# 1

Formulate and prove Riesz theorem

# 2a

Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator

$$\text{Let } [T]_S^S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Find $(\ker T)^{\perp}$

Solution:

$$[\ker T]_S = N([T]_S^S)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies \ker T = sp\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\implies (\ker T)^{\perp} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| z - x = 0 \right\} = \boxed{sp\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

# 2b

$$U = sp\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Determine whether $T[U] \subseteq U$

Solution:

$$[T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}]_S = [T]_S^S[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}]_S = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in U$$

$$[T\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}]_S = [T]_S^S[\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}]_S = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \in U$$

$$\implies \forall u \in U : u = \alpha\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \implies T(u) = T(\alpha v_1 + \beta v_2) = (2\alpha + \beta)v_1 + 2\beta v_2 \in U$$

$$\implies T[U] \subseteq U$$

# 2c

Determine whether there exists a basis $E$ of $\mathbb{R}^3$ such that:

$$[T]_E^E = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

Solution:

$$P_T(x) = P_{[T]_S^S}(x) = \begin{vmatrix} x-1 & -1 & -1 \\ 0 & x-2 & 0 \\ -1 & 0 & x-1 \end{vmatrix} = (x-1)^2(x-2) - (x-2) = x(x-2)^2$$

$$\text{Let } E = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, v \right\}$$

$$T[U] \subseteq U \implies \begin{cases} [T(u_1)]_E = [2u_1]_E = 2e_1 \\ [T(u_2)]_E = [u_1 + 2u_2]_E = e_1 + 2e_2 \end{cases}$$

$$\implies [T]_E^E = \begin{pmatrix} 2 & 1 & a \\ 0 & 2 & b \\ 0 & 0 & c \end{pmatrix}$$

$$\text{Let } v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \implies T(v) = 0 \implies [T]_E^E = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which also happens to be a normal Jordan form of $T$

# 3

Let $V = \mathbb{C}^{n \times n}$

Let $A \in V$ be invertible

Let $T : V \to V : T(B) = ABA^{-1}$

## 3a

Find $T^*$

Solution:

$$\forall B, C \in V : \langle T(B), C \rangle = \langle B, T^*(C) \rangle$$

$$\langle T(B), C \rangle = \langle ABA^{-1}, C \rangle = tr(ABA^{-1}C^*) = tr(BA^{-1}C^*A)$$

$$\langle B, T^*(C) \rangle = tr(B(T^*(C))^*)$$

$$\implies A^{-1}C^*A = (T^*(C))^* \implies \boxed{T^*(C) = A^*C(A^{-1})^*}$$

## 3b

Let $A$ be unitary

Determine whether $T$ is necessarily unitary

Solution:

$$A \text{ is unitary} \implies A^* = A^{-1}$$

$$T \text{ is unitary} \iff \forall B \in V : \|B\|^2 = \|T(B)\|^2 = \|ABA^{-1}\|^2$$

$$\|ABA^{-1}\|^2 = \langle ABA^{-1}, ABA^{-1} \rangle = \langle ABA^*, ABA^* \rangle = tr(ABA^*(ABA^*)^*) =$$

$$= tr(ABA^*AB^*A^*) = tr(ABB^*A^*) = tr(BB^*A^*A) = tr(BB^*) = \langle B, B \rangle = \|B\|^2$$

$$\implies \boxed{T \text{ is unitary}}$$

# 4a

Let $A \in \mathbb{R}^{n \times n}$

Prove or disprove: $A^2 = A \implies tr(A) = rank(A)$

Proof:

$$A^2 = A \implies A(A - I) = 0 \implies m_A(x) \mid x(x-1)$$

$$\implies m_A(x) = \begin{bmatrix} x \\ x - 1 \\ x(x-1) \end{bmatrix. \implies A \text{ is diagonalizable}$$

$$P_A(x) = x^k (x-1)^t$$

$$\implies tr(A) = tr(D) = \sum_{i=1}^{k} 0 + \sum_{i=1}^{t} 1 = t$$

$$g_0 = k_0 = k \implies \dim N(A) = k \implies rank(A) = n - k = t$$

$$\implies \boxed{tr(A) = rank(A)}$$

# 4b

Let $A \in \mathbb{R}^{n \times n}$ be symmetric

Prove or disprove: $C(A) = N(A)^{\perp}$

Proof:

$$A \text{ is symmetric} \implies C(A) = R(A)$$

Let $v \in N(A)$

$$Av = 0 \implies \forall i \in [1, n] : \langle R_i(A), v \rangle = 0$$

$$\implies R(A) \subseteq N(A)^{\perp}$$

$$N(A) + N(A)^{\perp} = \mathbb{R}^n$$

$$\implies \dim N(A)^{\perp} = n - \dim N(A)$$

$$\dim R(A) = rank(A) = n - \dim N(A)$$

$$\implies R(A) = N(A)^{\perp} \implies \boxed{C(A) = N(A)^{\perp}}$$

# 4c

Let $A, B \in \mathbb{R}^{7 \times 7}$

Let $A$ be invertible and $AB + BA = 0$

Prove or disprove: $B$ is not invertible

Proof?:

$$AB + BA = 0 \implies B + A^{-1}BA = 0$$

$$\implies B = A^{-1}(-B)A \implies B \sim -B$$

$$\implies \det(B) = \det(-B) = (-1)^7 \det(B) = -\det(B)$$

$$\implies \det(B) = 0 \implies \boxed{B \text{ is not invertible}}$$

# 5

Let $T : V \to V$ be idempotent, $T^2 = T$

# 5a

## 1

Prove: $T(v) - v \in \ker T$

Proof:

$$T(T(v) - v) = T^2(v) - T(v) = u - u = 0$$

## 2

$$\text{Prove: } \mathrm{Im}T \oplus \ker T = V$$

$$\text{Proof:}$$
$$\text{Let } v \in \mathrm{Im}T \cap \ker T$$
$$\implies \exists u \in V : T(u) = v, T(v) = 0$$
$$T(v) = T(T(u)) = T^2(u) = 0 \implies T(u) = 0 \implies v = 0$$
$$\implies \ker T \cap \mathrm{Im}T = \{0\}$$
$$\text{Let } v \in V$$
$$T(v) \in \mathrm{Im}T$$
$$T(v) - v \in \ker T \implies v - T(v) \in \ker T \implies v = \underbrace{T(v)}_{\in \mathrm{Im}T} + \underbrace{(v - T(v))}_{\in \ker T}$$
$$\implies \boxed{\mathrm{Im}T \oplus \ker T = V}$$

## 5b

$$\text{Let } \mathrm{Im}T = (\ker T)^{\perp}$$

## 1

$$\text{Prove: } T = T^*$$

$$\text{Proof:}$$
$$\forall v \in V : T(v) \in \mathrm{Im}T, T(v) - v \in \ker T$$
$$\implies \forall v \in V : \langle Tv, Tv - v\rangle = 0$$
$$\langle Tv, Tv - v\rangle = \langle v, T^*Tv - Tv\rangle = 0$$
$$\implies T^*Tv - Tv = 0 \implies T^*Tv = Tv = T^2v$$
$$\implies \boxed{T^* = T}$$

## 2

$$\text{Prove: } T = P_{\mathrm{Im}T}$$

$$\text{Proof:}$$
$$\text{Let } v \in \mathrm{Im}T$$
$$\implies \begin{cases} P_{\mathrm{Im}T}(v) = v \\ \exists u \in V : T(u) = v \implies T(v) = T(T(u)) = T^2(u) = T(u) = v \end{cases}$$
$$\text{Let } v \in \ker T$$
$$\implies \forall u \in (\ker T)^{\perp} = \mathrm{Im}T : \langle v, u\rangle = 0$$
$$\implies P_{\mathrm{Im}T}(v) = 0 = T(v)$$
$$\implies \forall v \in V : T(v) = P_{\mathrm{Im}T}(v) \implies \boxed{T = P_{\mathrm{Im}T}}$$

## 3

$$\text{Prove: } \forall v \in V : \|v\| \geq \|T(v)\| \text{ and } \|v\| = \|T(v)\| \implies v \in \mathrm{Im}T$$

$$\text{Proof:}$$
$$\|v\|^2 = \|\underbrace{v - T(v)}_{\in \ker T} + \underbrace{T(v)}_{\in \mathrm{Im}T = (\ker T)^{\perp}}\|^2 = \underbrace{\|v - T(v)\|^2}_{\geq 0} + \|T(v)\|^2$$
$$\implies \|v\|^2 \geq \|T(v)\|^2 \implies \boxed{\|v\| \geq \|T(v)\|}$$
$$\|v\| = \|T(v)\| \implies \|v - T(v)\|^2 = 0 \implies v - T(v) = 0 \implies \boxed{v = T(v) \in \mathrm{Im}T}$$