Example 0.1. *$X \sim U(0,1)$, $Y_1 = x^2$, $Y_2 = x^3$, $Y_3 = x^4$, $Y_4 = X^{\frac{1}{2}}$.*

$\rho_{XY_1} = 0.968$, $\rho_{XY_2} = 0.916$, $\rho_{XY_3} = 0.866$, $\rho_{XY_4} = 0.968$,

1 Covariance of linear combination

Example 1.1. *$X \sim N(\mu, \sigma)$, $Y \sim poisson(\lambda)$, $Z \sim Gamma(\alpha, \beta)$, they are i.i.d. $U = X + Y$, $V = Y + Z$, $?cov(U,V)$*

$cov(U,V) = \lambda$, $corr(U,V) = \frac{cov(U,V)}{\sqrt{(\sigma^2+\lambda)(\lambda+\alpha\beta)}}$
Either: $cov(UV) = cov(X+Y, Y+Z) = cov(X,Y) + cov(Y,Y) + cov(Y,Z) + cov(X,Z) = V(Y) = \lambda$

Theorem 1.2.

$$\begin{aligned}
&T_x = \sum_{i=1}^{m} a_i x_i \quad E\left(x_i\right) = \mu_i \quad V\left(x_i\right) = \sigma_i^2 \\
&T_y = \sum_{j=1}^{n} b_j Y_j \quad E\left(y_i\right) = \gamma_i \quad V\left(x_i\right) = \tau_i^2 \\
&E\left(T_x\right) = \sum_{i=1}^{m} a_i \mu_i \quad E\left(T_x\right) = \sum_{\hat{j}=1}^{n} b_j \gamma_1 \\
&\operatorname{Cov}\left(T_x, T_y\right) = \sum_{i=1}^{m} \sum_{i=1}^{n} a_i b_j \operatorname{Cov}\left(x_i, Y_i\right). \\
&V\left(T_x\right) = \operatorname{Cov}(Tx, Tx) = \sum_{i=1}^{m} a_i^2 \sigma_i^2 + \sum_{i \neq j} a_i a_j \operatorname{Cov}\left(x_i, x_j\right)
\end{aligned}$$

Example 1.3. *X and Z i.i.d., let Y= X+Z, $?\rho_{x,y}$*

$$\begin{aligned}
\operatorname{cov}(x, x+z) &= \operatorname{cov}(x,x) + \operatorname{cov}(x,z) \\
&= v(x) \\
\operatorname{cov}(x, x+z) &= \frac{v(x)}{\sqrt{v(x) \cdot (v(x) + v(z))}} = \frac{1}{\sqrt{1 + \frac{\sigma_x^2}{\sigma_z^2}}}
\end{aligned}$$

Example 1.4. *group N men and N women, T_N = # men who made a pair w/ the women had paired them with*

$$\begin{aligned}
x_i &\sim \operatorname{Ber}\left(\frac{1}{N}\right) \quad V\left(T_N\right) = 1 \\
V(Tn) = V\left(\sum_{i=1}^{n} x_i\right) &= \sum_{i=j} \sum_{j} \operatorname{cov}\left(x_i, x_j\right) + \sum_{i \neq j} \sum_{j} \operatorname{Cov}\left(x_i, x_j\right) \\
&= N\left(\frac{1}{N}\left(1 - \frac{1}{N}\right)\right) + N(N-1) \operatorname{cov}\left(x_1, x_2\right) \\
&= \left(1 - \frac{1}{N}\right) + N(N-1)\left(E\left(x, x_2\right) - Z\left(x_1\right) Z\left(x_2\right)\right) \\
\operatorname{corr}\left(x, x_2\right) &= \frac{\operatorname{Cov}\left(x_1, x_2\right)}{\sqrt{v\left(x_1\right) \cdot v\left(x_2\right)}} = \frac{\frac{1}{N(N-1)} - \frac{1}{N_v}}{\frac{1}{N}\left(1 - \frac{1}{N}\right)}
\end{aligned}$$

Example 1.5. *An orn contains 5 R, 10 W, 5 B, N = # of red blue pairs.*

$$Y_i = \mathrm{Ber}\left(P_2\right) \quad P = \frac{\left(\begin{array}{c}5\\1\end{array}\right)\left(\begin{array}{c}5\\1\end{array}\right)}{\left(\begin{array}{c}2\\2\end{array}\right)}$$

$$E(N) = \sum_{j=1}^{10} E\left(Y_j\right) = 10 \cdot \frac{50}{25 \cdot 19}$$

$$v(N) = \sum_{j=1}^{13} v\left(Y_j\right) + \sum_{i \neq j} \operatorname{cov}\left(Y_i, Y_j\right)$$

$$= 10 \frac{50}{20 \cdot 19}\left(1 - \frac{50}{20 \cdot 19}\right) + 10 \cdot 9 \cdot \operatorname{cov}\left(Y_1, Y_2\right) = 0.823$$