2

$$\text{Let } V = \mathbb{R}^{2\times 2} \text{ with a standrad inner product}$$
$$\text{Let } W = \mathbb{R}^2 \text{ with inner product:}$$
$$\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle = xx' - xy' - x'y + 2yy'$$
$$\text{Let } T : V \to W \text{ be a linear operator}$$
$$\forall A \in V : T(A) = C_1(A) + C_2(A)$$

2a

$$\text{Find } T^*$$

$$\text{Solution:}$$
$$\text{Standard basis of } \mathbb{R}^{2\times 2} \text{ is an orthonormal basis of } V$$
$$\forall w \in W : T^*(w) = \sum_{i=1}^{4} \langle w, T(e_i) \rangle e_i$$
$$T(e_1) = T(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$T(e_3) = T(e_4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\implies T^* \begin{pmatrix} x \\ y \end{pmatrix} = \langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle (e_1 + e_2) + \langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle (e_3 + e_4) =$$
$$= (x - y)(e_1 + e_2) + (2y - x)(e_3 + e_4) = \begin{pmatrix} x - y & x - y \\ 2y - x & 2y - x \end{pmatrix}$$

2b

$$\text{Find an orthonormal basis of } \ker T$$

$$\text{Solution:}$$
$$\text{Let } S = \{e_1, e_2, e_3, e_4\} \text{ be a standard basis of } V$$
$$\ker T \subseteq V$$
$$T(e_1) = T(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$T(e_3) = T(e_4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\implies \forall v \in V : T(v) = T(\alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4) = (\alpha + \beta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\gamma + \delta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\implies T(v) = 0 \iff \begin{cases} \alpha = -\beta \\ \gamma = -\delta \end{cases} \iff v = \alpha(e_1 - e_2) + \gamma(e_3 - e_4)$$
$$\iff v \in sp \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$
$$\langle \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \rangle = tr \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = tr \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0$$
$$\implies \text{This is an orthogonal basis of } \ker T$$
$$\left\| \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right\|^2 = tr \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = tr \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2$$
$$\left\| \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\|^2 = tr \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = tr \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 2$$
$$\implies \text{An orthonormal basis of } \ker T \text{ is } \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

3a.1

Let $J \in \mathbb{C}^{n\times n}$ be a Jordan form matrix

Prove: $J \sim J^T$

Proof:

Let A be a Jordan block of size m

$$A \in \mathbb{C}^{m\times m}, A = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

$$A^T = \begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$

$$\implies P_{A^T}(x) = (x-\lambda)^n$$

$$x = \lambda \implies xI - A^T = \begin{pmatrix} 0 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 0 \end{pmatrix} \implies rank(xI - A^T) = n-1$$

$$\implies g_\lambda = 1 \implies J_{A^T} = J_n(\lambda) = A$$

$$\implies A^T \sim J_{A^T} = A$$

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} \text{ where } \forall i \in [1,k] : J_i \text{ is a Jodran block}$$

$$J^T = \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_k^T \end{pmatrix}$$

$$\forall i \in [1,k] : J_i \sim J_i^T$$

$$\text{Let } P = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_k \end{pmatrix} \text{ such that } \forall i \in [1,k] : J_1 = P_i J_i^T P_i^{-1}$$

$$\implies P^{-1} = \begin{pmatrix} P_1^{-1} & & \\ & \ddots & \\ & & P_k^{-1} \end{pmatrix}$$

$$\implies PJP^{-1} = \begin{pmatrix} P_1 J_1 P_1^{-1} & & \\ & \ddots & \\ & & P_k J_k P_k^{-1} \end{pmatrix} = \begin{pmatrix} J_1^T & & \\ & \ddots & \\ & & J_k^T \end{pmatrix} = J^T$$

$$\implies \boxed{J \sim J^T}$$

3a.2

Prove: $\forall A \in \mathbb{C}^{n\times n} : A \sim A^T$

Proof:

$$\exists Q : A = QJ_AQ^{-1}$$
$$\implies A^T = (Q^{-1})^T J_A^T Q^T$$
By 3a.1: $\exists P : J_A^T = PJ_AP^{-1}$
$$\implies A^T = (Q^{-1})^T P J_A P^{-1} Q^T$$
$$(Q^{-1})^T P \cdot P^{-1} Q^T = (Q^{-1})^T Q^T = (QQ^{-1})^T = I^T = I$$
$$P^{-1}Q^T \cdot (Q^{-1})^T P = P^{-1}(QQ^{-1})^T P = P^{-1} I^T P = P^{-1}P = I$$
$$\implies A^T \sim J_A \sim A \implies \boxed{A^T \sim A}$$

3b

Let V be a finite-dimensional inner product space over $\mathbb{C}$

Let $T : V \to V$ be a diagonalizable linear operator

Prove: $\exists$ inner product on V such that T is normal

Proof:

T is diagonalizable $\implies \exists B$ basis of V such that $[T]_B^B$ is diagonal

Let $B = \{v_1, \ldots, v_n\}$

$$\text{Let } \forall i,j \in [1,n] : \langle v_i, v_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$
$$\implies G_B = I \implies \forall v,u \in V : \langle v,u\rangle = [v]_B^T G_B \overline{[u]_B} = [v]_B^T \overline{[u]_B}$$
$G_B = I \implies B$ is an orthonormal basis of V

$[T]_B^B$ is diagonal and B is orthonormal $\implies T$ is unitary diagonalizable $\implies$ $\boxed{T \text{ is normal}}$

4a

Let V be a finite-dimensional inner product space over $\mathbb{R}$

Let $T : V \to V$ be an anti-Hermitian linear operator

1

Prove or disprove: T is diagonalizable

Disproof:

Let $V = \mathbb{R}^2$ with standard inner product

$$\text{Let } T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$
$$\forall v,u \in V : \langle Tv, u\rangle = \langle T\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\rangle = \langle \begin{pmatrix} -y \\ x \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\rangle = -yz + xw$$
$$\langle Tv, u\rangle = \langle v, T^*u\rangle = \langle \begin{pmatrix} x \\ y \end{pmatrix}, T^*\begin{pmatrix} z \\ w \end{pmatrix}\rangle = x\cdot a + y \cdot b \implies T^*\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} w \\ -z \end{pmatrix}$$
$$\implies T^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = -T\begin{pmatrix} x \\ y \end{pmatrix} \implies T^* = -T$$
T is anti-Hermitian
$$[T]_S^S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \implies P_T(x) = P_{[T]_S^S}(x) = x^2 + 1$$
$\implies P_T(x)$ is not factorizable into linear factors over $\mathbb{R}$

$\implies T$ is not diagonalizable over $\mathbb{R}$

2

Prove or disprove: T^2 is diagonalizable

Proof:

$$\forall v \in V : \langle T^2 v, v\rangle = \langle v, (T^*)^2 c\rangle$$

$$(T^*)^2 = (-T)^2 = T^2$$

$$\implies (T^2)^* = (T^*)^2 = T^2 \implies T^2 \text{ is Hermitian}$$

Let B be an orthonormal basis of V

$$\implies [T^2]_B^B = [(T^2)^*]_B^B = ([T^2]_B^B)^* = ([T^2]_B^B)^T \implies [T^2]_B^B \text{ is symmetric}$$

$$\implies [T^2]_B^B \text{ is orthogonal diagonalizable} \implies \boxed{T^2 \text{ is diagonalizable}}$$

4b

Let V be a finite-dimensional inner product space over $\mathbb{C}$

Let W be a subspace of V

Prove or disprove: $\forall w \in W, \forall v \in V : \langle v, w\rangle = \langle P_W(v), w\rangle$

Proof:

Let $w \in W, v \in V$

$$v = \underbrace{P_W(v)}_{\in W} + \underbrace{(v - P_W(v))}_{\in W^\perp}$$

$$\implies \langle v, w\rangle = \langle P_W(v) + (v - P_W(v)), w\rangle = \langle P_W(v), w\rangle + \underbrace{\langle v - P_W(v), w\rangle}_{=0} = \langle P_W(v), w\rangle$$

5

Let V be a finite-dimensional inner product space over $\mathbb{R}$

Let $T : V \to V$ be an idempotent linear operator

5a

Determine whether T is necessrily diagonalizable

Solution:

$$T = T^2 \implies T(T - I) = 0 \implies m_T(x) \mid x(x-1)$$

$$\implies m_T(x) = \left[\begin{array}{c} x \\ (x-1) \\ x(x-1) \end{array}\right. \implies m_T(x) \text{ is factorizable into distinct linear factors}$$

$$\implies \boxed{T \text{ is diagonalizable}}$$

5b

Let U be a subspace of V

Prove: P_U is idempotent

Proof:

$$\forall v \in V : P_U(v) \in U$$

$$\forall u \in U : P_U(u) = u$$

$$\implies \forall v \in V : P_U^2(v) = P_U(\underbrace{P_U(v)}_{\in U}) = P_U(v) \implies P_U^2 = P_U$$

5c

$$\text{Prove: } \mathrm{Im}T = \ker(I-T)$$

$$\text{Proof:}$$

$$\text{Let } v \in \mathrm{Im}T$$

$$\implies \exists u \in V : T(u) = v \implies T(T(u)) = v \implies T(v) = v$$

$$\implies v - T(v) = 0 \implies I(v) - T(v) = 0 \implies (I-T)(v) = 0 \implies v \in \ker(I-T)$$

$$\implies \mathrm{Im}T \subseteq \ker(I-T)$$

$$\text{Let } v \in \ker(I-T)$$

$$\implies (I-T)(v) = 0 \implies v - T(v) = 0 \implies T(v) = v \implies v \in \mathrm{Im}T$$

$$\implies \ker(I-T) \subseteq \mathrm{Im}T \implies \boxed{\mathrm{Im}T = \ker(I-T)}$$

5d

$$\text{Let } T \text{ be normal}$$

1

$$\text{Prove: } T \text{ is Hermitian}$$

$$\text{Proof:}$$

$$T \text{ is diagonalizable} \implies \text{Its characteristic polynomial is factorizable into linear factors}$$

$$T \text{ is also normal} \implies T \text{ is unitary diagonalizable}$$

$$\implies \exists B \text{ orthonormal: } [T]_B^B \text{ is diagonal}$$

$$[T]_B^B \text{ is diagonal} \implies [T]_B^B \text{ is symmetric}$$

$$[T]_B^B \in \mathbb{R}^{n\times n} \implies [T]_B^B \text{ is Hermitian}$$

$$\implies \boxed{T \text{ is Hermitian}}$$

2

$$\text{Prove: } \exists W \text{ subspace of } V : \forall w \in W : T(w) = w \text{ and } \forall u \in W^\perp : T(u) = 0$$

$$\text{Proof:}$$

$$\mathrm{Im}T = \ker(I-T) \implies \boxed{\forall v \in \mathrm{Im}T : v - T(v) = 0 \implies T(v) = v}$$

$$\text{Let } W = \mathrm{Im}T$$

$$\text{Let } u \in W^\perp$$

$$\implies \forall w \in W : \langle w, u\rangle = 0$$

$$\forall v \in V : \langle \underbrace{T(v)}_{\in W}, u\rangle = \langle v, T^*(u)\rangle = 0$$

$$T \text{ is Hermitian} \implies \forall v \in V : \langle v, T^*(u)\rangle = \langle v, T(u)\rangle = 0$$

$$\implies T(u) = 0$$

$$\implies \boxed{\forall u \in W^\perp : T(u) = 0} \implies W^\perp \subseteq \ker T$$

$$\text{It is also possible to prove } W^\perp = \ker T \text{ if necessary}$$