Chapter: 7
Algebric Structure

☆ Set

→ N = Natural number = $\{1, 2, 3, 4, 5, \ldots\}$

→ I or Z = Positive / negative Integer number
$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

→ Q = Rational = Integer and Non-Integer

$= \{-3, -3/2, -5/2, -2, -1, 0, 1, 2, 1/2, 3/2, 5/3\}$

→ R = Real numer = In Q → Root
$= \{-3, -3/2, -1/2, 0, 1, \sqrt{3}, 1/2, \sqrt{2}, 3/2\}$

→ Complex Number $= \{-2i, 3+2i, 1/2, -3/2, -1, 0, 1, 1/2\}$

☆ Binary Operation

→ $a * b = a+b, a-b, a \times b, a \div b$

☆ Properties of Binary Operation

→ Let A be non empty set $a, b, c \in A$

(a) Closure property

→ $a, b \in A$, $a * b \in A$

(b) Associative Property

→ $a, b, c \in A$,
$(a * b) * c = a * (b * c)$

(c) Existance of Identity

→ Addition : $e = 0$
Multiplication : $e = 1$ | $a \cdot 1 = a$

(d) Existance of Inverse

→ $a * b = e$ $= b * a$

(e) Commutative law

→ $a * b = b * a$

☆ Groupoid

→ A non empty set G with binary operation "*" is called groupoid. If binary operation "*" satisfied closure property.

Ex. Consider set of Rational numer and binary operation of multiplication an prove that (Q, *) is a groupoid.

→ $Q = \{\ldots, -3, -3/2, -1/2, -2, -1, 0, 1/2, 3/2, 5/2, \ldots\}$

(a) Closure property :

$a, b \in Q$
$a \cdot b \in Q$

$a = -3, \; b = 1/2$

$a \cdot b = -3/2$

→ So, Q is a groupoid.

* Groupoid : $(N, +)$, $(N, \cdot)$, $(I +)$, $(I, -)$, $(I, \cdot)$, $(R, +)$, $(R, -)$, $(R, \cdot)$, $(R, \div)$, $(Q, +)$, $(Q, -)$, $(Q, \cdot)$, $(Q, \div)$

* Non-Groupoid : $(N, -)$, $(N, \div)$, $(I, \div)$

☆ Semigroup

→ Satisfies following property :
(a) Closure Propery (b) Associative Property

Ex. $G = \{2, 4, 6, 8, \ldots, 2n\}$ consider set of all even integers and binary operation of multiplication prove that $(G, \cdot)$ is semigroup.

→ $G = \{2, 4, 6, 8, \ldots, 2n\}$

(a) <u>Closure</u>

→ $a, b \in G$ $a \cdot b \in G$	$a = 2, \; b = 10$ $a \cdot b = 20$

(b) <u>Associative</u>

→ $a, b, c \in G$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$	$a = 2, \; b = 6, \; c = 8$ $(2 \cdot 6) \cdot 8 = 2 \cdot (6 \cdot 8)$ $96 = 96$

→ So, G is Semi-group.

Ex. $(A, *)$ where $A = \{0, 1\}$ check binary operation of Addition $(A, +)$.

→ $A = \{0, 1\}$

(a) <u>Closure</u>

→

+	0	1
0	0	1
1	1	2

$a, b \in A$
but, $a + b \notin A$

→ So, A is not semi-group.

* Semi-Group : $(N, +)$, $(N, \cdot)$, $(I, +)$, $(I, \cdot)$, $(Q, +)$, $(Q, \cdot)$

* Non-Semi Group : $(N, -)$, $(N, \div)$, $(I, \div)$

☆ Monoid

→ A non-empty set G with binary operation $(G, *)$ is called monoid If it satisfies

(a) Closure Property
(b) Associative Property
(c) Existance of Identity

Ex. Prove that Algebric structure $(A, +)$ is a monoid where a is a set of integer and "+" is a binary operation.
$A = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

→

(a) Closure

→ $a, b \in A$
$a+b \in A$

$a = -5, \quad b = 2$
$a+b = -3$

(b) Associative

→ $a, b, c \in A$
$(a+b)+c = a+(b+c)$

$a = 2, \quad b = 4, \quad c = 5$
$(2+4)+5 = 2+(4+5)$
$11 = 11$

(c) Existance of Identity

→ $e = 0, \quad a+e = a$

Ex. Check $(N, +)$ is Monoid
$N = \{1, 2, 3, 4, \ldots \infty\}$

→ (a) Closure

→ $a, b \in N$
$a+b \in N$

(b) Associative

→ $a, b, c \in N$
$a+(b+c) = (a+b)+c$

(c) Existance of Identity

→ $a \in N$
$a+e = a$ is true for $e = 0$

but, $0 \notin N$

→ So, Identity element doesn't exist.
So, $(N, +)$ is not monoid.

☆ Check Associative or Not.

* **Division :**

→ $\frac{(a/b)}{c} = \frac{a}{bc}$ $\qquad$ $\frac{(2/3)}{5} = \frac{2}{3(5)} = \frac{2}{15}$

$\frac{a}{(b/c)} = \frac{ac}{b}$ $\qquad$ $\frac{2}{(3/5)} = \frac{10}{3}$

→ $\frac{a/b}{c} \neq \frac{a}{b/c}$

So, It's not associative.

* **Exponent :**

→ $(a^b)^c = a^{b \cdot c}$ $\qquad$ $(3^2)^4 = 3^8$

→ $(a)^{(b^c)} = a^{(b^c)}$ $\qquad$ $(3)^{(2^4)} = 3^{16}$

$(a^b)^c \neq (a)^{(b^c)}$

So, It's not associative.

☆ Check Existance of Identity & Inverse

(1) $a * b = \frac{ab}{2}$

→ Existance of Identity

$a * e = a$

$\frac{a \cdot e}{2} = a$

$e = 2$

→ Existance of Inverse

$a * b = e$

$\frac{ab}{2} = 2$

$b = 4/a \qquad a \neq 0$

(2) $a * b = a + b - ab$

→ Existance of Identity

$a * e = a$

$a + e - ae = a$

$e - ae = 0 \qquad \therefore e(1-a) = 0 \quad \therefore e = 0$

→ Existance of Inverse

$a * b = e$

$a + b - ab = 0$

$b - ab = -a \qquad \therefore b(1-a) = -a$

$b = \frac{-a}{(1-a)} \qquad a \neq 1$

☆ Group

→ Satisfies following Property

(a) closure
(b) Associative
(c) Existance of Identity
(d) Existance of Inverse.

Ex. $(Q_0, \cdot)$ is group or not?
Q_0 means it doen't include 0.

→ (a) closure
→

$a, b \in Q_0$
$a \cdot b \in Q_0$

(b) Associative
→

$a, b, c \in Q_0$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(c) Existance of Identity [$e = 1$]

→ $a \cdot e = a$

(d) Existance of Inverse

→ $a \cdot b = e \quad \therefore a \cdot b = 1 \quad \therefore b = 1/a$

$a \neq 0$

→ So, Q_0 is Group.

Ex. If the algebric system $(Q, *)$ is a group?
where, $a * b = a + b - ab$
$Q = \{-\infty, -3, -2, -1, 0, 1, 2, 3, \cdots\}$

→ (a) Closure
→

$a, b \in Q$
$a * b \in Q$

(b) Associative

→ $(a*b)*c = a*(b*c)$
→ $L.H.S = (a*b)*c$
$= [a+b-ab]*c$
$= [(a+b-ab)+c] - [(a+b-ab)c]$
$= a+b-ab+c-ac-bc+abc$
$= a+b+c-ab-ac-bc+abc$

→ $R.H.S = a*(b*c)$
$= a*[b+c-bc]$
$= [a+(b+c-bc)] - [(b+c-bc)a]$
$= (a+b+c-bc) - (ba-ca+abc)$
$= a+b+c-ab-ac-bc+abc$

$a*(b*c) = (a*b)*c$

→ (c) Existance of Identity

→ $a * b = a+b - ab$

$$a * e = a$$
$$a+e - ae = a$$
$$e - ae = 0$$
$$e(1-a) = 0$$
$$e = 0$$

(d) Existance of Inverse

→ $a * b = e$

$$a+b - ab = e$$
$$a+b - ab = 0$$
$$b - ab = -a$$
$$b(1-a) = -a$$
$$b = \frac{-a}{(1-a)}$$

$$a \neq 1$$

→ Inverse doesn't exist for $a = 1$

So, its not group

☆ Abelian Group

→ Satisfies following Property

(a) closure
(b) Associative
(c) Existance of Identity
(d) Existance of Inverse
(e) Commutative

Ex. Q_+ be a set of positive rational numbers, $a * b = \frac{ab}{3}$ is an abelian group or not?

→ $Q_+ = \{1, 1/2, 2, 2/3, \dots\}$

(a) closure

→ $a, b \in Q_+$
$a * b \in Q_+$

$a = 2, b = 4$
$a * b = ab/3 = 8/3 \in Q_+$

(b) Associative

→ $(a * b) * c = a * (b * c)$

$L.H.S = (a * b) * c = \frac{ab}{3} * c = \frac{\left(\frac{ab}{3}\right)c}{3} = \frac{abc}{9}$

$R.H.S = a * (b * c) = a * \frac{bc}{3} = \frac{a\left(\frac{bc}{3}\right)}{3} = \frac{abc}{9}$

→ $(a * b) * c = a * (b * c)$

(c) Existance of Identity

$\rightarrow$ $a * e = a$

$\frac{ae}{3} = a$

$e = 3$

(d) Existance of Inverse

$\rightarrow$ $a * b = e$

$\frac{ab}{3} = 3$

$b = 9/a$

$a \neq 0$

So, Inverse exist.

(e) Commutative

$\rightarrow$ $a * b = b * a$

L.H.S $= a * b = ab/3$

R.H.S $= b * a = ba/3$

So, $a * b = b * a$

$\rightarrow$ So, Q_+ is an Abelian Group.

Ex. If $G = \{a + b\sqrt{2} \mid a, b \in Q\}$ than find out $(G, +)$ is an abelian group?

→ (a) Closure

$x, y \in G$
$x + y \in G$

(b) Associative

$x, y, z \in G$
$\cancel{(x+y)+z} \; (x+y)+z = x + (y+z)$

(c) Existance of Identity

$x + e = x$

$(a_1 + b_1\sqrt{2}) + 0 = x$
$(a_1 + 0) + (b_1\sqrt{2} + 0) = a_1 + b\sqrt{2}$
$= x$

(d) Existance of Inverse

$x + y = e$
$(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = 0$
$(a_1 + a_2) + \sqrt{2}(b_1 + b_2) = 0$
$a + \sqrt{2}b = 0$
$b = \frac{-a}{\sqrt{2}}$

Inverse exist

(e) Commutative

$x + y = y + x$

→ So, $G = \{a + b\sqrt{2}, a, b \in Q\}$ is an abelian group.

☆ Addition and Multiplication Module

- $4 +_7 8 = \frac{4+8}{7} = \frac{12}{7} = 5$
- $9 +_5 3 = \frac{9+3}{5} = \frac{12}{5} = 2$
- $3 +_5 12 = \frac{3+12}{5} = \frac{15}{5} = 0$
- $1 +_8 4 = \frac{4+1}{8} = \frac{5}{8} = 5$
- $1 +_{11} 5 = \frac{1+5}{11} = \frac{6}{11} = 6$

→ $4 \times_5 3 = \frac{4 \times 3}{5} = \frac{12}{5} = 2$

→ $5 \times_6 7 = \frac{5 \times 7}{6} = \frac{35}{6} = 5$

→ $2 \times_9 3 = \frac{2 \times 3}{9} = \frac{6}{9} = 6$

→ $3 \times_9 3 = \frac{3 \times 3}{9} = \frac{9}{9} = 0$

Ex. $G = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and operation is $+_8$ addition Modulo then $(G, +_8)$ is an abelian group?

(a) → Closure

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

→ $a, b \in G$

$a +_8 b \in G$

(b) Associative

→ $a, b, c \in G$

$(a +_8 b) +_8 c = a +_8 (b +_8 c) \in G$

$a = 2, b = 3, c = 4$

$(a +_8 b) +_8 c = (2 +_8 3) +_8 4 = 1$

$a +_8 (b +_8 c) = a +_8 (3 +_8 4) = 1$

(c) Existance of Identity.

→ $a +_8 e = a$

(d) Existance of Inverse

→ $a, b \in G$

$a +_8 b = e$

$a +_8 b = 0 = b +_8 a$

→ Inverse of 0 is 0.

1 is 7

2 is 6

3 is 5

4 is 4

5 is 3

6 is 2

7 is 1

(e) Commutative

→ $a, b \in G$

$a +_8 b = b +_8 a$

→ So, G is an abelian Group.

☆ Cyclic Group

→ A group G is called cyclic group if for some $a \in G$ every element $x \in G$ can be written as a^m where m is some integer i.e every element of G can be Genrated by some element of G.
Such Group is known as cyclic Group and elements are known as Genrator of group G.

Ex. Fourth roots of Unity form a cyclic Group.

→ $G = \{1, -1, i, -i\}$

For i: $(i)^1 = i$
$(i)^2 = -1$
$(i)^3 = i^2 \cdot i = -1 \cdot (+i) = -i$
$(i)^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

For $-i$: $(-i)^1 = (-i)$
$(-i)^2 = i^2 = (-1)$
$(-i)^3 = -i^2 \cdot i = (-1)(-i) = i$
$(-i)^4 = i^4 = 1$

→ So, G is cyclic Group
Genrators are, $G = \{1, -1\}$

Ex. Prove that $(G, +_6)$ is a cyclic group where $G = \{0, 1, 2, 3, 4, 5\}$

→ 1 is genrator

$(1)^2 = 1 +_5 1 = 2$

$(1)^3 = (1 +_5 1) +_5 1) = 3$

$(1)^4 = (1 +_5 1 +_5 1) +_5 1 = 4$

$(1)^5 = (1 +_5 1 +_5 1 +_5 1) +_5 1 = 0$

$(1)^6 = (1 +_5 1 +_5 1 +_5 1 +_5 1 +_5 1) +_5 1 = 1$

→ So G is a cyclic Group
Genrator are $\{1, 5\}$

☆ Rings

→ Suppose R is a non-empty set equipped with two binary operation called addition and Multiplication and denoted by $(+)$ and $(\cdot)$ respectively.
Then algebric structure $(R, +, \cdot)$ known as Ring.

→ Commutative Ring

$a, b \in R$
$a \cdot b = b \cdot a$
$a + b = b + a$

→ Ring with Unity

$a + e = e + a = a$
$a \cdot e = e \cdot a = a$

Ex. $M = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} / a, b \in R \right\}$ is not a ring under matrix Addition and Multiplication.

→ $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \in M$, $B = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \in M$

$C = \begin{bmatrix} e & 0 \\ f & 0 \end{bmatrix} \in M$

* Addition

(a) Closure

$\rightarrow$ $A, b \in M$

$$A+b = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ b+d & 0 \end{bmatrix} \in M$$

(b) Associative

$\rightarrow$ $(A+B)+C = A+(B+C)$:

$$L.H.S = (A+B)+C$$
$$= \left[\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \right] + \begin{pmatrix} e & 0 \\ f & 0 \end{pmatrix}$$
$$= \begin{bmatrix} a+c+e & 0 \\ b+d+f & 0 \end{bmatrix}$$

$$R.H.S = A+(B+C)$$
$$= \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} + \left[\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} + \begin{pmatrix} e & 0 \\ f & 0 \end{pmatrix} \right]$$
$$= \begin{bmatrix} a+c+e & 0 \\ b+d+f & 0 \end{bmatrix}$$

(c) Existance of Identity

$\rightarrow$ $A+e = A$

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$$

(d) Existance of Inverse

$\rightarrow$ $A + B = e$

$A + B = 0$

$B = -A$

$= \begin{bmatrix} -a & 0 \\ -b & 0 \end{bmatrix} \in M$

(e) Commutative

$\rightarrow$ $A + B = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$

$= \begin{bmatrix} a+c & 0 \\ b+d & 0 \end{bmatrix} = \begin{bmatrix} c+a & 0 \\ d+b & 0 \end{bmatrix}$

$= \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$

$= B + A$

* <u>Multiplication</u>

easy

$\rightarrow$ It is Closure, Associative, Existance of Identity, Existance of Inverse but not Commutative.

(e) Commutative

$B \cdot A = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} ca & 0 \\ da & 0 \end{bmatrix}$

$B \cdot A \neq A \cdot B$

$\rightarrow$ So, M is not Ring.

☆ Permutation

→ Suppose S is a non-empty set having n distinct elements and then one-one and onto function from S to S is called a permutation of degree n.

→ $f = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ f(a) & f(a_2) & f(a_3) & & f(a_n) \end{pmatrix}$

No. of element in set is known as degree of permutation.

Set of all such permutation is denoted by (P_n).

Ex. $S = \{1, 2, 3, 4\}$
$f(s) = 2s + 3$

→ $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 7 & 9 & 11 \end{pmatrix}$

☆ Permutation of 2 Product Sum

→ $f = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

$g = \begin{pmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$

→ $f \cdot g = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$

Ex. Find product of 2 permutation

$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

$\rightarrow$ $f \cdot g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$\rightarrow$ $g \cdot f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

☆ Identity of Permutation

$\rightarrow$ If I is a permutation of degree n such that I replace each element by the element it self.

Ex. $S = \{1, 2, 3, 4\}$

$I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Ex. $a = (3, 2, 1)$

$I = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

☆ Inverse of Permutation:

→ We know that every mapping which is one to one and onto is Inversible. Every permutation f on S is inversible.

$$f \cdot g = I$$

where $g = f^{-1}$.

Ex. $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ find inverse.

→ $f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix}$

$I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

→ $f \cdot f^{-1} = I$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & d & b & a \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

→ $a = 4$, $b = 3$, $c = 1$, $d = 2$

$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

Ex. Let $S = \{1, 2\}$ and S_2 be a permutation set of permutation on $S = \{P_1, P_2\}$. find S is abelian or not?

$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

→ (a) Closure

$\cdot$	P_1	P_2
P_1	P_1	P_2
P_2	P_2	P_1

- $P_1 \cdot P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = P_1$
- $P_1 \cdot P_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = P_2$
- $P_2 \cdot P_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = P_2$
- $P_2 \cdot P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = P_1$

So, its closure.

(b) Associative.

$P_1 \cdot P_2 = P_2 \cdot P_1$

It's associative.

(c) Existance of Identity

$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = I$

(d) Existance of inverse

$P \cdot P^{-1} = I$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$\rightarrow$ $P_2 \cdot P_2^{-1} = I$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$\therefore$

$$P_2^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Inverse exist.

(e) Commutative

$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Here, $P_1 \cdot P_2 = P_2 \cdot P_1$

$\rightarrow$ So, its an Abelian Group.

☆

(a) Closure Property → Groupoid

(b) Associative Property → Semi-Group

(c) Existance of Identity → Monoid

(d) Existance of Inverse → Group

(e) Commutative → Abelian Group

Chapter: 6 Counting

* Denumerable Set

→ An infinite set is said to be denumerable if its elements can be put in One-One correspondence with the set of natural number.

-10	10
-2	2
-1	1

$f(x) : A \to N$
where, $A = \{-\infty \text{ to } -1\}$

* Contable Set

→ A set which is either empty or finite or denumerable is called countable set.
$A = \{1, 2, 3, 4\}$ is countable $|A| = 4$

* Uncountable Set

→ A set which is denumerable is called uncountable set.

→ Set of Q (rational) number.

* Properties of countable and Uncontable sets.

-> A subset of countable is also countable

-> Union of countable set is countable.

-> Cartesian product of countable set is countable.

-> Union of countable collection of countable set is countable.

-> Super set of uncountable set is uncountable.

☆ Pigeon Hole Principle.

-> The pigeon hole principle states that if there are more pigeons than pigeon holes, then there must be at least one pigeonhole with at lest two pigeon in it.

-> Among any group of 31 peoples who have birthday in month of april atleast two people with same birthday.

→ If 11 shoes are selected from 10 pairs of shoes there must be a pair of match shoes along selected

→ In a group of office employee 13 clerks so that at least two of them must have birthday during same month.

☆ Extended Pigeon Hole Principle

→ If n pigeons are assign to m pigeon holes then one of the pigeon holes must be occupied by at least

n → pigeons
m → pigeon holes

$$\left[\frac{n-1}{m}\right] + 1$$

Ex. Prove that 7 colours are used to paint 50 bicycles, at least 8 will be same colour.

→ $n = 50$
$m = 7$

$$= \left[{}^{n-1}/_{m}\right] + 1$$

$$= \left[{}^{50-1}/_{7}\right] + 1$$

$$= 7 + 1$$
$$= 8$$

Ex. Find minimum number of students in a class be sure that three born in the same month.

→ $m = 12$

$$\left[\frac{n-1}{m}\right] + 1 = 3$$

$$\left(\frac{n-1}{12}\right) = 2$$

$n - 1 = 24$

$n = 25$

Ex. Prove that If seven distinct number from 1 to 12 choose then two of them will add upto 13.
Six different set such that sum of two number is 13.

→ $A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$, $A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$

$n = 7$.

$$= \left[\frac{n-1}{m}\right] + 1$$

$$= \left[\frac{7-1}{6}\right] + 1$$

$$= 2$$

☆ Permutation

→ An arrangment in sequence of element is called permutation.

Ex. 2 letter word from alphabate A, B and C.

→ We can select first alphabate from A, B and C in 3 ways,

A
B
C

→ We can select 2nd Alphabate in 2 ways

A B C } [2]
C B

B A C } [2]
C A

C A B } [2]
B A

→ Total 6 way is possible.

OR

$\boxed{\uparrow}\ \boxed{\uparrow}$

$3 \times 2 = 6$

→ 1st position can be filled by 3 alphabate.
2nd position can be filled by 2 alphabate.
Total = $3 \times 2 = 6$ ways

OR

→ $n = 3$, $r = 2$

$$^3P_2 = \frac{3!}{(3-2)!} = 3! = 6$$

☆ Theorem

→ The number of r permutation of a set with n distinct element.

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Ex. Find three letters word using only the given six letter without Repetetion.

→ $\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}$

$6 \times 5 \times 4 = 120$

Total $= 6 \times 5 \times 4 = 120$ way

[or]

→ $n = 6$, $r = 3$

$$^{6}P_{3} = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120 \text{ way}$$

Ex. Compute number of permutation of the set given $\{1, 2, 3, 4, 5\}$

→ 5 number can be fit by $5! = 120$

[or] $\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}$

$5 \times 4 \times 3 \times 2 \times 1 = 120$

→ Total $= 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

Ex. How many four digit number can be formed one of the digit 1, 2, 3, 4, 5, 6, 7, 8, 9.
(1) No repetition allowed
(2) How many of these will be greater than 3000.

→ (1) Out of 8 digit 4 digit number can be formed

8	7	6	5

= 1680

(2) 4 digit number formed must be greater than 3000.

Thousand	Hundread	Ten	Unit

→ first position can be filled by 3, 4, 5, 7, 8, 9 numers means 6 ways.
→ Second position can be filled by Remaining 7 way.
→ Third position can be filled by Remaining 6 ways.

→ Fourth position can be filled by remaining 5 ways.

Total way = $6 \times 7 \times 6 \times 5$ = 1260 way

Ex. (1) Suppose Repetition are not allowed how many four digits number can be formed from six digits 1, 2, 3, 5, 7, 8.

(2) How many such numbers are less than 4000?

(3) How many in (1) are even.

(4) How many in (1) are odd.

(5) How many in (1) contain both 3 and 5.

(6) How many in (1) are divisible by 10.

→ (1) Out of six numbers, 4 digit number can be formed in

$n = 6, \quad r = 4$

$$^6P_4 = \frac{6!}{2!} = 360 \text{ ways}$$

(2)

3	5	4	3

→ 1st position can be filled as 3 ways.

→ 2nd position can be filled as 5 digits.

→ 3rd position can be filled as 4 digits

→ 4th position can be filled as 3 digit.

Total ways $= 3 \times 5 \times 4 \times 3 = 180$ ways

(3) | 5 | 4 | 3 | 2 | = 120 ways

→ The number is even if last digit can be ending with 2 or 8.
So last digit can be choose at 2 ways.

→ First place can be filled remaining 5 way.
→ Second place can be filled remaining 4 way
→ Third place can be filled remaining 3 way

Total ways = $5 \times 4 \times 3 \times 2$ = 120 ways

(4) | 5 | 4 | 3 | 4 | = 240 ways

→ Number is odd if ending with odd digit.

last position can be filled by 4 way.
First position " " " by 5 way
Second " " " " by 4 way
Third " " " " by 3 way.

Total ways = $5 \times 4 \times 3 \times 4$ = 240 ways

(5) digit 3 can be occupy any one of 4 position.
→ Remaining 3 position occupid by digit 5.
→ So 2 position can be filled as 12 ways

→ Remaining two position can be filled by a number way = 12×12 = 144 ways

(6) From 1, 2, 3, 5, 7, 8 not a single number is divisible by 10 because there is no zero at unit place.

Ex. 2 Maths paper and 8 other papers can be arranged in examination Find total numbe of ways if
(1) Maths paper are consecutive
(2) Maths paper are not consecutive.

→ (1)

M_1	M_2					

→ Consider both M_1 and M_2 as a single paper.
→ Now 6 papers can be arranged in 6! ways and M_1 and M_2 arrang in 2! way

$$\begin{array}{r} 72 \\ 2 \\ \hline 144 \end{array}$$

Total way = $6!\,2! = 6\times5\times4\times3\times2\times2$
$= 1440$ ways

(2)

Ex. In how many permutation of the letters A B C D E F G H contain string ABC ?

→ Consider ABC as bolock.

$\boxed{A\ B\ C}$ D E F G H

→ Permutation of six object $= 6! \times 1!$
$= 720$ ways

Ex. How many multiple of 5 are there from 10 to 95 ?

| 9 | 2 | $= 18$ way

→ For multiple of 5 units place is either 0 or 5 last place can be filled by 2 ways

→ First place can be filled by 9 ways.

Total ways $= 9 \times 2 = 18$ ways

Ex. A bit is either 0 or 1. A byte is a sequence of 8 bits
find (1) Number of byte.
(2) Number of byte beigin and end with 11.

→ (1) Total byte $= 2^8$
$= 256$

→ (2)

1	1					1	1

→ Here first and last two digits are 1
→ Remaining 4 position can be filled with 0 or 1

Total way $= 1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1$
$= 16$ ways

Ex. How many auto license piolates can be made if each identified by 2 letter followed by 4 digits.

→

→ First two position can be choose in 26×26 ways.
→ Next four position can be choose from 0 to 9 digit. each position.

Total ways $= (26 \times 26) \times (10 \times 10 \times 10 \times 10)$
$= 6760000$ ways

Ex. In how many ways can trainer arrang 5 Lions and 4 tigers in a Row so that no two lines together.

-> | L_1 | T_1 | L_2 | T_2 | L_3 | T_3 | L_4 | T_4 | L_5 |

Lions can be arrange in 5! ways
Tigers can be arrange in 4! ways.

-> Total ways = 5! 4!
= 2880 ways

Ex. In how many ways letter ORGANISB can be arranged in such way that all vowels always come together.

-> There are 4 vowels [A, I, E, O]

A	E	I	O				

-> Consider 4 Vowels as a one object

-> So total number of letter = 5

-> 5 letters and 4 consonats can be arranged as 5! 4!

Total arrangment = 5! 4!
= 2880 ways

Ex. The number plates in pune start with PU, NA, NNNN where N represent number with ~~consider~~ condition the first N is not equal to zero Also last 4 N's cant be all equal to 0 and A Represent Alphabate. How many maximum number plate can be issued.

→

P	U	N	A	N	N	N	N
		9	26	10	10	10	10

→ 3 rd digit position N can be filled with 1 to 9 = 9 ways

→ 4th position A can be filled with 26 alphabate.

→ last 4 digit can be filled as $= (10 \times 10 \times 10 \times 10) - 1$
$= 9999$

→ Total number plate $= 9 \times 26 \times 9999$
$= 23339766$

Ex. Four friends go to city in which there are 10 hotles.

(1) In how many ways can they stay?

(2) In how many ways can they stay if no two friend stay together?

(3) In how many ways can they stay if aleast two friend stay together.

→ (1) Hotels = 10

Friends = 4

→ 1st, 2nd, 3rd, 4th friend can stay in 10 hotles.

Total way = $10 \times 10 \times 10 \times 10$ = 10,000 ways

→ (2)

→ 1st friend can stay in 10 hotles.

2nd " " " " 9 " .

3rd " " " " 8 " .

4th " " " " 7 " .

Total ways = $10 \times 9 \times 8 \times 7$ = 5040 ways

→ (3)

→ Total number of ways in which 4 friends can stay in 10 hotles = 10,000.

→ The number of ways in which no two friends stay together = 5040

→ So atleast two friend stay together = ~~501~~ 10,000 − 5040

= 4060

Ex. Consider the word ROTOR which how many ways you read it from left to right and right to left you get same word. Such word known as palindrom. find the maximum possible number of 5 letter palindromes.

->

26	26	26	↑	↑
			2nd	1st

-> 1st letter from left can be Choosen in 26 ways.

-> 2nd letter from left can be Choosen in 26 ways.

-> 3rd letter from left can be choosen in 26 ways.

-> Fourth and fifth letter is same as Second and first letter.

Total ways = $26 \times 26 \times 26$
= 17,576 ways

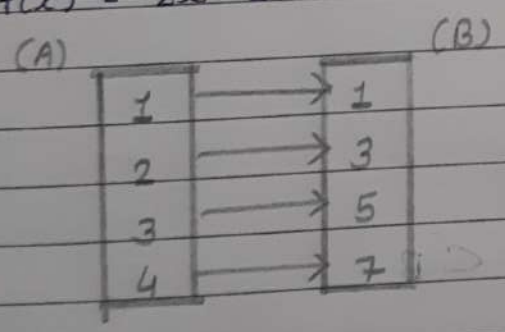
Functions

☆ Function

→ Let A and B two non empty sets then a function or mapping F from the set A to set B is called rule which assign to element $a \in A$ to unique element $b \in B$.

→ The notation that F maps a to b is,
$f(a) = b$ or $(a, b) \in f$

Ex. $f(x) = 2x - 1$



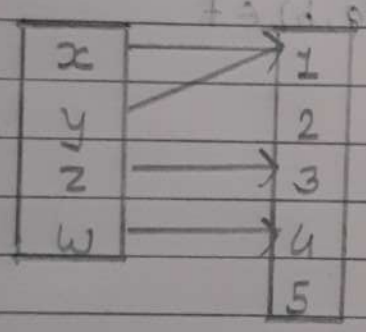
→ Domain (A) : $\{1, 2, 3, 4\}$
Co-Domain (B) : $\{1, 3, 5, 7\}$

☆ Range of Function

→ Let f be a function from set A to set B the element $y \in B$ which the function f associate to an element $x \in A$ is called image x or the value of function of x.

$$R_f = \{ y = f(x) \mid x \in A \}$$

Ex. $A = \{x, y, z, w\}$ $B = \{1, 2, 3, 4, 5\}$



→ Range $= \{1, 4, 3\} \subset B$

→ Range of f is a subset of codomain.

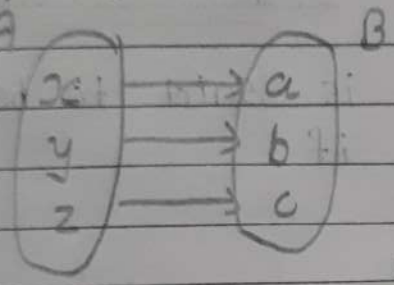
→ Element of A is called Pre Image.

→ Element of B is called Image

☆ Decide whether followings are function or not?

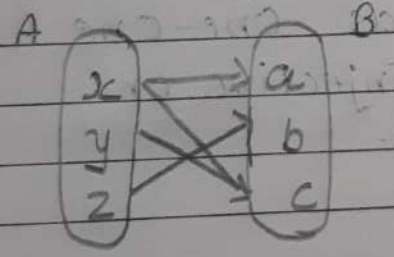
(1) $f = \{(x, a), (y, b), (z, c)\}$

→



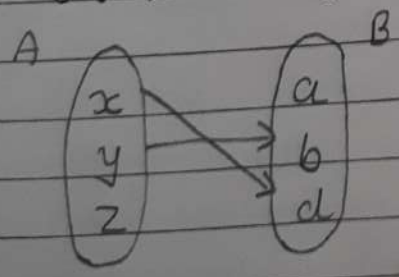
→ $f: A \to B$ is a function. |→ Range = $\{a, b, c\}$

(2) $f = \{(x, a), (y, c), (z, b), (x, c)\}$



→ f is not a function because x is connected with a and c.

(3) $f = \{(x, d), (y, b), (z,)\}$



→ f is not a function because (z) element of A is not connected to any element of B.

☆ Types of Function

* One-One Function (Injective Mapping)

→ Function $f: A \to B$ is said to be one-one mapping if

$\forall\ a_1, a_2 \in A,$

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

$\cancel{f(b_1) = f(b_2) = b_1 = b_2}$

$f(a_1) \neq f(a_2) \Rightarrow a_1 \neq a_2$

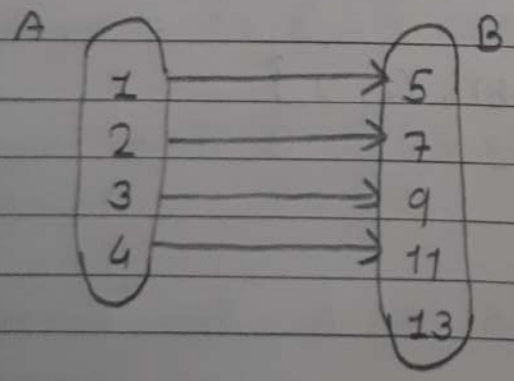
→ It is also known as One-One Correspondence or Injective or Equivalant function.

Ex. $A = \{1, 2, 3, 4\}$

$B = \{5, 7, 9, 11, 13\}$

$y = 2x + 3$

→

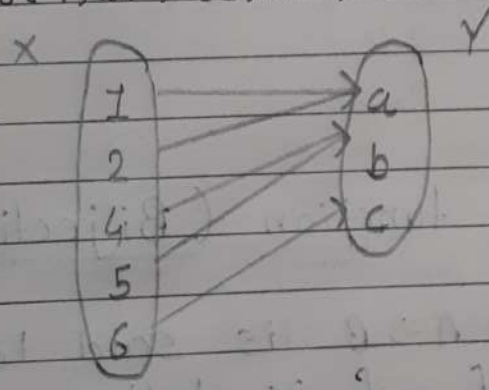


* Many to One function

→ Which relates two or more elements of domain set with single element of co-domain set is called Many to One function.

→ $f: x \to y$ such that two or more elements of set x are related to single element of set y.

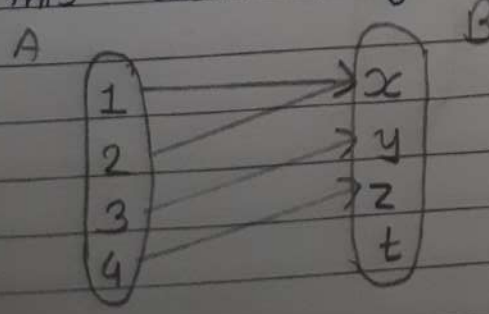
Ex. $f = \{(1,a), (2,a), (4,b), (5,b), (6,c)\}$



Domain = $\{1, 2, 4, 5, 6\}$
Range = $\{a, b, c\}$

* Into Function

→ If $f: A \to B$ then f is mapping from $A \to B$ is at least one element in B which is not associate of element A.
In this case Range of is subset of co-domain.

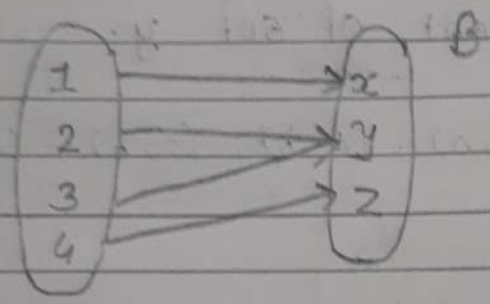


Co-domaing ≠ Range
Range ⊂ co-domain

* Onto Function (Surgective)

→ A function $f: A \to B$ is called onto function if every element of B is connected to any element of set A.

Ex. $A = \{1, 2, 3, 4\}$ $B = \{x, y, z\}$



* One One Onto function (Bijective)

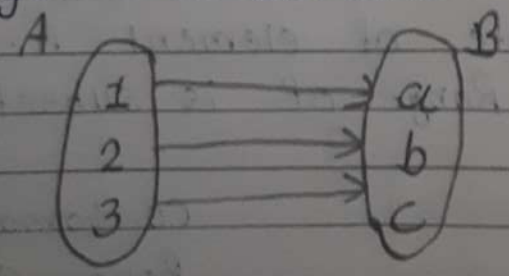
→ A function $f: A \to B$ is said to be bijective if f is bothe one-one and onto.

→ f is one-one onto if

(1) $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

(2) Every element of set B is connected to at least one element of set A.

→ Range = Co-domain

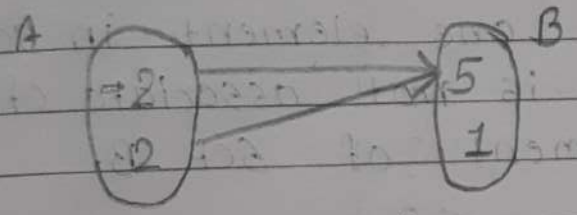


* Many One Into Mapping

→ If $f: A \to B$ is many one mapping and also f is into mapping then f is said to be Many one Into mapping.

(1) $f(a_1) = f(a_2)$ but $a_1 \neq a_2$

(2) $\exists$ at least one element in B which is not associated with any element of A.

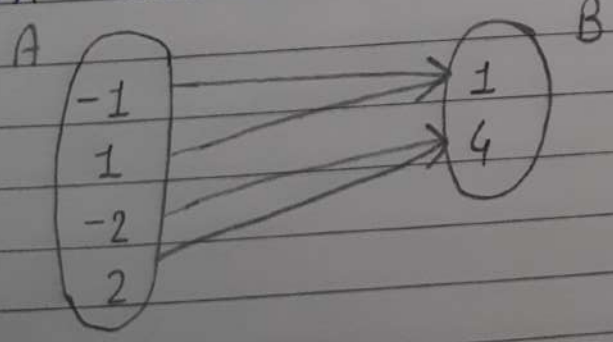


* Many One Onto function

→ If $f: A \to B$ is many one mapping and also f is Onto mapping then f is said to be Many One Onto function.

→ (1) $a_1 \neq a_2 \Rightarrow f(a_1) = f(a_2)$

(2) Every element of set B is associate with atleast one element of set A.

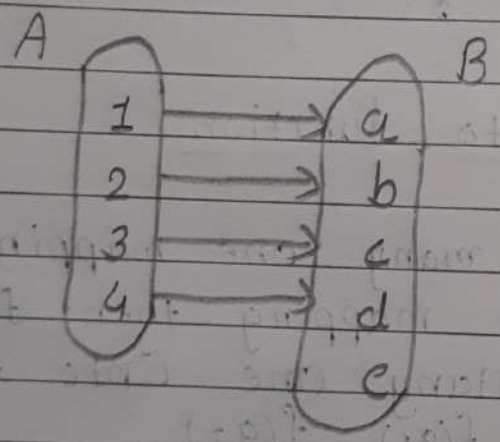


* One - One Into mapping

→ A mapping $f: A \to B$ is one one Into mapping If f is both one one and Into function.

→ (1) $f(a_1) \neq f(a_2) \Rightarrow a_1 \neq a_2$

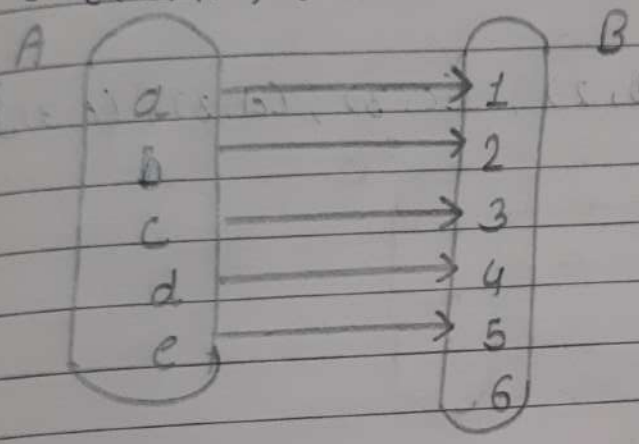
(2) At least one element in set B which is not associate of any element of set B.



☆ Examples

(1) Injective (one-one) but not Surjective (onto)

→ $f = \{(a,1), (b,2), (c,3), (d,4), (e,5)\}$



→ Injective : All the element from set A is associate with unique element of set B

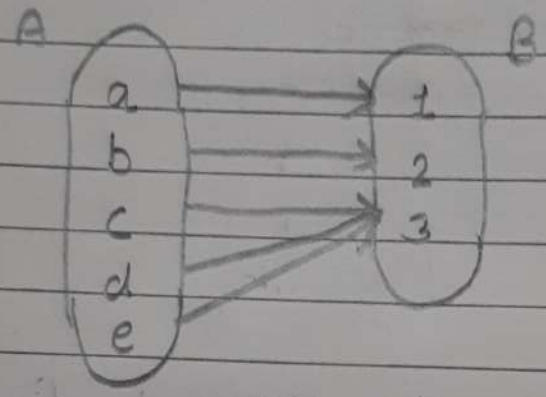
→ Surjective : Element 6 from set B is not associate with any element of set A so it's not surjective.

→ So, f is not surjective but Injective.

(2) Surgective but Not Injective.
(onto) (one-one)

→ $A = \{a, b, c, d, e\}$
$B = \{1, 2, 3\}$

$f = \{(a,1), (b,2), (c,3), (d,3), (e,3)\}$

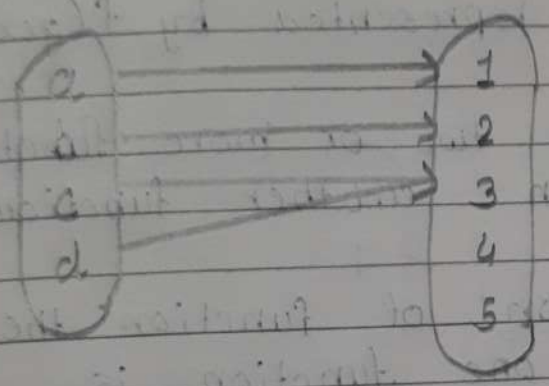


→ Injective : Here in Set A element c, d, e is connected to one element of set B. So it's not Injective.

→ Surgective : All the element of B is connected at least one element of A So it's Surgective.

(3) Neither Injective Nor Surgective.
(one-one) (onto).

→ $A = \{a, b, c, d\}$
$B = \{1, 2, 3, 4, 5\}$



→ Injective : Here in set of A element c and d is connected with same element of set B. so, it's not Injective.

→ Surgective : Here in set of B element 4 and 5 is not associated with any element of set A. so it's not surgective.

☆ Composition of Function

→ The composition function $f(x)$ and $g(x)$ where $g(x)$ acting first is represented by $f(g(x))$.

→ It combines two or more functions to result in another function.

→ In composition of function the output of one function is inside paranthesis and become input of outside function.

→ $fog(a) = f[g(a)]$

$gof(a) = g[f(a)]$

Ex. Let function f and g defined by $f(x) = 2x+1$, $g(x) = x^2 - 2$ find
(1) $fog(4)$ and $gof(4)$
(2) $fog(a+2)$ and $gof(a+2)$

→ $fog(a) = f[g(a)]$

$gof = f(2a+1)$

$= (2a+1)^2 - 2$

$= 2a^2 + 4a + 1 - 2$

$= 2a^2 + 4a - 1$

$gof(a) = g[f(a)]$

$= g[$

(a) → $fog(a) = f[g(a)]$
$= f[14]$
$= 28 + 1$
$= 29$

$gof(a) = g[f(a)]$
$= g[9]$
$= 81 - 2$
$= 79$

(b) → $fog(a+2) = f[g(a+2)]$
$= f[a^2 + 2a + 4 - 2]$
$= f[a^2 + 2a + 2]$
$= 2(a^2 + 2a + 2) + 1$
$= 2a^2 + 4a + 5$

→ $gof(a+2) = g[f(a+2)]$
$= g[2a + 5]$
$= 4a^2 + 20a + 25 - 2$
$= 4a^2 + 20a + 23$

Ex. Let $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$ for $x \in R$ where R is a set of Real number Find
(1) fog (2) gof (3) fof (4) hog (5) gogog
(6) foh (7) hof (8) fohog (9) hogof (2)

→ (1) $fog = f[g(x)]$
$= f[x - 2]$
$= x$

→ (2) $gof = g[f(x)]$
$= g[x + 2]$
$= x$

(3) $fof = f[f(x)]$
$= f[x + 2]$
$= x + 4$

→ (4) $hog = h[g(x)]$
$= h[x - 2]$
$= 3x - 6$

(5) $gogog = g[g(g(x))]$

$= g[g(x-2)]$

$= g[x-4]$

$= x-6$

(6) $foh = f[h(x)]$

$= f[3x]$

$= 3x+2$

(7) $hof = h[f(x)]$

$= h[x+2]$

$= 3x+6$

(8) $fohog = f[h(g(x))]$

$= f[h(x-2)]$

$= f[3x-6]$

$= 3x-4$

(9) $hogof(2) = h[g(f(2))]$

$= h[g(4)]$

$= h[2]$

$= 6$

Ex. Let $f(x)$ and $g(x) \rightarrow a$ are both even function prove that fog is also even function.

$\rightarrow$ given $f(-x) = f(x)$, $g(-x) = g(x)$.

$$\rightarrow fog(-x) = f[g(-x)]$$
$$= f[g(x)]$$
$$= fog(x)$$

So, fog is an even function.

Ex. Let $f(x)$ be any Real function show that $g_1(x) = f(x) + f(-x)/2$ is always an even function where as $g_2(x) = f(x) - f(-x)/2$ is always an odd function.

$\rightarrow g_1(x) = \dfrac{f(x) + f(-x)}{2}$

Let, $x = -x$

$$g_1(-x) = \frac{f(-x) + f(x)}{2}$$

$$g_1(-x) = g_1(x)$$

$\rightarrow$ So, $g_1(x)$ is always an even function

$\rightarrow g_2(x) = \dfrac{f(x) - f(-x)}{2}$

Let, $x = -x$

$$g_2(-x) = \frac{f(-x) - f(x)}{2}$$

$$= -\left[\frac{f(x) - f(-x)}{2}\right]$$

$$g_2(x) = -g_2(x)$$

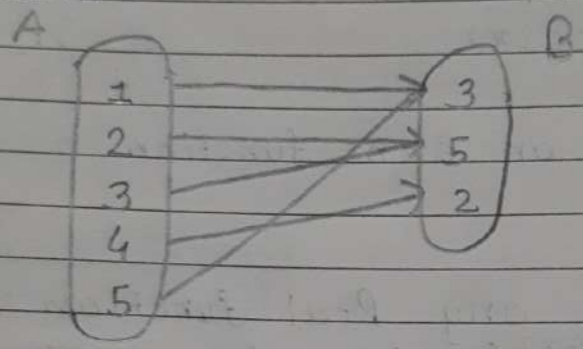
$\rightarrow$ So, $g_2(x)$ is an odd function.

Ex. Represent the given function in (1) Graphical (2) Tabular (3) Matrix.

$F = \{(1,3), (2,5), (3,5), (4,2), (5,3)\}$

→

(1) Function :



(2) Tabular :

f	1	2	3	4	5
1			✓		
2					✓
3					✓
4		✓			
5			✓		

(3) Matrix :

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Set Theory

☆ Types of Set

* **Finite Set :** A set which contains finite number of ~~set~~ element is known as finite set.

  $A = \{a, b, c, d\}$

* **Infinite set :** A set which contain infinite number of element called as Infinite set.

  $A = \{1, 2, 3, \ldots\}$

* **Singleton Set :** A set which only contain one element is known as singleton set.

  $A = \{1\}$

* **Pair Set :** A set which contains two element is called pair set.

  $A = \{(1, 2), (3, 4)\}$

* **Empty set :** A set which contain no elements is known as an empty set.

  $A = \{\ \}$

* **Subset** : Let A and B be two set if each element of set A is an element of set B then A is called subset of B.

$$A \subset B$$

Ex. $A = \{1, 3, 5\}$

$B = \{1, 2, 3, 4, 5, 6\}$

$\rightarrow A \subset B$

* **Proper Subset** : If a is subset of B and number of element in A and B are not same.

Ex. $A = \{a, b, c, d\}$

$B = \{x, y, z\}$

$C = \{a, b, c, d, x, y, z\}$

$\rightarrow A \subset C$

$B \subset C$

* **Equivalent Set** : Two sets A and B are said to be equivalent if they contains equal number of elements, which is also known as similar set and denoted by $A \approx B$

Ex. $A = \{1, 2, 3, 4, 5\}$

$B = \{a, b, c, d, e\}$

$\rightarrow |A| = |B| = 5$

$\rightarrow$ A and B are equivalent set.

* **Equal set** : Two sets A and B are said to equal if and only if every element of A is in B and B in A.

$A = B$ If $\{A \subseteq B$ and $B \subseteq A\}$
$A = B$ If $\{x \in A \rightarrow x \in B\}$

Ex. $A = \{1, 2, 3\}$
$B = \{3, 2, 1\}$

* **Power set** : If A be a set, the set of all subset of A is known as power set of A which is denoted by $P(A)$.

Ex. $A = \{a, b, c\}$
$\rightarrow P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

* **Universal set** : A set which is ~~subset~~ superset of all the set under consideration is known as universal set which is denoted by U.

Ex. $A = \{1, 2, 3\}$
$B = \{4, 5, 6\}$
$C = \{7, 8, 9\}$
$\rightarrow U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

☆ Operation On Set.

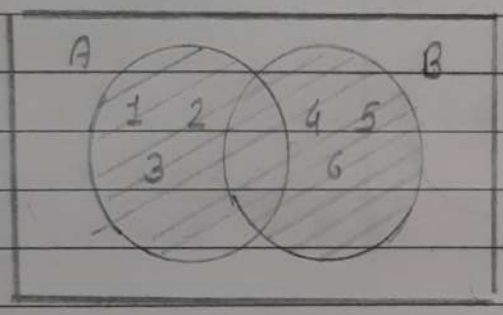
* **Union** : Let A and B be two sets then union $A \cup B$ is defined as set of all elements which are either in A or B or both.

$A \cup B = \{x / x \in A \text{ or } x \in B\}$

Ex. $A = \{1, 2, 3\}$

$B = \{3, 4, 5\}$

$\rightarrow A \cup B = \{1, 2, 3, 4, 5\}$

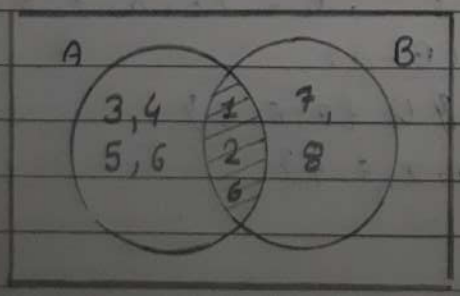


* **Interselection** : Let A and B be two sets then interselection A and B is defined as set of all those element which are common to both set.

$A \cap B = \{x / x \in A \text{ and } x \in B\}$

Ex. $A = \{1, 2, 3, 4, 5, 6\}$

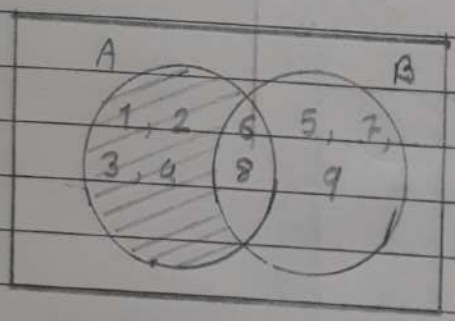
$B = \{1, 2, 6, 7, 8\}$



* **Difference** : If A and B are two sets then the difference (A − B) is defined as a set of all there elements which are not in B.

$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$

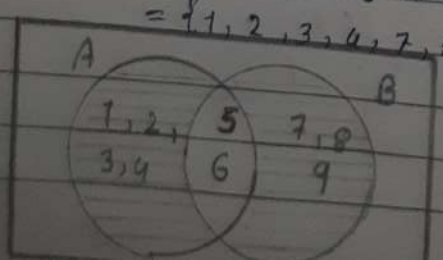
Ex. $A = \{1, 2, 3, 4, 6, 8\}$
$B = \{5, 6, 7, 8, 9\}$
$\rightarrow A - B = \{1, 2, 3, 4\}$



* **Symmetric Difference** : If A and B are two sets then the symmetric difference $(A \Delta B)$ or $(A \oplus B)$ is defined as a set of all those element which are either in A or B but not in common.

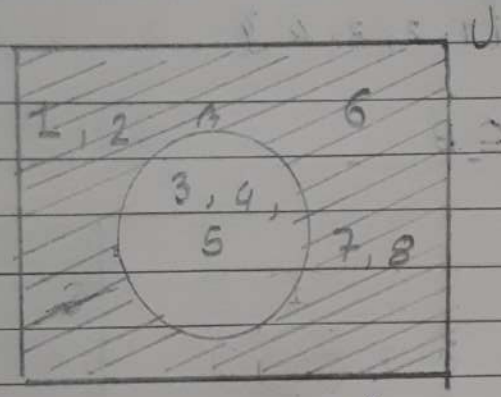
$A \oplus B = (A - B) \cup (B - A)$

Ex. $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{5, 6, 7, 8, 9\}$
$\rightarrow A \oplus B = (A - B) \cup (B - A)$
$= \{1, 2, 3, 4\} \cup \{7, 8, 9\}$
$= \{1, 2, 3, 4, 7, 8, 9\}$



* **Complement of Set:** If A is a set then complement of A is $\bar{A}$, A' and is defined as a set of all those element which are in Universal set but not in A.

$A' = \{x \mid x \in A \text{ and } x \notin A\}$

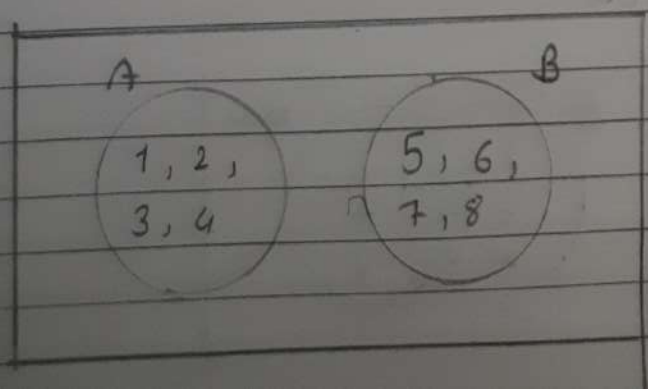


$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$A = \{3, 4, 5\}$
$A' = \{1, 2, 6, 7, 8\}$

* **Disjoint set:** Two set A and B are called disjoint if both sets have no common element.

$$A \cap B = \emptyset$$

Ex. $A = \{1, 2, 3, 4\}$
$B = \{5, 6, 7, 8\}$
$\rightarrow A \cap B = \emptyset$

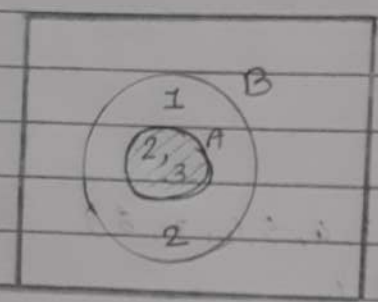


Ex. Let, A and B be sets such that $(A \cup B) \subseteq B$ and $B \nsubseteq A$ Draw ven diagram.

→ $A = \{2, 3\}$
$B = \{1, 2, 3, 4\}$

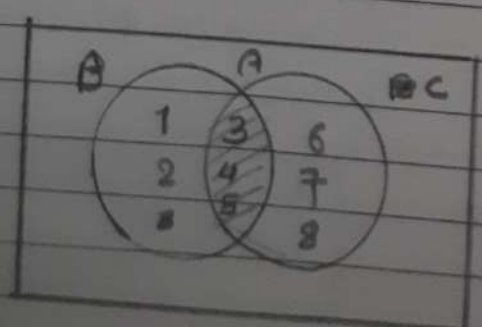
$A \cup B = \{1, 2, 3, 4\}$

$(A \cup B) \subseteq B$



Ex. Let, A, B and C b such that $A \subseteq B$, $A \subseteq C$, $(B \cap C) \subseteq A$ and $A \subseteq (B \cap C)$ Draw diagram.

→ $A = \{3, 4, 5\}$
$B = \{1, 2, 3, 4, 5\}$
$C = \{3, 4, 5, 6, 7, 8\}$
- $A \subseteq B$, $A \subseteq C$
$B \cap C = \{3, 4, 5\}$
- $A \subseteq (B \cap C)$



Ex. Let, A, B and C be sets such that $(A \cap B \cap C) = \emptyset$, $(A \cap B) \neq \emptyset$, $(A \cap C) \neq \emptyset$ $(B \cap C) \neq \emptyset$ Draw corresponding diagram.

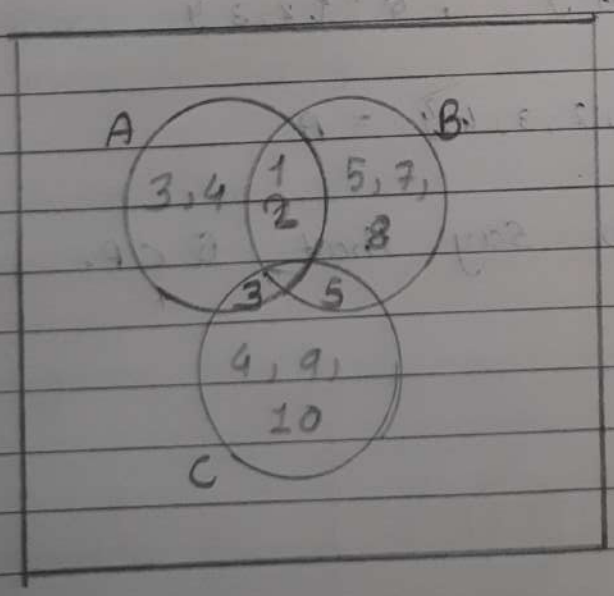
→ $A = \{1, 2, 3, 4\}$
$B = \{1, 2, 5, 7, 8\}$
$C = \{3, 4, 5, 9, 10\}$

$A \cap B = \{1, 2\}$

$B \cap C = \{5\}$

$A \cap C = \{3, 4\}$

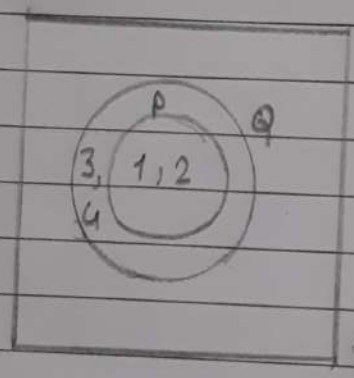
$A \cap B \cap C = \{\emptyset\}$



Ex. What can you say about sets P and Q if, (i) $P \cap Q = P'$ (ii) $P \cup Q = P'$

→ (1) $P = \{1, 2\}$, $Q = \{1, 2, 3, 4\}$

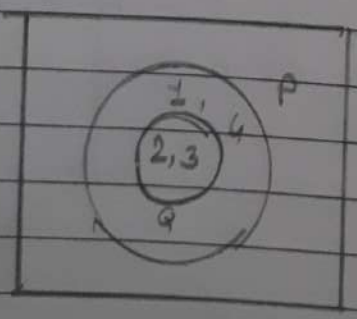
$P \cap Q = \{1, 2\}$
$P \cap Q = P'$
$P \subset Q$.



(2) $P = \{1, 2, 3, 4\}$, $Q = \{2, 3\}$

$P \cup Q = \{1, 2, 3, 4\} = P$

So, we can say that $Q \subset P$.



☆ Some Basic Theorem

(a) Commutative law

→ $A \cup B = B \cup A$

$A \cap B = B \cap A$

(b) Associative law

→ $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

(c) Idempotent law

→ $A \cup A = A$

$A \cap A = A$

(d) Distributive law

→ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

* Absorption law : $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

* De Morgan's law : $(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

* Complement law : $A \cup A' = U$

$A \cap A' = \emptyset$

* Inovolution law : $(A')' = A$

☆ Prove that

(a) $A \cup A = A$

→ Let, $x \in (A \cup A) = x \in A$ or $x \in A$

$\Rightarrow x \in A$

$\Rightarrow A \cup A \subset A$ — ①

→ Let, $y \in (A) \Rightarrow y \in A$ or $y \in A$

$\Rightarrow y \in (A \cup A)$

$\Rightarrow A \subset (A \cup A)$ — ②

→ From eq. ① and ②,

$A \cup A = A$

(b) $A \cap A = A$

→ Let, $x \in (A \cap B) = x \in A$ and $x \in A$

$\Rightarrow x \in A$

$\Rightarrow (A \cap B) \subset A$ — ①

→ Let, $y \in A = y \in A$ and $y \in A$

$\Rightarrow y \in (A \cap A)$

$\Rightarrow A \subset (A \cap A)$ — ②

→ From eq. ① and ②,

$(A \cap A) = A$

$\cup \rightarrow$ OR

$\cap \rightarrow$ And

(3) $A \cup B = B \cup A$

$\rightarrow$ Let, $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in (A \cup B)$

$\Rightarrow (A \cup B) \subset (B \cup A)$ — ①

$\rightarrow$ Let, $y \in B \cup A \Rightarrow x \in B$ or $y \in A$

$\Rightarrow y \in A$ or $y \in B$

$\Rightarrow y \in (B \cup A)$

$\Rightarrow (B \cup A) \subset (A \cup B)$ — ②

$\rightarrow$ From eq. ① and ②,

$A \cup B = B \cup A$

(4) $A \cup (B \cup C) = (A \cup B) \cup C$

$\rightarrow$ Let, $x \in [A \cup (B \cup C)] =$

$x \in A$ or $(x \in B$ or $x \in C)$

$(x \in A$ or $x \in B)$ or $x \in C$

$(x \in A$ or $x \in B) \cup C$

$x \in (A \cup B) \cup C$

$(A \cup B) \cup C \subset A \cup (B \cup C)$ — ①

$\rightarrow$ Let, $y \in [(A \cup B) \cup C] = y \in (A \cup B)$ or $y \in C$

$= (y \in A$ or $y \in B)$ or $y \in C$

$= y \in A$ or $(y \in B$ or $y \in C$

$= y \in A$ or $(B \cup C)$

$= y \in A \cup (B \cup C)$

$= A \cup (B \cup C) \subset (A \cup B) \cup C$ — ②

$\rightarrow$ From eq. ① and ②,

$A \cup (B \cup C) = (A \cup B) \cup C$

(6) Demorgan's law

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

(i) → Let, $x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in (A' \cap B')$

$\Rightarrow (A' \cap B') \subset (A \cup B)'$ — ①

→ Let, $y \in (A' \cap B') \Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

$\Rightarrow (A \cup B)' \subset (A' \cap B')$ — ②

→ From eq. ① and ②,

$(A \cup B)' = A' \cap B'$

(ii) → Let, $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in (A' \cup B')$

$\Rightarrow (A' \cup B') \subset (A \cap B)'$ — ①

→ Let, $y \in (A' \cup B')$,

$\Rightarrow y \in (A' \cup B')$

$\Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

$\Rightarrow (A \cap B)' \subseteq (A' \cup B')$ — ②

→ From eq. ① and ②,

$(A \cap B)' = A' \cup B'$

Ex. In GGH conference at valsad 500 deligates attend 200 of them could take tea, 350 could take coffe and 10 didn't take either tea or coffe.

(1) How many take both.

(2) How many take tea only.

(3) How many take coffe only.

→ $|U|$ = Total deligates = 500

$|T|$ = Who take tea = 200

$|C|$ = Who take coffe = 350

→ $|T \cup C|'$ = Delegates who didn't take either tea or coffe = 10

$|T \cup C|$ = Deligates who take either tea or coffe = 500 - 10 = 490

→ (1) No. of deligates who take both

$|T \cup C| = |T| + |C| - |T \cap C|$

$490 = 200 + 350 - |T \cap C|$

$|T \cap C| = 200 + 350 - 490$

$|T \cap C| = 60$

→ (2) No. of deligates who take tea only

$|T| - |T \cap C| = 200 - 60 = 140$

→ (3) No. of deligates who take coffe only

$|C| - |T \cap C| = 350 - 60 = 290$

Ex. Consider set of Integer from 1 to 250. Find how many of those numbers are divisible by 3 or 5 or 7? And also indicates how many are divisible by 3 or 5 but not 7?

→ $|A| = [250/3] = 83$

$|B| = [250/5] = 50$

$|C| = [250/7] = 35$

$|A \cap B| = [250/3 \times 5] = 16$

$|A \cap C| = [250/3 \times 7] = 11$

$|B \cap C| = [250/5 \times 7] = 7$

$|A \cap B \cap C| = [250/3 \times 5 \times 7] = 1$

→ How many are divisible by 3 or 5 or 7.

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 83 + 50 + 35 - 16 - 7 - 11 + 1$

$= 135$

→ How many are divisible by 3 or 5 but not 7,

$= |A \cup B \cup C| - |C|$

$= 135 - 35$

$= 100$

Ex: Among 100 students 32 study Maths, 20 study physics, 45 study bio. 15 study Maths and Bio, 7 study Maths and physics, 10 study physics and Bio. and 30 do not study any of three subject. Find how many numbers of student studying all three subject

→ $|U| = 100$

$|A| =$ Maths $= 32$

$|B| =$ Physics $= 20$

$|C| =$ Bio $= 45$

→ $|A \cap B| = 15$

$|B \cap C| = 10$

$|A \cap B| = 7$

→ $|A' \cap B' \cap C'| =$ Do not study any subject $= 30$

→ $A \cup B \cup C = 100 - |A' \cap B' \cap C'|$

$= 100 - 30$

$= 70$

→ Studying all three subject

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$70 = 32 + 20 + 45 - 15 - 7 - 10 + |A \cap B \cap C|$

$70 = 65 + |A \cap B \cap C|$

$|A \cap B \cap C| = 5$

→ Student Studying only Maths

$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$
$= 32 - 7 - 16 + 5$
$= 15$

→ Student who only study Physics

$= |B| - |A \cap B| + |A \cap B \cap C|$
$= 20 - 7 - 10 + 5$
$= 8$

→ Student who only study biology

$= |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$= 45 - 15 - 10 + 5$
$= 25$

→ Student who only study exacty one subject $= 15 + 8 + 25$
$= 48$

Ex. The 60,000 fans are arranged in home coming football game bought up all the paraphernalia for their cars.
All together 20,000 bumper sticker 36,000 windows dcard and 12,000 key rings were sold.
We know that 52,000 fan bought at least one item and no ~~more~~ one bought more than one of given iteam.
Also 6,000 fan bought bothe decard ~~or~~ and key rings. 9,000 bough for decards and bumper sticker. and 5,000 bought for key ring and bumber sticker.
(1) How many fan bought all item
(2) How many fan bought exuctly one item?
(3) Someone questioned the accuracy of the total number of purehose 52,000 this person claimed total number of purchase either 60,000 or 44,000

→ $|U| = 60,000$

$|A|$ = who bought bumper sticker = 20,000
$|B|$ = who bought decards = 36,000
$|C|$ = who bought key rings = 12,000

→ $|A \cup B \cup C| = 52,000$
$|B \cap C| = 6,000$
$|A \cap B| = 9,000$
$|A \cap C| = 5,000$

→ (1) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$52{,}000 = 20{,}000 + 36{,}000 + 12{,}000 - 9000 - 6000 - 5000 + |A \cap B \cap C|$

$52{,}000 = 48000 + |A \cap B \cap C|$

$|A \cap B \cap C| = 4000$

→ (2) Number of fan bought only bumper sticker

$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 20{,}000 - 9000 - 5000 + 4000$

$= 10{,}000$

- Number of fan bought only decards

$= |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$

$= 36{,}000 - 9{,}000 - 6{,}000 + 4000$

$= 29{,}000$

- Number of fan bought only key rings

$= |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 12{,}000 - 5{,}000 - 6{,}000 + 4{,}000$

$= 5{,}000$

→ Numbers of fan bought exactly one item

$= 10000 + 29000 + 5000$

$= 44{,}000$

→ (3) If $|A \cup B \cup C| = 60,000$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$60,000 = 48000 + |A \cap B \cap C|$

$|A \cap B \cap C| = 12,000$

Which is not possible, because, $A \cap B \cap C$ is always less than $A \cap B$, $B \cap C$, $A \cap C$

→ If $A \cup B \cup C = 40,000$

$|A \cup B \cup C| = 48,000 + |A \cap B \cap C|$

$40,000 = 48,000 + |A \cap B \cap C|$

$|A \cap B \cap C| = -4,000$

Which is not possible.

Recurrence Relation

→ For the sequence $\{a_n\}_{n=0}^{\infty}$ is an equation or formula that express as in terms of one or more of the previous terms $a_0, a_n, a_{n-1}, \ldots$ for any integer $n \geq 1$ it is also known as recurrence relation.

Ex. Fibonaci Series : $a_n = a_{n-2} + a_{n-1}$

$\boxed{\begin{matrix} a_0 = 0 \\ a_1 = 1 \end{matrix}}$

$n = 2, \quad a_2 = a_0 + a_1 = 1$

$n = 3, \quad a_3 = a_1 + a_2 = 2$

$n = 4, \quad a_4 = a_2 + a_3 = 3$

Ex. Exponential form: $a_n = 2^{n-1}$

$a_n = 2a_{n-1}$ $\boxed{a_1 = 1}$

$n = 2, \quad a_2 = 2a_1 = 2$

$n = 3, \quad a_3 = 2a_2 = 4$

$n = 4, \quad a_4 = 2a_3 = 8$

$n = 5, \quad a_5 = 2a_4 = 12$

☆ Degree of Recurrence Relation

→ The highest power of (a_n) or (f_x) is called degree of the recurrence relation

(i) $y_n^4 - 5y_{n-1}^3 + 6y_n = f(n)$ → Degree = 4

(ii) $a_{k+3}^3 + 3y_{k+2}^2 + 2y_{k+1} = 0$ → Degree = 3

(iii) $y_{k+3} - 2y_{k+2} + 2y_{k+1} = f(k)$ → Degree = 1

☆ Order of Reurrence Relation

→ The order of recurrence relation is defined by the difference between the highest and lowest subscript.

(i) $a_n = a_{n-2} + a_{n-1}$

→ Order of recurrence relation $= n-(n-2)$
$= 2$

(ii) $f(x+3) - 3f(x+2) + 4f(x)$

→ order $= (x+3) - x$
$= 3$

(iii) $a_n^3 = -a_n^3 - 1$

→ order $= n(n-1)$
$= 1$

☆ Homogeneous Equation

→ The linea recurrence relation is called liniear homogeneous of order K if $f(n) = 0$.

$$a_{n-2} + a_{n-1} = 0$$

→ $f(n) \neq 0$ is called as linear non-homogeneous recurrence ralation

* Case : 1 → When all roots are real and distinct.

→ Let, $\alpha_1, \alpha_2, \alpha_3$ are roots of equation then

$$(a_n)^h = a_r = A_1(\alpha_1)^n + A_2(\alpha_2)^n + A_3(\alpha_3)^n$$

Ex. $\alpha^2 - 5\alpha + 6 = 0$

→ $(\alpha - 2)(\alpha - 3) = 0$

$\alpha = 2, 3$

→ $(a_n)^h = a_r = A_1(2)^n + A_2(3)^n$

Ex. $\alpha(\alpha + 2)(\alpha - 5) = 0$

→ $\alpha = 0, -2, 5$

→ $(a_n)^h = a_r = A_1(0)^n + A_2(-2)^n + A_3(5)^n$

Ex. $a_n - 4a_{n-1} + 3a_{n-2} = 0$

→ $\alpha^2 - 4\alpha + 3 = 0$

$(\alpha - 1)(\alpha - 3) = 0$

$\alpha = 1, 3$

→ $(a_n)^h = a_r = A_1(1)^n + A_2(3)^n$.

* Case : 2 → When all roots are real and same

→ Let, α, α, α

$(a_n)^h = a_r = (A_1 n^0 + A_2 n^1 + A_3 n^2 + A_4 n^4)(\alpha)^n$

Ex $\alpha = 2, 2, 2$

→ $(a_n)^h = a_r = (A_1 + A_2 n + A_3 n^2)(2)^n$

Ex. $a_n - 6a_{n-1} + 9a_{n-2} = 0$

→ $\alpha^2 - 6\alpha + 9 = 0$

$(\alpha - 3)^2 = 0$

$\alpha = 3, 3$

→ $(a_n)^h = a_r = (A_1 + A_2 n)(3)^n$

* **Case : 3** → When roots are imaginary and distinct

→ Let, $\alpha_1 = a \pm ib$

$= a+ib, \quad a-ib$

$(a_n)^h = A_1 (a+ib)^n + A_2 (a-ib)^n$

Ex. $\alpha = 3 \pm 2i$

→ $(a_n)^h = A_1 (3+2i)^n + A_2 (3-2i)^n$

Ex. $\alpha = -5 \pm 3i$

→ $(a_n)^h = A_1 (-5+3i)^n + A_2 (-5-3i)^n$

Ex. $a_n - 4a_{n-1} + 5a_{n-2} = 0$

→ $\alpha^2 - 4\alpha + 5 = 0$

$\alpha^2 - 4\alpha = -5$

$\alpha^2 - 4\alpha + 4 = 4 - 5$

$(\alpha - 2)^2 = -1$

$(\alpha - 2)^2 = (i)^2$

$\alpha - 2 = \pm i$

$\alpha = i \pm 2$

$L.S = \frac{(M.S)^2}{4 \times F.S}$

$= \frac{(-4\alpha)^2}{4 \times \alpha^2}$

$= \frac{16\alpha^2}{4\alpha^2}$

$= 4$

$(a_n)^h = A_1 (2+i)^n + A_2 (2-i)^n$

* **Case : 4** → When roots are imaginary and Repeated.

→ $\alpha_1 = \alpha_2 = a + ib$

$\alpha_3 = \alpha_4 = a - ib$

→ $a \pm ib,\ a \pm ib$

$= a+ib,\ a-ib,\ a+ib,\ a-ib$

→ $(a_n)^h = (A_1 + A_2 n)(a+ib)^n + (A_3 + A_4 n)(a-ib)^n$

Ex. $\alpha = -3 \pm 2i,\ -3 \pm 2i,\ -3 \pm 2i$

→ $(a_n)^h = (A_0 + A_1 n + A_2 n^2)(-3+2i)^n + (A_3 + A_4 n + A_5 n^2)(-3-2i)^n$

Ex. $\alpha = 1/2 \pm 3/2\, i,\ 1/2 \pm 3/2\, i$

→ $(a_n)^h = (A_1 + A_2 n)(1/2 + 3/2\, i)^n + (A_3 + A_4 n)(1/2 - 3/2\, i)^n$

☆ Examples

(1) $a_n + 9a_{n-2} = 0$

$\rightarrow$ $\alpha^2 + 9 = 0$

$\alpha^2 = -9$

$\alpha = 9i^2$

$\alpha = \pm 3i \quad = 3i, -3i$

$\rightarrow$ $(a_n)^n = A_1 (3i)^n + A_2 (-3i)^n$

(2) $a_n - 5a_{n-1} + 6a_{n-2} = 0$ | where, $a_0 = 2$, $a_1 = 5$

$\rightarrow$ $\alpha^2 - 5\alpha + 6 = 0$

$(\alpha - 2)(\alpha - 3) = 0$

$\alpha = 2, 3$

$\rightarrow$ $a_n = A_1 (2)^n + A_2 (3)^n$

Let, $n=0$ $\quad a_0 = A_1 (2)^0 + A_2 (3)^0$

$2 = A_1 + A_2 \quad$ — (1)

Let, $n=1$ $\quad a_1 = A_1 (2) + A_2 (3)$

$5 = 2A_1 + 3A_2 \quad$ — (2)

$\rightarrow$ Solving equation (1) and (2),
Multiply eq. (1) by 2,

$$\begin{aligned} 2A_1 + 2A_2 &= 4 \\ 2A_1 + 3A_2 &= 5 \\ \hline -A_2 &= -1 \\ A_2 &= 1 \end{aligned}$$

Put value of A_2 in eq. (1)

$2 = A_1 + A_2$

$A_1 = 1$

$\rightarrow$ $a_n = (2)^n + (3)^n$

Ex. Solve $d_n = 2d_{n-1} - d_{n-2}$,
where, $d_1 = 1.5$, $d_2 = 3$

$\rightarrow$ $d_n - 2d_{n-1} + d_{n-2} = 0$

$\alpha^2 - 2\alpha + 1 = 0$

$(\alpha - 1)^2 = 0$

$\alpha = 1, 1$

$d_n = (A_1 + A_2 n)(1)^n$

$n = 1$ $\quad d_1 = (A_1 + A_2 n)(1)^1$

$1.5 = (A_1 + A_2)$ — ①

$n = 2$ $\quad d_2 = (A_1 + A_2(2))(1)^2$

$3 = A_1 + 2A_2$ — ②

$\rightarrow$ Solving eq. ① and ②,

$$\begin{aligned} A_1 + A_2 &= 1.5 \\ A_1 + 2A_2 &= 3 \\ - \quad - \quad & \quad - \\ \hline -A_2 &= -1.5 \\ A_2 &= 1.5 \end{aligned}$$

$\rightarrow$ Put value of A_2 in eq. ①,

$A_1 + A_2 = 1.5$

$A_1 + 1.5 = 1.5$

$A_1 = 0$

$\rightarrow$ $d_n = (0 + (1.5)A_2)(1)^n$

$= (1.5)n(1)^n$.

Ex. $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$

$\rightarrow \quad \alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$

✓ $\alpha = -2 \quad -8 + 24 - 24 + 8 = 0$

$\alpha = 2 \quad 8 + 24 + 24 + 8 = 64$ [not possible]

$\rightarrow \quad \alpha + 2$

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$
$$\alpha^3 + 2\alpha^2 + 4\alpha^2 + 8\alpha + 4\alpha + 8 = 0$$
$$\alpha^2(\alpha+2) + 4\alpha(\alpha+2) + 4(\alpha+2) = 0$$
$$(\alpha+2)(\alpha^2+4\alpha+4) = 0$$
$$(\alpha+2)(\alpha+2)^2 = 0$$
$$\alpha = -2, -2, -2$$

$$a_n = (A_1 + A_2 n + A_3 n^2)(-2)^n$$

Ex. Solve, $S_k = S_{k-1} + S_{k-2}$

$\rightarrow \quad \alpha^2 - \alpha - 1 = 0$

$$\alpha^2 - \alpha = 1$$
$$\alpha^2 - \alpha + \frac{1}{4} = 1 + \frac{1}{4}$$
$$\alpha^2 - \alpha + 1/4 = 5/4$$
$$(\alpha - 1/2)^2 = 5/4$$
$$(\alpha - 1/2) = \pm\sqrt{5}/2$$
$$\alpha = 1/2 \pm \sqrt{5}/2$$
$$= \frac{1}{2} + \frac{\sqrt{5}}{2}, \quad \frac{1}{2} - \frac{\sqrt{5}}{2}$$

$$L.S = \frac{(M.S)^2}{4 \times F.S}$$
$$= \frac{(-\alpha)^2}{4 \times \alpha^2}$$
$$= 1/4$$

$\rightarrow \quad S_k = A_1\left(1 - \sqrt{5}/2\right)^n + A_2\left(1 + \sqrt{5}/2\right)^n$

☆ Non-Homogeneous Equation

→ $a_n = (a_n)^h + (a_n)^P$ [$f(n)$ = constant]

* Case : 1 → When all roots are real and distincts.

Ex. Solve, $a_n - 5a_{n-1} + 6a_{n-2} = 9$

→ $\alpha^2 - 5\alpha + 6 = 9$

⇒ Homogeneous equation : $\alpha^2 - 5\alpha + 6 = 0$

$(\alpha - 2)(\alpha - 3) = 0$

$\alpha = 2, 3$

$(a_n)^h = A_1(2)^n + A_2(3)^n$

⇒ Let, $(a_n)^P = 9A$ — ①

$(a_{n-1})^P = 9A$

$(a_{n-2})^P = 9A$

→ Now, equation can be written as,

$(a_n)^P - 5(a_{n-1})^P + 6(a_{n-2})^P = 9$

$9A - 5(9A) + 6(9A) = 9$

$9A - 45A + 54A = 9$

$9A + 9A = 9$

$A = 9/18$

$A = 1/2$ — ②

Put value of A in eq. ①,

$(a_n)^P = 9(1/2) = 9/2$

→ $a_n = (a_n)^h + (a_n)^P$

$a_n = A_1(2)^n + A_2(3)^n + 9/2$

* Case : 2

$$f(n) = (\text{constant})^n$$

Ex. Solve, $a_n = 2a_{n-1} + 3a_{n-2} + 5^n$

$\rightarrow$ $a_n - 2a_{n-1} - 3a_{n-2} = 5^n$

$\alpha^2 - 2\alpha - 3 = 5^n$

$\Rightarrow$ Homogeneous equation

$\alpha^2 - 2\alpha - 3 = 0$

$(\alpha - 3)(\alpha + 1) = 0$

$\alpha = -1, 3$

$(a_n)^h = A_1(-1)^n + A_2(3)^n$

$\Rightarrow$ Let, $(a_n)^p = A(5)^n$

$(a_{n-1})^p = A(5)^{n-1}$

$(a_{n-2})^p = A(5)^{n-2}$

Equation can be written as,

$(a_n)^p - 2(a_{n-1})^p - 3(a_{n-2})^p = 5^n$

$A5^n - 2[A(5)^{n-1}] - 3[A(5)^{n-2}] = 5^n$

$A5^n - 2A\frac{5^n}{5} - 3A\frac{5^n}{25} = 5^n$

$A5^n[1 - 2/5 - 3/25] = 5^n$

$A \cdot 5^n (12/25) = 5^n$

$A = 25/12$

$(a_n)^p = \frac{25}{12}(5)^n$

$\rightarrow$ $a_n = [A_1(-1)^n + A_2(3)^n] + [\frac{25}{12}(5)^n]$

Ex. Solve $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 2^n$
where, $a_0 = 0$, $a_1 = 0$, $a_2 = 2$

$\rightarrow$ $\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 2^n$

$\Rightarrow$ Homogeneous Equation,

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$
$$(\alpha+2)^3 = 0$$
$$\alpha = -2, -2, -2$$

$$(a_n)^h = (A_1 + A_2 n + A_3 n)(-2)^n$$

$\rightarrow$ Let, $(a_n)^p = A2^n$
$(a_{n-1})^p = A2^{n-1}$
$(a_{n-2})^p = A2^{n-2}$
$(a_{n-3}) = A2^{n-3}$

$\rightarrow$ Equation can be written as,

$$(a_n)^p + 6(a_{n-1})^p + 12(a_{n-2})^p + 8(a_{n-3})^p = 2^n$$
$$A2^n + 6(A2^{n-1}) + 12(2^{n-2}) + 8(2^{n-3}) = 2^n$$
$$A2^n + 6\frac{A2^n}{2} + 12\frac{A2^n}{4} + 8\frac{A2^n}{8} = 2^n$$

$$A2^n + A3 \cdot 2^n + A3(2)^n + A(2)^n = 2^n$$
$$A2^n(1+3+3+1) = 2^n$$
$$A\,8 = 1$$
$$A = 1/8$$

$$(a_n)^p = \frac{1}{8}(2)^n$$

$\rightarrow$ $a_n = \left[(A_1 + A_2 n + A_3 n^2)(-2)^n\right] + \left[\frac{1}{8}(2)^n\right]$

$a_0 = 0$

$\rightarrow n = 0$ $\quad a_0 = (A_1 + 0 + 0)(-2)^n + \frac{1}{8}$

$0 = A_1(1) + \frac{1}{8}$

$A_1 = -1/8$ —— ①

$\rightarrow a_1 = 0$

$n = 1,$ $\quad a_1 = (A_1 + A_2 + A_3)(-2) + \frac{1}{8}(2)$

$0 = [A_1 + A_2 + A_3](-2) + 1/4$

$= \left(\frac{-1}{8} + A_2 + A_3\right)(-2) + 1/4$

$= \frac{1}{4} - 2A_2 - 2A_3 + \frac{1}{4}$

$= 1/2 - 2A_2 - 2A_3$

$2A_2 + 2A_3 = 1/2$

$2(A_2 + A_3) = 1/2$

$A_2 + A_3 = 1/4$ —— ②

$\rightarrow a_2 = 2$

$n = 2$ $\quad a_2 = (A_1 + A_2(2) + A_3(4))4 + 1/8(4)$

$2 = (A_1 + 2A_2 + 4A_3)4 + 1/2$

$2 = 4A_1 + 8A_2 + 16A_3 + 1/2$

$2 = -\frac{1}{8}(4) + 8A_2 + 16A_3 + 1/2$

$8A_2 + 16A_3 = 2$

$A_2 + 2A_3 = 1/4$ —— ③

$\rightarrow$ Solving equation ② and ③,

$A_2 + A_3 = 1/4$

$A_2 + 2A_3 = 1/4$

$- \quad - \quad -$

$A_3 = 0$

$A_2 = 1/4$

$\rightarrow (a_n) = [(A_1 + A_2 n + A_3 n^2)](-2)^n + [\frac{1}{8}(2)^n]$

$= [(-\frac{1}{8} + \frac{1}{4}n](-2)^n + \frac{(2)^n}{8}$

* Case : 3

Ex. Solve $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

$\rightarrow \alpha^2 + 5\alpha + 6 = 3n^2$

$\Rightarrow$ Homogeneous equation is,

$\alpha^2 + 5\alpha + 6 = 0$

$(\alpha+3)(\alpha+2) = 0$

$\alpha = -3, -2$

$(a_n)^h = A_1(-3)^n + A_2(-2)^n$

$\Rightarrow (a_n)^P = A_0 + A_1 n + A_2 n^2$

$(a_{n-1})^P = A_0 + A_1(n-1) + A_2(n-1)^2$

$(a_{n-2})^P = A_0 + A_1(n-2) + A_2(n-2)^2$

$(a_{n-1})^P = A_0 + nA_1 - A_1 + A_2(n^2 - 2n + 1)$

$= A_0 + nA_1 - A_1 + A_2 n^2 - 2A_2 n + A_2$

$= (A_0 - A_1 + A_2) + n(A_1 - A_2) - A_2 n^2$

$(a_{n-2})^P = A_0 + nA_1 - 2A_1 + A_2(n^2 - 4n + 4)$

$= A_0 + nA_1 - 2A_1 + A_2 n^2 - A_2 4n + 4A_2$

$= (A_0 - 2A_1 + 4A_2) + n(A_1 - 4A_2) + A_2 n^2$

$\rightarrow$ Equation can be written as,

$\rightarrow (a_n)^P + 5(a_{n-1})^P + 6(a_{n-2})^P = 3n^2$

$\therefore [A_0 + A_1 n + A_2 n^2] + 5[(A_0 - A_1 + A_2) + n(A_1 - A_2) - A_2 n^2]$
$+ 6[(A_0 - 2A_1 + 4A_2 + n(A_1 - A_2) - A_2 n^2] = 3n^2$

$\therefore A_0 + A_1 n + A_2 n^2 + [5A_0 - 5A_1 + 5A_2 + 5n(A_1 - A_2)$
$- 5A_2 n^2 + [6A_0 - 12A_1 + 24A_2 + 6n(A_1 - A_2) - 6A_2 n^2] = 3n^2$

$\therefore (12A_0 + (-17A_1) + 29A_2) + (12A_1 - 39A_2)n$
$+ (12A_2)n^2 = 3n^2 + 0n + 0$

→ Compare both side coefficient,

n^2	n	constant
$12A_2 = 3$ $A_2 = 1/4$	$12A_1 - 34A_2 = 0$ $12A_1 - \frac{34}{2} = 0$ $12A_1 = 17/2$ $A_1 = 17/24$	$12A_0 - 17A_1 + 29A_2 = 0$ $12A_0 = 17(17/24) - 29(\frac{1}{4})$

$$12A_0 = \frac{17(17)}{24} - \frac{29(6)}{24}$$

$$A_0 = 115/288$$

$$(a_n)^P = \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$$

$$a_n = (a_n)^h + (a_n)^P$$

$$= [A_1(-3)^n + A_2(-2)^n] + \left[\frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2\right]$$

Relation

* **Cartesian Product**: Let A and B two set then $(A \times B)$ is called cartesian product of A and B.

$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

* **Binary Relation**: Let A and B two sets then any subset R of the cartesian product $(A \times B)$ is a relation from the set A to set B.

$R \subseteq (A \times B)$

$R = \{(x, y) \mid x \in A \text{ and } y \in B\}$

- If $(x, y) \in R$ then we write xRy and say x is related to y.

* **Domain**: Let, R be a relation from set A to the set B. Then the set of all first element of the ordered pair present in the Relation R is known as domain of R.

$D(R) = \{x \mid (x, y) \in R \text{ for } x \in A\}$

* **Range**: Let, R be a relation from the set A to set B then the set of all second elements of pair present in Relation R is known as range of R.

$R(R) = \{y \mid (x, y) \in R \text{ for } y \in B\}$

* **Inverse Relation:** Let, R be a relation from the set A to set B Then the inverse of the relation R is a relation from the set B to set A.

$$R^{-1} = \{ (y, x) \mid (x, y) \in R \}$$

☆ **Examples**

(1) $R = \{ (x, y) \mid x = y \}$
$A = \{ 1, 2, 3, 4 \} = B$

→ $R = \{ (1, 1), (2, 2), (3, 3), (4, 4) \}$
$D(R) = \{ 1, 2, 3, 4 \} = R(R)$
$R^{-1} = R$

(2) $R = \{ (x, y) \mid x = 3y \}$
$A = \{ 1, 2, 3, 4 \}$

→ $R = \{ (1, 3), (2, 6), (3, 9), (4, 12) \}$
$D(R) = \{ 1, 2, 3, 4 \}$
$R(R) = \{ 3, 6, 9, 12 \}$
$R^{-1} = \{ (3, 1), (6, 2), (9, 3), (12, 4) \}$.

(3) $R = \{ (x, y) \mid x/y \}$
$A = \{ 2, 3, 5, 6 \}$

→ $R = \{ (2, 2), (3, 3), (5, 5), (6, 6), (6, 2), (6, 3) \}$
$D(R) = \{ 2, 3, 5, 6 \}$
$R(R) = \{ 2, 3, 5, 6 \}$
$R^{-1} = \{ (2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6) \}$

☆ Matrix Representation

→ Let, A and B finite sets, containing m and n number of element respectively R is a relation from A to B. The relation R can be represented by m×n Matrix $M = M_{ij}$.

$M_{ij} = 1$ if $a_i R b_j$ or $(a_i, b_j) \in R$
$= 0$ if $a_i R b_j$ or $(a_i, b_j) \notin R$

+ Operations

→ Relation matrix is known as boolean matrix as the entries are 0 or 1.

$A = [a_{ij}]$, $B = [b_{ij}]$

→ $A \times B = [a_{ij}]\,[b_{ij}]$
$= [c_{ij}]$

$c_{ij} = 1$ if $[a_{ij}] = 1$ and $[b_{ij}] = 1$
$c_{ij} = 0$ if $[a_{ij}] = 0$ and $[b_{ij}] = 0$

→ $A + B = [a_{ij}] + [b_{ij}]$
$= [c_{ij}]$

$c_{ij} = 1$ if $[a_{ij}] = 1$ and $[b_{ij}] = 1$
$c_{ij} = 0$ if $[a_{ij}] = 0$ and $[b_{ij}] = 0$

Ex. $A = \{1, 2, 3, 4, 6\} = B$ aRb if only if a/b then find Relation matrix M_R.

$\rightarrow$ $R = \{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (6,3), (4,4), (6,6), (3,3)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Ex. Addition and Multiplication operation on matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\rightarrow A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\rightarrow A \times B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 1+0+0 & 0 \\ 0+0 & 0+1+0 & 0+1 \\ 1+0+1 & 1+1+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

☆ Graphical Representation of Relation.

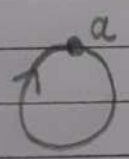
→ Let, A be finite set and R be a Relation defined on R, then R can be represented by graph.

→ The elements of set A are represented by circle or points. Those elements are known as nodes or vertices.

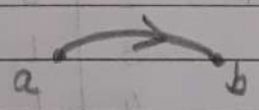
→ If aRa then this can be represented by loop around a.

→ If aRb then this can be represented by a to b with an arrow from a to b

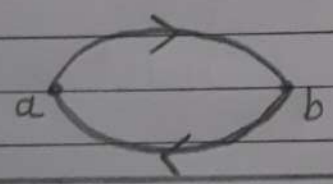
Ex. aRa



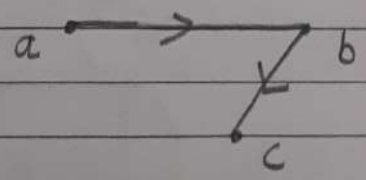
Ex. aRb



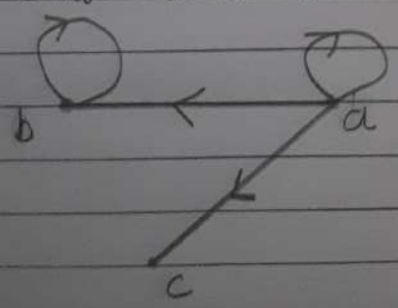
Ex. $aRb \wedge bRa$



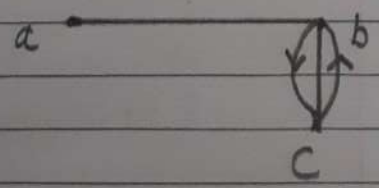
Ex. $aRb \wedge bRc$



Ex. $aRa \wedge bRb \wedge aRc \wedge aRb$

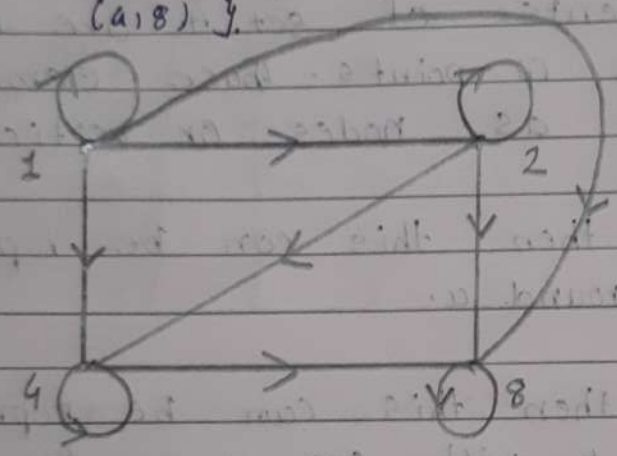


Ex. $aRb \wedge bRc \wedge cRb$



Ex. $A = \{1, 2, 4, 8\}$ aRb if b/a (a divides b) then find the graph representation.

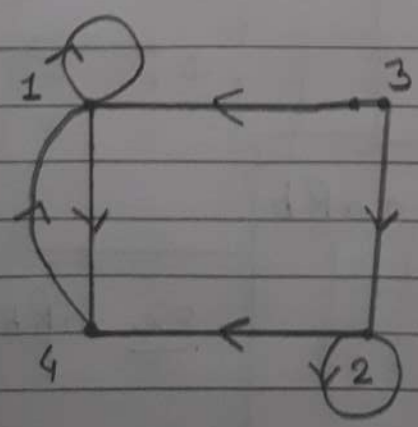
→ $R = \{(1,1), (2,2), (4,4), (8,8), (1,2), (1,4), (2,4), (1,8), (2,8), (4,8)\}$.



Ex.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

→ $R = \{(1,1), (1,4), (2,2), (2,4), (3,1), (3,2), (4,1)\}$



☆ Binary Relations

(1) **Reflexive Relation** : A Relation R defined on a set A is said to be reflexive if $(x,y) \in R$ for every element $x \in A$.

Ex. $A = \{a, b, c\}$
$R = \{(a,a), (a,b), (b,b), (b,c), (c,c)\}$

→ If in matrix all diagonals entries are 1 it's called Reflexive matrix.

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}$$

(2) **Symetric Relation** : Relation R defined on set A is said to be symetric if $(x,y) \in R$ then $(y,x) \in R$.

Ex. $A = \{1, 2, 3\}$
$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3)\}$

(3) **Compatible Relation** : A Relation R on a set S is said to be compatible if it's reflexive and symetric.

Ex. $A = \{1, 2, 3\}$
$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$

(4) Antisymetric : A Relation R defined on a set A is said to be antisymetric relation if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$.

Ex. $A = \{1, 2, 3\}$
$R = \{(1,1), (2,2), (2,3), (1,3)\}$

(5) Transistive Relation : A relation R defined on set A is said to be transistive if $(a, b) \in R$, $(b, c) \in R$ $\Rightarrow (a, c) \in R$.

aRb, $bRc \Rightarrow aRc$

Ex. $A = \{1, 3, 5, 7, 2\}$
$R = \{(1,1), (1,3), (3,2), (1,2), (2,5), (1,5), (3,5), (1,7), (7,2)\}$

$\rightarrow 1R3$, $3R2 \Rightarrow 1R2$
$1R3$, $3R5 \Rightarrow 1R5$
$1R2$, $2R5 \Rightarrow 1R5$
$1R1$, $1R7 \Rightarrow 1R7$

(6) <u>Anti-Reflexive</u> : A relation R defined on set A is said to be Anti-reflexive if $(x,x) \notin R$ for every element $x \in A$.

<u>Ex.</u> $A = \{1, 3, 5, 7\}$
$R = \{(1,3), (1,5), (1,7), (3,5), (3,7)\}$
$\boxed{(1,1), (3,3), (5,5), (7,7)}$ → not allowed

(7) <u>Asymetric</u> : A Relation R defined on set A is said to be asymmetric if $(x,y) \in R$ then $(y,x) \notin R$

<u>Ex.</u> $A = \{1, 3, 5, 7\}$
$R = \{(1,3), (3,5), (5,7), (3,7)\}$

$\boxed{(5,3), (7,5), (3,1), (7,3)}$ → not allowed

<u>Ex.</u> $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(1,2), (1,3), (2,2), (3,4)\}$ find Reflexive, Symetric, Transistive closure on given set A.

→ $A = \{1, 2, 3, 4, 5\}$

→ $R(R) = \{(1,2), (1,3), (2,2), (3,4), (1,1), (3,3), (4,4), (5,5)\}$

→ $S(R) = \{(1,2), (1,3), (2,2), (3,4), (2,1), (3,1), (4,3)\}$

→ $T(R) = \{(1,2), (1,3), (2,2), (3,4), (1,4)\}$

Ex. Let, N be a set of all natural numbers R be a relation in N defined by xRy if and only if $x+3y=12$ Examine relation for (1) Reflexive (2) Symmetric (3) Transistive

→ $xRy : x+3y=12$

(1) Let, $x+3y=12$ $[y=x]$

$x+3x=12$

$x=3$ xRx for only $x=3$

→ Here R is not reflexive for N.

(2) Symetric

$x+3y=12 \rightarrow xRy$

$y+3x=12 \rightarrow yRx$

$y+3x=12$ may or may not equal to 12.

→ So, Here R is not symmetric.

(3) Transistive :

Let, xRy , yRz

$x+3y=12$, $y+3z=12$

this are true for $x=y=z=3$

→ So, its transistive for 3 but not for all natural number

☆ Types of Relation Matrix.

(1) **Reflexive Relation** : The Relation R is said to be reflexive if $m_{ii} = 1$ all element of main diagonal in reflexive matrix will be 1.

Ex. $M(R) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = m_{ii} = 1.$

(2) **Symetric Relation** : The relation R is said to be symetric if $m_{ij} = m_{ji}$

$M(R) = [M(R)]^T$ where $[M(R)]^T$ represent the transpose of a relation matrix.

Ex. $M(R) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $[M(R)]^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$M(R) = [M(R)]^T$

(3) **Transistive Relation** : The relation R is said to be transistive if $m_{ij} = 1$ and $m_{jk} = 1$ then $m_{ik} = 1$.

$M(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$[M_{(R)}]^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0 & 1+1+0 & 1+1+0 \\ 1+1+0 & 1+1+0 & 1+1+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow [M_{(R)}]^2 = [M_{(R)}]$

(4) <u>Anti-Reflexive</u> : The relation R is said to be anti reflexive if $m_{ij} = 0$

- All element in main diagonal in matrix is zero.

$$M_{(R)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Ex. Consider a Relation R on $\{1,2,3,4\}$ as $R = \{(1,3), (1,4), (2,2), (3,3), (4,3)\}$ Examine the relation for reflexive, Symetric and transistive with help of relation matrix.

→ $R = \{(1,3), (1,4), (2,2), (3,3), (4,3)\}$

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Here, its not refxive because,
$m_{11} = 0$ and $m_{44} = 0$

→ $$[M(R)]^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$M(R) \neq [M(R)]^T$ So, its not Symetric.

→ $$[M(R)]^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$[M(R)]^2 \neq M(R)$, So, its not transistive

☆ Warshall's Algorithm

Ex. Find transistive closure using warshall's algorithm where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \{1, 2, 3\}.$$

→ $$W_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

→ We want to find W_1 so select 1st row and 1st column.

Column : $P_1 : (1, 1)$, $P_2 : (3, 1)$
Row : $q_1 : (1, 1)$, $q_2 : (1, 3)$
add $(3, 3)$ in W_0 so we get W_1,

$$W_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

→ We want to find W_2 so select 2nd row and column.

column : $P_1 : (2, 2)$, $P_2 : (3, 2)$
row : $q_1 : (2, 2)$

$$W_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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→ We want to find W_3 so select 3rd row and column.

column : P_1 (1, 3) , P_2 (3, 3)
row : q_1 (3, 1) , q_2 (3, 2) , q_3 (3, 3)

So, add (1, 1) and (1, 2) in W_2 so we get W_3.

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$R^* = \{(1,1), (1,2), (1,3), (2,2), (3,1), (3,2), (3,3)\}$

Ex. Find the transistive closure of relation R on $A = \{1, 2, 3, 4\}$ defined by $R = \{(1,2), (1,3), (1,4), (2,1), (2,3), (3,2), (3,4), (4,2), (4,3)\}$

→ $$W_0 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

→ We have to find W_1, so select 1st row and 1st column.

column : $P_1(2,1)$
row : $q_1(1,2)$, $q_2(1,3)$, $q_3(1,4)$

- add $(2,2)$, $(2,4)$, $(1,1)$, in W_0 so we get W_1

$$W_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

→ We have to find W_2, select 2nd column and row,

column : $P_1(1,2)$, $P_2(2,2)$, $P_3(3,2)$, $P_4(4,2)$
row : $q_1(2,1)$, $q_2(2,2)$, $q_3(2,3)$, $q_4(2,4)$

add $(1,1)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,1)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,3)$, $(4,4)$, $(2,2)$, $(2,4)$, in W_1 so we get W_2

$$w_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

→ All position in w_2 are 1 hence,
w_2 = Transistive closure.

$$\omega R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

Partial Ordering

☆ Partially order set or POSET

→ If A is any non-empty set and R is a partial order relation on set A then the ordered pair (A, R) is called partially ordered set or POSET.

Ex. S is a non-empty set and $P(S)$ is power set of S the relation $\subseteq$ is partial ordered relation on set $P(S)$.

→ $(P(S), \subseteq)$ is known as POSET.

→ aRb if $a|b$ (a divides b) is partial ordered relation on set of natural number.

→ The relation $\leq$ is a partial order relation on set of Real number. Hence genrally $\leq$ is used to denote relation on set A as partial order relation ~~and~~ are $(A \leq)$ is POSET

Ex. Let $A = \{4, 9, 16, 36\}$ relation divide is a partial relation on A.

→ (1) Reflexive

4 divides 4
9 divides 9
16 divides 16
36 divides 36

So, here aRa so it's reflexive.

→ (2) Antisymmetric

4 divides 16 but,
16 doesn't divide 4

aRb but,
b $\not R$ a

So, it's not antisymmetric

→ (3) Transistive

4 divides 16 → aRb
16 divides 64 → bRc
4 divides 64 → aRc

So, it's transistive.

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☆ Hash Diagram

→ POSET can be represented by Diagraphs A simple way of representing POSET is Hash diagram.

* Maximal and Minimul

→ An element $a \in A$ is known as Maximal element if there is no element c in a such that $a \leq c$.

→ An element of POSET which is not less than any other element of the POSET. or which is not related to any element Top element of Hash diagram.

→ An element of POSET which is not greater than any other element of POSET. or which element that is no other element related to this Bottom element of diagram.

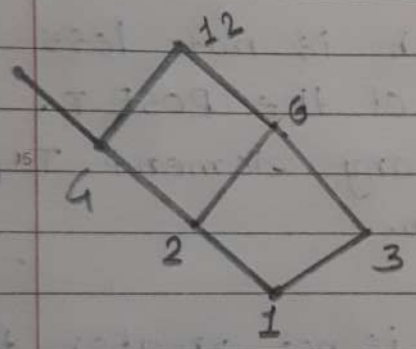
* Least upper bound and Greatest lower bound

→ C is known as least upper bound if c is an upper bound of a, b and if there is no other upper bound.

→ e is said to be lower bound of a and b if $e \leq a$ and $e \leq b$. and e is known as glb if if there is no other lower bound of a and b.

Ex. Draw the hash diagram representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$

→ Cover of 1 = 2, 3,
2 = 4, 6
3 = 6
4 = 8, 12
6 = 12
8 = –
12 = –



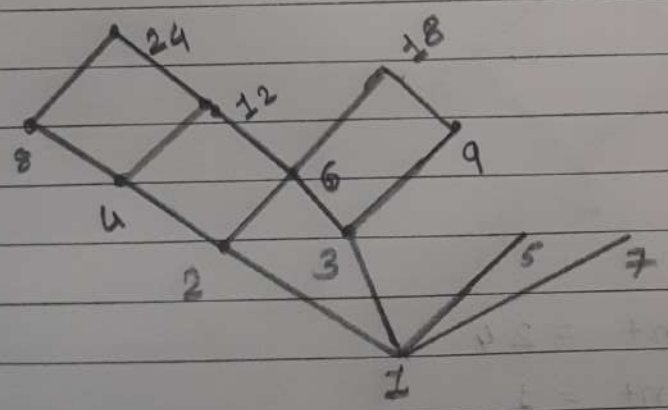
→ Maximal element = 12
Minimal element = 1

→ ~~Greatest~~ least upper bound of 2 and 3 = 6
Greatest lower bound of 2 and 3 = 1

→ Upper bound of 2 and 3 = 6, 12
lower bound of 2 and 3 = 1

Ex. 2 $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$ $[b/a]$

→ Cover of 1 = 2, 3, 5, 7,
2 = 4, 6,
3 = 6, 9
4 = 8, 12
5 = -
6 = 12, 18
7 = -
8 = 24
9 = 18
12 = 24
18 = -
24 = -



→ Maximal element = 24
Minimal element = 1

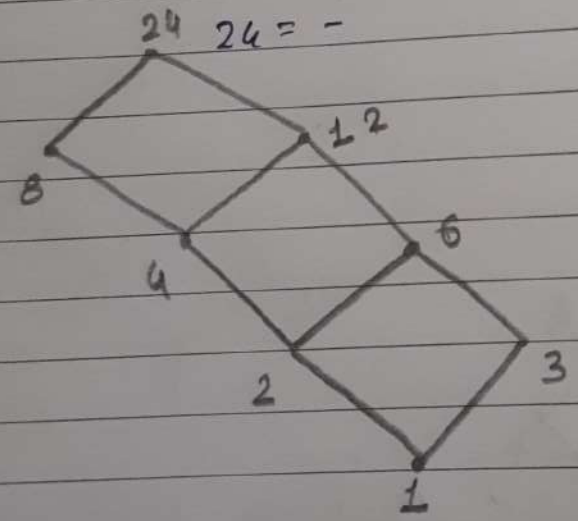
→ Upper bound of 4 and 6 = 12, 24
lower bound of 4 and 6 = 2, 1

→ lub of 4 and 6 = 12
glb of 4 and 6 = 2

Ex.3 $\langle S_{24}, D \rangle$ where D means a divide b.

→ $= \{1, 2, 3, 4, 6, 8, 12, 24\}$

cover of 1 = 2, 3,
2 = 4, 6
3 = 6
4 = 8, 12
6 = 12
8 = 24
12 = 24
24 = -



→ Maximal element = 24
Minimal element = 1

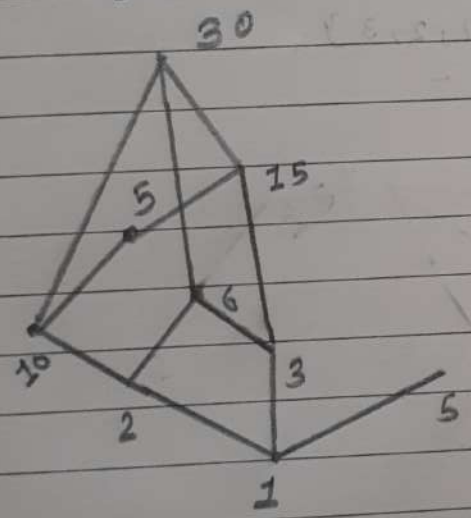
→ Upper bound of 2 and 3 = 6, 12, 24
lower bound of 2 and 3 = 1

→ glb of 2 and 3 = 1
lup of 2 and 3 = 6

Ex. $\langle S_{30}, \mathrm{D} \rangle$ where D means a divides b

→ $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Cover of 1 = 2, 3, 5,
2 = 6, 10
3 = 6, 15
5 = 10, 15
6 = 30
10 = 30
15 = 30
30 = –



→ Minimal element = 1
Maximal element = 30

→ Upper bound of 2 and 3 = 6, 30
lower bound of 2 and 3 = 5

→ glb of 2 and 3 = 1
lub of 2 and 3 = 6

Ex. $S = \{1, 2, 3\}$ $(P(S), \leq)$ is a poset and find Hash diagram.

→ $P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

Cover of $\phi = \{1\}, \{2\}, \{3\}$

$\{1\} = \{1,2\}, \{1,3\}$

$\{2\} = \{1,2\}, \{2,3\}$

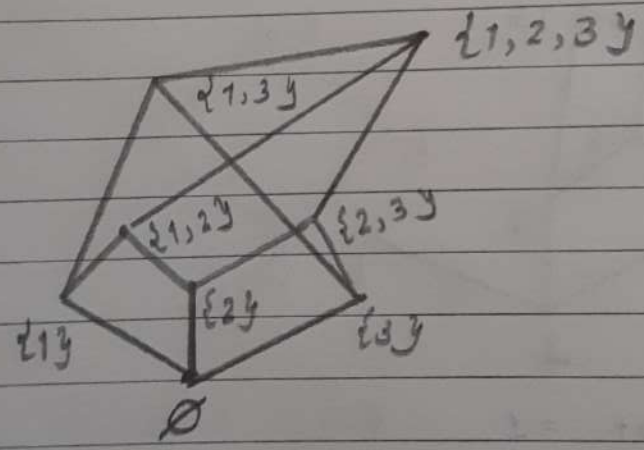
$\{3\} = \{1,3\}, \{2,3\}$

$\{1,2\} = \{1,2,3\}$

$\{2,3\} = \{1,2,3\}$

$\{1,3\} = \{1,2,3\}$

$\{1,2,3\} = -$



→ Maximal element $= \{1,2,3\}$

Minimal element $= \phi$

→ Upper bound of $\{1\}$ and $\{2\} = \{1,2\}, \{1,2,3\}$

lower bound of $\{1\}$ and $\{2\} = \phi$

→ lub of $\{1\}$ and $\{2\} = \{1,2\}$

glb of $\{1\}$ and $\{2\} = \phi$

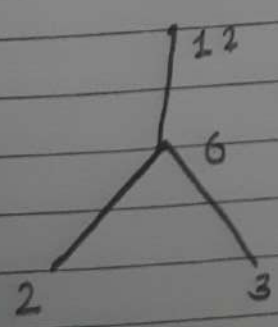
Lattice

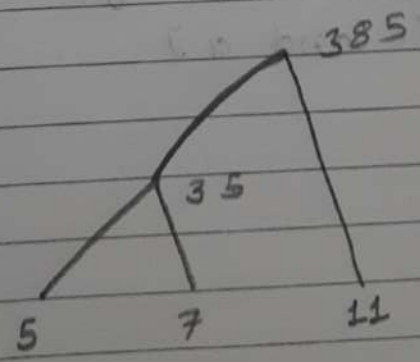
→ A lattice is a POSET $(A, \leq)$ if for all $x, y \in A$ then element lub(x, y) and glb(x, y) both exist on a.

→ A lattice is a POSET if $(A, \leq)$ in which every subset $\{a, b\}$ of A has a lub and a glb.

Ex. Determine POSET $\{2, 3, 5, 6, 7, 11, 12, 35, 385\}$ where x divides y.

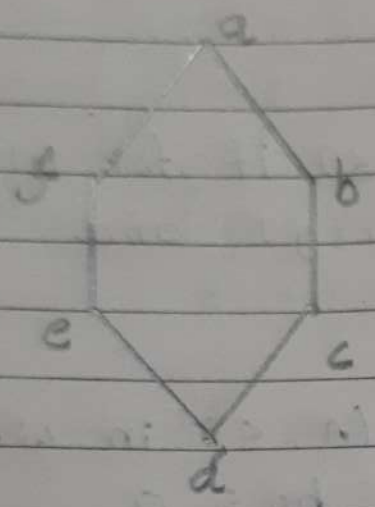
→ Cover of
2 = 6
3 = 6
5 = 35
6 = 12
7 = 35
11 = 385
35 = 385
385 = –





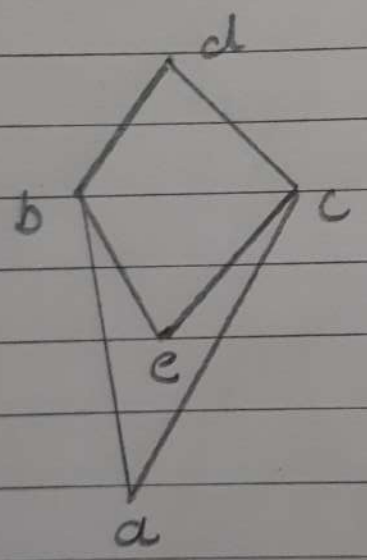
→ It's not lattices because we didn't get glb for (2, 3), (5, 7), (7, 11)

Ex.



→ It's lattice because each pair we get glb and lub.

Ex.



→ It's not lattice because for pair (b, c) we get two glb (e and a)