Commutativity of determinant #lemma

$$\forall A, B \in \mathbb{F}^{n\times n}$$
$$|AB| = |BA|$$

Proof:

$$|AB| = |A| \cdot |B| = |B| \cdot |A| = |BA|$$

Determinant of n matrix product #theorem

$$\forall \{A_i\}_{i\in I} \subseteq \mathbb{F}^{n\times n}$$
$$\left|\prod_{i\in I} A_i\right| = \prod_{i\in I} |A_i|$$

Proof:

By induction, starting from the identical lemma for two matrices

Determinant of transpose #lemma

$$\forall A \in \mathbb{F}^{n\times n}$$
$$|A| = |A^T|$$

Proof:

Let $P : \alpha R_i$

$P(I)$ is diagonal

$$\implies P(I) = P(I)^T \implies |P(I)| = |P(I)^T|$$

Let $P : R_i \leftrightarrow R_j$

$$P(I) = P(I)^T \implies |P(I)| = |P(I)^T|$$

Let $P : R_i + \alpha R_j$

$P(I)$ is triangular $\implies$ $P(I)^T$ is triangular

Diagonals of $P(I)$ and $P(I)^T$ are the same

$$\implies |P(I)| = |P(I)^T|$$

Let A be non-invertible

$$\implies rank(A^T) = rank(A) < n$$

$\implies$ A^T is non-invertible $\implies |A^T| = 0 = |A|$

Let A be invertible

$$\implies A = \prod_{i=1}^{k} P_i(I) \implies |A| = \left|\prod_{i=1}^{k} P_i(I)\right| = \prod_{i=1}^{k} |P_i(I)|$$
$$A^T = \left(\prod_{i=1}^{k} P_i(I)\right)^T = \prod_{i=1}^{k} P_{k+1-i}(I)^T$$
$$\implies |A^T| = \left|\prod_{i=1}^{k} P_{k+1-i}(I)^T\right| = \prod_{i=1}^{k} |P_{k+1-i}(I)^T| =$$
$$= \prod_{i=1}^{k} |P_{k+1-i}(I)| = \prod_{i=1}^{k} |P_i(I)|$$
$$\implies |A^T| = \prod_{i=1}^{k} |P_i(I)| = |A|$$

Corollary

$$\text{Elementary column operations affect determinant in the same way row operations do}$$

Determinant of an inverse #lemma

$$\begin{gathered}\text{Let } A \in \mathbb{F}^{n\times n} \\ \text{Let } \exists A^{-1} \\ \implies |A| \neq 0 \\ |AA^{-1}| = |A| \cdot |A^{-1}| = |I| = 1 \\ \implies |A^{-1}| = \frac{1}{|A|}\end{gathered}$$

Summary

$$\begin{gathered}\exists A^{-1} \iff |A| \neq 0 \\ |AB| = |A| \cdot |B| \\ |A| = |A^T| \\ |\alpha A| = \alpha^n |A| \\ |A^{-1}| = |A|^{-1}\end{gathered}$$

Matrix minor #definition

$$\begin{gathered}\text{Let } A \in \mathbb{F}^{n\times n} \\ \text{Let } i, j \in [1, n] \\ \text{Minor } M_{ij}(A) \text{ is a matrix,} \\ \text{obtained by removing row } i \text{ and column } j \text{ from matrix } A \\ A = \begin{pmatrix} 1 & 2 & \boxed{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \implies M_{13}(A) = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}\end{gathered}$$

Using minors to calculate determinant #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$

$$\forall i \in [1, n] : |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |M_{ij}(A)|$$

$$\forall j \in [1, n] : |A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |M_{ij}(A)|$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$|A| = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 + 12 - 9 = 0$$

$$|A| = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 60 + 48 = 0$$

Determinant of a linear operator (transformation) #lemma

$$\text{Let } V \text{ be a vector space over } \mathbb{F}$$
$$\text{Let } T : V \to V \text{ be a linear transformation}$$
$$\text{Let } B \text{ be a basis of } V$$
$$|T| := \left|[T]_B^B\right|$$
$$\forall B, C \text{ basis of } V : \left|[T]_B^B\right| = \left|[T]_C^C\right|$$

$$\text{Proof:}$$
$$[T]_B^B = [I]_B^C [T]_C^C [I]_C^B$$
$$\implies \left|[T]_B^B\right| = \left|[I]_B^C [T]_C^C [I]_C^B\right| = \left|[I]_B^C\right| \cdot \left|[T]_C^C\right| \cdot \left|[I]_C^B\right| =$$
$$= \left|[T]_C^C\right| \cdot \left|[I]_B^C [I]_C^B\right| = \left|[T]_C^C\right| \cdot \left|[I]_B^B\right| = \left|[T]_C^C\right|$$

Eigenvalues and eigenvectors #definition

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\lambda \in \mathbb{F} \text{ is called an Eigenvalue of } A \text{ if}$$
$$\exists v \neq 0 \in \mathbb{F}^n : Av = \lambda v$$
$$v \text{ is then called an Eigenvector of } A \text{ in respect to Eigenvalue } \lambda$$

Example

$$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix} = 4 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$\lambda = 4, v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Determinant in relation to eigenvalue #lemma

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\lambda \text{ is an Eigenvalue of } A \iff |\lambda I - A| = 0$$

$$\text{Proof:}$$
$$\lambda \text{ is an Eigenvalue of } A$$
$$\iff \exists v \neq 0 \in \mathbb{F}^n : Av = \lambda v$$
$$\iff \exists v \neq 0 \in \mathbb{F}^n : \lambda v - Av = 0$$
$$\iff \exists v \neq 0 \in \mathbb{F}^n : (\lambda I - A)v = 0$$
$$\iff N(\lambda I - A) \neq \{0\}$$
$$\iff \nexists (\lambda I - A)^{-1}$$
$$\iff |\lambda I - A| = 0$$

Corollary

$$\nexists A^{-1} \iff 0 \text{ is an Eigenvalue of A}$$

$$\exists v \neq 0 \in \mathbb{F}^n : Av = 0v = 0$$
$$\iff N(A) \neq \{0\}$$
$$\iff \nexists A^{-1}$$

$$|A| = 0 \iff |-A| = 0 \iff |0I - A| = 0 \iff 0 \text{ is an Eigenvalue of } A$$

Example

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 3 \\ 0 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 4) = 0$$

$$\implies \begin{bmatrix} \lambda = 2 \\ \lambda = 4 \end{bmatrix}$$

How do we find Eigenvectors in respect to these Eigenvalues?

$$N(\lambda I - A) = \{v | (\lambda I - A)v = 0\}$$

$\implies$ Eigenvectors in respect to Eigenvalue λ

is a set of non-zero solutions to homogeneous system of equations $(\lambda I - A)v = 0$

Let $\lambda = 2$

$$\left(\begin{array}{cc|c} 0 & -3 & 0 \\ 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v \in sp\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

Let $\lambda = 4$

$$\begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \implies v \in sp\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$$

Eigenspace #definition

Let $A \in \mathbb{F}^{n \times n}$

Let $\alpha \in \mathbb{F}$ be an Eigenvalue of A

Set of Eigenvectors in respect to α, and zero vector, is then called Eigenspace

$$E = \{v | (\lambda I - A)v = 0\} = N(\lambda I - A)$$

Characteristic polynomial #definition

Let $A \in \mathbb{F}^{n \times n}$

Let $\lambda \in \mathbb{F}$

$P_A(\lambda) = |\lambda I - A|$ is called a characteristic polynomial of A

Matrix similarity #definition

Let $A, B \in \mathbb{F}^{n \times n}$

Matrices A, B are called similar if

$$\exists P \in \mathbb{F}^{n \times n} : P^{-1}AP = B$$

Similarity is an equivalence relation:

Reflexive: $A = I^{-1}AI$

Symmetric: $P^{-1}AP = B \implies B = (P^{-1})^{-1}AP^{-1}$

Transitive: $B = P^{-1}AP, C = P_1^{-1}BP_1 \implies C = P_1^{-1}P^{-1}APP_1 = (PP_1)^{-1}A(PP_1)$

Similar matrix properties #lemma

Let $A, B \in \mathbb{F}^{n \times n}$

Let $A \sim B$

$$|A| = |B|$$

$$tr(A) = tr(B)$$

$$rank(A) = rank(B)$$

$$P_A(\lambda) = P_B(\lambda)$$

Diagonalizable matrix #definition

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\text{Let } D \text{ be a diagonal matrix } \in \mathbb{F}^{n \times n}$$
$$A \text{ is called diagonalizable}$$
$$\iff \exists P \in \mathbb{F}^{n \times n} : P^{-1}AP = D$$