Uniform convergence of function series

$$S_N \rightrightarrows S$$

$$\iff d_N = \sup_{x\in A} |S_N(x) - S(x)| = \sup_{x\in A} \left| \sum_{n=1}^{N} f_n(x) - \sum_{n=1}^{\infty} f_n(x) \right| =$$

$$= \sup_{x\in A} \left| \sum_{n=N+1}^{\infty} f_n(x) \right| = \sup_{x\in A} |r_N|$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad x \in (-1,1)$$

$$d_N = \sup_{x\in(-1,1)} \left| \sum_{n=N+1}^{\infty} x^n \right|$$

$$\sum_{n=N+1}^{\infty} x^n = x^{N+1} + x^{N+2} + \dots = x^{N+1} \cdot \sum_{n=0}^{\infty} x^n = \frac{x^{N+1}}{1-x}$$

$$\implies d_N = \sup_{x\in(-1,1)} \left| \frac{x^{N+1}}{1-x} \right|$$

$$x \to 1 \implies \frac{x^{N+1}}{1-x} \to \infty \implies d_N \nrightarrow 0 \implies \sum_{n=0}^{\infty} x^n \not\rightrightarrows \frac{1}{1-x}$$

$$x \in \left[0, \frac{1}{23}\right] \implies d_N = \sup_{x\in\left[0,\frac{1}{23}\right]} \left| \frac{x^{N+1}}{1-x} \right| = \frac{\left(\frac{1}{23}\right)^{N+1}}{1-\frac{1}{23}} \to 0$$

Weierstrass M-test #theorem

$$\text{Let } \sum_{n=1}^{\infty} f_n(x)$$

$$\text{Let } \sum_{n=1}^{\infty} a_n \to M$$

$$\text{Let } \forall n \in \mathbb{N}, \forall x \in A : |f_n| \le a_n$$

$$\text{Then } \sum_{n=1}^{\infty} f_n(x) \rightrightarrows S(x)$$

$$\text{And } \sum_{n=1}^{\infty} |f_n(x)| \rightrightarrows S(x)$$

$$\text{Proof:}$$

$$\sum_{n=1}^{\infty} a_n \text{ converges } \implies \forall \varepsilon > 0 : \exists N_\varepsilon : \forall M > N > N_\varepsilon : S_M - S_N < \frac{\varepsilon}{2}$$

$$\text{Let } \varepsilon > 0$$

$$\forall x \in A : \forall m > n > N_\varepsilon : |S_M(x) - S_N(x)| = \left| \sum_{n=N+1}^{M} f_n(x) \right| \le \sum_{n=N+1}^{M} |f_n(x)| \le$$

$$\le \sum_{n=N+1}^{M} a_n = S_M - S_N < \frac{\varepsilon}{2}$$

$$\text{Let } N > N_\varepsilon$$

$$|S(x) - S_N(x)| = \left| \lim_{M\to\infty} S_M(x) - S_N(x) \right| = \lim_{M\to\infty} |S_M(x) - S_N(x)| \le \frac{\varepsilon}{2} < \varepsilon$$

$$\implies S_N(x) \rightrightarrows S(x) \implies \boxed{\sum_{n=1}^{\infty} f_n(x) \rightrightarrows S(x)}$$

$$\text{Similar proof for } \sum_{n=1}^{\infty} |f_n(x)| \rightrightarrows S(x)$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\implies \sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\implies \sum_{n=0}^{\infty} n x^{n} = \frac{x}{(1-x)^2}$$

$$x = \frac{1}{2} \implies \sum_{n=0}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2$$

Why can we do this?

If

$$\sum_{n=0}^{\infty} (x^n)' \rightrightarrows g(x), x \in A$$

$$\exists x_0 \in A : \sum_{n=0}^{\infty} x_0^n \to M$$

$$\text{Then } \sum_{n=0}^{\infty} (x^n)' \rightrightarrows \left(\frac{1}{1-x}\right)'$$

$$\text{Let } A = \left[0, \frac{1}{2}\right]$$

$$\boxed{\forall x_0 \in A : \sum_{n=0}^{\infty} x_0^n \to M_{x_0}}$$

$$\forall n \in \mathbb{N}, \forall x \in A : \left|n x^{n-1}\right| = n x^{n-1} \le n \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{n}{2^{n-1}}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{2^{n-1}}} = \lim_{n\to\infty} \frac{\sqrt[n]{2} \cdot \sqrt[n]{n}}{\sqrt[n]{2^n}} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

$$\implies \sum_{n=0}^{\infty} \frac{n}{2^{n-1}} \to M \implies \text{By the Weierstrass M-test } \boxed{\sum_{n=0}^{\infty} n x^{n-1} \rightrightarrows g(x)}$$

$$\implies \sum_{n=0}^{\infty} (x^n)' \rightrightarrows \left(\frac{1}{1-x}\right)'$$