$$A = PDP^{-1} \iff \text{Eigenvectors of } A \text{ form a basis of } \mathbb{F}^{n\times n}$$
$$A = \underbrace{P}_{[I]_B^S}\underbrace{DP^{-1}}_{[I]_S^B}$$

Change of basis might change distance, angle, etc. between vectors
We don't really want that, we would like to preserve them
So what we'd like to do is to make P unitary

Gram-Schmidt matrix of two bases #lemma

Let $B, \hat{B}$ be bases of V

Let $C = [I]_B^{\hat{B}}$

Then $G_{\hat{B}} = C^T G_B \overline{C}$

Proof:

$$\langle v, u\rangle = [v]_B^T G_B \overline{[u]_B} = ([I]_B^{\hat{B}}[v]_{\hat{B}})^T G_B \overline{[I]_B^{\hat{B}}[u]_{\hat{B}}} = [v]_{\hat{B}}^T \cdot C^T G_B \overline{C} \cdot \overline{[u]_{\hat{B}}}$$
$$= \langle v, u\rangle = [v]_{\hat{B}}^T G_{\hat{B}} \overline{[u]_{\hat{B}}}$$
$$\implies [v]_{\hat{B}}^T \cdot C^T G_B \overline{C} \cdot \overline{[u]_{\hat{B}}} = [v]_{\hat{B}}^T \cdot G_{\hat{B}} \cdot \overline{[u]_{\hat{B}}}$$
$$\text{Let } \hat{B} = \{v_1, \dots, v_n\}$$
$$\forall i, j \in [1, n] : [v_i]_{\hat{B}}^T \cdot C^T G_B \overline{C} \cdot \overline{[v_j]_{\hat{B}}} = [v_i]_{\hat{B}}^T \cdot G_{\hat{B}} \cdot \overline{[v_j]_{\hat{B}}}$$
$$\implies \forall i, j \in [1, n] : e_i^T \cdot C^T G_B \overline{C} \cdot e_j = e_i^T \cdot G_{\hat{B}} \cdot e_j$$
$$\implies \forall i, j \in [1, n] : (C^T G_B \overline{C})_{ij} = (G_{\hat{B}})_{ij}$$
$$\implies \boxed{C^T G_B \overline{C} = G_{\hat{B}}}$$

Basis change matrix unitarity #lemma

Let $B, \hat{B}$ be orthonormal bases

Then $C = [I]_B^{\hat{B}}$ is unitary

Proof:

$$G_{\hat{B}} = C^T G_B \overline{C}$$
$$B, \hat{B} \text{ are orthonormal} \implies G_{\hat{B}} = G_B = I$$
$$\implies C^T \overline{C} = I \implies C^* C = I \implies \boxed{C \text{ is unitary}}$$

Matrix unitary triangularization #definition

Let $A \in \mathbb{F}^{n\times n}$
A is then called unitary triangularizable if exists P unitary such that
$A = PTP^{-1} = PTP^*$ where T is triangular

Linear operator unitary triangularization #definition

Let $T : V \to V$ be a linear operator
T is then called unitary triangularizable if exists
orthonormal basis B such that $[T]_B^B$ is triangular

Choice of basis for linear operator unitary triangularization #lemma

We can choose any basis for linear operator triangularization

Proof:

$\dots$

Normal triangular matrix is diagonal #lemma

Let $A \in \mathbb{F}^{n\times n}$

A is normal and triangular $\iff$ A is diagonal

Proof:

$\boxed{\Longleftarrow}$ Is trivial

$\boxed{\Longrightarrow}$ Let A be normal and triangular

Linear operator unitary triangularizability criterion #theorem

Let $T : V \to V$ be a linear operator

Then T is unitary triangularizable $\iff$ Its characteristic polynomial is factorizable into linear factors

Proof:

$\boxed{\Longrightarrow}$ Let T be unitary triangularizble

$\Longrightarrow$ T is triangularizable

$\Longrightarrow$ $\boxed{\text{Its characteristic polynomial is factorizable into linear factors}}$

$\boxed{\Longleftarrow}$ Let characteristic polynomial of T be factorizable into linear factors

$\Longrightarrow$ T is triangularizable

$\Longrightarrow$ $\exists B$ basis of $V : [T]_B^B$ is upper triangular

Let $B = \{v_1, \dots, v_n\}$

Let $\hat{B}$ be a basis obtained by Gram-Schmidt orthonormalization process on B

$$\hat{B} = \{u_1, \dots, u_n\}$$

$$[T]_{\hat{B}}^{\hat{B}} = [I]_{\hat{B}}^{B}[T]_B^B[I]_B^{\hat{B}}$$

Note: this follows by definition of Gram-Schmidt process

$$\forall i \in [1, n] : u_i = \sum_{j=1}^{i} \alpha_j v_j \implies [u_i]_B = \sum_{j=1}^{i} \alpha_j e_j$$

Or in other words: $\forall i \in [1, n] : sp\,\{v_1, \dots, v_i\} = sp\,\{u_1, \dots, u_i\}$

$\Longrightarrow$ $[I]_{\hat{B}}^{B}, [I]_B^{\hat{B}}$ are upper triangular

$\Longrightarrow$ $[T]_{\hat{B}}^{\hat{B}}$ is also triangular $\Longrightarrow$ $\boxed{T \text{ is unitary triangularizable}}$

Matrix unitary diagonalization #definition

Let $A \in \mathbb{F}^{n\times n}$

A is then called unitary triangularizable if exists P unitary such that

$A = PTP^{-1} = PDP^*$ where D is diagonal

Unitary linear operator diagonalization #definition

Let $T : V \to V$ be a linear operator
T is then called unitary triangularizable if exists
orthonormal basis B such that $[T]_B^B$ is diagonal

Choice of basis for linear operator diagonalization #lemma

We can choose any basis for linear operator diagonalization

Proof:
$\ldots$

Linear operator unitary diagonalizability criterion #theorem

Let $T : V \to V$ be a linear operator
Then T is unitary diagonalizable $\iff$ Its characteristic polynomial is factorizable
into linear factors and T is normal

Proof:
$\boxed{\implies}$ Let T be unitary digonalizable

$$\implies \exists B \text{ orthonormal basis of } V : [T]_B^B \text{ is diagonal}$$

$$\implies [T]_B^B \text{ is triangular } \implies T \text{ is also unitary triangularizable}$$

$$\implies \boxed{\text{Its characteristic polynomial is factorizable into linear factors}}$$

$$A = [T]_B^B = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}$$

$$\implies A^* = \overline{A^T} = \overline{A} = \begin{pmatrix} \overline{\alpha_1} & & & \\ & \overline{\alpha_2} & & \\ & & \ddots & \\ & & & \overline{\alpha_n} \end{pmatrix}$$

$$\implies AA^* = A^*A = \begin{pmatrix} \alpha_1\overline{\alpha_1} & & & \\ & \alpha_2\overline{\alpha_2} & & \\ & & \ddots & \\ & & & \alpha_n\overline{\alpha_n} \end{pmatrix}$$

$$\implies \boxed{T \text{ is normal}}$$

$\boxed{\impliedby}$ Let Characteristic polynomial of T is factorizable into linear factors
and T is normal
T is unitary triangularizable

$$\implies \exists B \text{ orthonormal basis of } V : [T]_B^B \text{ is normal and triangular } \implies [T]_B^B \text{ is diagonal}$$

$$\implies \boxed{T \text{ is unitary diagonalizable}}$$

Matrix unitary diagonalizability criterion #theorem

Let $A \in \mathbb{F}^{n \times n}$
Then A is unitary diagonalizable $\iff$ Its characteristic polynomial is factorizable
into linear factors and A is normal

Real-value matrix orthogonal diagonalization #definition

Let $A \in \mathbb{R}^{n\times n}$

A is then called unitary triangularizable if exists P orthogonal such that

$A = PTP^{-1} = PDP^*$ where D is diagonal

Reminder: matrix is called orthogonal if it is unitary and real-value

Real-value matrix orthogonal diagonalizability criterion #theorem

Let $A \in \mathbb{R}^{n\times n}$

Then A is orthogonal diagonalizable $\iff$ Its characteristic polynomial is factorizable into linear factors and A is normal

Proof:

$\boxed{\implies}$ Let A be orthogonal diagonalizable

$$A = PDP^{-1}$$

$$P \text{ is orthogonal} \implies P \text{ is unitary} \implies A \text{ is unitary orthogonal} \implies \boxed{\text{See theorem above}}$$

$\boxed{\impliedby}$ Let Characteristic polynomial of A is factorizable into linear factors and A is normal

$$\implies A \text{ is unitary diagonalizable}$$

$$\implies \exists P \text{ unitary} : A = PDP^{-1}$$

$$A \in \mathbb{R}^{n\times n}, D \in \mathbb{R}^{n\times n}$$

$$\forall v \text{ eigenvector of } A : v \in \mathbb{R}^n \implies P \in \mathbb{R}^{n\times n}$$

$$\implies P \text{ is orthogonal} \implies \boxed{A \text{ is orthogonal diagonalizable}}$$

Real-value matrix orthogonal diagonalizability alternative criterion #definition

$A \in \mathbb{R}^{n\times n}$

Then A is orthogonal diagonalizable $\iff$ A is symmetric

Proof:

$\boxed{\implies}$ Let A be orthogonal diagonalizable

$$\exists P \text{ orthogonal} : A = PDP^{-1} = PDP^T$$

$$\implies A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

$$\implies \boxed{A \text{ is symmetric}}$$

$\boxed{\impliedby}$ Let A be symmetric

$$\implies A = A^T = A^* \implies A \text{ is hermitian}$$

$$\implies AA^* = A^2 = A^*A \implies \boxed{A \text{ is normal}}$$

$$A \text{ is normal} \implies \text{All its eigenvalues are real} \implies \text{Characteristic polynomial of } A$$

$$\text{is factorizable into linear factors} \implies \boxed{A \text{ is orthogonal diagonalizable}}$$

Unitary diagonalization algorithm #definition

Let $A \in \mathbb{F}^{n\times n}$

1. Check if A is unitary diagonalizable
2. Find eigenvalues and eigenvectors of A
3. For each eigenvalue, use Gram-Schmidt orthonormalization process on E_λ
4. Construct columns of P with resulting eigenvectors
5. Proceed as with "regular" diagonalization

Orthogonal diagonalization algorithm #definition

Let $A \in \mathbb{R}^{n \times n}$

Orthogonal diagonalization algorithm is then identical to a unitary diagonalization one

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Note: can read about spectral decomposition
