Chapt 5 - property of a Random sample.

Def: The R.V. $X_1 \dots X_n$ are called a random sample of size n if they are iid.

Example 0.1. *Suppose x is a Random draw from a population and x has density f.*

if $x_i, i = 1, 2 \cdots n$ are iid and. $x_i z_i x$, then.

$$f(x, \dots x_n) = \prod_{i=1}^{n} f(x_i)$$

Comment: In most case we don't use joint density of the sample, but rather use the iid Property directly

Example 0.2. *X_i are ind, $X_i \sim \operatorname{Exp}(\text{ rate } = \lambda), y = \min(x_i)$*

$F_y(y) = P(Y \leq y) = 1 - P(Y > y) = 1 - \prod_{i=1}^{n} P(x_i > y) \quad x's$ are iid
$= 1 - e^{-\lambda n y}$, cdt of $\exp(\lambda n)$

$$f_y(y) = \frac{d}{dx} F_y(y)$$

Definition 0.3. *Sampling w/o replacement from a finite population is called simple random sampling.*

- In most Cases, Samples are not independent.

Definition 0.4. *Suppose x_1, x_n is random sample, Arr r.V.Y of the form $Y = T(X_1, X_n)$ is coiled a statistic. The dist of Y is called. its sampling distribution. The dist of Y is called. its Sampling distribution.*

Comment: the supply dit con be found anally tally for unity a few statistics. and a few populations (eg. exponential, normal.)

Theorem 0.5. *Suppose $x_i, 1 \leq i \leq n$, are $\text{iid}\, \omega \mid E(x_i) = \mu, v(x_i) = \sigma^2$.*
(a) $E[\bar{x}] = \mu$.
(b). $V(\bar{x}) = \sigma^2/n$
(c) $E(S_n^2) = \sigma^2$.

Proof. Define $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2, \S_n^2 = \frac{n}{n-1} \hat{\sigma}^2$

$$\begin{aligned} E(\hat{\sigma}^2) &= \frac{1}{n} E\left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right) = \frac{1}{n} E\left[\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2\right] \\ &= \frac{1}{n}\left[\sum_{i=1}^{n} (\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right] \\ &= \frac{n-1}{n} \bar{\sigma}^2 \end{aligned}$$

$\square$

5.4 Order statistic

Definition 0.6. *The order statistics of a random sample $X_1 m X_n$ core the sample values placed in ascending order and are denoted $X_{(1)}, X_{(2)} \sim X_{(n)}$ $X_{(n)} \leq X_{(2)} \leq m \leq X_{(n)}$*

In Particular, $X_{(1)} = min X_i \quad X_{(n)} = \max X_i$.
The Sample range is defined $R = x_{(n)} - x_{(1)}$

The sample median, denoted M, is defined. $M = \begin{cases} x_{(\frac{n+1}{2})} \text{ if n is odd} \\ (x_{\frac{n}{2}} + x_{\frac{n}{2}+1})/2 \text{ if n i even} \end{cases}$

The notation $\{b\}$ in a subscript is defined to be number b rounded. to the nearest number

Example 0.7. *$\{b\} = i$ where $i - 0.5 \leq b < i + 0.5$.*

Example 0.8. *uniform $(0,1)$ order statistics. wish to find $(a) f_{u(k)}, (b) f_{u(k),u(i)}$*

$$\text{(a). Let } 1_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{ow.} \end{cases}$$

$$1_{[0,t]}(u_k) \sim \text{Ber}(t)$$

Define $B_n(t) = \sum_{i=1}^n 1_{[0,t]}(U_i) = \text{Binomial}\,(n,t)$. Event identify $\left[u_{(k)} > t\right] = [B_n(t) < k]$

$$\begin{aligned} F_{u_{(k)}}(t) &= P\left(u_{(k)} \leq t\right) = P\left(B_n(t) \geqslant k\right) = \sum_{i=1}^n \left(\begin{array}{c} n \\ 1 \end{array}\right) t^i (1-t)^{n-i}, \\ f_{u(k)}(t) &= \frac{d}{dt} F_{u(k)}(t) \\ &= \sum_{i=1}^n \frac{n!}{(n-i)!i!} i t^{i-1} (1-t)^{k-i} - \sum_{i=1}^n \frac{n!}{(n-i)!i!} (n-i) t^i (1-t)^{n-i-1} \\ &= \frac{n!}{(n-k)!(1k-1)!} t^{k-1} (1-t)^{n-k}, 0 < t < 1 \end{aligned}$$

However, we can obtain $f_{u(k)}$ by a more important and elementary "think method" argnmen$t$.

$f_{u(k)}(t) \leftarrow$ prob density of having $u_k = t$. Then having $u_1 \ldots u_{k-1}$ *all* $< t$ and $u_{k+1} m u_n$ all $< t$

$$f_{n(k)}(t) = \left(\begin{array}{c} n \\ k-1, 1, n-k \end{array}\right) f_u(t) \cdot \left(F_n(t)\right)^{k-1} \cdot \left(1 - F_n(t)\right)^{n-k}.$$

$$\downarrow$$

$$\frac{n!}{(k-1)!1!(n-k)!}$$

(b) $f_{u(k),u_{(i)}}(s,t) = \left(\begin{array}{c} n \\ k-1,1,,i-k,1,n-i \end{array} \right) s^{k-1} \cdot (t-s)^{i-1-k}(1-t)^{n-i}$, $0 \leq s \leq t \leq 1$