Let $V = \mathbb{R}^{2 \times 2}$ with a standrad inner product

Let $W = \mathbb{R}^2$ with inner product:

$$\langle inom{x}{y}, inom{x'}{y'}
angle
angle = xx' - xy' - x'y + 2yy'$$

Let $T:V \to W$ be a linear operator

$$orall A \in V: T(A) = C_1(A) + C_2(A)$$

2a

Find T^*

Solution:

Standard basis of $\mathbb{R}^{2 \times 2}$ is an orthonormal basis of V

$$egin{aligned} orall w \in W: T^*(w) &= \sum_{i=1}^4 \langle w, T(e_i)
angle e_i \ &T(e_1) = T(e_2) = egin{pmatrix} 1 \ 0 \end{pmatrix} \ &T(e_3) = T(e_4) = egin{pmatrix} 0 \ 1 \end{pmatrix} \ &\Longrightarrow T^* egin{pmatrix} x \ y \end{pmatrix} = \langle inom{x} \ y \end{pmatrix}, inom{1} \ 0 \end{pmatrix}
angle (e_1 + e_2) + \langle inom{x} \ y \end{pmatrix}, inom{0} \ 1 \end{pmatrix}
angle (e_3 + e_4) = \ &= (x-y)(e_1 + e_2) + (2y-x)(e_3 + e_4) = inom{x-y}{2y-x} \ 2y-x \end{pmatrix}$$

2b

Find an orthonormal basis of $\ker T$

Solution:

Let
$$S = \{e_1, e_2, e_3, e_4\}$$
 be a standard basis of V

$$\ker T\subseteq V$$

$$T(e_1)=T(e_2)=egin{pmatrix}1\0\end{pmatrix}$$

$$T(e_3)=T(e_4)=egin{pmatrix}0\1\end{pmatrix}$$

$$egin{aligned} \Longrightarrow \ orall v \in V: T(v) = T(lpha e_1 + eta e_2 + \gamma e_3 + \delta e_4) = (lpha + eta) egin{pmatrix} 1 \ 0 \end{pmatrix} + (\gamma + \delta) egin{pmatrix} 0 \ 1 \end{pmatrix} \ \Longrightarrow \ T(v) = 0 \iff egin{cases} lpha = -eta \ \gamma = -\delta \iff v = lpha(e_1 - e_2) + \gamma(e_3 - e_4) \ \Leftrightarrow v \in sp \left\{ egin{pmatrix} 1 & -1 \ 0 & 0 \end{pmatrix}, egin{pmatrix} 0 & 0 \ 1 & -1 \end{pmatrix}
ight\} \end{aligned}$$

$$\langle egin{pmatrix} 1 & -1 \ 0 & 0 \end{pmatrix}, egin{pmatrix} 0 & 0 \ 1 & -1 \end{pmatrix}
angle = tr egin{pmatrix} 1 & -1 \ 0 & 0 \end{pmatrix} egin{pmatrix} 0 & 1 \ 0 & -1 \end{pmatrix} = tr egin{pmatrix} 0 & 2 \ 0 & 0 \end{pmatrix} = 0$$

 \implies This is an orthogonal basis of ker T

$$egin{pmatrix} 1 & -1 \ 0 & 0 \end{pmatrix}^{-2} = tr egin{pmatrix} 1 & -1 \ 0 & 0 \end{pmatrix} egin{pmatrix} 1 & 0 \ -1 & 0 \end{pmatrix} = tr egin{pmatrix} 2 & 0 \ 0 & 0 \end{pmatrix} = 2$$

$$egin{pmatrix} \left(egin{matrix} 0 & 0 \ 1 & -1 \end{matrix}
ight)^{-2} = tr \left(egin{matrix} 0 & 0 \ 1 & -1 \end{matrix}
ight) \left(egin{matrix} 0 & 1 \ 0 & -1 \end{matrix}
ight) = tr \left(egin{matrix} 0 & 0 \ 0 & 2 \end{matrix}
ight) = 2$$

 \implies An orthonormal basis of $\ker T$ is $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$