Uniform convergence of function series

$$S_N
ightharpoonup S$$
 $\iff d_N = \sup_{x \in A} |S_N(x) - S(x)| = \sup_{x \in A} \sum_{n=1}^N f_n(x) - \sum_{n=1}^\infty f_n(x) = \sup_{x \in A} \sum_{n=N+1}^\infty f_n(x) = \sup_{x \in A} \sum_{n=N+1}^\infty f_n(x) = \sup_{x \in A} |r_N|$
 $\sum_{n=0}^\infty x^n = \frac{1}{1-x}, \quad x \in (-1,1)$
 $d_N = \sup_{x \in (-1,1)} \sum_{n=N+1}^\infty x^n$
 $\sum_{n=N+1}^\infty x^n = x^{N+1} + x^{N+2} + \dots = x^{N+1} \cdot \sum_{n=0}^\infty x^n = \frac{x^{N+1}}{1-x}$
 $\implies d_N = \sup_{x \in (-1,1)} \frac{x^{N+1}}{1-x}$
 $x \to 1 \implies \frac{x^{N+1}}{1-x} \to \infty \implies d_N \nrightarrow 0 \implies \sum_{n=0}^\infty x^n \not \rightrightarrows \frac{1}{1-x}$
 $x \in \left[0, \frac{1}{23}\right] \implies d_N = \sup_{x \in \left[0, \frac{1}{123}\right]} \frac{x^{N+1}}{1-x} = \frac{\left(\frac{1}{23}\right)^{N+1}}{1-\frac{1}{23}} \to 0$

Weierstrass M-test #theorem

$$egin{aligned} \operatorname{Let} \ \sum_{n=1}^\infty f_n(x) \ &\operatorname{Let} \ \sum_{n=1}^\infty a_n o M \ &\operatorname{Let} \ orall n \in \mathbb{N}, orall x \in A: |f_n| \leq a_n \ &\operatorname{Then} \ \sum_{n=1}^\infty f_n(x)
ightrightarrows S(x) \ &\operatorname{And} \ \sum_{n=1}^\infty |f_n(x)|
ightrightarrows S(x) \end{aligned}$$

Proof:

$$\sum_{n=1}^{\infty} a_n ext{ converges} \implies orall arepsilon > 0: \exists N_arepsilon: orall M > N > N_arepsilon: S_M - S_N < rac{arepsilon}{2} \ ext{Let } arepsilon > 0 \ ext{ det } arepsilon$$

$$\sum_{n=0}^{\infty} x^n = rac{1}{1-x}$$
 $\Longrightarrow \sum_{n=0}^{\infty} nx^{n-1} = rac{1}{(1-x)^2}$
 $\Longrightarrow \sum_{n=0}^{\infty} nx^n = rac{x}{(1-x)^2}$
 $x = rac{1}{2} \implies \sum_{n=0}^{\infty} rac{n}{2^n} = rac{rac{1}{2}}{\left(rac{1}{2}
ight)^2} = 2$

Why can we do this?

$$\sum_{n=0}^{\infty} (x^n)'
ightrightarrows g(x), x \in A$$

$$\exists x_0 \in A: \sum_{n=0}^{\infty} x_0^n
ightarrow M$$

Then
$$\sum_{n=0}^{\infty} (x^n)' \Rightarrow \left(\frac{1}{1-x}\right)'$$

Let
$$A = \left[0, rac{1}{2}
ight]$$

$$orall x_0 \in A: \sum_{n=0}^\infty x_0^n o M_{x_0}$$

$$orall n\in \mathbb{N}, orall x\in A: \ nx^{n-1} \ = nx^{n-1} \leq n\cdot \left(rac{1}{2}
ight)^{n-1} = rac{n}{2^{n-1}}$$

$$\lim_{n o\infty}\sqrt[n]{rac{n}{2^{n-1}}}=\lim_{n o\infty}rac{\sqrt[n]{2}\cdot\sqrt[n]{n}}{\sqrt[n]{2^n}}=rac{1\cdot 1}{2}=rac{1}{2}$$

$$\implies \sum_{n=0}^{\infty} rac{n}{2^{n-1}} o M \implies ext{By the Weierstrass M-test} \left[\sum_{n=0}^{\infty} n x^{n-1}
ightrightarrows g(x)
ight]$$

$$\implies \sum_{n=0}^{\infty} (x^n)'
ightharpoons \left(rac{1}{1-x}
ight)'$$