Orthogonal decomposition theorem #theorem

$$\text{Let } W \subseteq V \text{ be a subspace of } V$$
$$\text{Then } W \oplus W^{\perp} = V$$

$$\text{Proof:}$$
$$\text{Let } v \in W \cap W^{\perp}$$
$$\implies \langle v, v \rangle = 0 \implies v = 0 \implies W \cap W^{\perp} = \{0\}$$
$$\text{Let } v \in V$$
$$v = P_W(v) + (v - P_W(v))$$
$$P_W(v) \in W$$
$$v - P_W(v) \in W^{\perp}$$
$$\implies v \in W + W^{\perp} \implies V \subseteq W + W^{\perp} \implies \boxed{W \oplus W^{\perp} = V}$$

$$W \oplus U = V \not\Longrightarrow U = W^{\perp}$$

$$\text{Example:}$$
$$V = \mathbb{R}^2$$
$$U = sp\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}, W = sp\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$$
$$W \oplus U = \mathbb{R}^2 = V$$
$$\begin{pmatrix}1\\-1\end{pmatrix} \in W^{\perp}, \notin U$$

$$\text{Let } W \text{ be a subspeace of } V$$
$$\text{Let } B_1, B_2 \text{ be orthogonal bases of } W$$
$$\text{Then } \forall v \in V : P_W^{B_1}(v) = P_W^{B_2}(v)$$

$$\text{Proof:}$$
$$\text{Let } v \in V$$
$$W \oplus W^{\perp} = V \iff \exists! w \in W, w^{\perp} \in W^{\perp} : v = w + w^{\perp}$$
$$v = \underbrace{P_W^{B_1}(v)}_{w_1} + \underbrace{(v - P_W^{B_1}(v))}_{w_1^{\perp}}$$
$$v = \underbrace{P_W^{B_2}(v)}_{w_2} + \underbrace{(v - P_W^{B_2}(v))}_{w_2^{\perp}}$$
$$W \oplus W^{\perp} = V \implies w_1 = w_2, w_1^{\perp} = w_2^{\perp}$$
$$\implies \boxed{P_W^{B_1}(v) = P_W^{B_2}(v)}$$

Gram-Schmidt orthonormalization #definition

$$\text{Let } B = \{v_1, \dots, v_n\} \text{ be a basis of } V$$
$$\text{Let } u_1 = v_1$$
$$\text{Let } U_k = sp\{u_1, \dots, u_k\}$$
$$\text{Let } \forall k \in [2, n] : \boxed{u_k = v_k - P_{U_{k-1}}(v_k)}$$
$$\forall i \in [1, k-1] : u_i \in U_{k-1}$$
$$u_k = v_k - \underbrace{P_{U_{k-1}}(v_k)}_{\in U_{k-1}} \in U_{k-1}^{\perp}$$
$$\implies \forall k \in [1, n] : U_k \text{ is on orthogonal set}$$
$$u_1 = v_1 \in sp\{v_1\} \implies U_1 = sp\{v_1\}$$
$$u_2 \in v_2 + U_1 = sp\{v_1, v_2\} \implies U_2 = sp\{v_1, v_2\}$$
$$\implies \text{By induction:}$$
$$\forall k \in [1, n] : u_k \in v_k + U_{k-1} = sp\{v_1, \dots, v_k\} \implies U_k = sp\{v_1, \dots, v_k\}$$
$$\forall k \in [1, n] : v_k \notin U_{k-1} \implies v_k \neq P_{U_{k-1}}(v_k) \implies u_k \neq 0 \implies \|u_k\| > 0$$
$$\implies \boxed{\{u_1, \dots, u_n\} \text{ is an orthogonal basis of } V}$$
$$\boxed{\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \dots, \frac{u_n}{\|u_n\|}\right\} \text{ is then an orthonormal basis of } V}$$

Example

$$V = \mathbb{R}^3$$
$$U = sp\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$$
$$\text{Find an orthonormal basis of } U$$

$$\text{Solution:}$$
$$u_1 = v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\|u_1\|^2}u_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} - 2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$
$$\langle u_1, u_2\rangle = 0$$
$$\implies \text{Orthonormal basis of } U \text{ is } \left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

$$\text{Note: we can extend any orthogonal basis by using Gram-Schmidt process}$$
$$\{u_1, \dots, u_k\} \to \{u_1, \dots, u_k, v_1, \dots, v_t\} \to \{\hat{u}_1, \dots, \hat{u}_k, \hat{v}_1, \dots, \hat{v}_t\}$$

$$\text{Even more than that:}$$
$$\forall i \in [1, t] : \hat{v}_i \in \{\hat{u}_1, \dots, \hat{u}_k\}^{\perp} = U^{\perp}$$
$$\implies \{\hat{v}_1, \dots, \hat{v}_t\} \subseteq U^{\perp}$$
$$U \oplus U^{\perp} = V$$
$$\implies \underbrace{\dim V}_{k+t} = \dim U + \dim U^{\perp} = k + \dim U^{\perp}$$
$$\implies \dim U^{\perp} = t$$
$$\implies sp\{\hat{v}_1, \dots, \hat{v}_t\} = U^{\perp}$$

Orthogonal projection norm #lemma

Let V be an inner product space
Let $W \subseteq V$ be a subspace of V
Then $\forall w \in W : \|v - w\| \geq \|v - P_W(v)\|$
And $\|v - w\| = \|P_W(v) - w\| \iff w = P_W(v)$

Proof:
Let $v \in V$
Let $w \in W$

$$\|v - w\|^2 = \|\underbrace{v - P_W(v)}_{\in W^\perp} + \underbrace{P_W(v) - w}_{\in W}\|^2 =$$

$$\overset{\text{By Pythagorean theorem}}{=} \|v - P_W(v)\|^2 + \|P_W(v) - w\|^2 \geq \|v - P_W(v)\|^2$$

$$\implies \|v - w\|^2 \geq \|v - P_W(v)\|^2 \implies \boxed{\|v - w\| \geq \|v - P_W(v)\|}$$

$$\|v - w\| = \|v - P_W(v)\| \iff \|P_W(v) - w\| = 0$$

$$\iff P_W(v) - w = 0 \iff \boxed{P_W(v) = w}$$

Bessel's inequality #lemma

Let V be an inner product space
Let $A = \{v_1, \ldots, v_k\}$ be an orthonormal set
Let $v \in V$

$$\|v\|^2 \geq \sum_{i=1}^{k} |\langle v, v_i \rangle|^2$$

$$\|v\|^2 = \sum_{i=1}^{k} |\langle v, v_i \rangle|^2 \iff v \in spA$$

Proof:
Let $v \in V$
Let $B = A \cup \{v_{k+1}, \ldots, v_{k+t}\}$ be an orthonormal basis of V

$$\implies v = \sum_{i=1}^{k+t} \alpha_i v_i$$

B is orthonormal $\implies \forall i \in [1, k+t] : \alpha_i = \langle v, v_i \rangle$

$$\implies v = \sum_{i=1}^{k+t} \langle v, v_i \rangle v_i$$

$$\|v\|^2 = \left\| \sum_{i=1}^{k+t} \langle v, v_i \rangle v_i \right\|^2 = \sum_{i=1}^{k+t} \|\langle v, v_i \rangle v_i\|^2 = \sum_{i=1}^{k+1} |\langle v, v_i \rangle|^2 \|v_i\|^2 = \sum_{i=1}^{k+t} |\langle v, v_i \rangle|^2 \geq \sum_{i=1}^{k} |\langle v, v_i \rangle|^2$$

$$\implies \boxed{\|v\|^2 \geq \sum_{i=1}^{k} |\langle v, v_i \rangle|^2}$$

$$\|v\|^2 = \sum_{i=1}^{k} |\langle v, v_i \rangle|^2 \iff \sum_{i=k+1}^{k+t} |\langle v, v_i \rangle|^2 = 0 \iff \forall i \in [k+1, k+t] : \langle v, v_i \rangle = 0$$

$$\iff v = \sum_{i=1}^{k+t} \langle v, v_i \rangle v_i = \sum_{i=1}^{k} \langle v, v_i \rangle v_i \iff \boxed{v \in spA}$$

Cauchy-Schwarz inequality #lemma

$$\text{Let } V \text{ be an inner product space}$$
$$\text{Let } v, u \in V$$
$$\|v\| \cdot \|u\| \geq |\langle v, u\rangle|$$
$$\|v\| \cdot \|u\| = |\langle v, u\rangle| \iff u = \alpha v$$

Proof:

$$\text{Case 1. } u = v = 0$$
$$\underbrace{\|v\|}_{0} \cdot \underbrace{\|u\|}_{0} \geq \underbrace{|\langle v, u\rangle|}_{0}$$
$$\text{Case 2. } \begin{bmatrix} v \neq 0 \\ u \neq 0 \end{bmatrix} \quad \text{Let } v \neq 0$$
$$\text{Let } A = \left\{ \frac{v}{\|v\|} \right\}$$
$$A \text{ is an orthonormal set}$$
$$\text{By Bessel's inequality: } \|u\|^2 \geq \left| \left\langle u, \frac{v}{\|v\|} \right\rangle \right|^2 = \frac{1}{\|v\|^2} |\langle u, v\rangle|^2 \underset{|z|=|\overline{z}|}{=} \frac{1}{\|v\|^2} |\langle v, u\rangle|^2$$
$$\implies \|u\|^2 \cdot \|v\|^2 \geq |\langle v, u\rangle|^2$$
$$\implies \boxed{\|v\| \cdot \|u\| \geq |\langle v, u\rangle|}$$

$$\|v\| \cdot \|u\| = |\langle v, u\rangle| \iff \underbrace{\|u\|^2 = \frac{1}{\|v\|^2} |\langle v, u\rangle|^2}_{\text{Bessel's equality case}}$$
$$\iff u \in spA \iff \boxed{u = \alpha v}$$

Root norm #lemma

$$\text{Let } v \in V$$
$$\sqrt{\langle v, v\rangle} = \|v\| \text{ (or } \|v\|^2 = \langle v, v\rangle)$$

Proof:

$$\sqrt{\langle v, v\rangle} \geq 0$$
$$\sqrt{\langle v, v\rangle} = 0 \iff v = 0$$
$$\sqrt{\langle \alpha v, \alpha v\rangle} = \sqrt{\alpha^2 \langle v, v\rangle} = |\alpha| \sqrt{\langle v, v\rangle}$$
$$\|v + u\|^2 = \langle v + u, v + u\rangle = \langle v, v + u\rangle + \langle u, v + u\rangle = \langle v, v\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle u, u\rangle =$$
$$= \|v\|^2 + \langle v, u\rangle + \overline{\langle v, u\rangle} + \|u\|^2 = \|v\|^2 + 2\mathrm{Re}(\langle v, u\rangle) + \|u\|^2 \leq$$
$$\leq \|v\|^2 + 2\left|\langle v, u\rangle\right| + \|u^2\| \overset{\text{By Cauchy-Schwarz inequality}}{\leq} \|v\|^2 + 2\|v\| \cdot \|u\| + \|u\|^2 = (\|v\| + \|u\|)^2$$
$$\implies \|v + u\| \leq \|v\| + \|u\|$$