1a

$$\int \frac{x \sin x}{\cos^2 x}\, dx$$

$$\text{Solution:}$$

$$f(x) = x \implies f'(x) = 1$$

$$g'(x) = \frac{\sin x}{\cos^2 x} \implies g(x) = \int \frac{\sin x}{\cos^2 x}\, dx$$

$$\int \frac{\sin x}{\cos^2 x}\, dx = \begin{Bmatrix} t = \cos x \\ dt = -\sin x dx \end{Bmatrix} = -\int \frac{1}{t^2}\, dt = \frac{1}{t} = \frac{1}{\cos x} + C$$

$$\text{Let } C = 0$$

$$\implies \int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx =$$

$$= \frac{x}{\cos x} - \int \frac{1}{\cos x}\, dx$$

$$\int \frac{1}{\cos x}\, dx = \begin{Bmatrix} t = \tan \frac{x}{2} \\ dt = \frac{1+t^2}{2} dx \\ \cos x = \frac{1-t^2}{1+t^2} \end{Bmatrix} = 2\int \frac{1}{1-t^2}\, dt = \int \frac{1}{1-t} + \frac{1}{1+t}\, dt =$$

$$= -\ln|1-t| + \ln|1+t| = \ln\left|1 + \tan\frac{x}{2}\right| - \ln\left|1 - \tan\frac{x}{2}\right| + C$$

$$\implies \int \frac{x \sin x}{\cos^2 x}\, dx = \frac{x}{\cos x} - \ln\left|1 + \tan\frac{x}{2}\right| + \ln\left|1 - \tan\frac{x}{2}\right| + C$$

1b

$$\int \ln(\sin x) \cos^3 x\, dx$$

$$\text{Solution:}$$

$$\int \ln(\sin x)\cos^3 x\, dx = \begin{Bmatrix} t = \sin x \\ dt = \cos x dx \end{Bmatrix} = \int \ln(t)(1-t^2)\, dt$$

$$f(t) = \ln(t) \implies f'(t) = \frac{1}{t}$$

$$g'(t) = 1 - t^2 \implies g(t) = t - \frac{t^3}{3}$$

$$\implies \int \ln(t)(1-t^2)\, dt = \ln(t)\left(t - \frac{t^3}{3}\right) - \int 1 - \frac{t^2}{3}\, dt =$$

$$= t\ln(t) - \frac{t^3 \ln(t)}{3} - t + \frac{t^3}{9} =$$

$$= \sin x \ln(\sin x) - \frac{\sin^3 x \ln(\sin x)}{3} - \sin x + \frac{\sin^3 x}{9} + C$$

2

$$\text{Let } f \text{ be a function defined on } [a, b]$$

$$\text{Let } S \in \mathbb{R} : \forall n \in \mathbb{N} : \forall \{x_0, \dots, x_n\} \text{ partition of } [a, b] : \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = S$$

2a

Prove or disprove: f is integrable on $[a,b] \implies \int_a^b f(x)\,dx = f(a)\cdot(b-a)$

Proof:

$$\text{Let } \{a,b\} \text{ be a partition of } [a,b]$$
$$\implies f(a)\cdot(b-a) = S$$
$$f \text{ is integrable} \implies \int_a^b f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(x_{k-1})\cdot \Delta x_k =$$
$$= \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = \lim_{\Delta x_k \to 0} S = S = f(a)\cdot(b-a)$$

2b

Prove or disprove: f is constant

Disproof:

$$\text{Let } a = 0, b = 1$$
$$\text{Let } S = 1$$
$$\text{Let } f(x) = \begin{cases} 1 & x \neq 1 \\ 0 & x = 1 \end{cases}$$
$$\forall \{x_k\} \text{ partitions of } [a,b] : \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = \sum_{k=1}^{n}(x_k - x_{k-1}) =$$
$$= x_n - x_0 = 1 - 0 = 1 = S$$

3a

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{k}{n^2 + nk}$$

Solution:

$$x_k = \frac{k}{n}$$
$$\Delta x_k = \frac{1}{n}$$
$$\lim_{n\to\infty} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$
$$\implies f\left(\frac{k}{n}\right) = \frac{k}{n + \underbrace{k}_{=\frac{nk}{n}}} = \frac{k}{n\left(1+\frac{k}{n}\right)} \implies f(x) = \frac{x}{1+x}$$
$$f \text{ is continuous and bounded on } [0,1] \implies f \text{ is integrable on } [0,1]$$
$$\implies \lim_{n\to\infty} \sum_{k=0}^{n} \frac{k}{n^2+nk} = \int_0^1 f(x)\,dx = \int_0^1 \frac{x}{1+x}\,dx =$$
$$= \int_0^1 1 - \frac{1}{1+x}\,dx = 1 - \int_0^1 \frac{1}{1+x}\,dx = \boxed{1 - \ln 2}$$

3b

Let $f(x) = \ln(\cos x)$

Find length of its graph on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

Solution:

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$\implies L(f) = \int_{\pi/3}^{\pi/2} \sqrt{1+\tan^2 x}\,dx = \int_{\pi/3}^{\pi/2} \sqrt{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}}\,dx = \int_{\pi/3}^{\pi/2} \frac{1}{\cos x}\,dx$$

$$\int \frac{1}{\cos x}\,dx = \left\{\begin{matrix} t = \tan\frac{x}{2} \\ dt = \frac{1+t^2}{2}dx \\ \cos x = \frac{1-t^2}{1+t^2} \end{matrix}\right\} = 2\int \frac{1}{1-t^2}\,dt = \int \frac{1}{1-t} + \frac{1}{1+t}\,dt =$$

$$= -\ln|1-t| + \ln|1+t| = \ln\left|1+\tan\frac{x}{2}\right| - \ln\left|1-\tan\frac{x}{2}\right| + C$$

$$\implies L(f) = \ln\left|1+\tan\frac{\pi}{4}\right| - \underbrace{\ln\left|1-\tan\frac{\pi}{4}\right|}_{=\ln 0 \implies \text{We will take limit instead}} - \ln\left|1+\tan\frac{\pi}{6}\right| + \ln\left|1-\tan\frac{\pi}{6}\right|$$

$$x \to \frac{\pi}{2}^- \implies \ln\left|1-\tan\frac{x}{2}\right| \underbrace{\to}_{\ln 0} -\infty$$

$$\implies L(f) = \ln 2 - \text{"}-\infty\text{"} - \ln\left(1+\frac{1}{\sqrt{3}}\right) + \ln\left(1-\frac{1}{\sqrt{3}}\right) = C + \infty = \infty$$

4

$$\sum_{n=0}^{\infty} \frac{1}{9^n(2n)!}$$

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Both of these converge absolutely on $\mathbb{R} \implies$ We can sum them and reorder terms:

$$\implies e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{(1+(-1)^n)x^n}{n!} = \sum_{n=0}^{\infty} \frac{2x^{2n}}{(2n)!}$$

$$\text{Let } x = \frac{1}{3} \implies \sum_{n=0}^{\infty} \frac{1}{9^n(2n)!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{2\left(\frac{1}{3}\right)^{2n}}{(2n)!} = \frac{e^{1/3}+e^{-1/3}}{2}$$

5

Find global extremums of $f(x,y) = 2x^4 + y^4 - x^2 - y^2$

Limited by $x^2 + y^2 \leq 2$

Solution:

First let's find critical points within the limits

$$f_x = 8x^3 - 2x = 0 \implies x(8x^2 - 2) = 0 \implies x(2x-1)(2x+1) = 0$$

$$f_y = 4y^3 - 2y = 0 \implies y(\sqrt{2}y - 1)(\sqrt{2}y + 1) = 0$$

$$\implies \begin{matrix} x = 0, \frac{1}{2}, -\frac{1}{2} \\ y = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \end{matrix}$$

All combinations of these are within the domain

$$f(0,0) = 0$$

$$f\left(0, \pm\frac{1}{\sqrt{2}}\right) = 2$$

$$f\left(\pm\frac{1}{2}, 0\right) = -\frac{1}{8}$$

$$f\left(\pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = -\frac{3}{8}$$

Let us now examine points on the border:

Let $g(x,y) = x^2 + y^2 - 2$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \implies \begin{cases} 8x^3 - 2x = 2\lambda x \implies x(4x^2 - 1 - \lambda) = 0 \\ 4y^3 - 2y = 2\lambda y \implies y(2y^2 - 1 - \lambda) = 0 \\ x^2 + y^2 - 2 = 0 \end{cases}$$

$x = 0, y = 0$ is not on the border

$$x = 0 \implies y^2 = 2 \implies y = \pm\sqrt{2}$$

$$y = 0 \implies x^2 = 2 \implies x = \pm\sqrt{2}$$

$$4x^2 - 1 - \lambda = 0, 2y^2 - 1 - \lambda = 0 \implies 4x^2 = 2y^2 \implies x^2 = \frac{1}{2}y^2$$

$$\implies \frac{3}{2}y^2 = 2 \implies y^2 = \frac{4}{3} \implies y = \pm\frac{2}{\sqrt{3}} \implies x = \pm\frac{\sqrt{2}}{\sqrt{3}}$$

All critical points on the border are then:

$$(0, \pm\sqrt{2}), (\pm\sqrt{2}, 0), \left(\frac{\pm\sqrt{2}}{\sqrt{3}}, \frac{\pm 2}{\sqrt{3}}\right)$$

$$f(0, \pm\sqrt{2}) = 2$$

$$f(\pm\sqrt{2}, 0) = 6$$

$$f\left(\frac{\pm\sqrt{2}}{\sqrt{3}}, \frac{\pm 2}{\sqrt{3}}\right) = \frac{8}{9} + \frac{16}{9} - \frac{6}{9} - \frac{12}{9} = \frac{6}{9}$$

$$\implies \begin{cases} \text{Global maximums are } f(\pm\sqrt{2}, 0) = 6 \\ \text{Global minimums are } f\left(\pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = -\frac{3}{8} \end{cases}$$