Mean value theorem for definite integrals #theorem

$$\text{Let } f \text{ be continuous on } [a,b]$$
$$\text{Then } \exists c \in [a,b] : \int_a^b f(x)dx = f(c)(b-a)$$

Definite Integral in the point #definition

$$\int_a^a f(x)dx = 0$$

Definite integral on inverse interval #theorem

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Area function #definition

$$\text{Let } f \text{ be Riemann-integrable on } [a,b]$$
$$S \text{ is called an area function of } f \text{ and defined as}$$
$$S(x) = \int_a^x f(t)dt$$

$$\text{Let } f(x) = x \text{ on } [0,1]$$
$$\implies S(x) = \frac{x^2}{2}$$
$$S'(x) = f(x)$$

Continuity of area function #theorem

$$\text{Let } f \text{ be a Riemann-integrable function on } [a,b]$$
$$\text{Let } S \text{ be an area function of } f$$
$$\text{Then } S \text{ is continuous}$$

$$\text{Proof:}$$
$$\text{Let } c \in [a,b]$$
$$\text{Let } x \to c$$
$$0 \le |S(x) - S(c)| = \left|\int_a^x f(t)dt - \int_a^c f(t)dt\right|$$
$$\int_a^x f(t)dt = \int_a^c f(t)d(t) + \int_c^x f(t)dt$$
$$\implies \left|\int_a^x f(t)dt - \int_a^c f(t)dt\right| = \left|\int_c^x f(t)dt\right| \le \int_c^x |f(t)|dt$$
$$f \text{ is Riemann-integrable} \implies |f| \text{ is Riemann-integrable}$$
$$\implies |f| \text{ is bounded} \implies \exists M : |f| \le M$$
$$\implies \int_c^x |f(t)|dt \le \int_c^x Mdt = M(x-c)$$
$$x \to c \implies x - c \to 0$$
$$0 \le |S(x) - S(c)| \le M(x-c) \to 0$$
$$\implies S(x) - S(c) \to 0 \implies \lim_{x \to c} S(x) = S(c) \implies \boxed{S \text{ is continuous on } [a,b]}$$

Fundamental theorem of Calculus (Part 1) #theorem

$$\text{Let } f \text{ be continuous on } [a,b]$$
$$\text{Then } S(x) = \int_a^x f(t)dt \text{ is differentiable and } S'(x) = f(x)$$

$$\text{Proof:}$$
$$\text{Let } c \in [a,b]$$
$$S'(c) = \lim_{h\to 0} \frac{S(c+h) - S(c)}{h} = \lim_{h\to 0} \frac{\left(\int_a^{c+h} f(t)dt - \int_a^c f(t)dt\right)}{h} =$$
$$= \lim_{h\to 0} \frac{\int_c^{c+h} f(t)dt}{h}$$
$$f \text{ is continuous } \implies \exists d \in [c, c+h] : \int_c^{c+h} f(t)dt = f(d)(c+h-c) = f(d)h$$
$$\implies \lim_{h\to 0} \frac{\int_c^{c+h} f(t)dt}{h} = \lim_{h\to 0} \frac{f(d)h}{h} = \lim_{h\to 0} f(d)$$
$$c \le d \le \underbrace{c+h}_{\to c+0=c} \implies d \to c$$
$$f \text{ is continuous } \implies f(d) \to f(c)$$
$$\implies S'(c) = \lim_{h\to 0} f(d) = f(c)$$

Fundamental theorem of Calculus (Part 2) aka Newton-Leibniz theorem

#theorem

$$\text{Let } f \text{ be Riemann-integrable on } [a,b]$$
$$\text{Let } F \text{ be continuous and a primitive of } f$$
$$\text{Then } \int_a^b f(x)dx = F(b) - F(a)$$

$$\text{Proof:}$$
$$\text{Let } f \text{ be continuous}$$
$$F' - S' = f - f = 0 \implies \exists C : \forall x \in [a,b] : F(x) = S(x) + C$$
$$\int_a^b f(t)dt = \int_a^b f(t)dt - 0 = \int_a^b f(t)dt - \int_a^a f(t)dt =$$
$$= S(b) - S(a) = F(b) + C - F(a) - C = F(b) - F(a)$$

$$\text{Let f be non-continuous}$$
$$F \text{ is continuous on } [a,b] \text{ and differentiable on } (a,b)$$
$$\implies \exists c \in (a,b) : F'(c)(b-a) = F(b) - F(a)$$
$$\text{Let } a = x_0 < x_1 < \cdots < x_n = b$$
$$F(b) - F(a) = F(x_n) - F(x_0) =$$
$$= F(x_n) + (-F(x_{n-1}) + F(x_{n-1})) + \cdots + (-F(x_1) + F(x_1)) - F(x_0) =$$
$$= (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \cdots + (F(x_2) - F(x_1)) + (F(x_1) - F(x_0))$$
$$F(b) - F(a) = \sum_{i=1}^n (F(x_i) - F(x_{i-1}))$$
$$\forall i \in [1,n] : \exists c_i \in [x_{i-1}, x_i] : F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$$
$$\implies \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = \sum_{i=1}^n f(c_i)\Delta x_i = S(f,P,C)$$
$$\text{Where } \begin{array}{c} P = \{x_0, x_1, \ldots, x_n\} \\ C = \{c_1, c_2, \ldots, c_n\} \end{array}$$
$$\implies \lim_{\lambda(P)\to 0} F(b) - F(a) = \lim_{\lambda(P)\to 0} S(f,P,C)$$
$$\implies F(b) - F(a) = \int_a^b f(x)dx$$

The following notation can be used:

$$F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}$$

$$\text{Let } f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases} \text{ on } [0,2]$$

f is Riemann-integrable by Lebesgue criterion, it is bounded and has one discontinuity

But, by Darboux theorem, there is no primitive of f because it has a jump discontinuity

Even more that that, if f has a primitive, it does not imply that f is Riemann-integrable

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^3}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$F'(0) = \lim_{h \to 0} \frac{F(h) - F(0)}{h} = \lim_{h \to 0} \frac{F(h)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h^3}\right) = 0$$

$$x \neq 0 \implies F'(x) = 2x \sin\left(\frac{1}{x^3}\right) + \left(-\frac{3}{x^2}\right) \cos\left(\frac{1}{x^3}\right)$$

$$\implies F'(x) = f(x) = \begin{cases} 2x \sin\left(\frac{1}{x^3}\right) - \frac{3}{x^2} \cos\left(\frac{1}{x^3}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f \text{ is not bounded because of } \frac{3}{x^2}, \text{ when } x \to 0$$

$$\implies f \text{ is not Riemann-integrable, but it does have a primitive} - F(x)$$

$$\lim_{x \to 0} \frac{\int_0^x \sin(t^2)dt}{x^3} = ???$$
