A mixed up-downwind scheme for solving a Heston stochastic volatility model on variable grids

Chong Sun

jointly with Dr. Q. Sheng

Department of Mathematics
Center for Astrophysics, Space Physics and Engineering Research
Baylor University

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Heston Stochastic Volatility Model

Heston proposed that stock prices $S(t)$ and the associated volatility $y(t)$ follow a Brownian motion,

$$\begin{aligned} dS(t) &= \mu S(t)dt + S(t)\sqrt{y(t)}dB(t) \\ dy(t) &= \kappa[\eta - y(t)]dt + \sigma\sqrt{y(t)}d\tilde{B}(t) \\ dB(t)d\tilde{B}(t) &= \rho dt \end{aligned}$$

Heston Partial Differential Equation

$$V_\tau = \frac{1}{2} y V_{xx} + \rho\sigma y V_{xy} + \frac{1}{2}\sigma^2 y V_{yy} - \left(\frac{1}{2} y - r\right) V_x + \kappa(\eta - y) V_y,$$

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$$\begin{aligned}
V(x,y,0) &= \max\left(1-e^x,0\right), \quad x\in\mathbb{R}, \quad y\in\mathbb{R}^+,\\
\lim_{x\to-\infty} V(x,y,\tau) &= 1, \quad y\in\mathbb{R}^+, \quad \tau\in\mathbb{R}^+,\\
\lim_{x\to\infty} V(x,y,\tau) &= 0, \quad y\in\mathbb{R}^+, \quad \tau\in\mathbb{R}^+,\\
V(x,0,\tau) &= \max\left(1-e^x,0\right), \quad x\in\mathbb{R}, \quad \tau\in\mathbb{R}^+.
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\lim_{y\to\infty} V_y(x,y,\tau) &= 0, \quad x\in\mathbb{R}, \quad \tau\in\mathbb{R}^+,
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Traditional Approaches and Limitation

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Limitation:

- von Neumann analysis can only be applied to Cauchy problems or periodic boundary conditions

Our Approach–Mixed Derivative

Mixed Derivative Term:

- Positive coefficient:

$$V_{xy}(x_m, y_n, \tau) \approx \frac{1}{2}(\Delta_{x,-}\Delta_{y,-} + \Delta_{x,+}\Delta_{y,+})V(x_m, y_n, \tau).$$

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- Negative coefficient:

$$V_{xy}(x_m, y_n, \tau) \approx \frac{1}{2}(\Delta_{x,+}\Delta_{y,-} + \Delta_{x,-}\Delta_{y,+})V(x_m, y_n, \tau).$$

Our Approach–Advection Terms

- Positive coefficient: Forward Difference Approximation
- Negative coefficient: Backward Difference Approximation

Semi-Discretised System

Semi-discretized system:

$$\mathbf{u}'(\tau) = \mathbf{M}\mathbf{u}(\tau) + \mathbf{f}(\tau),$$

The solution is

$$\mathbf{u}(\tau) = e^{\tau\mathbf{M}}\mathbf{u}(0) - \int_0^{\tau} e^{(\tau-s)\mathbf{M}}\mathbf{f}(s)ds.$$

Definition of Stability of Semi-Discretised Systems

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The semi-discretised system is stable if for every $\tau^* > 0$, there exists a constant $c(\tau^*) > 0$ such that

$$\|e^{\tau \mathbf{M}}\| \leq c(\tau^*), \quad \tau \in [0, \tau^*]. \tag{1}$$

where $\|\cdot\|$ is an appropriate matrix norm.

Gerschgorin's Circle and Exponential Behavior Theorems

Theorem (Gerschgorin's Circle Theorem/Brauer's Theorem)

Let M_s be the sum of the moduli of the elements along the sth row of matrix $\boldsymbol{M}$ excluding the diagonal element m_{ss}. Then each eigenvalue of $\boldsymbol{M}$ lies inside or on the boundary of at least one of the circles $|\lambda - m_{ss}| = M_s$.

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Theorem

For $\rho \in [-1, 1]$, the semi-discretised system is stable.

Domain Truncation

$$V_\tau = \frac{1}{2} y V_{xx} + \rho\sigma y V_{xy} + \frac{1}{2}\sigma^2 y V_{yy} - \left(\frac{1}{2} y - r\right) V_x + \kappa(\eta - y) V_y,$$

$$\begin{aligned}
V(x, y, 0) &= \max\left(1 - e^x, 0\right), \quad x \in [-X,\ X], \quad y \in [0,\ Y], \\
V(-X, y, \tau) &= 1, \quad y \in [0,\ Y], \quad \tau \in \mathbb{R}^+, \\
V(X, y, \tau) &= 0, \quad y \in [0,\ Y], \quad \tau \in \mathbb{R}^+, \\
V(x, 0, \tau) &= \max\left(1 - e^x, 0\right), \quad x \in [-X,\ X], \quad \tau \in \mathbb{R}^+. \\
V_y(x, Y, \tau) &= 0, \quad x \in [-X,\ X], \quad \tau \in \mathbb{R}^+,
\end{aligned}$$

Solution Surface

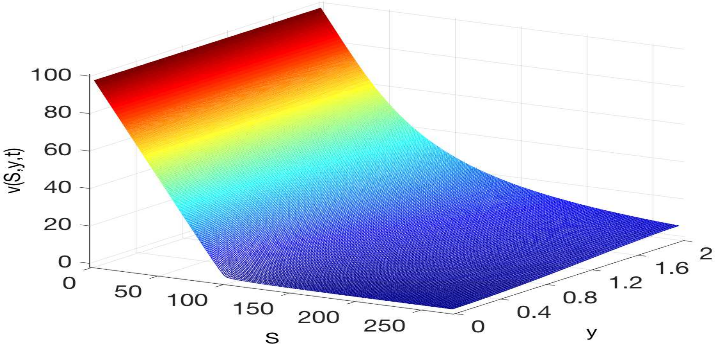

Figure: Price of an European put option

Convergence Surface

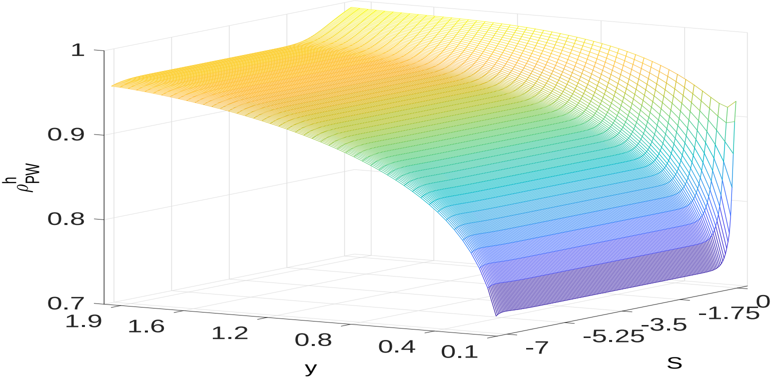

Figure: Rate of convergence ρ^h_{PW} surface at $T = 0.5$. The figure indicates approximately an order one rate of convergence.

Future Work

- Exponential Splitting
- Adaptive Grids
- Higher-Order Schemes
- Free Boundary Value Problems

Thank You