2

$$\text{Let } T : \mathbb{R}^4 \to \mathbb{R}^4 \text{ be a linear operator}$$

$$\text{Let } T\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\2\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\3\\0\\3\end{pmatrix}$$

$$T\begin{pmatrix}-2\\-1\\2\\1\end{pmatrix} = T\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

2a

$$\text{Find an orthonormal basis } B \text{ and diagonal matrix } D \text{ such that } [T]_B^B = D$$

$$\text{Solution:}$$

$$\text{Let } E = \left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}\right\} \text{ be a basis of } \mathbb{R}^4$$

$$T(v_1) = 2v_1$$

$$T(v_2) = 3v_2$$

$$T(v_3) = T(v_4) = 0$$

$$\implies v_1, v_2, v_3, v_4 \text{ are eigenvectors of } T \text{ and } P_T(x) = x^2(x-2)(x-3)$$

$$\dim E_0 = 2 \implies T \text{ is diagonalizable}$$

$$\text{Let us apply Gram-Schmidt to } E_0$$

$$u_1 = v_1 = \begin{pmatrix}-2\\-1\\2\\1\end{pmatrix} \implies \frac{u_1}{\|u_1\|} = \frac{u_1}{\sqrt{10}}$$

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2}u_1 = \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} - \frac{6}{10}\begin{pmatrix}-2\\-1\\2\\1\end{pmatrix} = \frac{1}{10}\begin{pmatrix}-10+12\\-10+6\\10-12\\10-6\end{pmatrix} = \frac{2}{10}\begin{pmatrix}1\\-2\\-1\\2\end{pmatrix}$$

$$\|u_2\| = \frac{2}{10}\cdot\sqrt{10} \implies \frac{u_2}{\|u_2\|} = \frac{\sqrt{10}}{2}u_2$$

$$\implies D = \begin{pmatrix}2 & & & \\ & 3 & & \\ & & 0 & \\ & & & 0\end{pmatrix}, B = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix}-2\\-1\\2\\1\end{pmatrix}, \frac{\sqrt{10}}{10}\begin{pmatrix}1\\-2\\-1\\2\end{pmatrix}\right\}$$

2b

Find the vector in $\ker T$ which is closest to $v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

Solution:

$$\forall u \in \mathbb{R}^4 : \|v - u\| \geq \|v - P_{\ker T}(v)\|$$

$$\text{An orthogonl basis of } \ker T \text{ is } \left\{ \begin{pmatrix} -2 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix} \right\}$$

$$\implies P_{\ker T}(v) = \frac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle v, v_2 \rangle}{\|v_2\|^2} v_2 = \frac{6}{10} \begin{pmatrix} -2 \\ -1 \\ 2 \\ 1 \end{pmatrix} + \frac{2}{10} \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

3a

Let $A, B \in \mathbb{F}^{n \times n}$

Prove or disprove: A is diagonalizable and $P_A(B) = 0 \implies B$ is diagonalizable

Disproof:

$$\text{Let } A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Let } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$P_A(B) = (B - I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0$$

But B is not diagonalizable

3b

Let $A, B \in \mathbb{C}^{2 \times 2}$

Let $C = AB - BA$

Prove or disprove: C is not nilpotent $\implies C$ is diagonalizable

Proof:

$$tr(C) = tr(AB - BA) = tr(AB) - tr(BA) = 0$$

Let C be not nilpotent

$$\implies P_C(x) = \begin{bmatrix} x(x-\lambda) & \lambda \neq 0 \\ (x-\lambda_1)(x-\lambda_2) & \lambda_1, \lambda_2 \neq 0 \\ (x-\lambda)^2 & \lambda \neq 0 \end{bmatrix}$$

First two options guarantee a diagonalizable matrix

Let $P_C(x) = (x - \lambda)^2$

$$\implies tr(C) = 2\lambda = 0 \implies \lambda = 0 - \text{Contradiction!}$$

$$\implies \boxed{C \text{ is diagonalizable}}$$

3c

Let $A \in \mathbb{F}^{n \times n}$

Prove or disprove: $(A - 3I)(A + 2I) = 0 \implies \exists v \neq 0 \in \mathbb{F}^n : Av = 3v$

Disproof:

$$\text{Let } A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \implies A + 2I = 0$$

3 is not an eigenvalue of $A \implies \forall v \in \mathbb{F}^n : Av \neq 3v$

3d

$$\text{Let } A \in \mathbb{F}^{n\times n} \text{ be invertible}$$

$$\text{Prove or disprove: } \forall v \in \mathbb{F}^n : v \text{ is an eigenvector of } A \implies v \text{ is an eigenvector of } A^{-1}$$

$$\text{Proof?:}$$

$$\text{Let } v \text{ be an eigenvector of } A$$

$$A \text{ is invertible} \implies \lambda \neq 0$$

$$\implies Av = \lambda v \implies v = \frac{1}{\lambda} Av$$

$$\implies A^{-1}v = A^{-1}\left(\frac{1}{\lambda}Av\right) = \frac{1}{\lambda}A^{-1}Av = \frac{1}{\lambda}v$$

$$\implies v \text{ is an eigenvector of } A^{-1}$$

4

$$\text{Let } A \in \mathbb{R}^{9\times 9}$$

$$\text{Let } A^3 = 0, rank(A^2) = 2$$

$$\text{Find all possible Jordan forms of } A$$

$$\text{Solution:}$$

$$A^3 = 0 \implies A \text{ is nilpotent} \implies P_A(x) = x^9$$

$$rank(A^2) = 2 \implies A^2 \neq 0 \implies m_A(x) = x^3$$

$$\implies \text{Largest block in Jordan form of } A \text{ is of size } 3$$

$$J_A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & B \end{pmatrix} \implies J_A^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & \\ & & 0 & \\ & & & B^2 \end{pmatrix}$$

$$A = PJ_AP^{-1} \implies A^2 = PJ_A^2P^{-1}$$

$$\implies rank(PJ_A^2P^{-1}) = 2 \implies rank(J_A^2) = 2$$

$$\implies rank(B^2) = 1$$

$$J_1(0)^2 = 0, J_2(0)^2 = 0 \implies B \text{ contains exactly one Jordan block of size } 3$$

$$\implies B = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & C \end{pmatrix} \implies B^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & \\ & & 0 & \\ & & & C^2 \end{pmatrix} \implies rank(C^2) = 0$$

$$\implies C \text{ can only contain blocks of size 2 or 1}$$

$$\implies \text{Possible Jordan forms are:}$$

$$J_A = J_3(0) \oplus J_3(0) \oplus \left[\begin{array}{c} J_2(0) \oplus J_1(0) \\ J_1(0) \oplus J_1(0) \oplus J_1(0) \end{array}\right.$$

5a

$$\text{Prove: } \forall M \in \mathbb{R}^{n\times n} : M \text{ is invertible} \implies M^TM \text{ is positive symmetric}$$

$$\text{Proof:}$$

$$(M^TM)^T = M^TM \implies M^TM \text{ is symmetric}$$

$$\text{Let } \lambda \text{ be an eigenvalue of } M^TM$$

$$\implies M^TMv = \lambda v$$

$$\implies \lambda\|v\| = \langle \lambda v, v\rangle = \langle M^TMv, v\rangle = \langle Mv, Mv\rangle = \|Mv\|$$

$$v \neq 0 \implies Mv \neq 0 \implies \|Mv\| > 0 \implies \boxed{\lambda > 0}$$

5b

$$\text{Let } B \in \mathbb{R}^{n\times n} \text{ be positive symmetric}$$
$$\text{Prove: } \exists M \in \mathbb{R}^{n\times n} \text{ invertible: } B = M^T M$$

$$\text{Proof:}$$
$$B \text{ is symmetric} \implies B \text{ is orthogonal diagonalizable}$$
$$\implies B = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^T$$
$$\forall i \in [1,n] : \lambda_i > 0 \implies \exists \sqrt{\lambda_i} \in \mathbb{R}$$
$$\text{Let } M = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} P^T$$
$$\begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} \text{ is invertible, } P^T \text{ is invertible} \implies M \text{ is invertible}$$
$$M^T M = P \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}^2 P^T = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^T = B$$

5c

$$\text{Let } B \in \mathbb{R}^{n\times n} \text{ be positive symmetric}$$
$$\text{Prove: } \forall P \in \mathbb{R}^{n\times n} \text{ invertible: } P^T B P \text{ is positive symmetric}$$

$$\text{Proof:}$$
$$\exists M \text{ invertible: } B = M^T M$$
$$\implies P^T B P = P^T M^T M P = (MP)^T M P \implies P^T B P \text{ is positive symmetric}$$

5d

$$\text{Let } A \in \mathbb{R}^{n\times n} \text{ be positive symmetric}$$
$$\text{Prove: } \exists P \in \mathbb{R}^{n\times n} \text{ invertible: } P^T A P = I$$

$$\text{Proof:}$$
$$A \text{ is orthogonal diagonalizable}$$
$$\implies \exists P : A = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^T$$
$$\text{Let } P = Q \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix}$$
$$Q \text{ is invertible, } \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix} \text{ is invertible} \implies P \text{ is invertible}$$
$$\implies P^T A P = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix} Q^T A Q \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix} =$$
$$= \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix} = I$$

5e

Let $A, B \in \mathbb{R}^{n\times n}$ be positive symmetric
Let $P \in \mathbb{R}^{n\times n}$ be invertible
Let $P^T A P = I$
Prove: $\det(A+B) \geq \det(A) + \det(B)$

Proof:

Let us fisrt prove $\det(I + P^T BP) \geq \det(I) + \det(P^T BP)$

B is positive symmetric $\implies$ $P^T BP$ is positive symmetric

$$\implies \begin{cases} \det(I + P^T BP) = \prod_{i=1}^n (\lambda_i + 1) \\ \det(I) + \det(P^T BP) = 1 + \prod_{i=1}^n \lambda_i \end{cases}$$

$$\prod_{i=1}^n (\lambda_i + 1) = \prod_{i=1}^n \lambda_i + \underbrace{\prod_{i=2}^n (\lambda_i + 1)}_{\geq 0} + \underbrace{\dots}_{\geq 0} + 1 \geq \prod_{i=1}^n \lambda_i + 1$$

$$\implies \det(I + P^T BP) \geq \det(I) + \det(P^T BP)$$

$$\det(I + P^T BP) = \det(P^T (A+B) P) = \det(P^T) \cdot \det(A+B) \cdot \det(P)$$

$$\det(I) + \det(P^T BP) = \det(P^T AP) + \det(P^T BP) =$$

$$= \dots = \det(P^T) \cdot (\det(A) + \det(B)) \cdot \det(P)$$

P is invertible $\implies$ $\det(P^T) = \det(P) = X > 0$

$$\implies X^2 \cdot \det(A+B) \geq X^2 (\det(A) + deT(B))$$

$$\implies \det(A+B) \geq \det(A) + \det(B)$$