3.1. Tích trg $\langle x, y \rangle$ trên R^2, 1 ánh xạ là tích trg khi thoả 3 tính chất:

Song tuyến tính (Bilinearity)

Đối xứng (Symmetry

Xác định dương (Positive Definiteness)

- kiểm tra $\langle \alpha x + \beta x', y \rangle = \alpha \langle x, y \rangle + \beta \langle x', y \rangle$

Giải

• Ta cho $\alpha, \beta \in R$:

$\langle \alpha x + \beta x', y \rangle = (\alpha x_1 + \beta x_1') y_1 - ((\alpha x_1 + \beta x_1') y_2 + (\alpha x_2 + \beta x_2') y_1) + 2(\alpha x_2 + \beta x_2') y_2$

$= \alpha (x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2) + \beta (x_1' y_1 - (x_1' y_2 + x_2' y_1) + 2 x_2' y_2) = \alpha \langle x, y \rangle + \beta \langle x', y \rangle$

$\rightarrow$ Thoả TC Song tuyến tính (1)

• kiểm tra tính đối xứng:

Xét: $\langle y, x \rangle = y_1 x_1 - (y_1 x_2 + x_1 y_2) + 2 y_2 x_2$

ta thấy $\langle y, x \rangle = \langle x, y \rangle$

Vậy tích trg này đối xứng (2)

• kiểm tra: $\langle x, x \rangle = x_1^2 - (x_1 x_2 + x_2 x_1) + 2 x_2^2$

$= x_1^2 - 2 x_1 x_2 + 2 x_2^2$

$= (x_1 - x_2)^2 + x_2^2 \geq 0$

$(x_1 - x_2)^2 + x_2^2 = 0$ khi $\begin{cases} (x_1 - x_2) = 0 \\ x_2 = 0 \end{cases}$

Suy ra: $x_1 = x_2 = 0$

$\Rightarrow \langle x, x \rangle = 0 \Rightarrow x = 0 \quad (3)$

Từ (1), (2), (3) suy ra ~~tích trg~~ $\langle \cdot, \cdot \rangle$ là một tích trg

3.3 Tính khoảng cách: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

a) $\langle x, y \rangle := x^T y$

$$x - y = \begin{bmatrix} 1-(-1) \\ 2-(-1) \\ 3-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

Xét tích trg: $\langle x-y ; x-y \rangle = 2^2 + 3^2 + 3^2 = 22$

$\Rightarrow d(x,y) = \sqrt{22}$

b) $\langle x, y \rangle := x^T A y$, với $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$Ay = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}$$

$$x^T A y = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} = -8$$

$$d(x,y) = \sqrt{\langle x-y , A(x-y) \rangle}$$

. $A(x-y) = A \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}$

. $(x-y)^T A(x-y) = \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix} = 47$

$\Rightarrow d(x,y) = \sqrt{47}$

3.4 Compute angle between $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

using:

a. $\langle x, y \rangle := x^T y$

$||x|| = \sqrt{x^T x} = \sqrt{1^2 + 2^2} = \sqrt{5}$

$||y|| = \sqrt{y^T y} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

Góc giữa 2 vecto: $\omega = \cos^{-1}\left(\frac{\langle x, y \rangle}{||x|| \, ||y||}\right)$

$= \cos^{-1}\left(\frac{-3}{\sqrt{10}}\right) \approx 2{,}8198$

b) $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. $By = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

$x^T By = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \end{bmatrix} = -11$

~~$\omega = \cos^{-1}\left(\dfrac{\langle x, y\rangle}{\|x\|_B \|y\|_B}\right) = \cos^{-1} \dfrac{-11}{\sqrt{5}.\sqrt{2}}$~~

$\|x\|_B = \sqrt{x^T B x} = \sqrt{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \sqrt{18}$

$\|y\|_B = \sqrt{y^T B y} = \sqrt{\begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}} = \sqrt{7}$

$\omega = \cos^{-1}\left(\dfrac{\langle x, y\rangle}{\|x\|_B \|y\|_B}\right) = \cos^{-1}\left(\dfrac{-11}{\sqrt{18}\sqrt{7}}\right) = 2,941$

~~3.8 Sử dụng Gram-Schmit, chuyển hệ cơ sở $B = (b_1, b_2)$ thành cơ sở trực chuẩn $C = (c_1, c_2)$ của U~~

~~Với: $b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$~~

3.9 a) C/M: $\sum_{i=1}^{n} x_i^2 \geq \dfrac{1}{n}$ với đk: $\sum_{i=1}^{n} x_i = 1$

Lấy 2 vecto: $x = (x_1, x_2, \ldots x_n)$, $y = (1, 1, 1 \ldots 1)$

Bất đẳng thức Cauchy-Schwarz:

$\langle x, y\rangle^2 \leq \|x\|^2 . \|y\|^2$ (*)

Tích vô hướng: $\langle x, y\rangle = \sum_{i=1}^{n} x_i = 1$

Chuẩn $\|x\|^2 = \sum_{i=1}^{n} x_i^2$, $\|y\|^2 = n$

Thay vào bất đẳng thức (*), ta được:

$$1^2 = \left(\sum_{i=1}^{n} x_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot n$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 \geqslant \frac{1}{n} \text{ (đpcm)}$$

b) CM: $\sum_{i=1}^{n} \frac{1}{x_i} \geqslant n^2$

Chọn 2 vectơ: $x = (\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, \ldots, \sqrt{x_n})$

$$y = \left(\frac{1}{\sqrt{x_1}}, \frac{1}{\sqrt{x_2}}, \ldots, \frac{1}{\sqrt{x_n}}\right)$$

Áp dụng Cauchy-Schwarz:

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right) \quad (*)$$

Tính: $\cdot \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} \sqrt{x_i} \cdot \frac{1}{\sqrt{x_i}} = \sum_{i=1}^{n} 1 = n$

$\cdot \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i = 1$

Thay vào (*) ta được:

$$n^2 \leq 1 \cdot \sum_{i=1}^{n} \frac{1}{x_i} \Leftrightarrow \sum_{i=1}^{n} \frac{1}{x_i} \geqslant n^2$$

3.10 Ta sd dung ma trên quay

$$R = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$R_{x_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cancel{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} - \frac{3}{2} \\ 1 + \frac{3\sqrt{3}}{2} \end{bmatrix}$$

$$R_{x_2} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

Sau khi quay 30° ta đc 2 vec tơ mới

$$R_{x_1} = \begin{bmatrix} \sqrt{3} - \frac{3}{2} \\ 1 + \frac{3\sqrt{3}}{2} \end{bmatrix}, \quad R_{x_2} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$