## Continuity of area function #lemma

$$\text{Let } f \text{ be Riemann-integrable on } [a,b]$$
$$\text{Then } S(x) = \int_a^x f(t)\,dt \text{ is continuous on } [a,b]$$

### Corollary

$$\lim_{x\to c} S(x) = S(c)$$

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$$\lim_{x\to 0} \frac{\int_0^x \sin(t^2)\,dt}{9x^3} \overset{L}{=} \lim_{x\to 0} \frac{\sin(x^2)}{27x^2} = \frac{1}{27}$$

## How do we differentiate definite integrals with various borders?

$$\int_{g(x)}^{h(x)} f(t)\,dt = \int_0^{h(x)} f(t)\,dt - \int_0^{g(x)} f(t)\,dt = S(h(x)) - S(g(x))$$
$$\implies \left(\int_{g(x)}^{h(x)} f(t)\,dt\right)' = (S(h(x)) - S(g(x)))' = h'(x)f(h(x)) - g'(x)f(g(x))$$

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$$\left(\int_{\sqrt{x}}^{x^2} \sin(t^2)\,dt\right)' = \left(\int_0^{x^2} \sin(t^2)\,dt - \int_0^{\sqrt{x}} \sin(t^2)\,dt\right)' = \sin(x^4)\cdot 2x - \sin x \cdot \frac{1}{2\sqrt{x}}$$

## Applications of definite integrals

$$\text{Let } F \text{ be a primitive of } f$$
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

1. Calculating area
   area between graphs of functions $f, g$ on $[a,b]$
   $$\text{is equal to } \int_a^b |f(x) - g(x)|\,dx$$
2. Calculating volume of a revolution (Pappus theorem)
   volume of a revolution is equal to
   $$\begin{cases} V_X(f) = \pi \int_a^b f^2(x)\,dx \text{ when rotating around axis } X \\ V_Y(f) = 2\pi \int_a^b x f(x)\,dx \text{ when rotating around axis } Y \end{cases}$$
3. Arc length
   length of the arc of continuously differentiable function on $[a,b]$
   $$\text{is equal to } L(f) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$
4. Revolution surface area
   Revolution surface area of continuously differentiable function on $[a,b]$
   $$\text{is equal to } A(f) = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Proof for 3.

$$\text{Let } P = \{x_0, \dots, x_n\}$$
$$L(f) \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$
$$\text{Let } \Delta x_i = x_i - x_{i-1}$$
$$\text{Let } \Delta f(x_i) = f(x_i) - f(x_{i-1})$$
$$\implies L(f) \approx \sum_{i=1}^{n} \Delta x_i \sqrt{1 + \left(\frac{\Delta f(x_i)}{\Delta x_i}\right)^2}$$
$$\text{By the Mean value theorem: } \exists c_i \in [x_{i-1}, x_i] : f'(c_i) = \frac{\Delta f(x_i)}{\Delta x_i}$$
$$\text{Let } C = \{c_1, \dots, c_n\}$$
$$\implies \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2} \Delta x_i = S(f, P, C)$$
$$\implies L(f) = \lim_{\lambda(P) \to 0} S(f, P, C) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

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$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\text{Let us calculate the arc length of hyperbolic cosine on } [0, 1]$$
$$(\cosh x)' = \frac{e^x - e^{-x}}{2} = \sinh x$$
$$L(\cosh) = \int_0^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx$$
$$\sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} = \sqrt{\frac{4 + e^{2x} - 2 + e^{-2x}}{4}} = \frac{\sqrt{e^{2x} + 2 + e^{-2x}}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$\implies L(\cosh) = \int_0^1 \cosh x\, dx = \int_0^1 \frac{e^x + e^{-x}}{2}\, dx = \left(\frac{e^x - e^{-x}}{2}\right)_{x=0}^{x=1} = \frac{e - e^{-1}}{2}$$

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