Chromatic symmetric functions

Soojin Cho
Ajou University

Part I

Chromatic polynomial: Birkhoff 1912

Given G with vertex set V a proper colouring κ of G in k colours is

$$\kappa : V \to \{1, 2, 3, \ldots, k\}$$

so if $v_1, v_2 \in V$ are joined by an edge then
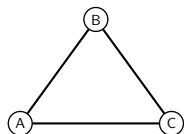
$$\kappa(v_1) \neq \kappa(v_2).$$

Example

Chromatic polynomial: Birkhoff 1912

Given G the chromatic polynomial $\chi_G(k)$ is the number of proper colourings with k colours.

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has $\chi_G(k) = k(k-1)(k-2)$.
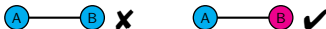
Chromatic symmetric function: Stanley 1995

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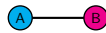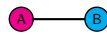
$$\kappa(v_1) \neq \kappa(v_2).$$

Example

Chromatic symmetric function: Stanley 1995

Given a proper colouring κ of vertices $v_1, v_2, \ldots, v_N$ associate a monomial in commuting variables $x_1, x_2, x_3, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}.$$

Example

(A)———(B) gives $x_1 x_2$.

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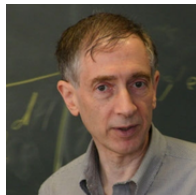
(A)———(B) gives $x_1 x_3$.

Chromatic symmetric function: Stanley 1995

Given G with vertices $v_1, v_2, \ldots, v_N$ the chromatic symmetric function is

$$X_G = \sum_\kappa x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

where the sum over all proper colourings κ. Then

$$\chi_G(k) = X_G(\underbrace{1, 1, \ldots, 1}_{k}, 0, 0, \cdots).$$

Chromatic symmetric function: Stanley 1995

ⓐ ⓑ has $X_G(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$.

Symmetric functions

A symmetric function is a formal power series f in commuting variables $x_1, x_2, \ldots$ such that for all permutations π
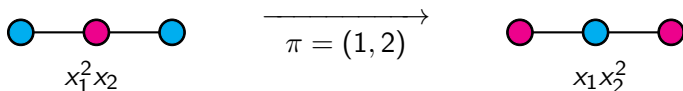
$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

Symmetric functions

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$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

X_G is a symmetric function.

$$\pi = (1, 2)$$

Let

$$\Lambda = \bigoplus_{N \geq 0} \Lambda^N \subset \mathbb{Q}[[x_1, x_2, \ldots]]$$

be the algebra of symmetric functions with Λ^N spanned by ...

Classical basis: monomial

A partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ of N is a list of positive integers whose sum is N: $3221 \vdash 8$.

The monomial symmetric function is for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$m_\lambda = \sum_{i_a \neq i_b} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

where all tuples $(i_1, \ldots, i_\ell)$ yield distinct monomials.

Example

$m_{21} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \cdots$

Classical basis: monomial

Given G, a stable partition of type $\lambda = \lambda_1 \cdots \lambda_\ell$ partitions vertices into sets of size $\lambda_1, \ldots, \lambda_\ell$ so each set is disconnected, and a_λ is the number thereof.

Theorem (Stanley 1995)

$$X_G = \sum_\lambda a_\lambda r_1! \cdots r_\ell! m_\lambda$$

where $r_i = \#\lambda_j s$ equal to i.

Example

$$G = \text{O}\!\!-\!\!\text{O}\!\!-\!\!\text{O}$$

$$X_G = 1 \cdot 1!1! m_{21} + 1 \cdot 3! m_{111} = m_{21} + 6m_{111}$$
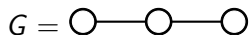
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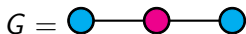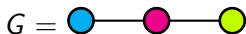
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$$X_G = 1 \cdot 1!1! m_{21} + 1 \cdot 3! m_{111} = m_{21} + 6m_{111}$$

Classical basis: power sum

The i-th power sum symmetric function is

$$p_i = x_1^i + x_2^i + x_3^i + \cdots$$

and for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

Example

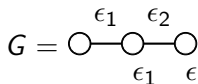$p_{21} = p_2 p_1 = (x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots)$

Classical basis: power sum
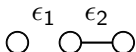
Given $S \subseteq E$, $\lambda(S)$ is the partition determined by the connected components of G restricted to S.

Theorem (Stanley 1995)

$$X_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}$$

Example

$$G = \underset{\epsilon_1 \quad \epsilon_2}{\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc}$$

G restricted to $S = \{\epsilon_2\}$ is $\bigcirc \quad \underset{\epsilon_1 \quad \epsilon_2}{\bigcirc\!\!-\!\!\bigcirc}$ and $\lambda(S) = 21$.

$$X_G = p_3 - 2p_{21} + p_{111}$$

Classical basis: elementary

The *i*-th elementary symmetric function is

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i}$$

and for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}.$$

Example

$$e_{21} = e_2 e_1 = (x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$$

$$G = \text{O—O—O} \qquad X_G = 3e_3 + e_{21}$$

Classical basis: elementary

Theorem (Stanley 1995)

If
$$X_G = \sum_\lambda c_\lambda e_\lambda$$

then

$$\sum_{\lambda \text{ with } k \text{ parts}} c_\lambda = \text{ number of acyclic orientations with } k \text{ sinks.}$$

Example

$$G = \bigcirc\!\!-\!\!-\!\!\bigcirc\!\!-\!\!-\!\!\bigcirc \qquad X_G = 3e_3 + e_{21}$$

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Example

$$G = \circ\!\!-\!\!-\!\!-\!\!\times\!\!-\!\!-\!\!-\!\!\circ \qquad X_G = 3e_3 + e_{21}$$

Classical basis: elementary

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Classical basis: elementary

Theorem (Stanley 1995)

If

$$X_G = \sum_\lambda c_\lambda e_\lambda$$

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Example

$$G = \bullet\!\longleftarrow\!\circ\!\longrightarrow\!\bullet \qquad X_G = 3e_3 + e_{21}$$

Classical basis: complete homogeneous

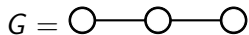The i-th complete homogeneous symmetric function is

$$h_i = \sum_{j_1 \leq \cdots \leq j_i} x_{j_1} \cdots x_{j_i}$$

and for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}.$$

Example

$h_{21} = h_2 h_1 = (x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$
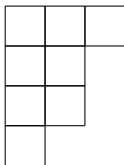
$$G = \text{O}\!-\!\!-\!\text{O}\!-\!\!-\!\text{O} \qquad X_G = 3h_3 - 7h_{21} + 4h_{111}$$

Partitions and diagrams

A partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ of N is a list of positive integers whose sum is N: $3221 \vdash 8$.

The diagram $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ is the array of boxes with λ_i boxes in row i from the top.

3221

Semi-standard Young tableaux

A semi-standard Young tableau (SSYT) T of shape λ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase.

1	1	1
2	4	
4	5	
6		

Given an SSYT T we have

$$x^T = x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \cdots .$$

$$x_1^3 x_2 x_4^2 x_5 x_6$$
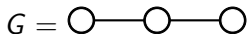
Classical basis: Schur

The Schur function is

$$s_\lambda = \sum_{T \text{ SSYT of shape } \lambda} x^T.$$

Example

$$s_{21} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3 + \cdots$$

$$G = \bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc \qquad X_G = s_{21} + 4s_{111}$$

Are these chromatic?

Question: Are classical symmetric functions ever examples of chromatic symmetric functions of a connected graph?
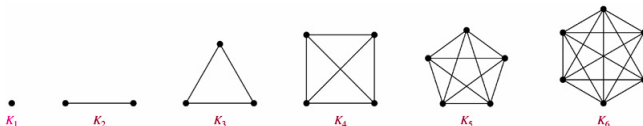
Are these chromatic?

Question: Are classical symmetric functions ever examples of chromatic symmetric functions of a connected graph?

Answer:

Theorem (C-van Willigenburg 2018)

Only the elementary symmetric functions, namely

$$e_n = \frac{1}{n!} X_{K_n}.$$

New bases

New bases

Pick favourite connected graph on 1 vertex:

New bases

Pick favourite connected graph on 1 vertex:

$$G_1 = \circ$$

New bases

Pick favourite connected graph on 1 vertex:

$$G_1 = \circ$$

Pick favourite connected graph on 2 vertices:

New bases

Pick favourite connected graph on 1 vertex:

$$G_1 = \circ$$

Pick favourite connected graph on 2 vertices:

$$G_2 = \circ\!\!-\!\!\circ$$

New bases

Pick favourite connected graph on 1 vertex:

$$G_1 = \circ$$

Pick favourite connected graph on 2 vertices:

$$G_2 = \circ\!\!-\!\!\circ$$

Pick favourite connected graph on 3 vertices:

New bases

Pick favourite connected graph on 1 vertex:

$$G_1 = \circ$$

Pick favourite connected graph on 2 vertices:

$$G_2 = \circ\!\!-\!\!\circ$$

Pick favourite connected graph on 3 vertices:

$$G_3 = \circ\!\!-\!\!\circ\!\!-\!\!\circ$$

And so on . . .

New bases

Pick favourite connected graph on 1 vertex:

$$G_1 = \circ$$

Pick favourite connected graph on 2 vertices:

$$G_2 = \circ\!\!-\!\!\circ$$

Pick favourite connected graph on 3 vertices:

$$G_3 = \circ\!\!-\!\!\circ\!\!-\!\!\circ$$

And so on ...

Let G_λ be the disjoint union $G_{\lambda_1} \cup \cdots \cup G_{\lambda_\ell}$.

Example

$$G_{211} = \text{O}\!\!-\!\!\text{O} \quad \text{O} \quad \text{O}$$

New bases

Theorem (C-van Willigenburg 2016)

$$\Lambda = \mathbb{Q}[X_{G_1}, X_{G_2}, \ldots] \qquad \Lambda^N = \mathrm{span}_{\mathbb{Q}}\{X_{G_\lambda} \mid \lambda \vdash N\}$$

where

$$X_{G_\lambda} = X_{G_{\lambda_1}} \cdots X_{G_{\lambda_\ell}}.$$

Example

$$G_{211} = \text{O———O} \quad \text{O} \quad \text{O}$$

$$X_{G_{211}} = X_{G_2} X_{G_1} X_{G_1}$$
$$= 2e_2 e_1 e_1 = 2e_{211}$$

e-positivity and Schur-positivity

G is *e-positive* if X_G is a positive linear combination of e_λ.

G is Schur-positive if X_G is a positive linear combination of s_λ.

e-positivity and Schur-positivity

G is e-positive if X_G is a positive linear combination of e_λ.

G is Schur-positive if X_G is a positive linear combination of s_λ.

 has $\begin{array}{l} X_G = e_{21} + 3e_3 \ \checkmark \\ X_G = 4s_{111} + s_{21} \ \checkmark \end{array}$

e-positivity and Schur-positivity

G is *e-positive* if X_G is a positive linear combination of e_λ.

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 has $\begin{aligned} X_G &= e_{21} + 3e_3 \ \checkmark \\ X_G &= 4s_{111} + s_{21} \ \checkmark \end{aligned}$

 has $\begin{aligned} X_G &= e_{211} - 2e_{22} + 5e_{31} + 4e_4 \ \times \\ X_G &= 8s_{1111} + 5s_{211} - s_{22} + s_{31} \ \times \end{aligned}$

e-positivity and Schur-positivity

G is *e*-positive if X_G is a positive linear combination of e_λ.

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K_{13} : Smallest graph that is not *e*-positive. Smallest graph that is not Schur-positive.

e-positivity and Schur-positivity

For $\lambda = \lambda_1 \cdots \lambda_\ell$

$$e_\lambda = \sum_\mu K_{\mu\lambda} s_{\mu^t}$$

where $K_{\mu\lambda} = \#$ SSYTs of shape μ filled with λ_1 1s, ..., λ_ℓ ℓs, and μ^t is the transpose of μ along the downward diagonal.

Hence $K_{\mu\lambda} \geq 0$ and

e-positivity implies Schur-positivity.

Example

$e_{21} = s_{21} + s_{111}$

1	1
2	

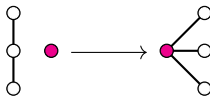| 1	1	2

e-positivity and Schur-positivity

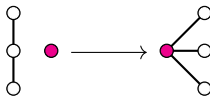### Conjecture (Stanley-Stembridge 1993)

If G is an incomparability graph of a $(3+1)$-free poset then X_G is e-positive.

e-positivity and Schur-positivity

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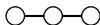
Theorem (Gasharov 1996)
If G is an incomparability graph of a $(3 + 1)$-free poset then X_G is Schur-positive.
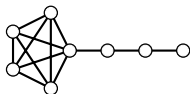
Known *e*-positive graphs

▶ Complete graphs K_m.

▶ Paths P_n (Stanley 1995).

▶ Lollipop graphs $L_{m,n}$ (Gebhard-Sagan 2001).

▶ Complement of G is bipartite (Stanley 1995).

▶ G avoids K_{13} and the paw (Hamel, Hoáng and Tuero 2017), copaw (HHT), or P_4 (Tsujie 2017).

Why e-positivity?

- ▶ Stanley-Stembridge conjecture.

- ▶ *e*-positivity implies Schur-positivity.

- ▶ If Schur-positive, then it arises as the Frobenius image of some representation of a symmetric group.

- ▶ If Schur-positive, then it arises as the character of a polynomial representation of a general linear group.

Part II

A Theorem by Guay-Paquet 2013

If X_G is e-positive for unit interval graphs (incomparability graphs of posets that are both (3+1)-free and (2+2)-free) then Stanley-Stembridge conjecture is true.

Chromatic *quasi* symmetric functions: Shareshian-Wachs 2012

For a simple graph $G = (V, E)$ which has a vertex set $V \subset \mathbb{P}$, the chromatic quasisymmetric function of G is

$$X_G(\mathbf{x}, t) = \sum_{\kappa} t^{\mathrm{asc}(\kappa)} \mathbf{x}_{\kappa} \ ,$$

where the sum is over all proper colorings κ and

$$\mathrm{asc}(\kappa) = \{\{i, j\} \in E \mid i < j \text{ and } \kappa(i) < \kappa(j)\} \ .$$

Example

①—②—③   gives $t\, x_1^2 x_2$

①—③—②   gives $t^2 x_1^2 x_2$

Chromatic *quasi* symmetric functions

$$G = \boxed{1} \!\!-\!\! \boxed{2} \!\!-\!\! \boxed{3}$$

then $X_G(\mathbf{x}, t) = e_3 + (e_3 + e_{21})t + e_3 t^2$

$$G = \boxed{1} \!\!-\!\! \boxed{3} \!\!-\!\! \boxed{2}$$

then $X_G(\mathbf{x}, t) = (e_3 + F_{12} + 2e_3)t + (e_3 + F_{21})t^2$

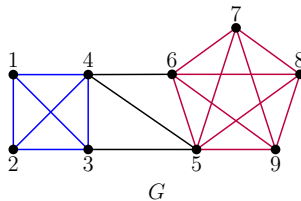where F_α is a fundamental quasisymmetric function for α a composition.
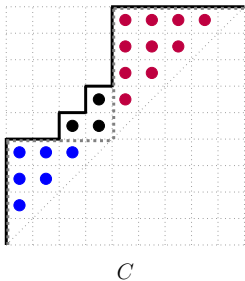
Naturally labeled unit interval graph (natural unit interval order)

$$G = \text{①——②——③} \quad \longleftrightarrow \quad P = \{(1 \prec 3)\}$$
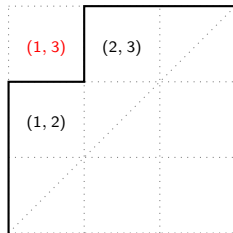
$h = (2, 3, 3),$ Hessenberg function

Naturally labeled unit interval graph (natural unit interval order)

C

G

$h = (4, 4, 5, 6, 9, 9, 9, 9)$, Hessenberg function

Chromatic quasisymmetric functions of natural unit interval order

Theorem (Shareshian-Wachs 2015)

If $G = inc(P)$ of a natural unit interval order P, then the coefficients of t^i, $i = 0, 1, \ldots, |E|$, in $X_G(\mathbf{x}, t)$ are symmetric functions and form a palindromic sequence.

Conjecture (Shareshian-Wachs 2015)

Let G be an incomparability graph of a natural unit interval order. Then $X_G(\mathbf{x}, t)$ is an e-positive and e-unimodal polynomial in t.

Example

$$G = \text{①——②——③} \qquad H = \text{①——②——③——④}$$

$$X_G(\mathbf{x}, t) = e_3 + (e_3 + e_{21})t + e_3 t^2$$

$$X_H(\mathbf{x}, t) = e_4 + (e_4 + e_{31} + e_{22})t + (e_4 + e_{31} + e_{22})t^2 + e_4 t^3$$

Geometric representation?

In Shareshian-Wachs 2015,

The palindromicity of $X_G(\mathbf{x}, t)$, when G is the incomparability graph of a natural unit interval order, suggests that $X_G(\mathbf{x}, t)$ might be the Frobenius characteristic of the representation of $\mathfrak{S}_n$ on the cohomology of some manifold...

Geometric representation: Hessenberg varieties

Conjecture (Shareshian-Wachs 2015)

Let P be a natural unit interval order with incomparability graph $G = (V, E)$, and let $\mathcal{H}(P)$ be the associated regular semisimple Hessenberg variety. Then

$$\omega(X_G(\mathbf{x}, t)) = \sum_{j=0}^{|E|} ch\left(H^{2j}(\mathcal{H}(P))\right) t^j,$$

where $chH^{2j}(\mathcal{H}(P))$ is the Frobenius characteristic of the representation of $\mathfrak{S}_n$ on the $2j$-th cohomology group of $\mathcal{H}(P)$ defined by Tymoczko using GKM theory.

▶ This conjecture is proved independently by Brosnan-Chow (2015) and Guay-Paquet (2016).

▶ $\omega : \Lambda \to \Lambda$ is an involution(algebra automorphism) defined as $\omega(e_\lambda) = h_\lambda$.

Hessenberg varieties: De Mari-Shayman 1988, De Mari-Procesi-Shayman 1992

For a positive integer n, a non-decreasing function $h : [n] \to [n]$ is called a Hessenberg function if $i \leq h(i)$ for all $i \in [n]$ where $[n] = \{1, 2, \ldots, n\}$.

natural unit interval orders (graphs) $\leftrightarrow$ Hessenberg functions

Type A regular semisimple Hessenberg variety associated with Hessenberg function h and a regular semisimple linear transformation $s : \mathbb{C}^n \to \mathbb{C}^n$ is

$$\mathcal{H}(h, s) = \{F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n \mid \dim F_i = i, \, sF_i \subseteq F_{h(i)} \, \forall i \in [n]\}.$$

* regular semisimple s is a diagonal matrix with distinct diagonal entries.

Cohomology ring of Hessenberg varieties

1. The maximal torus $T \subset GL(n, \mathbb{C})$ of diagonal matrics acts on $\mathcal{H}(h, s)$ by left multiplication and $\mathcal{H}(h, s)^T = Fl^T \cong \mathfrak{S}_n$.

2. By GKM(Goresky-Kottwitz-MacPherson 1998) theory T-equivariant cohomology ring of $\mathcal{H}(h, s)$ can be constructed; GKM graph of $\mathcal{H}(h, s)$:

$$\text{vertex set} = \mathfrak{S}_n$$
$$\text{edge set} = \{\{w, w(ij)\} \mid j < i \text{ and } i \leq h(j)\},$$
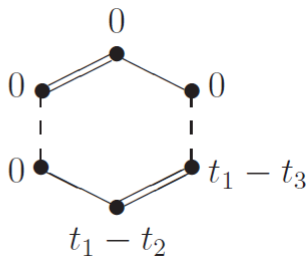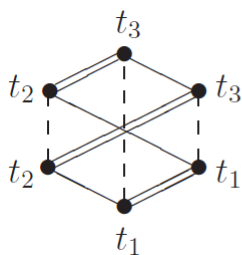where the edge $\{w, w(ij)\}$ is labeled $t_{w(i)} - t_{w(j)}$.
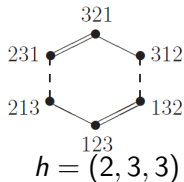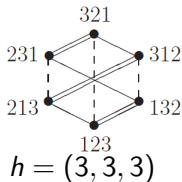
$$h = (3, 3, 3) \qquad h = (2, 3, 3)$$

Cohomology ring of Hessenberg varieties

$H_T^*(\mathcal{H}(h,s)) =$

$\{\alpha \in \bigoplus_{w \in \mathfrak{S}_n} \mathbb{C}[t_1, \ldots, t_n] \mid \alpha(w) - \alpha(w') \text{ is divisible by } t_{w(i)} - t_{w(j)}\}$

3. (Tymoczko 2008) $v \in \mathcal{S}_n$ acts on $\alpha = (\alpha(w)) \in H_T^*(\mathcal{H}(h,s))$ as

$$(v \cdot \alpha)(w) = v \cdot \alpha(v^{-1}w) \text{ for all } w \in \mathfrak{S}_n$$

Cohomology ring of Hessenberg varieties

4. From this, one gets ordinary cohomology ring;

$$H^*(\mathcal{H}(h,s)) \cong H_T^*(\mathcal{H}(h,s))/\langle t_1, \ldots, t_n \rangle H_T^*(\mathcal{H}(h,s))$$

and $\mathcal{S}_n$-action on $H^*(\mathcal{H}(h,s))$

****

Theorem (Brosnan-Chow, Guay-Paquet)

$$\omega(X_G(\mathbf{x}, t)) = \sum_{j=0}^{|E|} chH^{2j}(\mathcal{H}(P))t^j$$

Hence, the *e*-positivity of $X_G(\mathbf{x}, t)$ is equivalent to the *h*-positivity of $chH^{2j}(\mathcal{H}(P))$ for each j, which means that each $H^{2j}(\mathcal{H}(P))$ is a permutation representation of $\mathcal{S}_n$.

Known results: refined(graded) e-positivity

- $h(i) = n$ for all i, or $G = K_n$ the complete graph
- $h(i) = i + 1$ or $G = P_n$ the path graph (Stanley)
- $h = (h(1), h(2), n, \ldots, n)$ (Shareshian-Wachs 2015)
- bounce number of h is 2, or the complement of bipartitie graph (C-Huh 2017, Harada-Precup 2017) non refined case (Stanley 1995)
- $h = (h(1), n-1, n-1, \ldots, n, \ldots, n)$ (C-Huh 2017)
- melting lollipop graph (Huh-Nam-Yoo 2019) non refined lollipop case (Dahlberg-van Willigenburg 2018)

Known results: *e*-unimodality

- $h(i) = n$ for all i, or $G = K_n$ the complete graph
- $h(i) = i + 1$ or $G = P_n$ the path graph (Shareshian-Wachs 2015)
- $h = (h(1), h(2), n, n, \ldots, n)$ (Shareshian-Wachs 2015)
- some special cases of h with bounce number 2 or 3 (C-Huh 2017)
- melting lollipop graph (Huh-Nam-Yoo 2019)
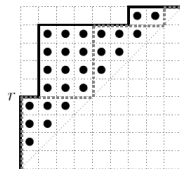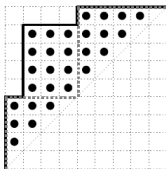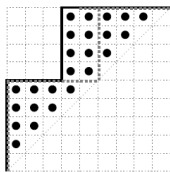
A recent result on Stanley-Stembridge conjecture

Theorem (C-Hong 2019)

$\omega(X_{inc(P(h))}(\mathbf{x}))$ *is h-positive when the bounce number of h is* 3, *that is when the longest chain in the poset $P(h)$ has length* 3.

Our method of proofs

- ▶ (Gasharov, Shareshian-Wachs)

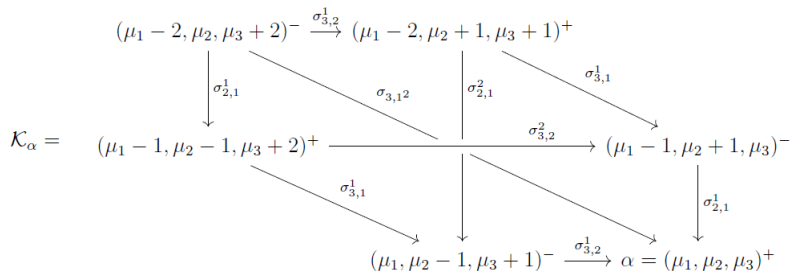$$\omega X_{G(h)}(\mathbf{x}, t) = \sum_T t^{inv_G(T)} s_{\lambda(T)}(\mathbf{x})$$

  where the sum is over all h-tableaux and $\lambda(T)$ is the shape of T.
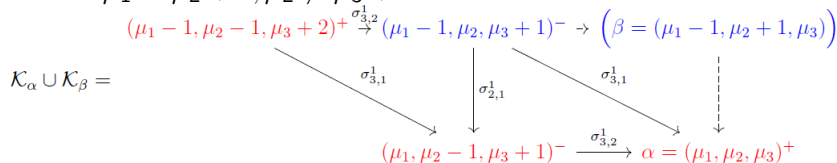
- ▶ Jacobi-Trudi identity

$$s_\lambda = \det(e_{\lambda'_i - i + j})_{\ell' \times \ell'} = \det(h_{\lambda_i - i + j})_{\ell \times \ell}$$

Our method of proofs

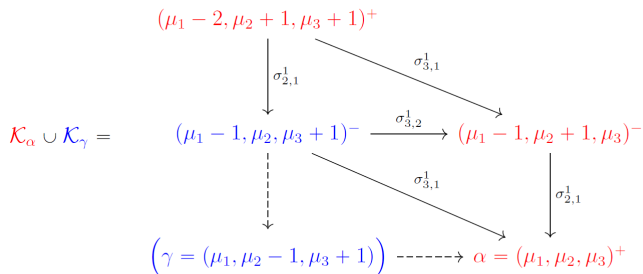**Case I.** $\mu_1 \neq \mu_2 + 1, \mu_2 \neq \mu_3 + 1$
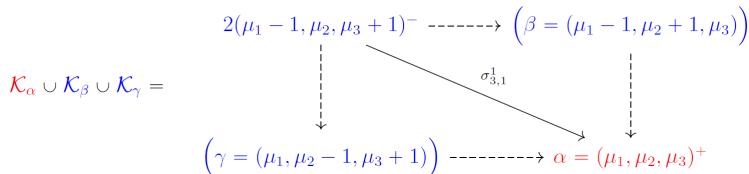

Case II. $\mu_1 = \mu_2 + 1, \mu_2 \neq \mu_3 + 1$

Our method of proofs

Case III. $\mu_1 \neq \mu_2 + 1, \mu_2 = \mu_3 + 1$

$\mathcal{K}_\alpha \cup \mathcal{K}_\gamma =$

$(\mu_1 - 2, \mu_2 + 1, \mu_3 + 1)^+$

$\sigma^1_{2,1}$

$\sigma^1_{3,1}$

$(\mu_1 - 1, \mu_2, \mu_3 + 1)^- \xrightarrow{\sigma^1_{3,2}} (\mu_1 - 1, \mu_2 + 1, \mu_3)^-$

$\sigma^1_{3,1}$

$\sigma^1_{2,1}$

$\left(\gamma = (\mu_1, \mu_2 - 1, \mu_3 + 1) \right) \dashrightarrow \alpha = (\mu_1, \mu_2, \mu_3)^+$

Case IV. $\mu_1 = \mu_2 + 1, \mu_2 = \mu_3 + 1$

$\mathcal{K}_\alpha \cup \mathcal{K}_\beta \cup \mathcal{K}_\gamma =$

$2(\mu_1 - 1, \mu_2, \mu_3 + 1)^- \dashrightarrow \left(\beta = (\mu_1 - 1, \mu_2 + 1, \mu_3) \right)$

$\sigma^1_{3,1}$

$\left(\gamma = (\mu_1, \mu_2 - 1, \mu_3 + 1) \right) \dashrightarrow \alpha = (\mu_1, \mu_2, \mu_3)^+$

Our method of proofs

- In each case, define (weight preserving) injections from negative set of h-tableaux to positive set of h-tableaux.

Thank you very much!

Special thanks to Stephanie van Willigenburg!