one

Example 0.1. *$y = \#$ of the egg laid, $x = \#$ of eggs that survive $y \sim poisson(\lambda)$, $x|y \sim binomial(Y, P)$*

1. pmf directly: $p(X = x) \to X \sim poission(\lambda P)$
2. mgfs:

$$M_x(t) = E(e_{tx}) = E(E(e^{tx}Y)) = E(pe_t + (1-p)^y) = \sum_{y=0}^{\infty} \frac{s^y \lambda^y e^{-y}}{y!} = \frac{e^{-\lambda}}{e^{-s\lambda}} \cdot \sum_{y=0}^{\infty} \frac{(sy)^\lambda}{y!} = e^{s-1}\lambda$$

recall that mgf of poisson(λ) is $M_x(t)e^{\lambda(e^t-1)}$, $E(x) = E_y(E(X|Y)) = E(Yp) = \lambda p$

Definition 0.2. *A r.v. X is said to have a mixture distribution if the dist of x depends on a quality that has a dist.*

* In general, hierarchical models led to mixture dist

Theorem 0.3. *For any two r.v.s X and Y, $V(Y) = E(V(Y|x)) + V(E(Y|X))$*

Proof.

$$\begin{aligned} V(Y) &= E(Y^2) - (EY)^2 \\ &= E(E(Y^2|X)) - E(m(x))^2 \\ &= E(E(Y^2|X) - m(x)^2) + E(m|x)^2 - E(m(x)^2) \\ &= E(V(Y|X)) + V(m(x)) \\ &= E(V(Y|X) + V(E(Y|X))) \end{aligned}$$

□

Example 0.4.

$$X|Y \sim Binomial(Y, P),\ Y \sim Poisson$$

$$V(x) = \lambda p$$

$$\begin{aligned} V(x) &= V(E(X|Y)) + E(V(X|Y)) \\ &= V(YP) + E(YP(1-P)) \\ &= P^2\lambda + P(1-P)\lambda \\ &= P^2\lambda + P\lambda - p^2\lambda \\ &= P\lambda \end{aligned}$$

Covariance and correction

Definition 0.5. $cov(X,Y) = E((x-\mu_x)(Y-\mu_y))$, $corr(X,Y) = \frac{cov(X,Y)}{\sigma_x \sigma_y}$

Basic Fact:

1. $cov(X,Y) = E(XY) - E(X)E(Y)$
2. $cov(X,Y) = 0$ if X,Y is i.d.

Proof.

$$\begin{aligned} cov(X,Y) &= E((X-\mu_x)(Y-\mu_Y)) \\ &= E(XY - \mu_x Y - X\mu_y + \mu_x\mu_y) \\ &= E(XY) - 2\mu_x\mu_y - \mu_x\mu_y \\ &= E(XY) - \mu_x\mu_y \end{aligned}$$

□