Gram-Schmidt matrix #definition

$$B = \{v_1, \dots, v_n\} \text{ is a basis of } V$$
$$(G_B)_{ij} = \langle v_i, v_j \rangle$$
$$B \text{ is orthonormal} \implies G_B = I$$

Inner product general form #lemma

$$\forall v, u \in V : \langle v, u \rangle = [v]_B^T G_B \overline{[u]_B}$$

Proof:

Let $B = \{v_1, \dots, v_n\}$

Let $v, u \in V$

$$v = \sum_{i=1}^{n} \alpha_i v_i$$
$$u = \sum_{i=1}^{n} \beta_i v_i$$
$$\langle v, u \rangle = \left\langle \sum_{i=1}^{n} \alpha_i v_i, u \right\rangle = \sum_{i=1}^{n} \alpha_i \langle v_i, u \rangle =$$
$$= \sum_{i=1}^{n} \alpha_i \left\langle v_i, \sum_{j=1}^{n} \beta_j v_j \right\rangle = \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i, v_j \rangle}$$

$$[v]_B^T G_B \overline{[u]_B} = (\alpha_1 \quad \dots \quad \alpha_n) G_B \begin{pmatrix} \overline{\beta_1} \\ \vdots \\ \overline{\beta_n} \end{pmatrix} =$$
$$= (\alpha_1 \quad \dots \quad \alpha_n) \begin{pmatrix} \sum_{i=1}^{n} \overline{\beta_i} (G_B)_{1i} \\ \vdots \\ \sum_{i=1}^{n} \overline{\beta_i} (G_B)_{ni} \end{pmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} (G_B)_{ij} =$$
$$= \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i, v_j \rangle}$$

Riesz Representation theorem #theorem

Let V be an inner product space over $\mathbb{F}$

Let $T : V \to \mathbb{F}$ be a linear transformation

Then $\exists! h \in V : \forall v \in V : T(v) = \langle v, h \rangle$

Proof:

Let B be an orthonormal basis of V

Let $C = \{1\}$ be a standard basis of $\mathbb{F}$

$$T(v) = [T(v)]_C = [T]_C^B [v]_B \in \mathbb{F} \implies T(v) = ([T]_C^B [v]_B)^T = [v]_B^T ([T]_C^B)^T$$
$$\langle v, h \rangle = [v]_B^T G_B \overline{[h]^B} = [v]_B^T \overline{[h]_B}$$
$$[\cdot]_B \text{ is surjective} \implies \boxed{\exists h \in V : [h]_B = \overline{([T]_C^B)^T} = (\overline{[T]_C^B})^T}$$

Let $\forall v \in V : \langle v, h_1 \rangle = T(v) = \langle v, h_2 \rangle$

$$\implies [v]_B^T \overline{[h_1]_B} = [v]_B^T \overline{[h_2]_B}$$
$$\implies [h_1]_B = [h_2]_B \implies \boxed{h_1 = h_2}$$

Adjoint linear transformation #definition

$$\text{Let } V, U \text{ be inner product spaces over } \mathbb{F}$$
$$\text{Let } T : V \to U \text{ be a linear transformation}$$
$$\text{Let } S : U \to V \text{ be a linear transformation}$$
$$\text{such that } \forall v \in V, u \in U : \langle T(v), u\rangle = \langle v, S(u)\rangle$$
$$S \text{ is then called a conjugate or adjoint linear transformation of } T$$
$$S \text{ is then denoted as } T^*$$

Existence and uniqueness of adjoint linear transformation #theorem

$$\text{Let } V, U \text{ be inner product spaces over } \mathbb{F}$$
$$\text{Let } T : V \to U \text{ be a linear transformation}$$
$$\text{Let } S : U \to V \text{ be a linear transformation}$$
$$\text{Then } \exists! S : S \text{ is a adjoint linear transformation of } T$$

$$\text{Proof:}$$
$$\text{Let } u \in U$$
$$\text{Let } K_u : V \to \mathbb{F}, K_u(v) = \langle T(v), u\rangle$$
$$\implies \text{ By Riesz theorem } \exists h_u \in V : \forall v \in V : K_u(v) = \langle v, h_u\rangle$$
$$\text{Let } S : U \to V, S(u) = h_u$$
$$\implies \forall v \in V, u \in U : \langle T(v), u\rangle = \langle v, h_u\rangle = \langle v, S(u)\rangle$$

$$\text{Let } v \in V, u_1, u_2 \in U, \alpha \in \mathbb{F}$$
$$\langle v, S(u_1 + \alpha u_2)\rangle = \langle T(v), u_1 + \alpha u_2\rangle = \langle T(v), u_1\rangle + \overline{\alpha}\langle T(v), u_2\rangle = K_{u_1}(v) + \overline{\alpha}K_{u_2}(v) =$$
$$= \langle v, h_{u_1}\rangle + \overline{\alpha}\langle v, h_{u_2}\rangle = \langle v, S(u_1)\rangle + \overline{\alpha}\langle v, S(u_2)\rangle = \langle v, S(u_1) + \alpha S(u_2)\rangle$$
$$\forall v \in V : \langle v, S(u_1 + \alpha u_2)\rangle = \langle v, S(u_1) + \alpha S(u_2)\rangle$$
$$\implies S(u_1 + \alpha u_2) = S(u_1) + \alpha S(u_2) \implies \boxed{S \text{ is a linear transformation}}$$
$$\text{Let } \hat{S} : U \to V, \forall v \in V, u \in U : \langle T(v), u\rangle = \langle v, S(u)\rangle$$
$$\implies \forall v \in V, u \in U : \langle v, S(u)\rangle = \langle v, \hat{S}(u)\rangle \implies \forall u \in U : S(u) = \hat{S}(u)$$
$$\implies \boxed{S = \hat{S}}$$

$$\text{Special note:}$$
$$A \in \mathbb{F}^{n\times m}, v \in \mathbb{F}^m, u \in \mathbb{F}^n$$
$$\langle Av, u\rangle = (Av)^T\overline{u} = v^TA^T\overline{u}$$
$$\langle v, A^*u\rangle = v^T\overline{A^*u} = v^TA^T\overline{u}$$
$$\implies \boxed{\langle Av, u\rangle = \langle v, A^*u\rangle}$$

Representation matrix of adjoint linear transformation #lemma

B orthonormal basis of V
C orthonormal basis of U
$T : V \to U$ is a linear transformation
$[T^*]^C_B = ([T]^B_C)^*$

Proof:
$\dim V = n$
$\dim U = m$
$[T]^B_C \in \mathbb{F}^{m \times n}$
$[T^*]^C_B \in \mathbb{F}^{n \times m}$
$\forall v \in V, u \in U : \langle T(v), u \rangle = \langle v, T*(u) \rangle$

$$\langle T(v), u \rangle = [T(v)]^T_C G_C \overline{[u]_C} = [T(v)]^T_C \overline{[u]_C} = [v]^T_B ([T]^B_C)^T \overline{[u]_C}$$

$$\langle v, T^*(u) \rangle = [v]^T_B G_B \overline{[T^*(u)]_B} = [v]^T_B \overline{[T^*]^C_B [u]_C} = [v]^T_B \overline{[T^*]^C_B} \overline{[u]_C}$$

$$\implies ([T]^B_C)^T = \overline{[T^*]^C_B} \implies \boxed{([T]^B_C)^* = [T^*]^C_B}$$

Properties of adjoint linear transformation #lemma

1. $(T^*)^* = T$
2. $(T + S)^* = T^* + S^*$
3. $(\alpha T)^* = \overline{\alpha} T^*$
4. $(S \circ T)^* = T^* \circ S^*$

Proof for 4:

$$[(S \circ T)^*]^D_B = ([S \circ T]^B_D)^* = ([S]^C_D [T]^B_C)^* = ([T]^B_C)^* ([S]^C_D)^* = [T^*]^C_B [S^*]^D_C = [T^* \circ S^*]^D_B$$

Special linear operators #definition

Let V be an inner product space over $\mathbb{F}$
Let $T : V \to V$ be a linear operator

Normal operator

$$TT^* = T^*T$$

Unitary operator

$$TT^* = I = T^*T$$
or $T^{-1} = T^*$

Hermitian (self-adjoint) operator

$$T^* = T$$

Special matrices #definition

Let $A \in \mathbb{F}^{m \times n}$

Normal matrix

$$AA^* = A^*A$$

Unitary matrix

$$AA^* = I = A^*A$$
or $A^{-1} = A^*$

Hermitian (self-adjoint) matrix

$$A = A^*$$