Homework of Chapter 8

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Ex. 8.1

Solution. Since log function is a strictly concave function. Using Jensen's inequality, we have

$$\begin{aligned}
&E_q log[r(Y)/q(Y)] \\
\leq &log E_q[r(Y)/q(Y)] \\
= &log \int [r(y)/q(y)]q(y)\mathrm{d}y \\
= &log \int r(y)\mathrm{d}y \\
= &log 1 \\
= &0
\end{aligned}$$

The Jensen's equality shows that $E_q log[r(Y)/q(Y)]$ is maximized as a function of $r(y)$ when $r(y) = q(y)$.
According to the definition of (8.46), we have

$$\begin{aligned}
R(\theta', \theta) = &E[\ell_1(\theta', \mathbf{Z}^M|\mathbf{Z})|\mathbf{Z}, \theta] \\
= &E[log Pr(\mathbf{Z}^M|\mathbf{Z}, \theta')|\mathbf{Z}, \theta] \\
R(\theta, \theta) = &E[\ell_1(\theta, \mathbf{Z}^M|\mathbf{Z})|\mathbf{Z}, \theta] \\
= &E[log Pr(\mathbf{Z}^M|\mathbf{Z}, \theta)|\mathbf{Z}, \theta]
\end{aligned}$$

Then, using the equality we have just proved above, we can get

$$\begin{aligned}
R(\theta', \theta) - R(\theta, \theta) = &E[log Pr(\mathbf{Z}^M|\mathbf{Z}, \theta')|\mathbf{Z}, \theta] - E[log Pr(\mathbf{Z}^M|\mathbf{Z}, \theta)|\mathbf{Z}, \theta] \\
= &E[log(Pr(\mathbf{Z}^M|\mathbf{Z}, \theta')/Pr(\mathbf{Z}^M|\mathbf{Z}, \theta))|\mathbf{Z}, \theta] \\
\leq &0
\end{aligned}$$

Thus, we have proved $R(\theta, \theta) \geq R(\theta', \theta)$.

$\square$

Ex. 8.3

Solution.

$$
\begin{aligned}
&E_{U_\ell^{(t)}}[Pr(u|U_\ell^{(t)}, \ell \neq k)] \\
=&\int Pr(u|u_\ell^{(t)}, \ell \neq k) Pr(u_\ell^{(t)}, \ell \neq k) \mathrm{d}\ell \neq k \\
=&Pr_{U_k}(u)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
E[\hat{P}r_{U_k}(u)] =&E[\frac{1}{M-m+1} \sum_{t=m}^{M} Pr(u|U_\ell^{(t)}, \ell \neq k)] \\
=&\frac{1}{M-m+1} \sum_{t=m}^{M} E[Pr(u|U_\ell^{(t)}, \ell \neq k)] \\
=&Pr_{U_k}(u)
\end{aligned}
$$

Thus, we have justified the estimate (8.50).

$\square$