2a

$$\text{Determine whether } A = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \text{ is unitary diagonalizable}$$

Solution:

$$P_A(x) = \begin{vmatrix} x & -i & 0 \\ -i & x & 0 \\ 0 & 0 & x-i \end{vmatrix} = (x^2 - i^2)(x-i) = (x-i)^2(x+i)$$

$$\implies \text{Characteristic polynomial is factorizable into linear factors}$$

$$A^* = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} = -A$$

$$\implies A \text{ is anti-Hermitian} \implies A \text{ is normal} \implies A \text{ is unitary diagonalizale}$$

$$x = i \implies \begin{pmatrix} i & -i & 0 \\ -i & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} i & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies E_i = sp\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} = sp\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$$

$$x = -i \implies \begin{pmatrix} -i & -i & 0 \\ -i & -i & 0 \\ 0 & 0 & -2i \end{pmatrix} \to \begin{pmatrix} i & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{pmatrix} \implies E_{-i} = sp\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} = sp\left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$$

$$\implies A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} i & & \\ & i & \\ & & -i \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}^*$$

2b

$$\text{Let } A \in \mathbb{R}^{n \times n} \text{ be symmetric}$$

$$\text{Let } \exists k \in \mathbb{N} : A^k = I$$

$$\text{Prove: } A \text{ is orthogonal}$$

Proof:

$$A = A^T \underbrace{\implies}_{A \in \mathbb{R}^{n \times n}} A = A^* \implies \text{Eigenvalues of } A \text{ are all real}$$

$$A \text{ is symmetric} \implies A \text{ is orthogonal diagonalizable}$$

$$\implies \exists P : A = PDP^T$$

$$\implies A^k = PD^kP^T = I$$

$$\implies D^k = P^TP = I \implies \forall i \in [1, n] : D_{ii} = \sqrt[k]{1} \in \mathbb{R} = \pm 1$$

$$A = A^T \implies AA^T = A^2 = PD^2P^T = P \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix} P^T$$

$$\forall i \in [1, n] : \lambda_i = \pm 1 \implies \lambda_i^2 = 1 \implies D = I$$

$$\implies AA^T = PD^2P^T = PP^T = I \implies \boxed{A \text{ is orthogonal}}$$

3

$$\text{Let } V \text{ be a finite-dimensional inner product space}$$

$$\text{Let } T : V \to V \text{ be a linear operator: } T = T^2$$

3a

Determine whether T is necessrily diagonalizable

Solution:

$$T = T^2 \implies T(T-I) = 0 \implies m_T(x) \mid x(x-1)$$

$$\implies m_T(x) = \begin{bmatrix} x \\ (x-1) \\ x(x-1) \end{bmatrix} \implies m_T(x) \text{ is factorizable into distinct linear factors}$$

$$\implies \boxed{T \text{ is diagonalizable}}$$

3b

Prove: $\text{Im}T = \ker(I-T)$

Proof:

Let $v \in \text{Im}T$

$$\implies \exists u \in V : T(u) = v \implies T(T(u)) = v \implies T(v) = v$$

$$\implies v - T(v) = 0 \implies I(v) - T(v) = 0 \implies (I-T)(v) = 0 \implies v \in \ker(I-T)$$

$$\implies \text{Im}T \subseteq \ker(I-T)$$

Let $v \in \ker(I-T)$

$$\implies (I-T)(v) = 0 \implies v - T(v) = 0 \implies T(v) = v \implies v \in \text{Im}T$$

$$\implies \ker(I-T) \subseteq \text{Im}T \implies \boxed{\text{Im}T = \ker(I-T)}$$

3c

Let T be normal

1

Prove: T is Hermitian

Proof:

$$T \text{ is diagonalizable} \implies \text{Its characteristic polynomial is factorizable into linear factors}$$

$$T \text{ is also normal} \implies T \text{ is unitary diagonalizable}$$

$$\implies \exists B \text{ orthonormal: } [T]_B^B \text{ is diagonal}$$

$$T = T^2 \implies T(T-I) = 0 \implies m_T(x) \mid x(x-1) \text{ and } P_T(x) \mid x^n(x-1)^n$$

$$\implies \text{The only eigenvalues are } \{0, 1\}$$

$$\implies [T]_B^B = \begin{pmatrix} I_k & \\ & 0_{n-k} \end{pmatrix}$$

$$\implies ([T]_B^B)^* = [T]_B^B \implies [T]_B^B \text{ is Hermitian}$$

$$\implies \boxed{T \text{ is Hermitian}}$$

2

Prove: $\exists W$ subspace of $V : \forall w \in W : T(w) = w$ and $\forall u \in W^\perp : T(u) = 0$

Proof:

$$\mathrm{Im}T = \ker(I - T) \implies \boxed{\forall v \in \mathrm{Im}T : v - T(v) = 0 \implies T(v) = v}$$

Let $W = \mathrm{Im}T$

Let $u \in W^\perp$

$$\implies \forall w \in W : \langle w, u \rangle = 0$$

$$\forall v \in V : \langle \underbrace{T(v)}_{\in W}, u \rangle = \langle v, T^*(u) \rangle = 0$$

$$T \text{ is Hermitian} \implies \forall v \in V : \langle v, T^*(u) \rangle = \langle v, T(u) \rangle = 0$$

$$\implies T(u) = 0$$

$$\implies \boxed{\forall u \in W^\perp : T(u) = 0} \implies W^\perp \subseteq \ker T$$

It is also possible to prove $W^\perp = \ker T$ if necessary

4

Let $A \in \mathbb{C}^{5\times 5}$

Let $rank(A - 2I) = 3, rank(A) = 4$

Let $A(A - 2I)(A - 5I)^2 = 0$

Find all possible Jordan forms of A

Solution:

$$rank(A - 2I) = 3 \implies 2 \text{ is an eigenvalue of } A \text{ with } g_2 = 5 - 3 = 2$$

$$rank(A) = 4 \implies 0 \text{ is an eigenvalue of } A \text{ with } g_0 = 5 - 4 = 1$$

$m_A(x)$ contains all eigenvalue-factors of A at least once

$$\implies m_A(x) = x(x - 2) \cdot f(x)$$

$$A(A - 2I)(A - 5)^2 = 0 \implies m_A(x) \mid x(x - 2)(x - 5)^2$$

$$\implies m_A(x) = \left[\begin{array}{c} x(x-2) \\ x(x-2)(x-5) \\ x(x-2)(x-5)^2 \end{array}\right.$$

$$g_2 = 2 \implies \text{Jordan form of } A \text{ has two blocks of eigenvalue 2 of size 1}$$

$$g_0 = 1 \implies \text{Jordan form of } A \text{ has one block of eigenvalue 0 of size 1}$$

The only other eigenvalue of A can be 5

$$\implies \text{Jordan form of } A \text{ has one block of eigenvalue 5 of size 2 or two blocks of size 1}$$

$$\implies J_A = J_1(2) \oplus J_1(2) \oplus J_1(0) \oplus \left[\begin{array}{c} J_2(5) \\ J_1(5) \oplus J_1(5) \end{array}\right.$$

5a

Let $A \in \mathbb{R}^{2\times 2}$

Let $|A| = 0, A^2 \neq 0$

Prove or disprove: A is diagonalizable

Proof:

$$|A| = 0 \implies 0 \text{ is an eigenvalue of } A$$

$$\implies P_A(x) = x(x - \lambda) \implies A \text{ is at least triangularizable}$$

$$\text{Let } A \text{ be nilpotent} \implies A^2 = 0 - \text{ Contradiction!}$$

$$\implies A \text{ is not nilpotent} \implies \lambda \neq 0 \implies m_A(x) = x(x - \lambda) \implies A \text{ is diagonalizable}$$

5b

Let V be a vector space

Let $\langle , \rangle_1 ; \langle , \rangle_2$ be two different inner products on V

Prove or disprove: $\exists B$ basis of $V : B$ is orthonormal in relation to both inner products

Disproof:

Let B be an orthonormal basis in relation to both inner products

$$\implies G_{1_B} = I = G_{2_B}$$

$$\forall v, u \in V : \begin{cases} \langle v, u \rangle_1 = [v]_B^T G_{1_B} \overline{[u]_B} = [v]_B^T \overline{[u]_B} \\ \langle v, u \rangle_2 = [v]_B^T G_{2_B} \overline{[u]_B} = [v]_B^T \overline{[u]_B} \end{cases}$$

$$\implies \forall v, u \in V : \langle v, u \rangle_1 = \langle v, u \rangle_2 \implies \langle , \rangle_1 = \langle , \rangle_2 - \text{Contradiction!}$$

$$\implies B \text{ cannot be orthonormal in relation to both inner products}$$

5c

Let $A \in \mathbb{C}^{n \times n}$ be Hermitian

Prove or disprove: $A - (i+1)I$ is invertible

Proof:

$$A \text{ is Hermitian} \implies \text{All its eigenvalues are real}$$

$$(i+1) \notin \mathbb{R} \implies (i+1) \text{ is not an eigenvalue of } A$$

$$\implies A - (i+1)I \text{ is invertible}$$

5d

Let V be an inner product space over $\mathbb{R}$

Let U be a subspace of V

Let $v \in V$ and $p = P_U(v)$

Prove or disprove: $\|v + p\| = \|v - p\| \implies v \in U^\perp$

Proof:

$$\|v+p\|^2 = \langle v+p, v+p \rangle = \|v\|^2 + \|p\|^2 + 2\langle v, p \rangle$$

$$\|v-p\|^2 = \langle v-p, v-p \rangle = \|v\|^2 + \|p\|^2 - 2\langle v, p \rangle$$

$$\implies 2\langle v, p \rangle = -2\langle v, p \rangle \implies \langle v, p \rangle = 0$$

$$v = \underbrace{v-p}_{\in U^\perp} + \underbrace{p}_{\in U}$$

$$\implies \langle v, p \rangle = \langle v - p + p, p \rangle = \underbrace{\langle v-p, p \rangle}_{=0} + \|p\|^2$$

$$\implies \|p\|^2 = 0 \implies p = 0 \implies \boxed{v \in U^\perp}$$