Homework W3

1. Prove that $1 + x + x^2 + ... + x^n = \frac{1-x^{n+1}}{1-x}$

2. Show that $n! < n^n$ for all $n \geq 2$

3. Prove that $6 \mid 4n^3 + 2n$ for all $n \in \mathbb{N}$. Hint: if $f(n) = 4n^3 + 2n$, prove that 6 divides $f(n+1) - f(n)$

4. A **palindrome** is a string which is spelled the same backwards or forwards. For example, $RACECAR$, 11111, 5, and *abcdeedcba* are all palindromes while 123, $aabc1$, and $xxxy$ are not. The set of palindromes over the alphabet $\{A, B, C\}$ with length 4 is

$$\{AAAA, ABBA, ACCA, BAAB, BBBB, BCCB, CAAC, CBBC, CCCC\}$$

   Determine the number of palindromes over the alphabet $\{A, B, C\}$ of length 2019, and explain your reasoning.

5. The divisors of $45 = 3^2 \cdot 5$ are $\{1, 3, 9, 5, 15, 45\} = \{1, 3, 3^2, 5, 3 \cdot 5, 3^2 \cdot 5\}$. Determine the number of divisors of $1389150 = 2 \cdot 3^4 \cdot 5^2 \cdot 7^3$, and explain your reasoning. Hint: try formulating this problem in terms of a product of sets.

6. Find the flaw(s) in the following inductive "proof":
   Untrue Statement: All people have the same name.
   'Proof': We will prove this by induction. Let $P(n)$ be the statement 'for any set of n people, they all have the same name'. We will prove this for all $n \geq 1$.

   Take the base case, $n = 1$. In a set of 1 person, that person has the same name as themself. Therefore the statement 'for any set of 1 person, they all have the same name' is true, so the base case holds.

   Now for the induction step. Assume that $P(n)$ is True. Consider a set of $n + 1$ people. Put these people in a line. If we remove the first person, we are left with a set of n people, all of whom have the same name since $P(n)$ is true. If we remove the last person, we get another set of n people all of whom have the same name. This implies that the first and last person have the same name, since they both have the same name as the second person in the line. Therefore all $n+1$ people have the same name, so $P(n)$ implies $P(n+1)$.

   We conclude that $P(n)$ holds for all $n \geq 1$.