On the Soundness of Algebraic Attacks against Code-based Assumptions

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Syndrome Decoding

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Extract e out of

$$(H, s = H \cdot e)$$

where $H \in \mathbb{F}^{(n-k)\times n}$ is a **parity-check matrix** and $e \in \mathbb{F}^n$ a noise vector of **Hamming-weight** w $\quad (\mathrm{hw}(e) = \# \{i \in [n] \mid e_i \neq 0\})$.

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Definition (Learning Parity with Noise)

Extract x out of

$$(G, y = Gx + e)$$

where $G \in \mathbb{F}^{n\times k}$ is a **generator matrix**, $e \in \mathbb{F}^n$ of **Hamming-weight** w, and $x \in \mathbb{F}^k$.

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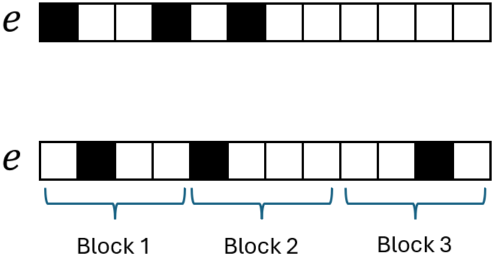
Non-regular noise
$(n = 12, w = 3)$

Regular noise
$(n = 12, w = 3, b = 4)$



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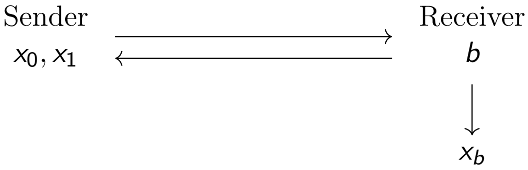

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$$\begin{tikzcd} \text{Sender} & & \text{Receiver} \\ x_0, x_1 \arrow[rr, shift left] & & b \arrow[ll, shift left] \arrow[d] \\ & & x_b \end{tikzcd}$$

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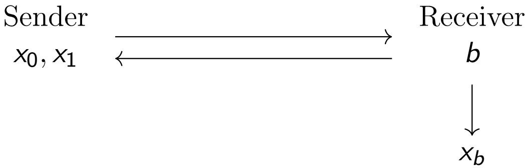

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Pseudorandom Correlation Generators (PCGs) allow for OT-extension, i.e., they stretch short strings of correlated random bits to longer strings of correlated pseudorandom bits [BCGI18, BCG+19, BCG+20].

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There are also other applications (signature schemes and local pseudorandom generators).

The Attack of Briaud and Øygarden [BØ23]

Consider $(H, s = He)$, $H \in \mathbb{F}^{(n-k)\times n}$, $e = (e^{(1)}, \ldots, e^{(w)}) \in \mathbb{F}^n$, $e^{(1)}, \ldots, e^{(w)} \in \mathbb{F}^b$ of HW ≤ 1.

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1. Variables $E_\alpha^{(i)}$, $i \in [w]$, $\alpha \in [b]$, for e with equations

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2. For every row

$$(h_{j,1}^{(1)}, \ldots, h_{j,b}^{(1)}, \ldots, h_{j,1}^{(w)}, \ldots, h_{j,b}^{(w)},)$$

of H and $s = He$, a parity check equation

$$s_j = \sum_{i\in[w],\alpha\in[b]} h_{j,\alpha}^{(i)} \cdot E_\alpha^{(i)} \qquad \text{for } j \in [n-k].$$

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Strategy: Use an XL-style algorithm to solve this equation system.

XL-Algorithm (Macaulay Matrices)

Consider a set of equations over $\mathbb{F}[X_1, \ldots, X_k]$

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2. **Relinearize**: Write coefficients of all equations into a matrix

$$M_D = \begin{pmatrix} \mathrm{coeff}(g_1) \\ \vdots \\ \mathrm{coeff}(g_L). \end{pmatrix}$$

   Columns correspond to monomials and are sorted according to some degree-preserving monomial ordering.

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$$X_i \cdot g \notin \mathrm{span}\{X_1^{a_1} \cdots X_k^{a_k} \cdot f_j(X)\} \text{ for some } i \in [k],$$

   then add $g(X) = 0$ to the set of initial equations and go back to step 1.
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(For example, $R = \mathbb{F}[X_1, \dots, X_k]$ is graded with

$$\mathbb{F}[X]^d = \text{Space of homogeneous polynomials of degree} d.)$$

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Given a series $\mathcal{H} = \sum_d c_d \cdot T^d$, its truncation is given by

$$[\mathcal{H}]_+ = \sum_{d=0}^{\mathrm{d_{reg}}(\mathcal{H})-1} c_d \cdot T^d.$$

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Theorem ([Sal25])

The above algorithm solves[a] the system $f_1(X) = \cdots = f_m(X) = 0$ in time $O(k^{4D})$ where $D = \mathrm{d}_{\mathrm{reg}}(f_1^{\mathrm{top}}, \ldots, f_m^{\mathrm{top}}) + 1$.

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$\mathrm{d}_{\mathrm{reg}}$ constant $\implies$ poly-time attack.
$\mathrm{d}_{\mathrm{reg}}$ sublinear $\implies$ subexponential-time attack.
Problem: How do we bound $\mathrm{d}_{\mathrm{reg}}(f_1^{\mathrm{top}}, \ldots, f_m^{\mathrm{top}})$?

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Definition (Mathematical Semi-Regularity Over Large Fields)

A sequence of homogeneous elements $f_1, \ldots, f_m \in R$ is **semi-regular** if

$$\begin{aligned} (R/(f_1, \ldots, f_{i-1}))^d &\longrightarrow (R/(f_1, \ldots, f_{i-1}))^{d+\deg(f_i)} \\ z &\longmapsto z \cdot f_i \end{aligned}$$

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Lemma

If $f_1, \dots, f_m \in R$ is semi-regular, we have

$$\mathcal{H}_{R/(f_1,\dots,f_m)}(T) = \left[(1 - T^{\deg(f_1)}) \cdots (1 - T^{\deg(f_m)}) \cdot \mathcal{H}_R(T)\right]_+ .$$

The Attack of Briaud and Øygarden [BØ23]

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Solution: Semi-Regularity Heuristic! Just assume that $h_1, \ldots, h_{n-k}$ are semi-regular.

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A sequence of polynomials $f_1, \ldots, f_m$ from **some** distribution in **some** graded ring R will be semi-regular with **some** probability for **some** parameters.

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Correctness of heuristic depends on various factors!

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Theorem

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Corollary

There is a PPT algorithm for the Regular Syndrome Decoding problem whenever $w \cdot \binom{b}{2} \geq 2\binom{k+1}{2}$.

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Lemma (Schwartz-Zippel)

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$$\begin{aligned}
\mathcal{H}_{\mathbb{F}[E]/(h_1,\ldots,h_{n-k},E_\alpha^{(i)}\cdot E_\beta^{(i)})}(T) =& (1-T)^{n-k} \cdot \mathcal{H}_{\mathbb{F}[E]/(E_\alpha^{(i)}\cdot E_\beta^{(i)})}(T) \\
=& (1-T)^{n-k} \cdot (1 + bT + bT^2 + \cdots)^w \\
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According to semi-regularity hypothesis, we have

$$\mathrm{d}_{\mathrm{reg}} \leq 2 \iff \binom{k+1}{2} \leq w \cdot \binom{b}{2}.$$

Primal-Dual Equivalence

For $(H, s = He)$, [BØ23] considered a **dual** modeling

$$\begin{aligned} E_\alpha^{(i)} \cdot E_\beta^{(i)} &= 0 && \text{for } i \in [w], 1 \leq \alpha < \beta \leq b, \\ s_j = \sum_{i \in [w], \alpha \in [b]} h_{j,\alpha}^{(i)} \cdot E_\alpha^{(i)} & && \text{for } j \in [n-k]. \end{aligned}$$

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$$(y_\alpha^i - \sum_{j=1}^{k} g_{\alpha,j}^{(i)} \cdot X_j) \cdot (y_\beta^i - \sum_{j=1}^{k} g_{\beta,j}^{(i)} \cdot X_j) = 0 \quad \text{for } i \in [w], 1 \le \alpha < \beta \le b$$

where every row of G is given by

$$g_\alpha^{(i)} = (g_{\alpha,1}^{(i)}, \ldots, g_{\alpha,k}^{(i)}).$$

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$$\mathrm{d}_{\mathrm{reg}}(\mathcal{H}_{\mathbb{F}[E]/(E_\alpha^{(i)} \cdot E_\beta^{(i)}, h_1(E), \ldots, h_{n-k}(E))}) = \mathrm{d}_{\mathrm{reg}}(\mathcal{H}_{\mathbb{F}[X]/(g_\alpha^{(i)}(X) \cdot g_\beta^{(i)}(X))}).$$

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For the Regular LPN problem $(G, y = Gx + e)$, consider the polynomial system over $\mathbb{F}[X_1, \ldots, X_k]$

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Idea: Prove hypothesis for $w = a^2$ and $b \geq \frac{k}{a} + 1$ by constructing linear forms $(g_1^{(i)}, \ldots, g_b^{(i)})_{i=1}^{w} \in \mathbb{F}[X]$ such that

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Prove hypothesis for $w \cdot \binom{b}{2} \geq 2\binom{k+1}{2}$ by noticing that

$$w \geq a^2 \text{ and } b \geq \frac{k}{a} + 1$$

for $a = \left\lceil \frac{k}{b-1} \right\rceil$.

Conclusion

Theorem

The semi-regularity hypothesis of [BØ23] is wrong for $w \in \{2, 3\}$*. However, if* $w \cdot \binom{b}{2} \geq 2\binom{k+1}{2}$*, then the dual and primal modelings of RSD have a degree of regularity of 2.*

Corollary

If $w \cdot \binom{b}{2} \geq 2\binom{k+1}{2}$*, then we can solve an RSD instance* (H, He) *in time* $O(n^{12})$ *with probability* $1 - O(k^2/|\mathbb{F}|)$ *over the randomness of* $H \leftarrow \mathbb{F}^{(n-k)\times n}$*.*

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3. Relinearize and solve the system once enough equations have been collected.

Arora-Ge: We need $n = O(\log(q) \cdot q \cdot k^d)$ samples.

Arora-Ge: Primal Modeling

Let $G \in \mathbb{Z}_q^{n \times k}$, $x \in \mathbb{Z}_q^k$, $e \in \{0, \ldots, d-1\}^n$.
Arora-Ge [AG11] algorithm for solving $(G, y = Gx + e)$:

1. Define $f(Z) := Z \cdot (Z-1) \cdot (Z-2) \cdots (Z-d+1)$.
2. Collect degree-d equations

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3. Relinearize and solve the system once enough equations have been collected.

Arora-Ge: We need $n = O(\log(q) \cdot q \cdot k^d)$ samples.
Our Work [NMSÜ25]: If $q > d$ is prime, $n = \binom{k+d-1}{d}$ samples are enough.

Semiregularity by Vandermonde Matrices

Lemma

Let $\operatorname{char} \mathbb{F} > d$ *and* $n = \binom{k+d-1}{d}$*. When sampling* $g_1, \ldots, g_n \leftarrow \mathbb{F}[X]^1$*, we have wop.*

$$\operatorname{span}_{\mathbb{F}}\{g_1(X)^d, \ldots, g_n(X)^d\} = \mathbb{F}[X]^d.$$

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Let $\alpha(1), \ldots, \alpha(n)$ be an enumeration of

$$\left\{\beta \in \mathbb{N}_0^{k-1} \;\middle|\; \beta_1 + \cdots + \beta_{k-1} \leq d\right\}.$$

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$$V = \begin{pmatrix} \alpha(1)_1^{\alpha(1)_1} \cdots \alpha(1)_{k-1}^{\alpha(1)_{k-1}} & \cdots & \alpha(1)_1^{\alpha(n)_1} \cdots \alpha(1)_{k-1}^{\alpha(n)_{k-1}} \\ \vdots & \ddots & \vdots \\ \alpha(n)_1^{\alpha(1)_1} \cdots \alpha(n)_{k-1}^{\alpha(1)_{k-1}} & \cdots & \alpha(n)_1^{\alpha(n)_1} \cdots \alpha(n)_{k-1}^{\alpha(n)_{k-1}} \end{pmatrix} \in \mathbb{F}^{n \times n}.$$

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For $f \in \mathbb{F}[X_1, \ldots, X_{k-1}]^d$: $\quad V \cdot \mathrm{coeff}(f) = (f(\alpha(1)), \ldots, f(\alpha(n)))$

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We have

$$(g_i(X))^d = \sum_{j=1}^{n} \gamma_j \cdot \alpha(i)^{\alpha(j)} \cdot X_1^{\alpha(j)_1} \cdots X_{k-1}^{\alpha(j)_{k-1}} \cdot X_k^{d-\alpha(j)_1-\cdots-\alpha(j)_{k-1}}$$

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Lemma

Let char $\mathbb{F} > d$ *and* $n = \binom{k+d-1}{d}$. *When* ***sampling*** $g_1, \ldots, g_n \leftarrow \mathbb{F}[X]^1$, *we have with probability* $\geq 1 - nd/|\mathbb{F}|$

$$\mathrm{span}_{\mathbb{F}}\{g_1(X)^d, \ldots, g_n(X)^d\} = \mathbb{F}[X]^d.$$

Overview of Runtimes

		Samples	Time	Heuristical?
[AG11]	**d**	$O\left(\log(q)\cdot q\cdot k^{d}\right)$	$O\left(\log(q)\cdot q\cdot k^{\omega d}\right)$	NO
[ACFP14]	2	$O\left(k\log\log k\right)$	$O\left(k^2\cdot 2^{\frac{\omega k\log\log\log k}{8\log\log k}}\right)$	NO
[ACFP14]	2	$c\cdot k$	$2^{O(k)}$	YES
[STA20]	2	$c\cdot k^2$	$k^{O(1/c)}$	YES
[STA20]	2	$k^{1+\alpha}$	$2^{\widetilde{O}\left(n^{1-\alpha}\right)}$	YES
[Ste24]	**d**	$> k$	$2^{O(k)}$	NO
[Ste24]	**d**	$O\left(\binom{k+d-1}{d}\right)$	$O\left(d^3\cdot c_d^{(k-1)^{1-1/\ln(4)}}\right)$	YES
[Ste24]	2	$O\left(k^2\right)$	$O\left(k^2\cdot\binom{k+3}{3}^{\omega+2}\right)$	YES
[NMSÜ25]	**d**	$\binom{k+d-1}{d}$	$O\left(dk^{1+d\omega}\right)$	NO

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