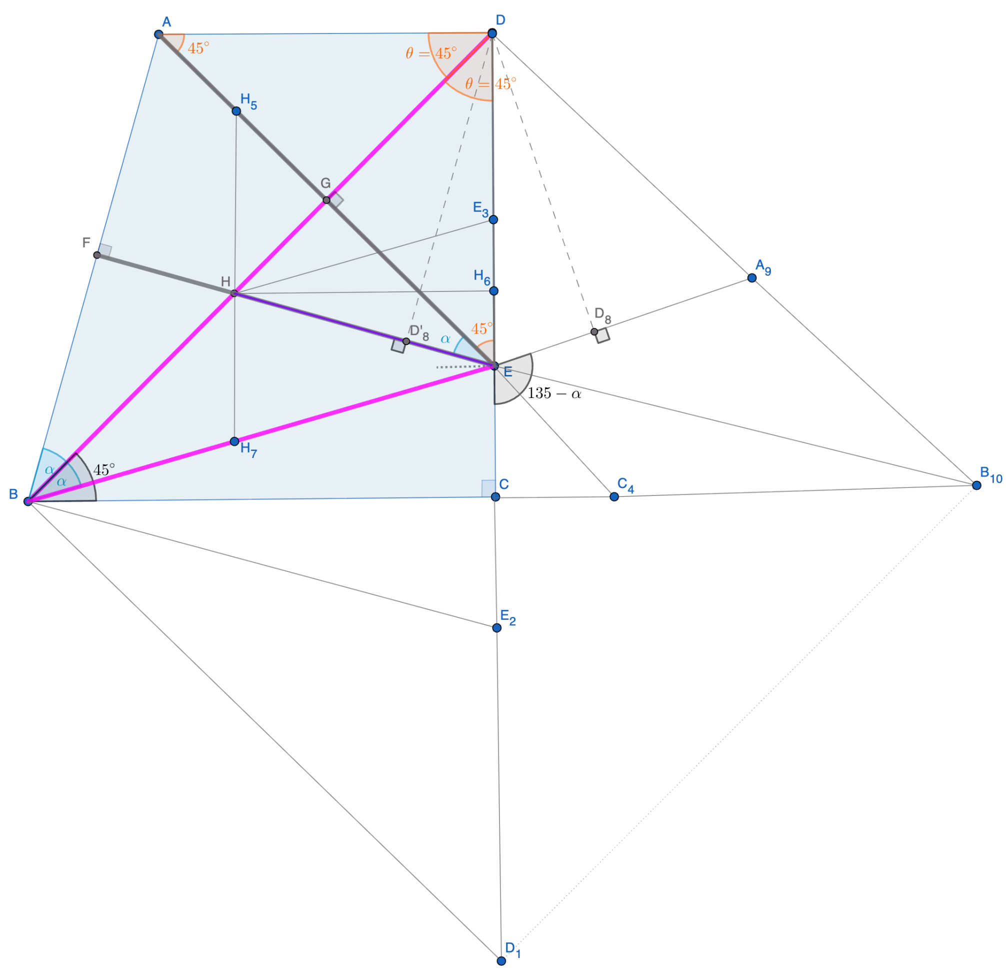


Figure 4: All auxiliary lines used in the synthetic proofs.

Below are brief proof sketches. Each can be completed as a standard middle-school geometry exercise.

Proof 1

> Extend DC to D_1 so that D_1 is symmetric to D with respect to BC. Then $\triangle EBD_1 \sim \triangle EHD$, so the desired ratio follows.

Proof 2

> Extend DC to E_2 so that E_2 is symmetric to E with respect to BC. Then $\triangle E_2BD \sim \triangle EHD$, so the desired ratio follows.

Proof 3

> From H draw a line parallel to BE and intersect DE at E'. Then $HE = HE'$. The rest follows.

Proof 4

> Extend AE to meet BC at C_4. Then $AC_4 = BD$. Also $\triangle HED \sim \triangle ABC_4$, so the ratio follows. (Credit: Prof. Xueheng Lan.)

Proof 4.1

> Use the reflection property: the ray from B to E reflects across DE and passes through H. This gives the same angle relation at DE, so the ratio follows.

Proof 5

> Draw a line from H parallel to DE and intersect EA at H_5. Then $DH = EH_5$, and $\triangle EHH_5 \sim \triangle BED$, which leads to the result.

Proof 6

> Draw a line from H parallel to BC and intersect DE at H_6. Then $DH/DB = HH_6/BD = EH/BD$.

Proof 7

> Draw a line from H parallel to DE and intersect BE at H_7. Then $HE : BE = H_7E : BE = HD : BD$.

Proof 8

> The distances from D to BE and EH are equal (use DD_8 and DD_8'; these are non-essential since $\angle BEC = \angle HED = 45^\circ + \alpha$). The distances from E

> to BD and HD are also equal. So area BDE to area DHE can be expressed as $BE \cdot DD_8/2 : EH \cdot DD'_8/2 = BE : EH$, and also as $BD \cdot GE/2 : DH \cdot GE/2 = BD : DH$. Hence they are equal.

Proof 9

> Extend BE to A_9 where $EA_9 = EH$. Then $DH = DA_9$, $EH = EA_9$, and $\angle EDA_9 = \angle EDH = 45^\circ$. Since DE bisects $\angle BDA_9$, we get $BD : DA_9 = BE : EA_9$. The rest is trivial.

Proof 10

> Extend EH to B_{10} where B_{10} intersects the extension of line BC. Then B_{10} is the reflection of B across BC (since HE is the reflection path of light BE at mirror DC). One can show $\angle CDB_{10} = \angle BDC = 45^\circ$, so $DH : HE = DB_{10} : EB_{10} = DB : BE$.

Proof 11

> This one also needs no auxiliary lines: notice $HD = HG + GE$ and $BD = BG + GE$, and $\triangle HGE \sim \triangle EGB$, so the result follows.

A note on degrees of freedom: the only adjustable angle in the original configuration is $\alpha < 45^\circ$. If $\alpha = 45^\circ$, then H and B overlap and the result is trivial. If $\alpha > 45^\circ$, then H lies on the extension of DB, and E lies on the extension of DC. The roles of B and H swap, and the same style of arguments still works. Similarly, one can show that a ray from B to E reflects across DE and reaches H.

Other proofs

By now we should be done. However, if you followed the solutions so far, you likely noticed two patterns: (1) we have essentially drawn a square $BDB_{10}D_1$ in Figure 4, and many proofs are based on various points on this square; (2) polygon $ABED$ is symmetric about BD, which is why we could leverage $AB = BE$ and $AH = HE$ in several steps. It follows that any proof obtained by extending lines in the original diagram has a mirror proof across BD. These are indicated again by subscripts, and they give Proofs 12, 13, 14, 15, and 16. They form the symmetric rectangle $BDB_{15}D_{12}$ and are shown in Figure 5.

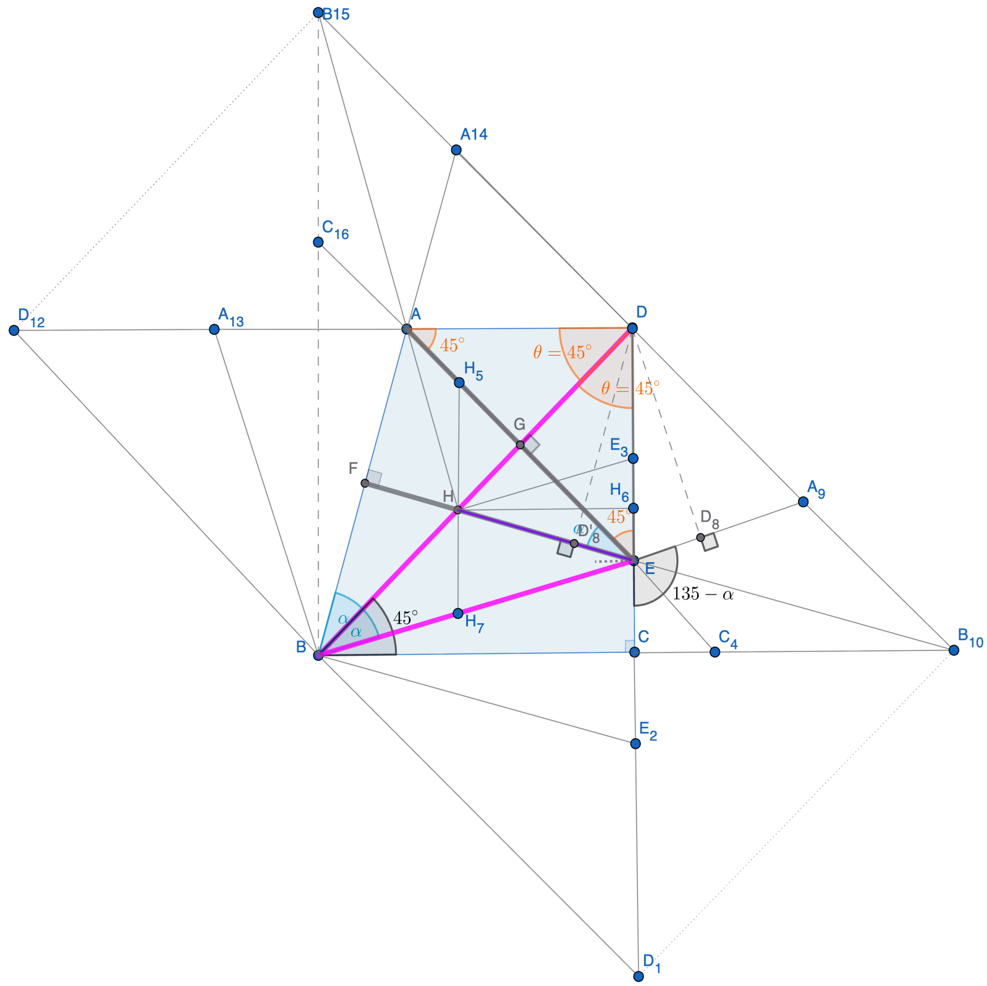


Figure 5: Five additional symmetric proofs.

Finally, several proofs such as Proofs 3, 5, 6, and 7 focus on what happens inside polygon $ABED$. Each has a symmetric counterpart (Proofs 17, 18, 19, 20) reflected across BD. For clarity, these are not displayed.

Part 2. Analytic geometry proof

This is a short coordinate proof.

Goal

$$BE \cdot DH = BD \cdot EH.$$

Proof

> Setup. Let $C = (0,0)$, $B = (-1,0)$, $D = (0,1)$. Then $AD \parallel BC$ implies $A = (t-1, 1)$ when $E = (0,t)$ and $DE = AD$ with $0 < t < 1$.
>
> Lines. $BD : y = x + 1$. The slope of AB is $1/t$, so $EF \perp AB$ has slope $-t$ and passes through E: $EF : y = t - tx$.

> Intersection. Solve $x + 1 = t - tx$ to get
>
> $$H = (\frac{t-1}{t+1},\ \frac{2t}{t+1}).$$
>
> Common factor.
>
> $$H - D = \frac{t-1}{t+1}(1,1), \qquad H - E = \frac{t-1}{t+1}(1,-t).$$
>
> Hence
>
> $$DH = \frac{1-t}{t+1}\sqrt{2}, \qquad EH = \frac{1-t}{t+1}\sqrt{1+t^2}.$$
>
> Also $BD = \sqrt{2}$ and $BE = \sqrt{1+t^2}$.
>
> Finish.
>
> $$BE \cdot DH = \sqrt{1+t^2} \cdot \frac{1-t}{t+1}\sqrt{2} = BD \cdot EH.$$

Conclusion: $BE \cdot DH = BD \cdot EH$.

Part 3. One more degree of freedom (ellipse view)

The 45° angle is special. If we relax that angle, the clean right-angle structure disappears, but the problem still has a simple geometric core: reflection on an ellipse. The key insight is to generalize the reflection: the ray from B to E reflects across DE and heads to H. This reveals an ellipse that reflects light from one focus to the other at any point E on it. That reflection property is equivalent to the constant-sum-of-distances definition of an ellipse.

Construction

Let O be the ellipse with foci B and H. Take a point A on O, with A closer to H. Draw the tangent line AD at A, and let it intersect BH at D. Take a point E on O, the reflection of A with respect to the major axis. Through E draw the line $D - E - C$, and choose C so that $BC \parallel AD$. Draw EH, and let it intersect AB at F. In general, $\angle BFE$ is not 90°, but when $\theta = 45^\circ$, $\angle BFE = 90^\circ$. The resulting shape is shown in Figure 6.

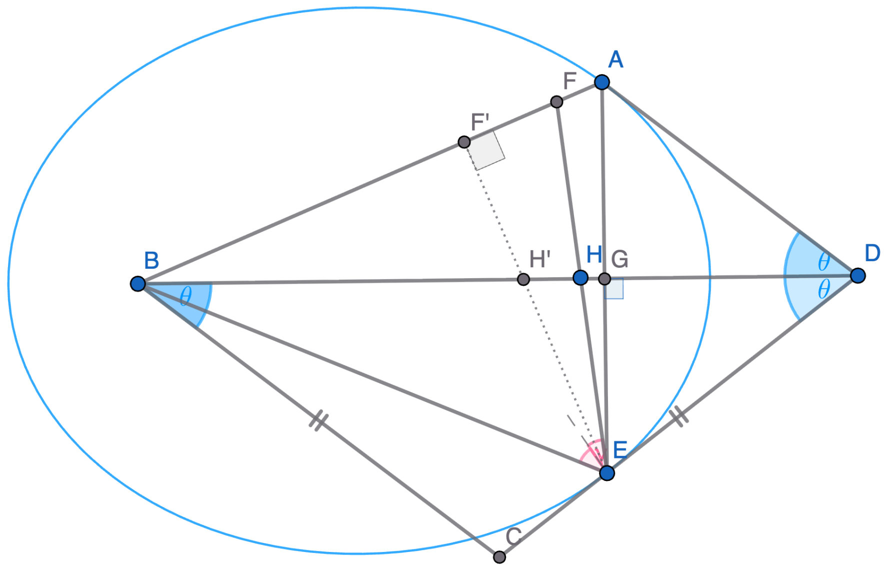


Figure 6: Ellipse construction with foci B and H.

Proof 3.1 (reflection property)

By the reflection property of the ellipse (a light ray from B to E reflects and then passes through H), we obtain

$$\frac{EH}{HD} = \frac{BE}{BD}.$$

Reflect H across line DE to H_m. Then

$$\frac{DB}{BE} = \frac{DH_m}{EH_m} = \frac{DH}{EH},$$

which is the same proportion.

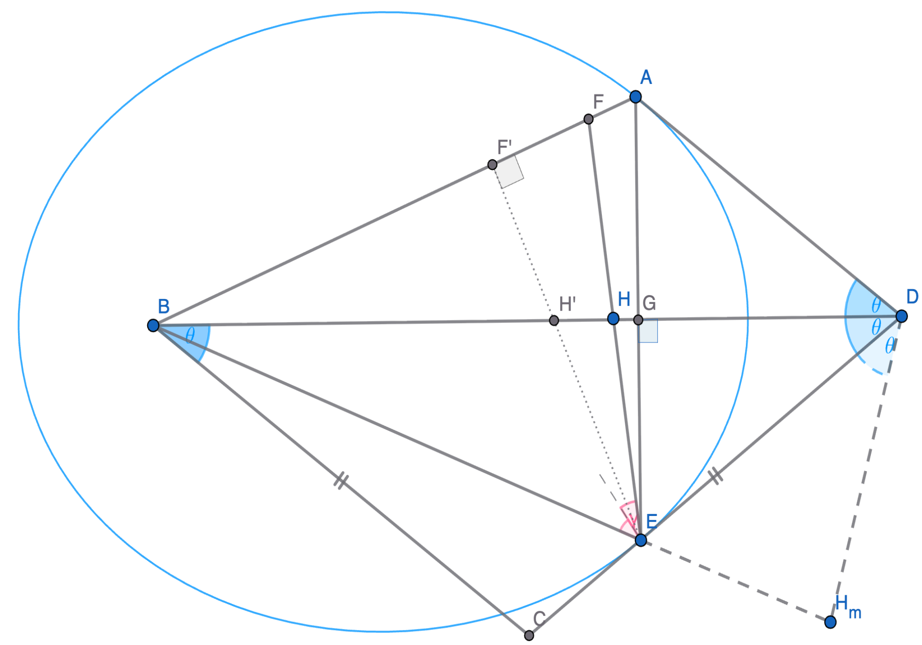


Figure 7: Reflection proof for the ellipse construction.

In this generalized setup, EF is no longer perpendicular to AB. In fact, the perpendicular line through E intersects AB and DB at F' and H', as shown in Figure 7. Here H' differs from H and does not satisfy the ratio property. The original problem is special because the 45° angles create a surplus of right angles. The single degree of freedom here is essentially the ellipse eccentricity, with A and C determined by the right angle $\angle ADC$. When the ellipse degenerates to a circle (foci coincide), the identity becomes obvious.

In this setup, points A, F, C, and G, and the lines through them, are not even essential; only the foci B and H, and the tangent line DE, are necessary, with D on the major axis. This shows the true simplicity of the original problem; Figure 8 shows the pure geometric approach.

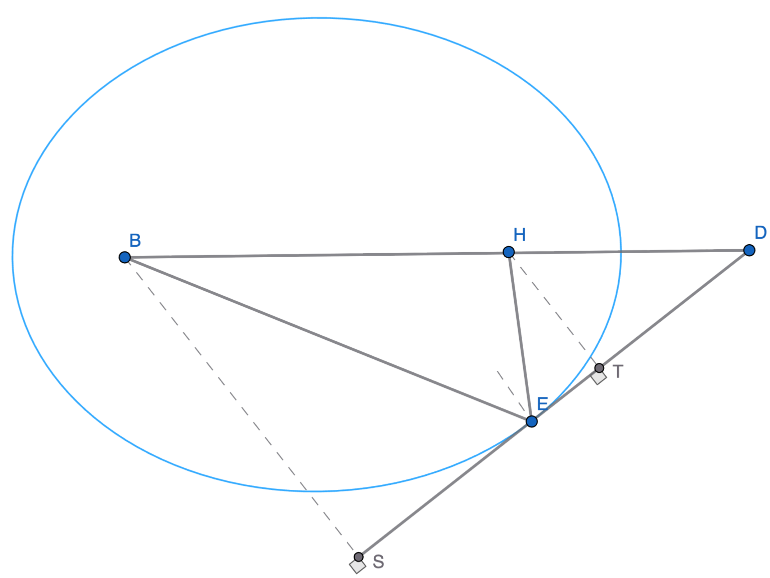


Figure 8: The simplified ellipse setup, which reveals the essence of the original problem.
The diagram also yields a second, similarity-based proof.

Proof 3.2, similarity

> Let T and S be the feet of the perpendiculars from H and B to the tangent line DE. By the reflection property, $\triangle HET \sim \triangle BES$, so $HE : BE = HT : BS$. Likewise, $\triangle HDT \sim \triangle BDS$, so $HD : DB = HT : BS$. Combining gives $HD/DB = HE/BE$.

Interestingly, we can also compute the ratio analytically in the ellipse model. It turns out to be simplest using the de La Hire (eccentric-angle) parameter, revealing the geometry transparently.

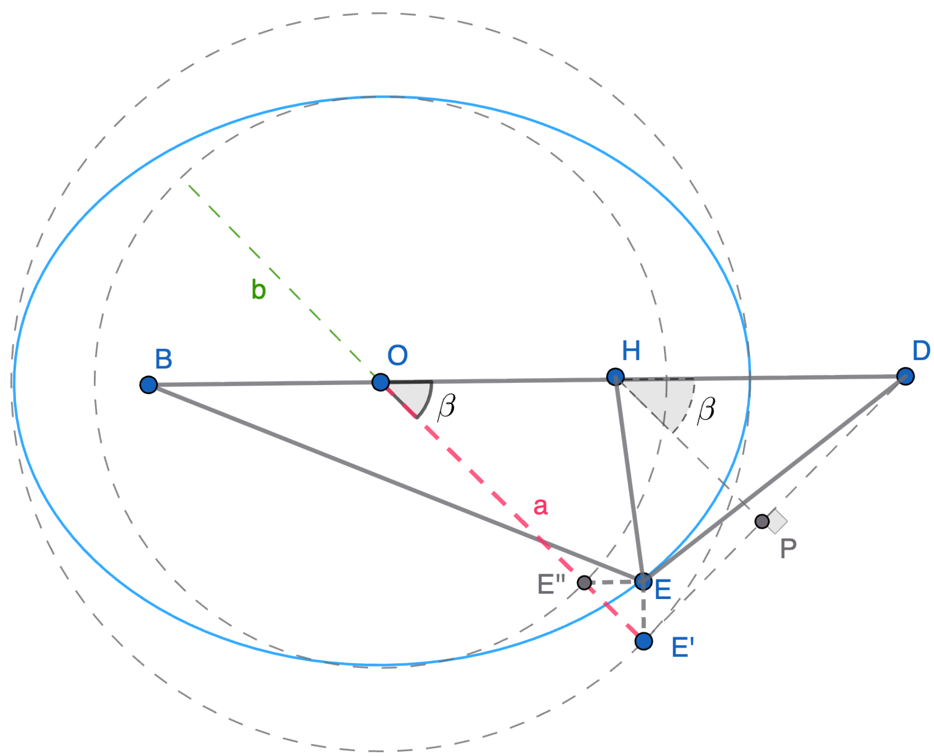


Figure 9: de La Hire parameters and outer-circle view with E, E', and E''.

Proof 3.3 (de La Hire + outer circle)

> Let the ellipse be $x^2/a^2 + y^2/b^2 = 1$ with center O, foci $B = (-c, 0)$ and $H = (c, 0)$, and $c^2 = a^2 - b^2$. Use the de La Hire (eccentric-angle) parameter β:
>
> $$E = (a\cos\beta,\, b\sin\beta), \qquad EB = a + c\cos\beta, \qquad EH = a - c\cos\beta$$
>
> (up to swapping B, H). Let ℓ be the tangent at E and $D = \ell \cap BH$.
>
> From the tangent formula

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1 \quad \Rightarrow \quad \frac{x\cos\beta}{a} + \frac{y\sin\beta}{b} = 1,$$

setting $y = 0$ gives $x_D = a/\cos\beta$.

Geometrically, the affine map $(x, y) \mapsto (x, \frac{a}{b}y)$ sends the ellipse to the outer circle $x^2 + y^2 = a^2$ and sends E to $E' = (a\cos\beta, a\sin\beta)$. Since only y is scaled, E and E' lie on a line perpendicular to BH and share the same x-coordinate. Let E'' be the intersection of the line through E parallel to BH with the outer circle, so E and E'' share the same y-coordinate (Figure 9). The map preserves tangency, so $\angle OE'D$ is right and $OD = a/\cos\beta$.

Therefore

$$DB = \frac{a}{\cos\beta} + c, \qquad DH = \frac{a}{\cos\beta} - c,$$

and

$$\frac{DH}{DB} = \frac{a - c\cos\beta}{a + c\cos\beta} = \frac{EH}{EB}.$$

Corollary 1 (axis-focus scaling; inner structure of the ratio)

From $DH = \frac{a}{\cos\beta} - c = \frac{a - c\cos\beta}{\cos\beta}$ and $EH = a - c\cos\beta$, we get

$$\boxed{EH = DH\cos\beta}, \qquad \boxed{EB = DB\cos\beta}.$$

This shows that the tangent-axis lengths DB, DH are just the focal lengths EB, EH scaled by the same factor $1/\cos\beta$ (see Figure 9). The main identity $DH/DB = EH/EB$ is exactly the cancellation of that common scale.

Corollary 2 (focus-to-outer-tangent distances)

Corollary 1 gives $EH = DH\cos\beta$, and geometrically $DH\cos\beta$ is exactly the perpendicular component of DH to ℓ', i.e., $\mathrm{dist}(H, \ell')$ (see Figure 9). Hence

$$\boxed{\mathrm{dist}(H, \ell') = EH}, \qquad \boxed{\mathrm{dist}(B, \ell') = EB}.$$

The simple consistency check

$$\mathrm{dist}(B, \ell') + \mathrm{dist}(H, \ell') = 2a = 2\ \mathrm{dist}(O, \ell')$$

accounts only for the sum. The real content is that each focus-to-tangent distance equals its matching focus-to-point distance; this rigidity is encoded by the de La Hire parameter and the ellipse's affine relation to the circle.

Proof 3.4 (outer circle proof)

> Once Corollary 2 is in hand, let P and P' be the feet of the perpendiculars from H and B to ℓ' (for simplicity Figure 9 only shows P). Then $\triangle DHP \sim \triangle DBP'$ (right angles at P, P' and a common angle at D), so $DH/DB = HP/BP' = \operatorname{dist}(H, \ell')/\operatorname{dist}(B, \ell')$, recovering the same ratio.

Finally, there is a fun fact. Note that the feet of the perpendiculars drawn in Proof 3.2, S and T, are exactly the intersection points of the tangent line DE with the ellipse's outer circle. A quick proof is to show that $OT = a$ (Figure 10). Let H' be the reflection of H across DE; then T is the midpoint of HH'. Since O is the midpoint of BH, we have $OT = \frac{1}{2}BH'$. But $BH' = BE + EH = 2a$, hence $OT = a$. The same reflection with B gives $OS = a$, so the fun fact is that S and T are precisely the two intersections of DE with the outer circle.

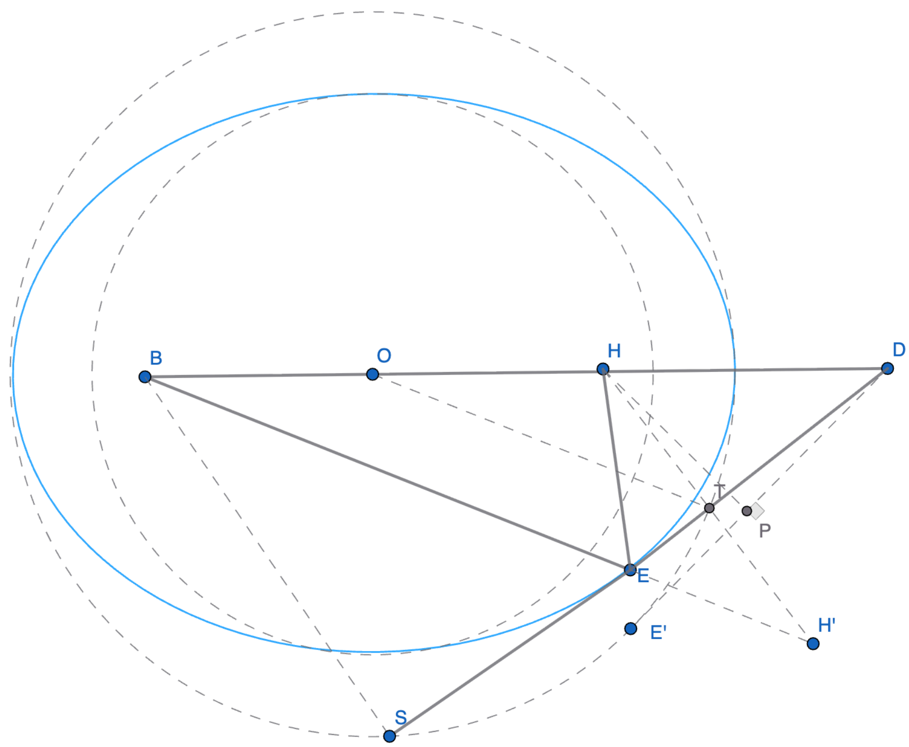


Figure 10: additional properties of T and S

Part 4. Vector analysis

Once one length is fixed, the trapezoid configuration has a single degree of freedom: the angle α. In the broader ellipse construction, a second parameter θ (the tangent point/slope) appears, and the same ratio follows from the reflection property. The special case $\theta = 45^\circ$ is exactly the original problem; that is why so many proofs exist, and why each auxiliary line exposes the same invariant ratio in a different way. Part 3 frees the θ dimension while keeping the reflection property, and that is what ultimately shows the simplicity behind the original picture.

In this part of the note, we focus on other mathematical tools that are useful in geometry. The target ratio is almost trivial here: $DH : DB = EH : EB$, given that the ray BE reflects across the mirror DE and reaches H. Part 3 already explained the reflection viewpoint, but it is still instructive to see how these tools express the same invariant.

Since Euclid (~300 BCE), geometry was developed in a synthetic style: one reasons directly with the figure using congruence, similarity, tangency, and auxiliary lines. That is the mindset behind Part 1. A major second language arrived with Descartes (1637): analytic geometry turns geometric constraints into algebra once you choose coordinates, matching the style of Part 2.

This arc also mirrors the historical split between geometry and algebra in physics and mechanics. Newton, in the *Principia* [NEWTON-PRINCIPIA], favored coordinate-free arguments because they were the clearest language of rigor in 1687, while modern physics often relies on coordinates for universal computation. Analytic geometry existed by Newton's time, but the algebraic language for differential equations matured later; Newton's methods therefore look Euclidean in flavor, mixing auxiliary lines with limiting arguments. Several later authors also gave elementary, coordinate-free proofs (see [KEPLER-TEACH], including Feynman's famous "lost lecture"). It is good for students to appreciate both techniques.

Physics naturally blends geometry with differential ideas. Here we keep that discussion light and focus on the geometric tools that sit just below calculus.

In the 1800s, geometry gained an especially efficient algebraic interface: vector analysis, where the dot and cross products encode the two most basic geometric primitives--projection/angle and oriented area. The modern $\cdot$ and $\times$ notation was introduced by Gibbs (and independently Heaviside) around 1881 and then standardized in physics usage. Later, divergence and curl became standard because they compactly describe fields and their integral laws--but that differential layer is not our focus here.

That same "area backbone" explains the next conceptual step. Grassmann (1844) introduced the wedge product and exterior algebra, where the fundamental geometric object is no longer just a vector but also a bivector (directed area), trivector (directed volume), and so on. Clifford (1878) then built geometric algebra by unifying these objects into one coherent multiplication system. In our 2D Euclidean setting,

$$|\mathbf{u} \times \mathbf{t}|, \qquad \|\mathbf{u} \wedge \mathbf{t}\|, \qquad |\omega(\mathbf{u}, \mathbf{t})|$$

are simply three notations for the same invariant: the area spanned by $\mathbf{u}$ and the tangent direction $\mathbf{t}$. So the proofs are interchangeable across these languages with no change in geometric content.

Finally, modern physics is "geometric" in a deeper sense: it naturally lives on spaces that are not just 2D pictures--3D space, 4D spacetime, and even higher-dimensional configuration/phase spaces. We don't always call those "geometry" in everyday language because they're hard to visualize, but mathematically they are geometry: they carry intrinsic notions of angle, area/flux, invariance, and coordinate-free meaning. In our ellipse problem, we see a toy version of the same idea: at each point E on the ellipse, there is a tangent line DE. Thinking of the tangent line as the attached object that moves with E is the simplest concrete example of the bundle/fiber mindset.

A brief timeline of tooling relevant to this note:

- ~300 BCE: Euclid -- synthetic geometry
- 1637: Descartes -- analytic geometry
- 1687: Newton -- Euclidean geometry + limiting arguments
- 1827: Gauss -- intrinsic geometric invariants (surfaces)
- 1844: Grassmann -- wedge / bivectors (area as an object)
- 1878: Clifford -- geometric algebra (bivectors/trivectors as first-class)
- 1881: Gibbs/Heaviside -- standard dot/cross notation
- 1899: Cartan -- exterior derivative/forms (beyond our needs here)

For readers familiar with Maxwell's equations, the table below shows how mathematical language for geometry evolved over time: from coordinate-expanded component equations, to vector analysis, to geometric algebra, and then to differential forms. The tools change, but the physics does not, even as the expressions become more compact.

Formulation	Coordinate status (what's really true)	Maxwell equations written in that language	What you gain / lose	Good reference
Maxwell (1865, component-heavy)	Coordinate-invariant laws, but written in components (explicit x, y, z bookkeeping) and bundled with extra relations (constitutive laws, Ohm, forces, potentials).	Not packaged as the modern four; a larger coupled component system (often summarized as ~20 equations/unknowns, plus auxiliaries).	Gain: historically close to derivations. Lose: compactness; structure is obscured by bookkeeping.	Maxwell's 1865 paper [MAXWELL-1865]; Deschamps [FORMS-EM].

Formulation	Coordinate status (what's really true)	Maxwell equations written in that language	What you gain / lose	Good reference
Vector calculus (Heaviside/Gibbs)	Coordinate-free in content; coordinates enter only when writing component formulas.	(SI, vacuum) $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \, \partial_t \mathbf{E}$.	Gain: four clean equations; practical in 3D Euclidean space. Caveat: $\nabla\cdot, \nabla\times$ depend on the chosen metric + orientation, though invariantly defined once fixed.	Wilson [VECTOR-ANALYSIS]; cross product [CROSS-PRODUCT]; vector analysis [VECTOR-ANALYSIS-WIKI].
Geometric algebra (Clifford; dot + wedge unified)	Coordinate-free once metric + orientation are fixed. Vectors are bold (e.g., $\mathbf{a}, \mathbf{b}$); higher-grade objects (bivectors, trivectors) are not vectors in disguise.	Core algebra: $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$. Maxwell can be written compactly (common spacetime GA convention): $\boxed{\nabla F = \mu_0 J}$, with the grade split $\nabla F = \nabla \cdot F + \nabla \wedge F$ giving $\nabla \wedge F = 0$ (homogeneous) and $\nabla \cdot F = \mu_0 J$ (sources).	Gain: cross product becomes derived (dual of $\wedge$); div/curl unify as one operator split by grade; rotations often cleaner. Lose: conventions vary (3D GA vs spacetime GA; units/signatures); less standard in mainstream EM courses.	Peeter Joot [GA-EM].
Differential forms (exterior calculus on spacetime)	Manifestly coordinate-invariant; works naturally on curved manifolds (GR). Metric enters via the Hodge star.	$\boxed{dF = 0}$, $\boxed{d{*}F = J}$. Here $*$ maps k-forms to $(n-k)$ -forms using the metric + orientation (in 4D, $2 \leftrightarrow 2$).	Gain: topology/relativity are transparent; $d^2 = 0$ exposes structure. Lose: abstraction up front; computations often unpack to components.	Deschamps [FORMS-EM]; Lindell [LINDELL-DF]; Epstein [EPSTEIN-DG].

Table 1 summarizes the evolution of Maxwell's equations across these languages.

Notes:

- Geometric algebra (Clifford; dot + wedge unified): metric + wedge fused immediately via $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$.
- Differential forms (exterior calculus on spacetime): forms are wedge-first, metric-later via $*$.

So the concept of vector analysis has evolved significantly since Newton's time. Maxwell's equations were first written in a coordinate-expanded form, which has the flavor of our solution in Part 2, along with differential operators that were by then well understood. Only later did divergence and curl become standard, giving a more compact, coordinate-free expression using $\nabla\cdot$ and $\nabla\times$. Those operators are coordinate-free in meaning, which parallels the shift from the coordinate solution in Part 2 back to the coordinate-free constructions in Part 1. Vector analysis keeps the geometric insight while still enabling the algebraic computations of the coordinate approach. To explain the power of this toolbox, I include a couple of short proofs below.

Since our problem is 2D, both dot and cross products can be used to solve it naturally. We emphasize how these tools enable coordinate-free proofs while still allowing analytic methods beyond the purely Euclidean geometry used in Part 1, yet more direct than the coordinate approach in Part 2.

Proof 4.1 (reflection on DE)

> Use the reflection property: a ray from B to E reflects across DE and passes through H. Let $p = |\overrightarrow{EB}|$, $q = |\overrightarrow{EH}|$, and define unit vectors $\hat{u} = \overrightarrow{EB}/p$, $\hat{v} = \overrightarrow{EH}/q$. Then $\overrightarrow{EB} = p\hat{u}$ and $\overrightarrow{EH} = q\hat{v}$, with $|\hat{u}| = |\hat{v}| = 1$. The tangent at E is perpendicular to $\hat{u} + \hat{v}$, so its direction is parallel to $\hat{u} - \hat{v}$. Hence
>
> $$\overrightarrow{ED} = \tau(\hat{u} - \hat{v})$$
>
> for some scalar τ. Because $D \in BH$,
>
> $$\overrightarrow{BH} = \overrightarrow{EH} - \overrightarrow{EB} = q\hat{v} - p\hat{u}, \quad \overrightarrow{BD} = \mu\overrightarrow{BH}.$$
>
> Using $\overrightarrow{ED} = \overrightarrow{EB} + \overrightarrow{BD}$ gives
>
> $$\tau(\hat{u} - \hat{v}) = p(1-\mu)\hat{u} + \mu q\hat{v}.$$
>
> Matching coefficients yields $\tau = p(1-\mu)$ and $-\tau = \mu q$, so $\mu = \frac{p}{p-q}$. Finally, $\overrightarrow{HD} = \overrightarrow{BD} - \overrightarrow{BH} = (\mu - 1)\overrightarrow{BH}$, so
>
> $$\frac{BD}{HD} = \frac{\mu}{\mu - 1} = \frac{p}{q} = \frac{BE}{EH},$$
>
> which is equivalent to $\frac{BE}{BD} = \frac{EH}{HD}$.

Proof 4.2 (using the cross product)

> Let the tangent at E meet BH at D, and take any nonzero tangent vector $\overrightarrow{ED}$. The reflection law says the tangent bisects the angle between $\overrightarrow{EB}$ and $\overrightarrow{EH}$, so the perpendicular components match:

$$\frac{|\overrightarrow{EH} \times \overrightarrow{ED}|}{EH} = \frac{|\overrightarrow{EB} \times \overrightarrow{ED}|}{EB}.$$

Now $\overrightarrow{EH} = \overrightarrow{ED} + \overrightarrow{DH}$ and $\overrightarrow{EB} = \overrightarrow{ED} + \overrightarrow{DB}$. Cross with $\overrightarrow{ED}$ and drop the $\overrightarrow{ED} \times \overrightarrow{ED}$ term:

$$\overrightarrow{EH} \times \overrightarrow{ED} = \overrightarrow{DH} \times \overrightarrow{ED}, \qquad \overrightarrow{EB} \times \overrightarrow{ED} = \overrightarrow{DB} \times \overrightarrow{ED}.$$

So

$$\frac{|\overrightarrow{DH} \times \overrightarrow{ED}|}{EH} = \frac{|\overrightarrow{DB} \times \overrightarrow{ED}|}{BE}.$$

Because B, D, H are collinear, $\overrightarrow{DB} \parallel \overrightarrow{DH}$, hence

$$|\overrightarrow{DB} \times \overrightarrow{ED}| : |\overrightarrow{DH} \times \overrightarrow{ED}| = BD : HD.$$

Substitute to get

$$\boxed{\frac{BD}{HD} = \frac{BE}{EH}},$$

which is equivalent to $\frac{BE}{BD} = \frac{EH}{HD}$.

I explored several alternative constructions which give various other proofs that I won't detail here since these two are the simplest and representative. For instance, in the original configuration, the condition $EH \perp AB$ immediately yields the dot-product constraint $\overrightarrow{EH} \cdot \overrightarrow{AB} = 0$. This orthogonality can be used to reduce the degrees of freedom without appealing to any reflection argument. More generally, one can start from essentially any point in the diagram, parameterize the remaining configuration by a small set of length (or scaling) parameters, and then drive the argument to the target length-ratio identity. There are many such routes, and they do not require introducing any auxiliary lines.

In this 2D Euclidean setting, the geometric-algebra approach does not introduce additional structure compared with the cross product. Both encode the oriented (signed) area of the parallelogram spanned by two vectors. The $\times$ operator only makes traditional sense in 3D space (it produces a vector perpendicular to the plane), whereas the $\wedge$ operator gives a bivector that is defined in any dimension. Likewise, the differential-forms formulation is not fundamentally different here: a 2-form takes two vectors as input and returns a scalar, and a 1-form is essentially a linear functional (often represented by a dot product with a fixed vector). The identities therefore reduce to the same area or determinant computations already used in the cross-product argument. For completeness, Proof 4.3 shows how they are unified in one proof. In modern math/physics, where the dimensionality is often beyond 3, the dot/cross picture becomes awkward, which is why spacetime formulations separate time from space and why geometric algebra and differential forms are preferred. That is also why the unified Maxwell equations are usually written without an explicit time coordinate.

Proof 4.3: Unified simple solution (area language: cross = wedge = 2-form = 1-form)

> Let DE be the tangent at E and let $\mathbf{t} = \overrightarrow{ED} \neq 0$. For any vector $\mathbf{u}$ based at E (here $\overrightarrow{EB}, \overrightarrow{EH}, \overrightarrow{DB}, \overrightarrow{DH}$), the area with the tangent is the same in all languages:
>
> $$A(\mathbf{u}) = |\mathbf{u} \times \mathbf{t}| = \|\mathbf{u} \wedge \mathbf{t}\| = |\omega(\mathbf{u}, \mathbf{t})|,$$
>
> where ω is an oriented area 2-form.
>
> Fix ω and define the 1-form $\alpha = \iota_{\mathbf{t}}\omega$, so $\alpha(\mathbf{u}) = \omega(\mathbf{u}, \mathbf{t})$ and $|\alpha(\mathbf{u})| = A(\mathbf{u})$. This is the perpendicular component of $\mathbf{u}$ to the tangent.
>
> Reflection at E gives equality of perpendicular components:
>
> $$\frac{A(\overrightarrow{EH})}{EH} = \frac{A(\overrightarrow{EB})}{EB}.$$
>
> Since $D \in BH$,
>
> $$\overrightarrow{EH} = \overrightarrow{ED} + \overrightarrow{DH}, \qquad \overrightarrow{EB} = \overrightarrow{ED} + \overrightarrow{DB}.$$
>
> Using bilinearity and $\omega(\mathbf{t}, \mathbf{t}) = 0$ gives
>
> $$A(\overrightarrow{EH}) = A(\overrightarrow{DH}), \qquad A(\overrightarrow{EB}) = A(\overrightarrow{DB}).$$
>
> Hence
>
> $$\frac{A(\overrightarrow{DH})}{EH} = \frac{A(\overrightarrow{DB})}{EB}.$$
>
> Because B, D, H are collinear, $A(\overrightarrow{DB})/A(\overrightarrow{DH}) = BD/HD$, so
>
> $$\frac{BD}{HD} = \frac{BE}{EH}, \quad \text{i.e.} \quad \frac{BE}{BD} = \frac{EH}{HD}.$$

By now I hope this part of the note has conveyed an appreciation for coordinate-free methods in our mathematical toolbox. I also hope it gives a preview of what modern physics use to solve geometry problems that are more than 3-dimensional, admittedly in a weak sense.

So far in this part we have discussed the simple case of how the reflection property implies the ratio equality. Below we show why the original problem in Part 1 gives the reflection property of EF and BE, using a vector argument.

Proof 4.4 Vector analysis: EF is the reflection of BE across CD in Figure 3

> Let $\mathbf{u} = \overrightarrow{CB}$ and $\mathbf{d} = \overrightarrow{CD}$. Then $\mathbf{u} \perp \mathbf{d}$ and $\|\mathbf{u}\| = \|\mathbf{d}\|$. Fix the unit normal $\mathbf{w} = \dfrac{\mathbf{u} \times \mathbf{d}}{\|\mathbf{u} \times \mathbf{d}\|}$. Since $AD \parallel CB$, $E \in CD$, and $DE = AD$, there exists a scalar k such that
>
> $$\overrightarrow{AB} = -\mathbf{d} + (1-k)\mathbf{u}, \qquad \overrightarrow{BE} = -(k-1)\mathbf{d} - \mathbf{u}.$$
>
> Reflection across CD is equivalent to
>
> $$\frac{\|\overrightarrow{BE} \times \mathbf{d}\|}{\|\overrightarrow{BE}\|} = \frac{\|\overrightarrow{EF} \times \mathbf{d}\|}{\|\overrightarrow{EF}\|}.$$
>
> Because $EF \perp AB$, the direction of EF is parallel to $\overrightarrow{AB} \times \mathbf{w}$, so the right-hand side equals
>
> $$\frac{\|(\overrightarrow{AB} \times \mathbf{w}) \times \mathbf{d}\|}{\|\overrightarrow{AB} \times \mathbf{w}\|}.$$
>
> Now compute the two normalized quantities. First,
>
> $$\overrightarrow{BE} \times \mathbf{d} = (-(k-1)\mathbf{d} - \mathbf{u}) \times \mathbf{d} = -\mathbf{u} \times \mathbf{d},$$
>
> so $\|\overrightarrow{BE} \times \mathbf{d}\| = \|\mathbf{u} \times \mathbf{d}\| = \|\mathbf{u}\|\,\|\mathbf{d}\| = \|\mathbf{d}\|^2$. Next, by Lagrange's formula $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ and $\mathbf{w} \cdot \mathbf{d} = 0$,
>
> $$(\overrightarrow{AB} \times \mathbf{w}) \times \mathbf{d} = \mathbf{w}(\overrightarrow{AB} \cdot \mathbf{d}).$$
>
> But
>
> $$\overrightarrow{AB} \cdot \mathbf{d} = (-\mathbf{d} + (1-k)\mathbf{u}) \cdot \mathbf{d} = -\|\mathbf{d}\|^2,$$
>
> hence $\|(\overrightarrow{AB} \times \mathbf{w}) \times \mathbf{d}\| = \|\mathbf{d}\|^2$. Finally, $\|\overrightarrow{AB} \times \mathbf{w}\| = \|\overrightarrow{AB}\|$, and
>
> $$\|\overrightarrow{AB}\|^2 = \|\mathbf{d}\|^2 + (k-1)^2\|\mathbf{u}\|^2 = (k-1)^2\|\mathbf{d}\|^2 + \|\mathbf{u}\|^2 = \|\overrightarrow{BE}\|^2,$$
>
> so $\|\overrightarrow{AB}\| = \|\overrightarrow{BE}\|$. Therefore the normalized cross magnitudes match, and the reflection condition holds:

$$\frac{\|\overrightarrow{BE} \times \overrightarrow{CD}\|}{\|\overrightarrow{BE}\|} = \frac{\|\overrightarrow{EF} \times \overrightarrow{CD}\|}{\|\overrightarrow{EF}\|}.$$

So BE reflects across CD into EF.

We used Lagrange's formula (the vector triple product identity) in the proof above. There are many known proofs, some classic and purely algebraic, but this one that I came up with is purely geometric and makes the identity feel almost unavoidable. I haven't seen this particular "Thales-circle + chord" intuition presented in such a compact form elsewhere, so it seems worth sharing though it can hardly be original, given that this is elementary. In a sense, it brings the argument back to a Euclid-style picture: analysis and geometry are not rivals, but two languages for the same structure, constantly translating into each other.

Proof of Lagrange's formula based on only geometric properties

Let O be the origin and $\overrightarrow{OU} = \mathbf{u}$, $\overrightarrow{OV} = \mathbf{v}$, $\overrightarrow{OW} = \mathbf{w}$. Write $\mathbf{w} = \mathbf{w}_{\parallel} + \mathbf{w}_{\perp}$ relative to the plane (OU, OV). Then $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}_{\perp} = 0$ (since $\mathbf{u} \times \mathbf{v} \parallel \mathbf{w}_{\perp}$) and $\mathbf{u} \cdot \mathbf{w}_{\perp} = \mathbf{v} \cdot \mathbf{w}_{\perp} = 0$, so it suffices to assume U, V, W are coplanar. By linearity of both $\times$ and $\cdot$ in each argument, scaling $\mathbf{u}, \mathbf{v}, \mathbf{w}$ by a, b, c multiplies both sides by abc; hence it suffices to prove the unit case $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$. Let $\alpha = \angle UOV$.

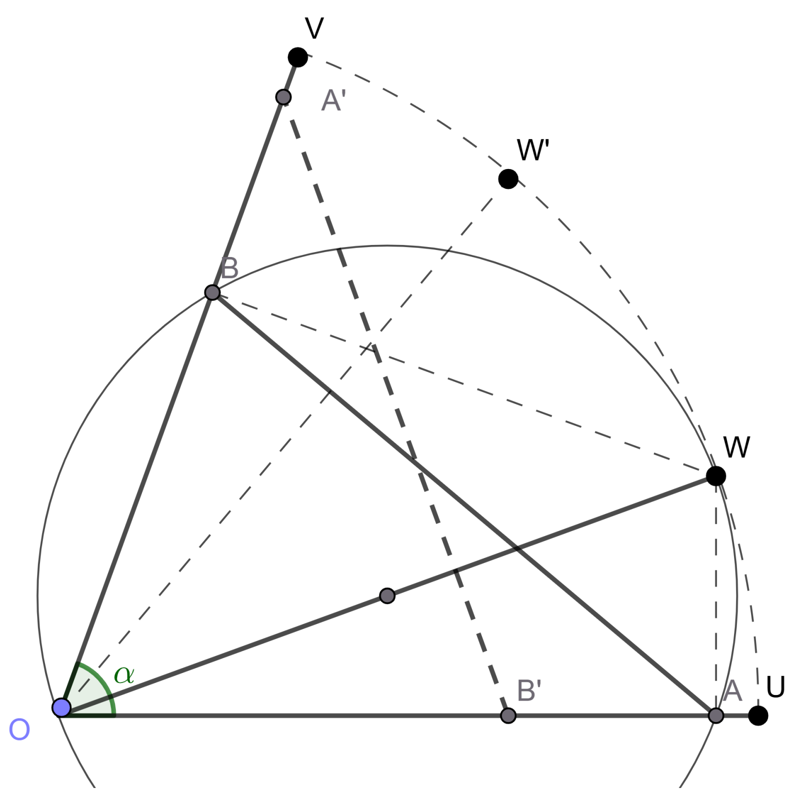


Figure 11: Geometric property of the triple vector formula.

> For the plane geometry, refer to Figure 11. Draw the circle with diameter OW. Let it meet rays OU, OV again at A, B. By Thales, $\angle OAW = \angle OBW = 90^\circ$, hence $OA = \mathbf{u}\cdot\mathbf{w}$ and $OB = \mathbf{v}\cdot\mathbf{w}$. Define $A' \in OV$ and $B' \in OU$ by $OA' = OA$ and $OB' = OB$. Then
>
> $$\overrightarrow{A'B'} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u} - (\mathbf{u}\cdot\mathbf{w})\mathbf{v} = -\big((\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}\big),$$
>
> so $|A'B'|$ is the magnitude of the right-hand side. It is easy to show $A'B' \perp OW$ by checking $\overrightarrow{A'B'}\cdot\mathbf{w} = 0$, but here we want a pure geometric proof. Since $\angle WOA' + \angle A'B'O = 90^\circ$, we get $A'B' \perp OW$, matching the left-hand side direction. Since $\triangle OAB \cong \triangle OA'B'$, we have $AB = A'B'$. In the diameter-1 circle, $AB = \sin\alpha$. But
>
> $$|(\mathbf{u}\times\mathbf{v})\times\mathbf{w}| = |\mathbf{u}\times\mathbf{v}|\,|\mathbf{w}| = \sin\alpha,$$
>
> so both sides are the same vector:
>
> $$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}.$$
>
> In short, we used the multilinearity of the dot and cross products (it is enough to prove the identity for coplanar unit vectors). We also used that dot and cross magnitudes correspond to cos and sin of the angle. The rest relies on a circle fact: two chords have the same length if and only if their inscribed angles are equal.

Postscript

It can be interesting to see how the following software would approach the problem:

1. Medal-level AI geometry solvers (closest to "gold/silver" performance in the International Math Olympiad)

- AlphaGeometry / AlphaGeometry 2 (research systems that can solve many Olympiad problems)
- Open-source software such as Newclid/Yuclid by the startup company Harmonic

2. Geometry theorem provers (strong automation, not always Olympiad-style proofs)

- GCLC / WinGCLC (area method, Wu's method, Gr?bner bases)
- JGEX and SymPy (Wu/Grobner/full-angle methods with visualization)
- OpenGeoProver (Wu/Gr?bner style)
- GeoLogic (interactive Euclidean prover)

3. Proof assistants (formal, but manual)

- Lean 4 + mathlib

- LeanGeo

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