1

Formulate and prove diagonalizability criterion

2

Let $\mathbb{R}^3$ be an inner product space with standard inner product

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}, T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

2a

Find an orthonormal basis B of $\mathbb{R}^3$ such that $[T]_B^B$ is diagonal

Solution:

$$T(e_1) = T\begin{pmatrix}1\\1\\0\end{pmatrix} - T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix} = 2e_1$$

$$T(e_2) = 2e_2$$

$$T(e_3) = 0$$

$$\implies [T]_S^S = \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 0\end{pmatrix}$$

$$\boxed{B = S,\ \text{standard basis is orthonormal under standard inner product}}$$

2b

Find a vector in $(\ker T)^\perp$ that is the closest to $\begin{pmatrix}0\\1\\1\end{pmatrix}$

Solution:

$$\ker T = sp\{e_3\}$$

$$\implies (\ker T)^\perp = sp\{e_1, e_2\}$$

$$\implies \text{Closest vector to } v = \begin{pmatrix}0\\1\\1\end{pmatrix} \text{ in } (\ker T)^\perp \text{ is an orthogonal projection } P_{(\ker T)^\perp}(v)$$

$$\boxed{P_{(\ker T)^\perp}(v) = \frac{\langle v, e_1\rangle}{\|e_1\|^2}e_1 + \frac{\langle v, e_2\rangle}{\|e_2\|^2}e_2 = e_2}$$

3a

Let V be a finite-dimensional inner product space over $\mathbb{C}$

Let $\vec{t}, \vec{s} \in V$

Let $T : V \to V$ be a linear operator

$$T(v) = \vec{t}\langle v, \vec{s}\rangle$$

Find explicitly T^*

Solution:

$$\forall v, u \in V : \langle T(v), u\rangle = \langle v, T^*(u)\rangle$$

$$\langle T(v), u\rangle = \langle \vec{t}\langle v, \vec{s}\rangle, u\rangle = \langle v, \vec{s}\rangle \cdot \langle \vec{t}, u\rangle = \langle v, \vec{s}\overline{\langle \vec{t}, u\rangle}\rangle = \langle v, T^*(u)\rangle$$

$$\implies \forall u \in V : \boxed{T^*(u) = \vec{s}\overline{\langle \vec{t}, u\rangle}}$$

3b

Let V, W be finite-dimensional inner product spaces over $\mathbb{F}$

We say that V, W are "inner product-isomorphic" if

$\exists$ invertible linear operator $T : V \to W$ such that

$$\forall v, u \in V : \langle v, u\rangle = \langle T(v), T(u)\rangle$$

Prove: V, W are "inner product-isomorphic" $\iff \dim V = \dim W$

Proof:

$\boxed{\implies}$ This direction is trivial, as existence of an invertible linear operator from V to W implies $\boxed{\dim V = \dim W}$

$\boxed{\impliedby}$ Let $\dim V = \dim W = n$

Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V

Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis of W

Let $T : V \to W : \forall i \in [1, n] : T(v_i) = w_i$

$$\implies \forall v \in V : v = \sum_{i=1}^{n} \langle v, v_i\rangle v_i$$

$$\implies \forall v, u \in V : \langle T(v), T(u)\rangle = \left\langle T\left(\sum_{i=1}^{n} \langle v, v_i\rangle v_i\right), T\left(\sum_{i=1}^{n} \langle u, v_i\rangle v_i\right)\right\rangle =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle v, v_i\rangle \overline{\langle u, v_j\rangle} \langle T(v_i), T(v_j)\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle v, v_i\rangle \overline{\langle u, v_j\rangle} \langle w_i, w_j\rangle =$$

$$= \sum_{i=1}^{n} \langle v, v_i\rangle \overline{\langle u, v_i\rangle} = \left\langle v, \sum_{i=1}^{n} \langle u, v_i\rangle v_i\right\rangle = \langle v, u\rangle$$

$$\implies \boxed{\forall v, u \in V : \langle v, u\rangle = \langle T(v), T(u)\rangle}$$

4a

Let $A \in \mathbb{C}^{n\times n}$

Let $\exists k : A^k$ is diagonalizable

Prove or disprove: A is diagonalizable

Disproof:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which is diagonal and diagonalizable by I

A is a Jordan block of size 2, which is not diagonalizable

4b

Let $A \in \mathbb{C}^{n\times n}$ be unitary
Prove or disprove: $|trA| \le n$

Proof:

$$A \text{ is unitary} \implies \text{Columns of } A \text{ form an orthonormal basis of } \mathbb{C}^n$$
$$\implies \forall i \in [1,n] : \|C_i(A)\| = 1$$
$$\text{Let } \exists i \in [1,n] : |A_{ii}| > 1$$
$$\implies \|C_i(A)\|^2 = \langle C_i(A), C_i(A)\rangle = \alpha_1^2 + \cdots + \underbrace{A_{ii}^2}_{>1} + \cdots + \alpha_{n-1}^2 > 1 - \text{Contradiction!}$$
$$\implies \forall i \in [1,n] : |A_{ii}| \le 1 \implies |trA| = \left|\sum_{i=1}^{n} A_{ii}\right| \le \sum_{i=1}^{n} |A_{ii}| \le n$$

4c

Let V be an inner product space over $\mathbb{C}$
Prove or disprove: $\{T : V \to V | T = T^*\}$ is a vector space

Disproof:

$$\text{Let } K = \{T : V \to V | T = T^*\}$$
$$0^* = 0 \implies 0 \in K$$
$$\text{Let } T_1, T_2 \ne 0 \in K$$
$$(T_1 + iT_2)^* = T_1^* - iT_2^* = T_1 - iT_2 \ne T_1 + iT_2$$
$$\implies T_1 + zT_2 \notin K \implies \boxed{K \text{ is not a vector space}}$$

5

Let V be an inner product space over $\mathbb{F}$
Let $\dim V = n$
Let $U, W \le V$ subspaces of V
Let $U \oplus W = V$
Let $P : V \to V$ be a linear operator
$\forall v \in V : \exists u \in U, w \in W : v = u + w$
Let $\forall v \in V : P(v) = P(u + w) = u$
P is then called projection and denoted $P_U^{U,W}$ or $\pi_U^{U,W}$

5a

Let $U \le V$ be a subspace of V
Prove: $P_U^{U,U^\perp} = P_U$

Proof:

$$\text{Let } v \in V$$
$$\exists u \in U, w \in U^\perp : v = u + w$$
$$P(v) = P(u + w) = u$$
$$U \oplus U^\perp = V$$
$$\forall u \in U : P(u) = P(u + 0) = u = P_U(u)$$
$$\forall w \in U^\perp = P(w) = P(0 + w) = 0 = P_U(w)$$
$$\implies \forall v \in V : P(v) = P_U(v) \implies \boxed{P = P_U^{U,U^\perp} = P_U}$$

5b

Let V be a finite-dimensional vector space over $\mathbb{F}$

Let $P : V \to V$ be a linear operator

Prove: P is a projection $\iff P = P^2$

Proof:

$\boxed{\implies}$ Let P be a projection

$$\text{Let } U, W \text{ such that } U \oplus W = V, \forall v \in V : v = \underbrace{u}_{\in U} + \underbrace{w}_{\in W} \implies P(v) = P(u+w) = u$$

Let $v \in V$

Let $u \in U, w \in W : v = u + w$

$$P(v) = P(u+w) = u$$

$$P^2(v) = P(P(v)) = P(P(u+w)) = P(u) = u$$

$$\implies \boxed{P^2 = P}$$

$\boxed{\impliedby}$ Let $P^2 = P$

Let $U = \mathrm{Im}P$

$$\forall u \in U : \exists v \in V : P(v) = u \implies P^2(v) = P(P(v)) = P(u) \implies P(u) = u$$

Let $W = \ker P$

$$\forall v \in V : P^2(v) = P(v) \implies P(P(v) - v) = 0 \implies P(v) - v \in \ker P$$

$$\forall v \in V : \underbrace{P(v)}_{u \in U} - v \in W \implies w = u - v \in W \implies v = u + w \implies U + W = V$$

Let $v \in U \cap W$

$$\implies x \in \mathrm{Im}P \cap \ker P$$

$$\implies \exists v \in V : P(v) = x, P(x) = 0 \implies P^2(v) = P(P(v)) = P(x) = 0 \implies x = 0$$

$$\implies U \cap W = \{0\} \implies \boxed{U \oplus W = V}$$

$$\boxed{\forall v \in V : v = u + w \implies P(v) = P(u+w) = P(u) + P(w) = u + 0 = u}$$

5c

Let V be a finite-dimensional vector space over $\mathbb{F}$

Let $P : V \to V$ be a projection

Prove: P is diagonalizable

Proof:

$$\text{Let } U, W \text{ such that } U \oplus W = V, \forall v \in V : v = \underbrace{u}_{\in U} + \underbrace{w}_{\in W} \implies P(v) = P(u+w) = u$$

$$P \text{ is a projection } \implies P^2 = P$$

$$\implies P^2 - P = 0 \implies P(P - I) = 0$$

$$\implies m_P(x) \mid x(x-1) \implies \{0, 1\} \text{ are eigenvalues of } P \text{ (not necessarily the only ones)}$$

$$\forall u \in U : P(u) = u \implies k_1 \geq g_1 = \dim U = k$$

$$\forall w \in W : P(w) = 0 \implies k_0 \geq g_0 = \dim W = t$$

$$U \oplus W = V \implies \dim U + \dim W = \dim V = n$$

$$\implies P_P(x) = x^{t+\alpha}(x-1)^{k+\beta}$$

$$\begin{cases} k + t = n \\ k + t + \alpha + \beta = n \\ \alpha \geq 0 \\ \beta \geq 0 \end{cases} \implies \alpha = \beta = 0 \implies P_P(x) = x^t(x-1)^t$$

$$\implies g_0 = k_0, g_1 = k_1 \implies \boxed{P \text{ is diagonalizable}}$$

5d

Let V be a finite-dimensional vector space over $\mathbb{F}$

Let $T : V \to V$ be a linear operator

T is then called a mirror if $\exists U, W$ subspaces of $V : U \oplus W = V : \forall u \in U, \forall w \in W :$

$$T(u + w) = u - w$$

And denoted as $T = R_U^{U,W}$

Prove: T is a mirror $\iff T^2 = I$

Proof:

$\boxed{\implies}$ Let T be a mirror

Let U, W such that $U \oplus W = V, \forall v \in V : v = \underbrace{u}_{\in U} + \underbrace{w}_{\in W} \implies T(v) = T(u + w) = u - w$

$$\forall v \in V : v = u + w, T(v) = T(u + w) = u - w$$

$$T^2(v) = T(T(v)) = T(T(u + w)) = T(u - w) = u + w = v \implies \boxed{T^2 = I}$$

$\boxed{\impliedby}$ Let $T^2 = I$

$$T^2 - I = (T - I)(T + I) = 0 \implies m_T(x) \mid (x - 1)(x + 1)$$

$$\implies m_T(x) = \left[\begin{array}{c} x - 1 \\ x + 1 \\ (x - 1)(x + 1) \end{array}\right. \implies T \text{ is diagonalizable} \implies E_1 \oplus E_{-1} = V$$

Let $U = E_1, W = E_{-1}$

$$\boxed{\forall v \in V : v = u + w \implies T(v) = T(u + w) = T(u) + T(w) = u - w}$$