4.1

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} - 3 \begin{vmatrix} 2 & 6 \\ 0 & 4 \end{vmatrix} + 5 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$

$$= 16 - 12 - 24 + 0 + 20 - 0$$

$$= 36 - 36 = 0$$

Vậy $\det(A) = 0$

4.2

$$C = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ -2 & 0 & 2 & -1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\det(C) = 2 \cdot 1 \cdot \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 2 \cdot 3 = 6$$

Vậy $\det(C) = 6$

4.3 Compute the eigen spaces

$$A = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

Cho $P_{(A)}(\lambda) = 0 \Rightarrow \det(A) - tr(A) \cdot \lambda + \lambda^2 = 0$

$\Leftrightarrow 1 - 2\lambda + \lambda^2 = 0$

$\Rightarrow \lambda = 1 = (\lambda - 1)^2 = 0$

Bội đại số là 2

Ứng với $\lambda = 1$: $(A - I)v = 0$

tức: $\begin{bmatrix} \lambda-1 & 0 \\ 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} 0x+0y=0 \\ 1x=0 \end{matrix}$

$\rightarrow \begin{cases} x=0 \\ y \in R \end{cases} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}.$

Với $B = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$ Cho $P_B(\lambda) = 0 \Leftrightarrow -6 + 1\lambda + \lambda^2 = 0$

$\Leftrightarrow (\lambda+3)(\lambda-2) = 0$

$\Rightarrow \left[\begin{matrix} \lambda = -3 \\ \lambda = 2 \end{matrix}\right.$

Ứng với $\lambda = -3$: $(B-I)v = 0$

tức: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{cases} 1x + 2y = 0 \\ 2x + 4y = 0 \end{cases} \Leftrightarrow x = -2y$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow E_3 = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}.$

Ứng với $\lambda = 2$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow 2x - y = 0 \Leftrightarrow y = 2x$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow E_2 = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$

Vậy $E_2 = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, $E_3 = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}.$

MGP

4.4.5. $A \in \mathbb{R}^2$: $P_A(\lambda) = \det(A) - tr(A)\lambda + \lambda^2$

$A_1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ $\det(A) = 1 \neq 0 \rightarrow$ có nghịch đảo

$P_A(\lambda) = \lambda^2 - 2\lambda + \lambda \Rightarrow 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \end{array}\right.$

Là ma trận đơn vị $\Rightarrow$ chéo hoá đc

$A_2 = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$ $\det(A_2) = 0 \rightarrow$ k° có nghịch đảo

$P_A(\lambda) = (\lambda - 1)(\lambda - 0) = 0$

Với $\lambda = 1 \Leftrightarrow \begin{cases} 0x = 0 \\ -y = 0 \end{cases} \Leftrightarrow \begin{cases} x \in R \\ y = 0 \end{cases} \Rightarrow x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow E_1 = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\lambda = 0 \Leftrightarrow \begin{cases} x = 0 \\ y \in R \end{cases} \Leftrightarrow y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow E_0 = span\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

Vectơ riêng độc lập $\rightarrow$ chéo hoá đc

$A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\det(A_3) = 1 \neq 0 \rightarrow$ Có nghịch đảo

$P_{A_3}(\lambda) = 0 \rightarrow \lambda = 1 \Rightarrow \begin{cases} 0x + y = 0 \\ 0x + 0y = 0 \end{cases} \Leftrightarrow \begin{cases} x \in R \\ y = 0 \end{cases} \Rightarrow$

$\Rightarrow Span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \Rightarrow$ không chéo hoá đc

$A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\det(A) = 0 \Rightarrow$ k° nghịch đảo

$P_{(A_4)}(\lambda) = 0 \Leftrightarrow \lambda^2 - 0\lambda + 0 = 0 \Rightarrow \lambda = 0$

$\Leftrightarrow \begin{cases} y = 0 \\ x \in R \end{cases} \Leftrightarrow x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow E_0 = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \Rightarrow$ k° chéo hoá đc

4.8 $A = U\Sigma V^T$

$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$ $\det(A^T A) = 0$

$\det(A^T A - \lambda I) = 0$

$= -\lambda^3 + 34\lambda^2 - 225\lambda + 0 = 0$

$= (\lambda - 25)(\lambda - 9)(\lambda - 0) = 0 \Rightarrow \begin{cases} \lambda = 0 \\ \lambda = 9 \\ \lambda = 25 \end{cases}$

$\Rightarrow D = \begin{vmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

$V_{\delta}, \lambda = 25$

$\Rightarrow \begin{cases} -12x + 12y + 2z = 0 \\ 12x - 12y - 2z = 0 \\ 2x - 2y - 17z = 0 \end{cases} \Leftrightarrow \begin{cases} z = 6x - 6y = 0 \\ -100x + 100y = 0 \\ x = y \end{cases}$

$\Leftrightarrow x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow E_{25} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

V_3: $\lambda = 9$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 4 & 12 & 2 & 0 \\ 12 & 4 & -2 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 32 & 0 & -8 & 0 \\ 12 & 4 & -2 & 0 \\ 0 & 16 & 4 & 0 \end{array}\right] \Leftrightarrow \begin{array}{l} 4x = z \\ 4y = -z \\ x = y \end{array}$$

$$\Leftrightarrow z \begin{bmatrix} -1/4 \\ 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} . V_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

V_3: $\lambda = 0$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 13 & 12 & 2 & 0 \\ 12 & 13 & -2 & 0 \\ 2 & -2 & 8 & 0 \end{array}\right] \Leftrightarrow \begin{cases} 15x + 15y = 0 \\ 12x + 13y - 2z = 0 \\ x - y + 4z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = -y \\ 2y = 4z \\ -12y + 13y - y = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x \\ y = 2z \end{cases} \Leftrightarrow \begin{cases} x = -y \\ y \in \mathbb{R} \\ z = \frac{1}{2} y \end{cases} \Rightarrow y \begin{bmatrix} -1 \\ 1 \\ 1/2 \end{bmatrix} = V_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

P: $V_3 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix}$

Σ: Tìm Σ : $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

~~λ — λ λ λ λλ λ~~

U:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{3} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Vậy: $A = U \Sigma V^T$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix}$$

3. Neutral ele: Find e:

$\exists e \mid a^{*}e = e^{*}a = a,\ \forall a \in R\setminus\{-1\}$

$a^{*}e = ae + a + e = a$

$\Rightarrow ae + e = 0$

$\Leftrightarrow e(a+1) = 0$

$\Rightarrow e = 0 \quad (e \in R\setminus\{-1\})$

Vậy e là Neutral ele

4. Phần tử nghịch đảo in b:

$\exists e \mid a^{*}b = b^{*}a = 0,\ \forall a,b \in R\setminus\{-1\}$

$a^{*}b = ab + a + b = 0$

$\Leftrightarrow b(1+a) = -a$

$\Rightarrow b = \dfrac{-a}{1+a}$

Đk: $b \neq -1 \Leftrightarrow \dfrac{-a}{1+a} \neq -1$

$\dfrac{-a}{1+a} = -1 \Leftrightarrow -a = -1-a$

$\Leftrightarrow 0 = -1$ (vô lí)

$\Rightarrow b \neq -1,\ b \in R\setminus\{1\}$

5. Tính giao hoán

CM: $a^{*}b = b^{*}a;\ a,b \in R\setminus\{-1\}$

$a^{*}b = ab + a + b \ (1)$

$b^{*}a = ab + a + b \ (2)$

$\Rightarrow (1) = (2)$ (đpcm)