1

$$\text{Let } A = J_4(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Find all values of λ such that Jordan form of A^2 and A is the same

Solution:

Let $J_{A^2} = J_A$

$$P_A(x) = (x-\lambda)^4$$

$$x = \lambda \implies (xI - A) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies rank(xI - A) = 3 \implies g_\lambda = 1$$

$$\implies J_A \text{ has one Jordan block with eigenvalue } \lambda \implies J_A = J_4(\lambda)$$

$$A^2 = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 & 0 \\ 0 & \lambda^2 & 2\lambda & 1 \\ 0 & 0 & \lambda^2 & 2\lambda \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}$$

$$\implies P_{A^2}(x) = (x-\lambda^2)^4$$

$$A^2 \sim J_{A^2} = J_A \sim A \implies A^2 \sim A \implies P_{A^2}(x) = P_A(x) \implies \lambda^2 = \lambda$$

$$\implies \left[\begin{array}{l} \lambda = 0 \\ \lambda = 1 \end{array}\right.$$

Let $\lambda = 0$

$$\implies A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies rank(\lambda I - A^2) = 2 \implies g_\lambda = 2$$

$$\implies J_{A^2} \text{ has two Jordan blocks} \implies J_{A^2} \neq J_4(\lambda) = J_A$$

$$\implies \lambda = 1$$

$$\lambda = 1 \implies (\lambda I - A^2) = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies rank(\lambda I - A^2) = 3 \implies g_\lambda = 1$$

$$\implies J_{A^2} \text{ has one block with eigenvalue } \lambda \implies \boxed{J_{A^2} = J_4(\lambda) = J_A}$$

$$\implies \boxed{\lambda = 1 \text{ is the only value such that } J_{A^2} = J_A}$$

2a

Let $A, B \in \mathbb{R}^{n\times n}$

Let $(A+B)^2 = A^2 + B^2$

Let n be odd

Prove: A is not invertible or B is not invertible

Proof:

$$(A+B)^2 = A^2 + AB + BA + B^2$$

$$(A+B)^2 = A^2 + B^2 \implies AB + BA = 0 \implies AB = -BA$$

$$\implies \det(AB) = \det(-BA) \implies \det(A)\cdot\det(B) = (-1)^n \cdot \det(A)\cdot\det(B)$$

Let $\det(A) \neq 0$ and $\det(B) \neq 0$

$$\implies 1 = (-1)^n = -1 \text{ Contradiction!}$$

$$\implies \det(A) = 0 \text{ or } \det(B) = 0$$

$$\implies \boxed{A \text{ is not invertible or } B \text{ is not invertible}}$$

2b

$$\text{Let } A, B \in \mathbb{R}^{n\times n}$$
$$\text{Let } (A+B)^2 = A^2 + B^2$$
$$\text{Find } A \neq 0, B \neq 0 \text{ such that } \ |A+B| \neq |A-B|$$

$$\text{Solution:}$$
$$\text{Let } A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$
$$(A+B)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = I$$
$$A^2 + B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = I = (A+B)^2$$
$$|A+B| = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$
$$|A-B| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$
$$\implies \boxed{|A+B| \neq |A-B|}$$

3a

$$\text{Let } A = J_n(\lambda) \in \mathbb{C}^{n\times n}$$
$$\text{Prove: } A \sim A^T$$

$$\text{Proof:}$$
$$A = J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$
$$\implies A^T = \begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \lambda & \\ & & 1 & \lambda \end{pmatrix} \implies P_{A^T}(x) = (x-\lambda)^n$$
$$x = \lambda \implies (xI - A^T) = \begin{pmatrix} 0 & & & \\ -1 & \ddots & & \\ & \ddots & 0 & \\ & & -1 & 0 \end{pmatrix} \implies rank(xI - A^T) = n-1 \implies g_\lambda = 1$$
$$\implies A^T \sim J_{A^T} = J_n(\lambda) = A$$
$$\implies \boxed{A^T \sim A}$$

3b

Let $A \in \mathbb{C}^{n\times n}$ be a Jordan matrix

Prove: $A \sim A^T$

Proof:

$$A \text{ is a Jordan matrix} \implies A = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} \implies A^T = \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_k^T \end{pmatrix}$$

As proved in 3a, $\forall i \in [1,k] : J_i \sim J_i^T \implies \exists P_i : J_i = P^i J_i^T P^{-1}$

$$\text{Let } P = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_k \end{pmatrix} \implies P^{-1} = \begin{pmatrix} P_1^{-1} & & & \\ & P_2^{-1} & & \\ & & \ddots & \\ & & & P_k^{-1} \end{pmatrix}$$

$$\implies PA^TP = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_k \end{pmatrix} \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_k^T \end{pmatrix} \begin{pmatrix} P_1^{-1} & & & \\ & P_2^{-1} & & \\ & & \ddots & \\ & & & P_k^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} P_1 J_1^T P_1^{-1} & & & \\ & P_2 J_2 P_2^{-1} & & \\ & & \ddots & \\ & & & P_k J_k P_k^{-1} \end{pmatrix} = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix} = A$$

$$\implies \boxed{A^T \sim A}$$

3c

Let $A \in \mathbb{C}^{n\times n}$

Prove: $A \sim A^T$

Proof:

$$A \in \mathbb{C}^{n\times n} \implies P_A(x) \text{ is factorizable into linear factors} \implies \exists J_A : A \sim J_A$$

$$\implies \exists P : A = P J_A P^{-1}$$

$$\text{As proved in 3b, } J_A = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_k \end{pmatrix} J_A^T \begin{pmatrix} P_1^{-1} & & & \\ & P_2^{-1} & & \\ & & \ddots & \\ & & & P_k^{-1} \end{pmatrix}$$

$$\text{Let } \hat{P} = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_k \end{pmatrix}$$

$$\implies A = P\hat{P}J_A^T\hat{P}^{-1}P^{-1}$$

$$\implies A^T = (P^{-1})^T(\hat{P}^{-1})^T J_A \hat{P}^T P^T = (\hat{P}^{-1}P^{-1})^T J_A (P\hat{P})^T$$

$$\text{Let } Q = (P\hat{P})^T$$

$$(P\hat{P})^T \cdot (\hat{P}^{-1}P^{-1})^T = (\hat{P}^{-1}\underbrace{P^{-1}P}_{I}\hat{P})^T = (\hat{P}^{-1}\hat{P})^T = I^T = I$$

$$(\hat{P}^{-1}P^{-1})^T \cdot (P\hat{P})^T = (P\underbrace{\hat{P}\hat{P}^{-1}}_{I}P^{-1})^T = (PP^{-1})^T = I^T = I$$

$$\implies (\hat{P}^{-1}P^{-1})^T = Q^{-1}$$

$$\implies A^T = Q^{-1}J_A Q \implies A^T \sim J_A$$

$$\implies A^T \sim J_A \sim A \implies \boxed{A^T \sim A}$$