Improper integral of the second type

What if interval is fine, but the function is not bounded on it?

For example, $\int_0^1 \frac{1}{x}\,dx$

Integrability on non-closed interval #definition

Function f is called Riemann-integrable on $(a, b]$

If $\forall c \in (a, b] : f$ is Riemann-integrable on $[c, b]$

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

Function f is called Riemann-integrable on $[a, b)$

If $\forall c \in [a, b) : f$ is Riemann-integrable on $[a, c]$

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

$$\int_0^{1/2} \frac{1}{x \ln x}\,dx = \lim_{c \to 0^+} \int_c^{1/2} \frac{1}{x \ln x}\,dx =$$

$$= \lim_{c \to 0^+} \ln |\ln x| \Big|_c^{1/2} = \ln \left| \ln \left(\frac{1}{2} \right) \right| - \ln |\ln c| = -\infty$$

$\implies$ Integral diverges

If integral has multiple "problems", or they are in the middle of the interval,

integral is to be calculated as a sum of integrals

Integral then converges iff each additive converges

Complex integrals might need a lot of sub-intervals:

$$\int_{-\infty}^{\infty} \frac{1}{(x-6)(x-23)}\,dx =$$

$$= \int_{-\infty}^{0} + \int_0^6 + \int_6^8 + \int_8^{23} + \int_{23}^{24} + \int_{24}^{\infty}$$

Comparison tests

p-Integral test for improper integrals of the first type #lemma

$$\int_a^b \frac{1}{(x-a)^p}\,dx \text{ converges} \iff p < 1$$

Comparison test for improper integrals of the second type #lemma

Let f, g be Riemann-integrable on $(a, b]$

Let $0 \le f \le g$

Then $\int_a^b g(x)\,dx$ converges $\implies \int_a^b f(x)\,dx$ converges

Limit comparison text for improper integrals of the second type #lemma

$$\text{Let } f, g \text{ be Riemann-integrable on } (a, b]$$

$$\text{Let } 0 \leq f, g$$

$$L = \lim_{x \to a^+} \frac{f(x)}{g(x)}$$

$$L = \infty \implies \quad \left[\int_a^b f(x)\,dx \text{ converges} \implies \int_a^b g(x)\,dx \text{ converges}\right]$$

$$L = 0 \implies \quad \left[\int_a^b f(x)\,dx \text{ converges} \impliedby \int_a^b g(x)\,dx \text{ converges}\right]$$

$$0 < L < \infty \implies \quad \left[\int_a^b f(x)\,dx \text{ converges} \iff \int_a^b g(x)\,dx \text{ converges}\right]$$