Normal/unitary/Hermitian linear operator and its representation matrix #lemma

$$\text{Let } V \text{ be an inner space over } \mathbb{F}$$
$$\text{Let } B \text{ be an orthonormal basis of } V$$
$$\text{Then } \begin{array}{rrcl} 1. & T \text{ is normal} & \iff & [T]_B^B \text{ is normal} \\ 2 & T \text{ is unitary} & \iff & [T]_B^B \text{ is unitary} \\ 3. & \text{T is hermitian} & \iff & [T]_B^B \text{ is hermitian} \end{array}$$

$$\text{Proof for 1:}$$
$$T \text{ is normal} \iff TT^* = T^*T \iff [T]_B^B[T^*]_B^B = [T^*]_B^B[T]_B^B$$
$$\iff [T]_B^B([T]_B^B)^* = ([T]_B^B)^*[T]_B^B \iff [T]_B^B \text{ is normal}$$

$$\text{Proof for 2:}$$
$$T \text{ is unitary} \iff TT^* = I = T^*T \iff [T]_B^B[T^*]_B^B = [I]_B^B = [T^*]_B^B[T]_B^B$$
$$\iff [T]_B^B([T]_B^B)^* = I = ([T]_B^B)^*[T]_B^B \iff [T]_B^B \text{ is unitary}$$

$$\text{Proof for 3:}$$
$$T \text{ is Hermitian} \iff T = T^* \iff [T]_B^B = [T^*]_B^B$$
$$\iff [T]_B^B = ([T]_B^B)^* \iff [T]_B^B \text{ is Hermitian}$$

Polar norm equations #lemma

$$\mathrm{Re}(\langle v, u\rangle) = \frac{1}{2}(\|v+u\|^2 - \|v\|^2 - \|u\|^2)$$
$$\mathrm{Im}(\langle v, u\rangle) = i\frac{1}{2}(\|v+iu\|^2 - \|v\|^2 - \|u\|^2)$$

Normal linear operator criterion #theorem

Let V be a finitely generated inner product space over $\mathbb{F}$
Let B be an orthonormal basis of V
Let $T: V \to V$ be a linear operator
Then T is normal $\iff \forall v \in V: \|T(v)\| = \|T^*(v)\|$

Proof:

$\boxed{\implies}$ Let T be normal

$$TT^* = T^*T$$

Let $v \in V$

$$\|T(v)\| = \sqrt{\langle Tv, Tv \rangle} = \sqrt{\langle v, T^*Tv \rangle}$$

$$\|T^*(v)\| = \sqrt{\langle T^*v, T^*v \rangle} = \sqrt{\langle v, TT^*v \rangle} = \sqrt{\langle v, T^*Tv \rangle} = \|T(v)\|$$

$$\implies \boxed{\forall v \in V: \|T^*(v)\| = \|T(v)\|}$$

$\boxed{\impliedby}$ Let $\forall v \in V: \|T(v)\| = \|T^*(v)\|$

Let $v, u \in V$

$$\langle u, TT^*v \rangle = \langle T^*u, T^*v \rangle =$$

$$= \frac{1}{2}(\|T^*u + T^*v\|^2 - \|T^*u\|^2 - \|T^*v\|^2) + i\frac{1}{2}(\|T^*u + iT^*v\|^2 - \|T^*u\|^2 - \|T^*v\|^2)$$

$$\langle u, T^*Tv \rangle = \langle Tu, Tv \rangle =$$

$$= \frac{1}{2}(\|Tu + Tv\|^2 - \|Tu\|^2 - \|Tv\|^2) + i\frac{1}{2}(\|Tu + iTv\|^2 - \|Tu\|^2 - \|Tv\|^2)$$

$$\implies \langle u, TT^*v \rangle - \langle u, T^*Tv \rangle =$$

$$= \frac{1}{2}(\|T^*u + T^*v\|^2 - \|Tu + Tv\|^2) + i\frac{1}{2}(\|T^*u + iT^*v\|^2 - \|Tu + iTv\|^2) =$$

$$= \frac{1}{2}(\|T^*(u + v)\|^2 - \|T(u + v)\|^2) + i\frac{1}{2}(\|T^*(u + iv)\|^2 - \|T(u + iv)\|^2) = 0 - 0i = 0$$

$$\implies \forall v, u \in V: \langle u, TT^*v \rangle = \langle u, T^*Tv \rangle \implies \boxed{TT^* = T^*T}$$

Unitary linear operator criterion #theorem

Let V be a finitely generated inner product space over $\mathbb{F}$
Let B be an orthonormal basis of V
Let $T: V \to V$ be a linear operator
Then the following are equivalent:

1. $TT* = I = T^*T$ (T is unitary)
2. $\forall v, u \in V: \langle v, u \rangle = \langle Tv, Tu \rangle$ (preserves inner product)
3. $\forall v \in V: \|v\| = \|Tv\|$ (preserves norm)
4. $\forall v, u \in V: p(v, u) = p(Tv, Tu)$ (preserves metric)

$$\text{Proof for } 1 \implies 2:$$
$$\text{Let } T \text{ be unitary}$$
$$\text{Let } v, u \in V$$
$$\langle Tv, Tu\rangle = \langle v, T^*Tu\rangle = \langle v, Iu\rangle = \langle v, u\rangle$$
$$\implies \boxed{\forall v, u \in V : \langle v, u\rangle = \langle Tv, Tu\rangle}$$

$$\text{Proof for } 2 \implies 1:$$
$$\text{Let } \forall v, u \in V : \langle v, u\rangle = \langle Tv, Tu\rangle$$
$$\forall u, v \in V : \langle u, T^*Tv\rangle = \langle Tu, Tv\rangle = \langle u, v\rangle = \langle u, Iv\rangle$$
$$\implies T^*T = I \implies \boxed{T \text{ is unitary}}$$

$$\text{Proof for } 2 \implies 3:$$
$$\text{Let } \forall v, u \in V : \langle v, u\rangle = \langle Tv, Tu\rangle$$
$$\text{Let } v \in V$$
$$\implies \|v\| = \sqrt{\langle v, v\rangle} = \sqrt{\langle Tv, Tv\rangle} = \|Tv\|$$
$$\implies \boxed{\forall v \in V : \|v\| = \|Tv\|}$$

$$\text{Proof for } 3 \implies 2:$$
$$\text{Let } \forall v \in V : \|v\| = \|Tv\|$$
$$\text{Let } v, u \in V$$
$$\langle v, u\rangle =$$
$$\langle Tv, Tu\rangle = \frac{1}{2}(\|Tv + Tu\|^2 - \|Tv\|^2 - \|Tu\|^2) + i\frac{1}{2}(\|Tv + iTu\|^2 - \|Tv\|^2 - \|Tu\|^2) =$$
$$= \frac{1}{2}(\|T(v + u)\|^2 - \|v\|^2 - \|u\|^2) + i\frac{1}{2}(\|T(v + iu)\|^2 - \|v\|^2 - \|u\|^2) =$$
$$= \frac{1}{2}(\|v + u\|^2 - \|v\|^2 - \|u\|^2) + i\frac{1}{2}(\|v + iu\|^2 - \|v\|^2 - \|u\|^2) = \langle v, u\rangle$$
$$\implies \boxed{\forall v, u \in V : \langle v, u\rangle = \langle Tv, Tu\rangle}$$

$$\text{Proof for } 3 \implies 4:$$
$$\text{Let } \forall v \in V : \|v\| = \|Tv\|$$
$$\text{Let } v, u \in V$$
$$p(v, u) = \|v - u\| = \|T(v - u)\| = \|Tv - Tu\| = p(Tv, Tu)$$
$$\implies \boxed{\forall v, u \in V : p(v, u) = p(Tv, Tu)}$$

$$\text{Proof for } 4 \implies 3:$$
$$\text{Let } \forall v, u \in V : p(v, u) = p(Tv, Tu)$$
$$\|Tv\| = \|Tv - 0\| = \|Tv - T(0)\| = p(Tv, T(0)) = p(v, 0) = \|v - 0\| = \|v\|$$
$$\implies \boxed{\forall v \in V : \|v\| = \|Tv\|}$$

Unitary linear operator and angles #lemma

$$T : V \to V$$
$$T \text{ is unitary} \implies T \text{ preserves angles}$$

$$\text{Proof:}$$
$$\text{Let } \alpha \text{ be an angle between } v, u$$
$$\cos\alpha = \frac{\langle v, u\rangle}{\|v\| \cdot \|u\|} = \frac{\langle Tv, Tu\rangle}{\|Tv\| \cdot \|Tu\|} = \cos\beta$$
$$\implies \text{Angle between } Tv, Tu \text{ is } \beta = \alpha$$

Examples of unitary operators

$$I, -I, iI, -iI$$

Operations preserving matrix unitarity #lemma

$$A \text{ is unitary } \iff A^* \text{ is unitary}$$
$$\boxed{\implies} AA^* = I = A^*A$$
$$\boxed{\impliedby} (A^*)^* = A$$

$$A \text{ is unitary } \iff A^T \text{ is unitary}$$
$$\boxed{\implies} A^*A = I \implies \overline{A^*A} = A^T\overline{A} = \overline{I} = I$$
$$A^T(A^T)^* = A^T\overline{A} = I$$
$$\boxed{\impliedby} (A^T)^T = A$$

$$A, B \text{ are unitary } \implies AB \text{ are unitary}$$
$$AB(AB)^* = ABB^*A^* = AIA^* = AA^* = I$$

Unitary matrix and its row/column spaces #lemma

Let $A \in \mathbb{F}^{n \times n}$

The following are equivalent:

1. A is unitary
2. $\{R_1(A)^T, \dots, R_n(A)^T\}$ is an orthonormal basis of $\mathbb{F}^n$ by standard inner product
3. $\{C_1(A), \dots, C_n(A)\}$ is an orthonormal basis of $\mathbb{F}^n$ by standard inner product

Proof:

$$1 \implies 2 \text{ and } 1 \implies 3$$

Let A be unitary

$$\forall i, j \in [1, n] : R_i(A)C_j(A^*) = R_i(A) \cdot \overline{R_j(A)}^T = \langle R_i(A)^T, R_j(A)^T \rangle = I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\implies \boxed{\{R_1(A)^T, \dots, R_n(A)^T\} \text{ is an orthonormal set } \implies \text{ it is an orthonormal basis of } \mathbb{F}^n}$$

$$\forall i, j \in [1, n] : R_i(A^*)C_j(A) = \overline{C_i(A)}^T \cdot C_j(A) \underbrace{=}_{=I_{ij} \in \{0,1\}} \overline{\overline{C_i(A)^T} \cdot C_j(A)} =$$

$$= C_i(A)^T \cdot \overline{C_j(A)} = \langle C_i(A), C_j(A) \rangle = I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\implies \boxed{\{C_1(A), \dots, C_n(A)\} \text{ is an orthonormal set } \implies \text{ it is an orthonormal basis of } \mathbb{F}^n}$$

$$2 \implies 1 \text{ and } 3 \implies 1$$

Let $\{R_1(A)^T, \dots, R_n(A)^T\}$ be an orthonormal basis of $\mathbb{F}^n$

$$\implies \forall i, j \in [1, n] : R_i(A) \cdot C_j(A^*) = R_i(A) \cdot \overline{R_j(A)}^T = \langle R_i(A)^T, R_j(A)^T \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\implies AA^* = I \implies \boxed{A \text{ is unitary}}$$

Let $\{C_1(A), \dots, C_n(A)\}$ be an orthonormal basis of $\mathbb{F}^n$

$$\implies \forall i, j \in [1, n] : R_i(A^*) \cdot C_j(A) = \overline{C_i(A)}^T C_j(A) = \overline{C_i(A)^T \cdot \overline{C_j(A)}} =$$

$$= \overline{\langle C_i(A), C_j(A) \rangle} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\implies A^*A = I \implies \boxed{A \text{ is unitary}}$$

Orthogonal matrix #definition

A is called orthogonal iff A is real and unitary

Following lemmas are also correct for matrices under standard inner product

Normality after addition with scalar linear operator #lemma

$$T \text{ is normal } \iff T - \lambda I \text{ is normal}$$

$$\text{Proof:}$$

$$\boxed{\implies} \text{Let } \lambda \in \mathbb{F}$$

$$T \text{ is normal } \implies \begin{cases} (T-\lambda I)(T-\lambda I)^* = TT^* - \lambda T^* - \overline{\lambda} T + \lambda^2 \\ (T-\lambda I)^*(T-\lambda I) = T^*T - \overline{\lambda} T - \lambda T^* + \lambda^2 \end{cases} \implies (T-\lambda I) \text{ is normal}$$

$$\boxed{\impliedby} (T-\lambda I) \text{ is normal } \implies (T-\lambda I) - (-\lambda)I = T \text{ is normal}$$

Eigenvalues of adjoint linear operator #lemma

$$\text{Let } T : V \to V \text{ be a normal linear operator}$$

$$\text{Let } v \text{ be an eigenvector of } T \text{ with eigenvalue } \lambda$$

$$\text{Then } \lambda \text{ be an eigenvalue of } T \iff \overline{\lambda} \text{ is an eigenvalue of } T^* \text{ with the same eigenvector}$$

$$\text{Proof:}$$

$$\exists v \neq 0 \in V : Tv = \lambda v$$

$$\iff Tv - \lambda v = 0 \iff (T-\lambda I)v = 0$$

$$\iff \|(T-\lambda I)v\| = 0 \iff \|(T-\lambda I)^* v\| = 0$$

$$\iff (T-\lambda I)^* v = 0 \iff T^* v = (\lambda I)^* v = \overline{\lambda} v$$

Orthogonality of eigenvectors of normal linear operator #lemma

$$\text{Let } T : V \to V \text{ be a normal linear operator}$$

$$\text{Let } \lambda \text{ be an eigenvalue of } T \text{ with eigenvector} v$$

$$\text{Let } \alpha \neq \lambda \text{ be an eigenvalue of } T \text{ with eigenvector} u$$

$$\text{Then } v, u \text{ are orthogonal, } \langle v, u \rangle = 0$$

$$\text{Proof:}$$

$$\alpha \neq \lambda$$

$$\exists v \neq 0 \in V : Tv = \lambda v$$

$$\exists u \neq 0 \in V : Tu = \alpha u \implies T^* u = \overline{\alpha} u$$

$$\lambda \langle v, u \rangle = \langle \lambda v, u \rangle = \langle Tv, u \rangle = \langle v, T^* u \rangle = \langle v, \overline{\alpha} u \rangle = \alpha \langle v, u \rangle$$

$$\implies (\lambda - \alpha)\langle v, u \rangle = 0 \underset{\lambda \neq \alpha}{\implies} \boxed{\langle v, u \rangle = 0}$$

Unitary linear operator eigenvalues #lemma

$$\text{Let } T : V \to V \text{ be a unitary linear operator}$$

$$\text{Let } \lambda \text{ be an eigenvalue of } T$$

$$\text{Then } |\lambda| = 1$$

$$\text{Proof:}$$

$$\|v\| = \|Tv\| = \|\lambda v\| = |\lambda| \cdot \|v\|$$

$$\implies \boxed{|\lambda| = 1}$$