1 one

Example 1.1. *y = # of the egg laid, x = # of eggs that survive $y \sim poisson(\lambda)$, $x|y \sim binomial(Y, P)$*

1. pmf directly: $p(X = x) \to X \sim poission(\lambda P)$
2. mgfs:

$$M_x(t) = E(e_{tx}) = E(E(e^{tx}Y)) = E(pe_t + (1-p)^y) = \sum_{y=0}^{\infty} \frac{s^y \lambda^y e^{-y}}{y!} = \frac{e^{-\lambda}}{e^{-s\lambda}} \cdot \sum_{y=0}^{\infty} \frac{(sy)^\lambda}{y!} = e^{s-1}\lambda$$

recall that mgf of poisson(λ) is $M_x(t)e^{\lambda(e^t-1)}$, $E(x) = E_y(E(X|Y)) = E(Yp) = \lambda p$

Definition 1.2. *A r.v. X is said to have a mixture distribution if the dist of x depends on a quality that has a dist.*

* In general, hierarchical models led to mixture dist

Theorem 1.3. *For any two r.v.s X and Y, $V(Y) = E(V(Y|x)) + V(E(Y|X))$*

Proof.

$$\begin{aligned}
V(Y) &= E(Y^2) - (EY)^2 \\
&= E(E(Y^2|X)) - E(m(x))^2 \\
&= E(E(Y^2|X) - m(x)^2) + E(m|x)^2 - E(m(x)^2) \\
&= E(V(Y|X)) + V(m(x)) \\
&= E(V(Y|X) + V(E(Y|X)))
\end{aligned}$$

□

Example 1.4.

$$X|Y \sim Binomial(Y, P),\ Y \sim Poisson$$

$$V(x) = \lambda p$$

$$\begin{aligned}
V(x) &= V(E(X|Y)) + E(V(X|Y)) \\
&= V(YP) + E(YP(1-P)) \\
&= P^2\lambda + P(1-P)\lambda \\
&= P^2\lambda + P\lambda - p^2\lambda \\
&= P\lambda
\end{aligned}$$

Covariance and correction

Definition 1.5. $cov(X, Y) = E((x - \mu_x)(Y - \mu_y))$, $corr(X, Y) = \frac{cov(X,Y)}{\sigma_x \sigma_y}$

Basic Fact:

1. $cov(X, Y) = E(XY) - E(X)E(Y)$
2. $cov(X, Y) = 0$ if X,Y is i.d.

Proof.

$$\begin{aligned}
cov(X, Y) &= E((X - \mu_x)(Y - \mu_Y)) \\
&= E(XY - \mu_x Y - X\mu_y + \mu_x\mu_y) \\
&= E(XY) - 2\mu_x\mu_y - \mu_x\mu_y \\
&= E(XY) - \mu_x\mu_y
\end{aligned}$$

□

For any constand a.b.c.d
3. $corr(X,Y) = 0$ if X,Y i.d.
4. $cov(X,X) = V(X)$
5. $cov(aX+b, cY+d) = ac \cdot cov(X,Y)$
6. $corr(X,Y) = sign(ac) \cdot corr(X,Y)$
7. $corr(X,Y) = corr(\frac{x-\mu_x}{\sigma_x}, \frac{y-\mu_y}{\sigma_y})$
if $\rho_{xy} = \rho$ then (1) $-1 \leq \rho \leq 1$ (2) $P = \pm 1$ if and only if $Y - \mu_y = \pm \frac{\sigma_y}{\sigma_x}(X - \mu_x)$