Gradient is the direction of maximum ascent #lemma

$$\text{Prove: } \max_{u\in\mathbb{R}^n}\{f_u(a)\} = \nabla f(a)$$

$$\text{Proof:}$$

$$\text{By Cauchy-Schwarz inequality: } |\langle u, \nabla f(a)\rangle| \leq \|u\|\cdot\|\nabla f(a)\| \implies |f_u(a)| \leq \|\nabla f(a)\|$$

$$f_{\nabla f(a)}(a) = \frac{\langle \nabla f(a), \nabla f(a)\rangle}{\|\nabla f(a)\|} = \|\nabla f(a)\|$$

$$\implies \boxed{\max_{u\in\mathbb{R}^n}\{f_u(a)\} = \nabla f(a)}$$

$$f_{-\nabla f(a)}(a) = \frac{\langle -\nabla f(a), \nabla f(a)\rangle}{\|-\nabla f(a)\|} = -\frac{\|\nabla f(a)\|^2}{\|\nabla f(a)\|} = -\|\nabla f(a)\|$$

$$\implies \boxed{\min_{u\in\mathbb{R}^n}\{f_u(a)\} = -\nabla f(a)}$$

$$\text{For functions of one variable:}$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(x) - f(a) - f'(a)(x-a) \approx 0$$

$$x \to a \implies f(x) - f(a) - f'(a)(x-a) \to 0$$

$$\text{But is the following true: } \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)} \overset{?}{\to} 0$$

$$\text{The answer is yes:}$$

$$\lim_{x\to a}\frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)} = \lim_{x\to a}\frac{f(x)-f(a)}{(x-a)} - f'(a) = f'(a) - f'(a) = 0$$

$$\lim_{x\to a}\frac{R_1(x)}{x-a} = 0 \text{ or } R_1(x) = o(x-a)$$

$$\text{And in general:}$$

$$\lim_{x\to a}\frac{R_k(x)}{(x-a)^k} = 0 \text{ or } R_k(x) = o((x-a)^k)$$

Differentiability of functions of multiple variables #definition

$$\text{Let } f : \mathbb{R}^n \to \mathbb{R}$$

$$f \text{ is then called differentiable at point } a \text{ if}$$

$$\lim_{h\to 0}\frac{f(a+h) - f(a) - \nabla f(a)\cdot h}{\|h\|} = 0$$

1. $\text{Differentiable} \implies \text{Partial derivatives exist}$
2. $\text{Differentiable} \implies \text{Continuous}$
3. $\text{All partial derivatives are continuous at } a \implies \text{Differentiable at } a$

Chain rule for functions of multiple variables #lemma

$$\text{Let } f(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$
$$f : \mathbb{R}^n \to \mathbb{R}$$
$$\forall i \in [1, n] : g_i : \mathbb{R}^m \to \mathbb{R}, m < n$$

Then, if f and all of g_i are differentiable at a,

$$f_{x_j} = \sum_{n=1}^{n} f_{g_i} \cdot g_{x_j}$$

$$e.g.$$
$$f(x, y, z) = f(x(u, v), y(u, v), z(u, v))$$
$$\implies f_u = f_x \cdot x_u + f_y \cdot y_u + f_z \cdot z_u$$