Variable substitution in integrals #lemma

When there is a pattern in the integral, being function $f(x)$
and it's derivative
We will use variable substitution:
Let $t = f(x)$

$$t'dt = f'(x)dx \implies dt = f'(x)dx$$

For example:

$$\int \frac{7\arctan(x)}{x^2+1}dx$$

$$(\arctan(x))' = \frac{1}{x^2+1}$$

$$\implies \int \frac{7\arctan(x)}{x^2+1}dx \underbrace{=}_{f(x)=\arctan(x)} 7\int f(x)f'(x)dx =$$

$$\underbrace{=}_{t=f(x)} 7\int tdt = \frac{7t^2}{2} + C = \frac{7\arctan^2(x)}{2} + C$$

Another example:

$$\int 6\ln(\sin x)\cos^3(x)dx$$

$$\text{Let } t = \sin x \implies dt = \cos(x)dx$$

$$\implies \int 6\ln(\sin x)\cos^3(x)dx = 6\int \ln(t)(1-t^2)dt$$

$$\text{Let } f(t) = \ln(t) \implies f'(t) = \frac{1}{t}$$

$$\text{Let } g'(t) = 1 - t^2 \implies g(t) = t - \frac{t^3}{3}$$

$$\implies 6\int \ln(t)(1-t^2)dt = 6\left(\ln(t)\left(t - \frac{t^3}{3}\right)\right) - 6\int\left(1 - \frac{t^2}{3}\right)dt =$$

$$= 6t\ln(t) - 2t^3\ln(t) - 6t + \frac{2t^3}{3} + C =$$

$$= 6\sin(x)\ln(\sin x) - 2\sin^3(x)\ln(\sin x) - 6\sin(x) + \frac{2\sin^3(x)}{3} + C$$

We can also substitute in a more complex way:

$$g(t) = f(x) \implies g'(t)dt = f'(x)dx$$

For example:

$$\int \sqrt{7-x^2}dx$$

$$\int \sqrt{7-x^2}dx = \sqrt{7}\int\sqrt{1-\left(\frac{x}{\sqrt{7}}\right)^2}dx$$

$$\text{Let } \sin t = \frac{x}{\sqrt{7}} \implies \cos(t)dt = \frac{dx}{\sqrt{7}} \implies dx = \sqrt{7}\cos(t)dt$$

$$\implies \sqrt{7}\int\sqrt{1-\left(\frac{x}{\sqrt{7}}\right)^2}dx = \sqrt{7}\int\sqrt{1-\sin^2(x)}dx = \sqrt{7}\int\sqrt{\cos^2(x)}dx =$$

$$= 7\int\cos^2(t)dt = 7\int\frac{1+\cos(2t)}{2}dt = \frac{7}{2}\left(t + \frac{\sin(2t)}{2}\right) + C =$$

$$= \frac{7}{2}\left(\arcsin\left(\frac{x}{\sqrt{7}}\right) + \frac{\sin\left(2\arcsin\left(\frac{x}{\sqrt{7}}\right)\right)}{2}\right) + C$$

Linear function composition integral #lemma

$$\int f(x)dx = F(x) + C \implies \int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

Proof:

Let $t = ax + b \implies dt = adx$

$$\implies \int f(ax+b)dx = \int \frac{f(t)}{a}dt = \frac{1}{a}F(t) + C = \frac{1}{a}F(ax+b) + C$$

Example:

$$\int 3^x dx = \frac{3^x}{\ln(3)} + C$$

$$\implies \int 3^{18x-5}dx = \frac{\frac{1}{18}3^{18x-5}}{\ln(3)} + C$$

Integral of a rational function (polynomial fractions) #lemma

$$\int \frac{p(x)}{q(x)}dx$$

There are usually three steps:

1. Division of polynomials
2. Partial fraction decomposition
3. Integration of partial fractions

Division of polynomials:

1.1 Divide highest degree of nominator by highest degree of denominator

1.2 Multiply result by denominator

1.3 Subtract result from nminator

1.4 Repear until degree of nominator is greater or equal to degree of denominator

For example:

$$\frac{x^9 + 6x^2 + 6}{x^4 - 1}$$

$$\frac{x^9}{x^4} = \boxed{x^5} \cdot (x^4 - 1) \implies x^9 - x^5$$

$$x^9 + 6x^2 + 6 - x^9 - x^5 = x^5 + 6x^2 + 6$$

$$\frac{x^5}{x^4} = \boxed{x} \cdot (x^4 - 1) \implies x^5 - x$$

$$x^5 + 6x^2 + 6 - x^5 - x = 6x^2 + x + 6$$

$$\implies x^9 + 6x^2 + 6 = x^5 + x + \frac{6x^2 + x + 6}{x^4 - 1}$$

Partial fraction decomposition:

Any polynomial can be decomposed into product of non-decomposable polynomials of smaller degree

Non-decomposable polynomial is a polynomial that cannot be represented as a product of polynomials of smaller degrees

A non-trivial example would be:

$$x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-2x^2 = (x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x)$$

In our case:

$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

$$\implies \frac{6x^2+x+6}{x^4-1} = \frac{6x^2+x+6}{(x-1)(x+1)(x^2+1)} = \frac{.}{x-1}+\frac{.}{x+1}+\frac{.}{x^2+1}$$

How do we calculate nominators now?

If denominator is linear, nominator will be a constant

If denominator is of second degree, nominator will be linear

$$\implies \frac{6x^2+x+6}{x^4-1} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$6x^2+x+6 = A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

Let $x=1$

$$13 = 4A \implies A = \frac{13}{4}$$

Let $x=-1$

$$11 = -4B \implies B = -\frac{11}{4}$$

$$A+B+C=0 \implies C = -\frac{2}{4}$$

$$\frac{p(x)}{(x-1)^3(x+1)} = \frac{Ax^2+Bx+C}{(x-1)^3}+\frac{D}{x+1}$$

$$\frac{Ax^2-2Ax+A}{(x-1)^3} = \frac{A(x-1)^2}{(x-1)^3} = \frac{A}{x-1}$$

$$\implies \frac{Ax^2+Bx+C}{(x-1)^3} = \frac{A}{x-1}+\frac{Bx+2Ax+C-A}{(x-1)^3}$$

$$\frac{Bx+2Ax-B-2A}{(x-1)^3} = \frac{B+2A}{(x-1)^2}$$

$$\implies \frac{Bx+2Ax+C-A}{(x-1)^3} = \frac{B+2A}{(x-1)^2}+\frac{-B-C-A}{(x-1)^3}$$

$$\implies \frac{Ax^2+Bx+C}{(x-1)^3} = \frac{A}{x-1}+\frac{B+2A}{(x-1)^2}+\frac{-(A+B+C)}{(x-1)^3} =$$

$$= \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{(x-1)^3}$$

Table of common denominators in partial fractions #lemma

This results in a following table of denominators and their decompositions

$\frac{p(x)}{ax+b}$	$\frac{A}{ax+b}$
$\frac{p(x)}{(ax+b)^k}$	$\sum_{i=1}^{k} \frac{A_i}{(ax+b)^i}$
$\frac{p(x)}{ax^2+bx+c}$	$\frac{Ax+B}{ax^2+bx+c}$
$\frac{p(x)}{(ax^2+bx+c)^k}$	$\sum_{i=1}^{k} \frac{A_ix+B_i}{(ax^2+bx+c)^i}$

Integration of partial fractions:

$$\int \frac{A}{ax+b}dx = \frac{A}{a}\ln|ax+b| + C$$

$$\int \frac{A}{(ax+b)^k}dx = \frac{A(ax+b)^{1-k}}{1-k} + C$$

$$\int \frac{Ax+B}{ax^2+bx+c}dx = \text{Generally not in this course, but it is written below}$$

$$\int \frac{Ax+b}{(ax^2+bx+c)^k}dx = \text{Generally not in this course}$$

$$\text{Specific cases of } \frac{Ax+B}{ax^2+bx+c}:$$

$$\int \frac{1}{(x+a)^2+b^2}dx = \frac{1}{b}\int \frac{1}{\left(\frac{x+a}{b}\right)^2+1} = \frac{1}{b}\arctan\left(\frac{x+a}{b}\right) + C$$

$$\int \frac{p'(x)}{p(x)}dx = \int \frac{1}{p}dp = \ln|p(x)| + C$$

$$\int \frac{3x+1}{x^2+2x+6}dx = \frac{3}{2}\int \frac{\left(2x+\frac{2}{3}\right)}{x^2+2x+6}dx =$$

$$= \frac{3}{2}\int \frac{2x+2}{2x^2+2x+6}dx - 2\int \frac{1}{x^2+2x+6}dx =$$

$$= \frac{3}{2}\ln|2x^2+2x+6| - 2\int \frac{1}{x^2+2x+6}dx$$

$$x^2+2x+5 = (x+1)^2+5$$

$$\implies \int \frac{1}{x^2+2x+6}dx = \int \frac{1}{(x+1)^2+(\sqrt{5})^2}dx = \frac{1}{\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right) + C$$

$$(ax^2+bx+c)' = 2ax+b$$

$$\frac{Ax+B}{ax^2+bx+c} = \frac{A}{2a}\cdot\frac{2ax+b+\frac{2aB}{A}-b}{ax^2+bx+c} = \frac{A}{2a}\frac{2ax+b}{ax^2+bx+c} + \left(B-\frac{Ab}{2a}\right)\frac{1}{ax^2+bx+c}$$

$$\frac{1}{ax^2+bx+c} = \frac{1}{\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2+\left(c-\frac{b^2}{4a}\right)} = \dots$$

$$\boxed{\int \frac{Ax+B}{ax^2+bx+c}dx = \frac{A}{2a}\ln|ax^2+bx+c| + \frac{(Ab-2Ba)\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{a\sqrt{4ac-b^2}} + C}$$