Function sequences #definition

Function sequence is a sequence in which each element is a function

For example:

$$f_n(x) = x^n \ f_1(x) = x, f_2(x) = x^2, f_3(x) = x^3, \ldots$$

Pointwise convergence #definition

 $\{f_n(x)\}$ is called point-convergent on setA if $orall x_0 \in A: orall arepsilon > 0: \exists N_arepsilon: orall n > N_arepsilon: |f_n(x_0) - f(x_0)| < arepsilon \ f(x) ext{ is then called a pointwise limit of } f_n \ f_n o f$

Note: $N_{arepsilon}$ depends both on arepsilon and $x_0 \in A$ Hence pointwise convergence

$$f_n(x)=x^n ext{ on } [0,1] \ x<1 \implies f_n(x)=x^n o 0 \ x=1 \implies f_n(x)=1^n=1 o 1 \ f(x)=egin{cases} 0 & x\in [0,1) \ 1 & x=1 \ f_n o f \end{cases}$$

$$f_n(x)=x^2+rac{x}{n}+rac{7}{n^2}\ \lim_{n o\infty}f_n(x)=\lim_{n o\infty}\left(x^2+rac{x}{n}+rac{7}{n^2}
ight)=x^2\ f(x)=x^2\ f_n o f$$

$$\lim_{n o\infty}rac{n^2x^6}{n^2+x^6}=\lim_{n o\infty}rac{x^6}{1+\underbrace{rac{x^6}{n^2}}}=x^6$$

$$egin{aligned} \lim_{n o\infty}\sqrt{n^2x^2+x^4}-nx&=\lim_{n o\infty}rac{x^4}{\sqrt{n^2x^2+x^4}+nx}\ &x>0\implies f_n o0\ &x=0\implies f_n(x)=0\ &x<0\implies f_n(x)=rac{\sqrt{n^2x^2+x^4}+nx}{\sqrt{n^2x^2+x^4}-\sqrt{nx}} o\infty\ &f(x)=0 ext{ on } [0,\infty)\ &f_n o f \end{aligned}$$

$$f_n(x) = n \arctan\left(rac{x}{n}
ight) \ \lim_{n o\infty} rac{\arctan\left(rac{x}{n}
ight)}{rac{1}{n}} \stackrel{t=rac{1}{n}}{=} \lim_{t o0} rac{\arctan(tx)}{t} \stackrel{L}{=} \lim_{t o0} rac{x}{1+(tx)^2} = x$$

$$f_n(x) = n^2 \ln \left(1 + \sin\left(rac{x^9}{n^2}
ight)
ight) \ \lim_{n o\infty} n^2 \ln \left(1 + \sin\left(rac{x^9}{n^2}
ight)
ight) = \lim_{n o\infty} rac{x^9 \cdot \ln \left(1 + \sin\left(rac{x^9}{n^2}
ight)
ight)}{rac{x^9}{\sin\left(rac{x^9}{n^2}
ight)} \cdot \sin\left(rac{x^9}{n^2}
ight)} = \ = \lim_{t o0} x^9 \cdot rac{\ln(1+\sin(t))}{\sin(t)} \cdot rac{\sin(t)}{t} = x^9 \ x = 0 \implies f_n(x) = 0 = x^9$$

$$f_n(x)=\sin^{4n}(x) \ \lim_{n o\infty}\sin^{4n}(x)=\lim_{n o\infty}(\sin^4(x))^n=egin{cases} 1 & \sin^4(x)=1 \ 0 & \sin^4(x)
eq 1 \end{bmatrix}=egin{cases} 1 & x=rac{\pi}{2}+\pi k \ 0 & ext{otherwise} \end{cases}$$

$$egin{aligned} f_n(x) &= egin{cases} 1 & x \in \left[0,rac{1}{n}
ight] ext{ on } \left[0,1
ight] \ x &= 0 \implies f_n(x) = 1
ightarrow 1 \ x &> 0 \implies orall n > rac{1}{x}: f_n(x) = 0
ightarrow 0 \ \implies f &= egin{cases} 1 & x = 0 \ 0 & ext{otherwise} \end{cases} ext{ on } \left[0,1
ight] \end{aligned}$$

Let
$$f_n o f$$

Let f_n be continuous

Is f necessarily continuous?

$$ext{No: } f_n(x)=x^n, f(x)=egin{cases} 1 & x=1 \ 0 & x\in [0,1) \end{cases}$$

Let f_n , f be differentiable

Does necessarily $f_n' \to f'$?

$$ext{No: } f_n(x) = rac{\sin(n^8 x)}{n} o 0$$

$$f_n'(x) = n^7 \cos(n^8 x)$$
 doesn't converge

Let f_n, f be Riemann-integrable on [a, b]

$$\begin{array}{c} \text{Does necessarily } \int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx?\\ \text{No: } f_n(x) = \begin{cases} n & x \in \left[0,\frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases} \to 0\\ \int_0^1 f_n(t)\,dt = \int_0^{1/n} f_n(t)\,dt + \int_{\frac{1}{n}}^1 f_n(t)\,dt = nt \int_{t=0}^{t=1/n} + C \int_{t=\frac{1}{n}}^{t=1} = 1 \end{array}$$