$$\det : \mathbb{F}^{n\times n} \to \mathbb{F}$$

$$\det(A) = |A| = \sum_{\sigma\in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

$$\text{For example: } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\text{Let } i \in [1, n]$$

$$\det(A) = |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \left|M_{ij}(A)\right|$$

$$\text{Let } j \in [1, n]$$

$$\det(A) = |A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \left|M_{ij}(A)\right|$$

$$\text{For example:}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 5 & 7 & 6 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 7 & 6 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} + 2 \begin{vmatrix} 3 & 1 \\ 5 & 7 \end{vmatrix} = 6 + 32 = 38$$

Exercises

$$\text{Let } A, B \in \mathbb{R}^{n\times n}$$

$$|A| = 7, |B| = 4$$

$$C = A^T (B^{-1})^5$$

$$\text{Find } |C|$$

$$\text{Solution:}$$

$$|C| = \left|A^T (B^{-1})^5\right| = \left|A^T\right| \cdot \left|(B^{-1})^5\right| = |A| \cdot \left|B^{-1}\right|^5 =$$

$$|A| \cdot |B|^{-5} = 7 \cdot \frac{1}{4^5} = \frac{7}{4^5}$$

$$\text{Let } n \in \mathbb{N} \text{ be odd}$$

$$\text{Let } A \in \mathbb{R}^{n\times n} \text{ be anti-symmetric}$$

$$\text{Prove: } A \text{ is non-invertible}$$

$$\text{Proof:}$$

$$A = -A^T$$

$$|A| = \left|-A^T\right| = (-1)^n \left|A^T\right| = -|A|$$

$$\implies |A| = 0 \implies A \text{ is non-invertible}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$$

$$\text{Find } \begin{vmatrix} i-4c & f & 2i+f \\ g-4a & d & 2g+d \\ h-4b & e & 2h+e \end{vmatrix} = \begin{vmatrix} i-4c & f & 2i \\ g-4a & d & 2g \\ h-4b & e & 2h \end{vmatrix} = 2\begin{vmatrix} i-4c & f & i \\ g-4a & d & g \\ h-4b & e & h \end{vmatrix} =$$

$$= 2\begin{vmatrix} -4c & f & i \\ -4a & d & g \\ -4b & e & h \end{vmatrix} = -8\begin{vmatrix} c & f & i \\ a & d & g \\ b & e & h \end{vmatrix} = -8\begin{vmatrix} c & a & b \\ f & d & e \\ i & g & h \end{vmatrix} =$$

$$8\begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix} = -8\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -16$$

$$\text{Let } A = \begin{cases} 2 & i=j \\ -1 & i=j+1 \\ -1 & i=j-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Find } |A|$$

$$\text{Let } A_n = A \in \mathbb{R}^{n\times n}$$

$$|A_n| = 2\cdot|A_{n-1}| - (-1)\,|M_{12}(A_n)| =$$

$$= 2\,|A_{n-1}| + (-1)\,|A_{n-2}| = 2\,|A_{n-1}| - |A_{n-2}|$$

$$|A_1| = 2$$

$$|A_2| = 3$$

$$|A_3| = 4$$

$$2\,|A_{n-1}| - |A_{n-2}| = 2(2A_{n-2} - A_{n-3}) - A_{n-2} = 3A_{n-2} - 2A_{n-3} = 4A_{n-3} - 3A_{n-4} = \cdots =$$

$$= (n-1)A_2 - (n-2)A_1 = 3n - 3 - 2n + 4 = n+1$$

$$\text{Let } A = \begin{cases} 2i & i \neq j \\ 0 & i = j \end{cases}$$

$$\text{Let } A' = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases}$$

$$|A| = (2n)!!\,|A'| = 2^n n!\,|A'|$$

$$|A'| = \begin{vmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \dots & 1 & 0 \end{vmatrix} \overset{\forall i\in[2,n]:R_i-R_1}{=} \begin{vmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \dots & 1 & -1 \end{vmatrix} =$$

$$\overset{\forall i\in[2,n]:C_1+C_i}{=} \begin{vmatrix} n-1 & 1 & \dots & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & -1 \end{vmatrix} = (n-1)\cdot(-1)^{n-1}$$

$$\implies |A| = (-1)^n (n-1) 2^n n!$$
