Singular Value Decomposition (SVD) #definition

$$\text{Let } A \in \mathbb{F}^{m \times n}$$

$$A = \underbrace{\overset{m\times m}{U}}_{\text{Unitary}} \underbrace{\overset{m\times n}{\Sigma}}_{\text{"Almost diagonal"}} \underbrace{\overset{n\times n}{V^*}}_{\text{Unitary}}$$

$$\text{Diagonal entries of } \Sigma \text{ are singular values of } A, \text{ which are all real and positive}$$

SVD existence #theorem

$$\text{Let } A \in \mathbb{F}^{m\times n}$$
$$\text{Then } \exists U, \Sigma, V : A = U\Sigma V^*$$

$$\text{Proof:}$$
$$\text{Let } m < n$$
$$\implies A^* \in \mathbb{F}^{m\times n}$$
$$A = U\Sigma V^* \implies A^* = V\Sigma^* U^* \implies \text{It is enough to prove for } m \geq n$$
$$\text{Let } m \geq n$$
$$(A^*A)^* = A^*A \implies A^*A \text{ is Hermitian} \implies \text{Its eigenvalues are real}$$
$$A^*Av = \lambda v \implies \lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle A^*Av, v\rangle = \langle Av, Av\rangle \geq 0$$
$$\langle v, v\rangle \geq 0 \implies \lambda \geq 0 \implies \text{Eigenvalues of } A^*A \text{ are real and positive}$$
$$A^*A \text{ is Hermitian} \implies A^*A \text{ is normal}$$
$$\text{All eigenvalues are real} \implies \text{Its characteristic polynomial is factorizable into linear factors}$$
$$\implies \exists V \text{ unitary: } A^*A = V \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{D} V^*$$
$$\text{Let } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$
$$\text{Let } v_i \text{ be } i\text{-th column of } V$$
$$\text{Let } W = AV \in \mathbb{F}^{m\times n}$$
$$\forall i \neq j \in [1, n] : \overline{\langle w_i, w_j\rangle} = \overline{\langle Av_i, Av_j\rangle} = \overline{(Av_i)^T\overline{Av_j}} = v_i^* A^* A v_j = (V^*A^*AV)_{ij} = D_{ij} = 0$$
$$W = \begin{pmatrix} | & & | \\ w_1 & \dots & w_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \tilde{w_1} & \dots & \tilde{w_n} \\ | & & | \end{pmatrix} \begin{pmatrix} \|w_1\| & & \\ & \ddots & \\ & & \|w_n\| \end{pmatrix}$$
$$\forall i \in [1, n] : \|w_i\|^2 = \langle w_i, w_i\rangle = D_{ii} = \lambda_i \implies \|w_i\| = \sqrt{\lambda_i}$$
$$\implies W = \begin{pmatrix} | & & | \\ \tilde{w_1} & \dots & \tilde{w_n} \\ | & & | \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$$
$$\lambda_i = 0 \implies \|w_i\| = 0 \implies w_i = 0$$
$$\implies \tilde{w_i} \text{ is some vector, orthogonal to } \{\tilde{w_1}, \dots, \tilde{w_{i-1}}\}, \text{ e.g. calculated by Gram-Schmidt}$$
$$\text{Let } w_1, \dots, w_r \neq 0$$
$$\implies \forall i \in [1, n] : \tilde{w_i} = \frac{w_i}{\|w_i\|}$$
$$\text{Let } B = \{\tilde{w_1}, \dots, \tilde{w_r}, \dots \tilde{w_m}\} \text{ be an orthonormal basis of } \mathbb{F}^{\mathbb{m}}$$
$$W = \underbrace{\begin{pmatrix} | & & | & & | \\ \tilde{w_1} & \dots & \tilde{w_r} & \dots & \tilde{w_m} \\ | & & | & & | \end{pmatrix}}_{U \in \mathbb{F}^{m\times m}, \text{unitary}} \underbrace{\begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}}_{\Sigma \in \mathbb{F}^{m\times n}}$$
$$W = AV = U\Sigma \implies \boxed{A = U\Sigma V^*}$$

Algorithm for calculating SVD

Let $A \in \mathbb{F}^{m \times n}, m \geq n$

1. Unitary diagonalization of A^*A by matrix V
2. Reorder columns of V and D such that $\lambda_1 \geq \dots \geq \lambda_n$
3. $$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$
4. Let $\forall i \in [1, n] : w_i = Av_i$
5. $\{w_1, \dots, w_n\}$ is orthogonal, remove zeroes, add vectors up to orthogonal basis of $\mathbb{F}^m$
6. Let $\forall i \in [1, r] : \tilde{w}_i = \frac{w_i}{\|w_i\|} = \frac{w_i}{\sqrt{\lambda_i}}$ where r is the number of non-zero vectors w_i
7. Let $\forall i \in [r+1, m] : \tilde{w}_i$ be orthogonal to $\{\tilde{w}_1, \dots, \tilde{w}_i\}$ and $\|\tilde{w}_i\| = 1$
8. $$U = \begin{pmatrix} | & & \\ \tilde{w}_1 & \dots & \tilde{w_m} \\ | & & \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\implies A^*A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 13 \end{pmatrix}$$

$$P_{A^*A}(x) = \begin{vmatrix} x-5 & -3 \\ -3 & x-13 \end{vmatrix} = (x-5)(x-13) - 9 = x^2 - 18x + 56 = (x-14)(x-4)$$

$$\Sigma = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$x = 14 \implies \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} \to \begin{pmatrix} 9 & -3 \\ 0 & 0 \end{pmatrix} \implies E_{14} = sp\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\} = sp\left\{\begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}\right\}$$

$$x = 4 \implies \begin{pmatrix} -1 & -3 \\ -3 & -9 \end{pmatrix} \to \begin{pmatrix} -1 & -3 \\ 0 & 0 \end{pmatrix} \implies E_4 = sp\left\{\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right\} = sp\left\{\begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix}\right\}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix}$$

$$AV = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 10 & 0 \\ 2 & 6 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} \frac{10}{\sqrt{10}} & 0 \\ \frac{2}{\sqrt{10}} & \frac{6}{\sqrt{10}} \\ \frac{6}{\sqrt{10}} & -\frac{2}{\sqrt{10}} \end{pmatrix}$$

Gram-Schmidt:

$$u_1 = w_1 \implies \frac{u_1}{\|u_1\|} = \frac{w_1}{\sqrt{14}}$$

$$u_2 = w_2 - P_{u_1}(w_2) = w_2 \implies \frac{w_2}{\|u_2\|} = \frac{u_2}{2}$$

$$\text{Let } u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_3 = u - \frac{\langle u, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle u, u_2 \rangle}{\|u_2\|^2} u_2 = u - \frac{\frac{6}{\sqrt{10}}}{14} u_1 + \frac{2}{4\sqrt{10}} u_2 =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{7\sqrt{10}} \begin{pmatrix} \frac{10}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{6}{\sqrt{10}} \end{pmatrix} + \frac{1}{2\sqrt{10}} \begin{pmatrix} 0 \\ \frac{6}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} \\ -\frac{6}{70} + \frac{6}{20} \\ 1 - \frac{18}{70} - \frac{1}{10} \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} \\ \frac{3}{14} \\ \frac{9}{14} \end{pmatrix}$$

$$\implies \frac{u_3}{\|u_3\|} = \frac{u_3}{\sqrt{\frac{36}{49 \cdot 4} + \frac{9}{49 \cdot 4} + \frac{81}{49 \cdot 4}}} = \frac{\sqrt{14} u_3}{3} = \begin{pmatrix} -\frac{\sqrt{14}}{7} \\ \frac{\sqrt{14}}{14} \\ \frac{3\sqrt{14}}{14} \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{5}{\sqrt{35}} & 0 & -\frac{\sqrt{14}}{7} \\ \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{\sqrt{14}}{14} \\ \frac{3}{\sqrt{35}} & -\frac{1}{\sqrt{10}} & \frac{3\sqrt{14}}{14} \end{pmatrix}$$

Pseudo-inverse matrix #definition

$$A = U \Sigma V^* \in \mathbb{F}^{m \times n}$$
$$A^+ = V \Sigma^+ U^* \in \mathbb{F}^{n \times m}$$

$$\text{Where } \Sigma^+ = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & & 0 & \dots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \frac{1}{\sqrt{\lambda_n}} & 0 & \dots & 0 \end{pmatrix}$$