Let V be a finitely generated vector space over $\mathbb{F}$

Let $T, S : V \to V$ be linear transformations

Prove: Eigenvalues of ST are equal to eigenvalues of TS

Proof:

Let $\lambda \neq 0$ be an eigenvalue of TS

$$\exists v \neq 0 : TSv = \lambda v$$

$$\implies ST(Sv) = S(TSv) = S(\lambda v) = \lambda Sv$$

$$\implies \lambda \text{ is an eigenvalue of } ST$$

Let λ be an eigenvalue of ST

Let $\lambda = 0$ be an eigenvalue of TS

$$\implies TSv = 0v = 0$$

$$\implies TS \text{ is not invertible} \implies \left[\begin{array}{l} T \text{ is not invertible} \\ S \text{ is not invertible} \end{array}\right. \implies ST \text{ is not invertible}$$

$$\implies ker(ST) \neq \{0\} \implies \lambda \text{ is an eigenvalue of } ST$$

Let $A \in \mathbb{R}^{n \times n}$ of rank 1

Prove: $\forall x \neq y \in \mathbb{R} \setminus \{0\} : \left[\begin{array}{l} xI - A \text{ is invertible} \\ yI - A \text{ is invertible} \end{array}\right.$

Determine whether there is always $x \neq 0 \in \mathbb{R} : xI - A$ is not invertible

Proof:

Let $xI - A$ be non-invertible and $yI - A$ be non-invertible

$$\implies \begin{cases} E_x = N(xI - A) \neq \{0\} \\ E_y = N(yI - A) \neq \{0\} \end{cases}$$

$$rank(A) = 1 \implies \gamma_A(0) = n - 1 \implies \mu_A(0) \geq n - 1$$

$$\implies \begin{cases} \mu_A(x) + \mu_A(y) = 1 \\ \mu_A(x) \geq 1 \\ \mu_A(y) \geq 1 \end{cases} \quad - \text{Contradiction!}$$

Solution:

No

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies P_A(\lambda) = \lambda^2 \implies \forall \lambda \neq 0 \in \mathbb{R} : \det(\lambda I - A) \neq 0$$

Let V be a finitely generated vector space over $\mathbb{F}$

Let $T : V \to V$ be an idempotentic linear transformation

Find eigenvalues of T

Determine whether T is diagonalizable

Solution:

$$E_0 = ker(T)$$

$$E_1 = Im(T)?$$

$$\text{Let } Tv = v \implies v \in Im(T)$$

$$\text{Let } v \in Im(T)$$

$$\exists u \in V : Tu = v \implies T(Tu) = Tv = v \implies v \in E_1$$

$$\implies E_1 = Im(T)$$

$$\dim(E_0) + \dim(E_1) = n \implies \text{There are no other eigenvalues and } T \text{ is diagonalizable}$$

$$\begin{cases} 1 \text{ is the only eigenvalue} & T \text{ is invertible} (T = I) \\ 0 \text{ is the only eigenvalue} & T = 0 \\ 0, 1 \text{ are the only eigenvalues} & \text{otherwise} \end{cases}$$

$$a_n = \begin{cases} 1 & n = 1, 2 \\ a_{n-1} + 2a_{n-2} & n > 2 \end{cases}$$

$$1, 1, 3, 5, 11, 21, \ldots$$

$$\text{Let } A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\forall n > 2 : A \cdot \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$$

$$\implies A^{n-2} \cdot \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$$

$$P_A(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

$$E_2 = sp\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$$

$$E_{-1} = sp\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$$\text{Let } P = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = PDP^{-1} \implies A^n = PD^nP^{-1}$$

$$\implies \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = P \begin{pmatrix} 2^{n-2} & 0 \\ 0 & (-1)^{n-2} \end{pmatrix} P^{-1} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$A^{n-2} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{n-2} & 0 \\ 0 & (-1)^{n-2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} =$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{n-2} & 2^{n-2} \\ (-1)^{n-2} & -2 \cdot (-1)^{n-2} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2^{n-1} + (-1)^{n-2} & 2^{n-1} - 2 \cdot (-1)^{n-2} \\ * & * \end{pmatrix}$$

$$\implies a_n = \frac{1}{3}(2^n - (-1)^{n-2})$$

$$\implies a_n = \frac{4}{3}2^{n-2} - \frac{1}{3}(-1)^{n-2}$$

$$a_n = \alpha\lambda_1^{n-2} + \beta\lambda_2^{n-2}$$

$$a_n = \sum_{i=1}^{k} \alpha_i \lambda_i^{n-k}$$

$$\text{Let } A \in \mathbb{R}^{n \times n}$$

$$P_A(\lambda) = \lambda^3(\lambda^2 + 4)$$

$$\text{Not diagonalizable over } \mathbb{R}$$

$$\text{Over } \mathbb{C}?$$

$$P_A(\lambda) = \lambda^3(\lambda - 2i)(\lambda + 2i)$$

$$A \text{ is diagonalizable} \iff rank(A) = 2$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & 0 & -2i \end{pmatrix} \text{ Incorrect :(, } A \text{ must only have real values}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix} \text{ Correct! :)}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix} \text{ Correct! :) And a Jordan 3x3 block!!}$$