Minimal polynomial #definition

Let $A \in \mathbb{F}^{n\times n}$

Let $f(x) \in \mathbb{F}[x]$

f is called a minimal polynomial of A if

A is it's root and there are no such polynomials of smaller degree

Minimal matrix polynomial is denoted as $m_A(x)$

Note: minimal polynomial is always monic

Existence and uniqueness of minimal polynomial #lemma

Let $A \in \mathbb{F}^{n\times n}$

$\exists! m_A(x) : m_A(A) = 0$

Proof:

By Cayley-Hamilton theorem: $P_A(A) = 0$

$P_A(x)$ is of degree n

Let $f = P_A(x)$

Let us make n choices:

Choice 1. $\nexists \deg m_A < \deg f$

Choice 2. $\exists m_A \in \mathbb{F}[x] : \deg m_A < \deg f \implies f = m_A$

After n choices, f definitely contains the minimal polynomial

Let f, g be minimal polynomials of A

$$f(x) = x^k + \sum_{i=1}^{k-1} \alpha_i x^i$$

$$g(x) = x^k + \sum_{i=1}^{k-1} \beta_i x^i$$

$$\implies f(x) - g(x) \in \mathbb{F}_t[x] : t \leq k-1, \frac{1}{\alpha_t - \beta_t}(f-g)(A) = 0 - \text{Contradiction!}$$

$$\implies \boxed{\exists! f \text{ minimal polynomial of } A}$$

Minimal polynomial divides any polynomial with matrix as a root #lemma

Let $A \in \mathbb{F}^{n\times n}$

Let $f(x) \in \mathbb{F}[x] : f(A) = 0$

Then $m_A(x) \mid f(x)$

Proof:

Case 0. $f = 0$ and we are done

Case 1. $f \neq 0$

$\deg f \geq \deg m_A$

$\implies \exists q, r \in \mathbb{F}[x] : f(x) = q(x) m_A(x) + r(x)$

$\deg r(x) < \deg m_A$

$$f(A) = q(A)\underbrace{m_A(A)}_{=0} + r(A) = 0$$

$$\implies r(A) = 0 \implies \begin{cases} r = 0 \\ \frac{1}{\alpha} r \text{ is a minimal polynomial} \end{cases}$$

$$\implies r = 0 \implies \boxed{m_A \mid f}$$

Corollary:

$m_A \mid P_A$

$\implies$ Roots of $m_A(x)$ are roots of $P_A(x)$ and eigenvalues of A

Characteristic polynomial divides any polynomial to the power of n with matrix as a root #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Let } f(x) \in \mathbb{F}[x] : f(A) = 0, \deg f \leq n$$
$$\text{Then } P_A \mid f^n$$

$$\text{Proof:}$$
$$\text{Let } f(x) = b_n x^n + b_{n-1}x^{n-1} + \dots$$
$$\text{Let } f(A) = 0$$
$$P_A \mid f^n \iff f^n(x) = P_A(x) \cdot q(x)$$
$$f^n(x) = \det(f(x)I)$$
$$P_A(x) = \det(xI - A)$$
$$\text{If exists } B(x) : (xI - A)B(x) = f(x)I$$
$$\implies \det((xI - A)B(x)) = \det(f(x)I)$$
$$\implies P_A(x)\underbrace{\det(B(x))}_{q(x)} = f^n(x)$$
$$\text{Let } B(x) = x^{n-1}B_{n-1} + \dots + xB_1 + B_0 \in \mathbb{F}[x]$$
$$(xI - A)(x^{n-1}B_{n-1} + \dots + xB_1 + B_0) =$$
$$= x^n B_{n-1} + x^{n-1}B_{n-2} + \dots + xB_0 - x^{n-1}AB_{n-1} - x^{n-2}AB_{n-2} - \dots =$$
$$= x^n B_{n-1} + x^{n-1}(B_{n-2} - AB_{n-1}) + x^{n-2}(B_{n-3} - AB_{n-2}) + \dots$$
$$\begin{cases} \text{Let } B_{n-1} & = b_n I \\ \text{Let } B_{n-2} & = AB_{n-1} + b_{n-1}I \\ \quad \dots \end{cases}$$
$$\implies x^n B_{n-1} + x^{n-1}(B_{n-2} - AB_{n-1}) + x^{n-2}(B_{n-3} - AB_{n-2}) + \dots =$$
$$= x^n(b_n I) + x^{n-1}\underbrace{(AB_{n-1} + b_{n-1}I - AB_{n-1})}_{b_{n-1}I} + \dots =$$
$$= f(x) \cdot I$$
$$\implies \boxed{\exists B(x) : P_A(x)\det(B(x)) = f^n(x)}$$

$$\text{Corollary:}$$
$$P_A \mid m_A^n$$

Corollary of two lemmas above

$$\text{Minimal polynomial contains all irreducible factors of } P_A$$
$$\text{at least once and at most algebraic multiplicity of each factor}$$
$$\implies \text{All eigenvalues of } A \text{ are roots of } m_A$$

Jordan block #definition

Matrix A is called a Jordan block with element α if

$$A \in \mathbb{F}^{n\times n} : A_{ij} = \begin{cases} \alpha & i = j \\ 1 & i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

Jordan block is denoted as $J_n(\alpha)$

$$P_{J_n(\alpha)}(\lambda) = (\lambda - \alpha)^n$$
$$\mu_{J_n(\alpha)}(\alpha) = n$$
$$\gamma_{J_n(\alpha)}(\alpha) = 1$$

Useful property:

$$(J_n(0)^k)_{ij} = \begin{cases} 1 & i = j-k \\ 0 & \text{otherwise} \end{cases}$$

$$m_{J_n(\alpha)}(\lambda) = (\lambda - \alpha)^k, 1 \le k \le n$$
$$\implies m_{J_n(\alpha)}(J_n(\alpha)) = J_n(0)^k$$
$$m_{J_n(\alpha)} = 0 \implies k = n \implies m_{J_n(\alpha)} = P_{J_n(\alpha)}$$

Jordan form #definition

Let $A \in \mathbb{F}^{n\times n}$

A is said to be a matrix in Jordan form if

A can be written as a diagonal block matrix

where each block on the diagonal is a Jordan block and all other blocks are 0

$$\text{e.g. } A = \begin{pmatrix} J_2(3) & 0 & 0 \\ 0 & J_1(3) & 0 \\ 0 & 0 & J_3(5) \end{pmatrix} \in \mathbb{F}^{6\times 6}$$

The common notation is: $A = J_2(3) \oplus J_1(3) \oplus J_3(5)$

Jordan decomposition theorem #theorem

Let $A \in \mathbb{F}^{n\times n}$

Then

1. $A \sim A_J \iff P_A$ is factorizable into linear factors over $\mathbb{F}$
2. A_J is unique up to the order of Jordan blocks
3. $\mu_A(\alpha)$ is the sum of sizes of Jordan blocks corresponding to eigenvalue α
4. $\gamma_A(\alpha)$ is the number of Jordan blocks corresponding to eigenvalue α
5. Algebraic multiplicity of α in the minimal polynomial is the largest size of Jordan block corresponding to eigenvalue α

Example:

$$P_A(\lambda) = (\lambda - 3)^5(\lambda - 1)$$
$$m_A(\lambda) = (\lambda - 3)^2(\lambda - 1)$$
$$\gamma_A(3) = 3$$
$$\implies A_J = \begin{pmatrix} J_1(1) & 0 & 0 & 0 \\ 0 & J_1(3) & 0 & 0 \\ 0 & 0 & J_2(3) & 0 \\ 0 & 0 & 0 & J_2(3) \end{pmatrix}$$

Diagonalization and minimal polynomial #theorem

$$\text{Let } A \in \mathbb{F}^{n\times n}$$

$$A \sim D \iff m_A \text{ is factorizable into distinct linear factors}$$

Proof:

$$\boxed{\implies} \text{Let } A \sim D$$

$$\implies P_A(\lambda) = \prod_{i=1}^{k} (\lambda - \alpha_i)^{\mu_A(\alpha_i)}$$

$$\implies m_A(\lambda) = \prod_{i=1}^{k} (\lambda - \alpha_i)^{t_i}, t_i \leq \mu_A(\alpha_i)$$

$$A_J = D$$

$$\implies \text{Largest Jordan block corresponding to any eigenvalue of } A \text{ is of size } 1$$

$$\implies \forall i \in [1,k] : t_i = 1 \implies \boxed{m_A(\lambda) = \prod_{i=1}^{k} (\lambda - \alpha_i)}$$

$$\boxed{\impliedby} \text{Let } m_A \text{ be factorizable into distinct linear factors}$$

$$\implies m_A(\lambda) = \prod_{i=1}^{k} (\lambda - \alpha_i)$$

$$\implies P_A(\lambda) = \prod_{i=1}^{k} (\lambda - \alpha_i)^{\mu_A(\alpha_i)} \implies A \sim A_J$$

$$\implies \text{Largest Jordan block corresponding to any eigenvalue of } A \text{ is of size } 1$$

$$\implies A_J = D \implies \boxed{A \sim D}$$