Directions

Directions: Read these directions carefully. Read the W1 Supplemental on BBLearn to answer questions 1 and 2. See chapter 0 of the Levin textbook for reference, definitions, and additional examples.

Put your answers on a separate sheet of paper, then turn in your Homework at the beginning of class in Week 2. Read the section on Explanations.

Explanations: If a question asks you to **explain**, fully explain your answer. As a rule of thumb, assume your reader is very literal and doesn't want to believe you. Some tips for proving stuff: to show that a statement is false, give a counterexample. If you want to prove it is true, you need to show that it is true in general. Here are some examples for how to prove/explain stuff.

Examples

Example 1 Suppose that $P \vee Q$ is True. Is the statement $P \implies Q$ necessarily True?
Answer: No, we will give a counterexample. Suppose that P is True but Q is False. Then $P \vee Q$ is True, but $P \implies Q$ is False.

Example 2 Prove that the empty set is a subset of every set.
Answer: Let A, B be sets. Then $A \subseteq B$ if

$$\forall x(x \in A \implies x \in B)$$

So $\emptyset \subseteq B$ if

$$\forall x(x \in \emptyset \implies x \in B)$$

Choose any x. The statement $x \in \emptyset$ is False since $\emptyset$ contains no elements, so the implication

$$x \in \emptyset \implies x \in B$$

Is automatically true. Since this is true for any x, it is true for all x. Hence

$$\forall x(x \in \emptyset \implies x \in B)$$

Is True. Therefore $\emptyset \subseteq B$.

Example 3 Show that a function f is surjective if its range equals its codomain.
Answer: Suppose that the range of f equals its codomain. We will show that f satisfies the definition of a surjective function. For concreteness, let $f : A \to B$ so the domain of f is A and both the codomain and range of f is B. Let $b \in B$. Then b is in the range of f, so there exists an $a \in A$ for which $b = f(a)$. Since this is true for any $b \in B$, it is true for all $b \in B$. This is the definition of being a surjective function, so f is surjective.

Homework W1

1. Read the supplemental on BBLearn. For each statement below, assume x, y are real numbers. Write a brief interpretation of each statement, and determine if the statement is True or False.
   (a) $\forall x(\exists y(x + y = 0))$
   (b) $\exists y(\forall x(x + y = 0))$
   (c) $\exists x(\exists y(xy = 0))$
   (d) $\forall x(\forall y(xy \in \mathbb{R}))$

2. Let $E(x)$ be the predicate 'x is even', let $O(x)$ be the predicate 'x is odd', and let $P(x)$ be the predicate 'x is prime'. Then the statement 'Every number is either even or odd' can be written symbolically as
   $$\forall x(E(x) \vee O(x))$$
   Write the following statements symbolically. Note that you may need multiple free variables/quantifiers.
   (a) No number is both even and odd
   (b) Every prime number is greater than one
   (c) Every number is less than a prime number
   (d) One more than any even number is an odd number
   (e) There is an even prime number
   (f) Between any two numbers there is a third number
   (g) There is a pair of prime numbers which differ by two

3. Use a truth table to prove De Morgan's Laws:
   (a) $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$
   (b) $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$

4. Let $D = \{1, \{2, 3\}\}$.
   (a) Find the powerset $P(D)$
   (b) Find $|P(D)|$
   (c) Is $\{1\} \in P(D)$?
   (d) Is $\{2, 3\} \in P(D)$?
   (e) Is $D \subseteq P(D)$?
   (f) Is $D \in P(D)$?

5. Let A, B be sets with $A \subseteq B$. Find each, and **explain** your reasoning
   (a) $A \cap B$

(b) $A \cup B$

(c) $A \cup \emptyset$

(d) $A \cap \emptyset$

6. Let $D = \{1, 2, 3, 4, 5\}$. Consider the function $f : P(D) \to \mathbb{N}$ which maps each subset of D to its cardinality. For example, $f(\{1, 3\}) = |\{1, 3\}| = 2$.

   (a) What is the domain of f?

   (b) What is the codomain of f?

   (c) What is the range of f?

   (d) Is f surjective, injective, bijective or none? **Explain** your reasoning.

   (e) Find $f^{-1}(1)$, and $f^{-1}(5)$

   (f) Find $f^{-1}(0)$ and $f^{-1}(6)$

7. Let $N^+ = \mathbb{N} \backslash \{0\}$ be the positive natural numbers and let $2\mathbb{N} = \{0, 2, 4, ...\}$ be the even natural numbers. Give an example of a function which satisfies each criterion.

   You can define your function by a formula, a "table", or a description such as "f sends even numbers to 0 and odd numbers to 1" or similar.

   (a) A bijection $f : \mathbb{N} \to \mathbb{N}^+$

   (b) A bijection $f : \mathbb{N} \to 2\mathbb{N}$

   (c) A bijection $f : \mathbb{N}^+ \to 2\mathbb{N}$

   (d) An injection (injective function) $f : \mathbb{N} \to \mathbb{N}$ which is not surjective

   (e) An surjection $f : \mathbb{N} \to \mathbb{N}$ which is not injective