Partial derivative #definition

Let $f : \mathbb{R}^n \to \mathbb{R}$

Partial derivative of f at point $a = (a_1, \ldots, a_n)$ in respect to variable x_i is defined as

$$f_{x_i}(a) = \frac{\partial}{\partial x_i} f(a) = \lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h} =$$

$$= \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

$$f(x,y) = \begin{cases} \frac{13x^3 + \pi y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{13h^3}{h^3} = 13$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\pi h^3}{h^3} = \pi$$

$$f(x,y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x.y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0h}{h^3} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0h}{h^3} = 0$$

Note: this example shows that existence of all partial derivatives, and even their equality, does not imply that the function is continuous

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Higher order derivatives

For higher order derivatives, order of differentiation matters

e.g.

$$f_{xx}(a) = (f_x(a))_x$$
$$f_{xy}(a) = (f_x(a))_y$$
$$f_{yx}(a) = (f_y(a))_X$$
$$f_{yy}(a) = (f_y(a))_y$$

Schwarz theorem #theorem

If all partial derivatives of order k are continuous

Then we can change order of differentiation up to order k

For functions of one variable:

$$y - f(a) = f'(a)(x - a)$$
$$y = f(a) + f'(a)(x - a)$$

This is a Taylor polynomial of order 1 around point a

Directional derivative #definition

Let $u \in \mathbb{R}^n$

Directional derivative along vector u is then defined as

$$f_u(a) = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h\|u\|}$$

Gradient #definition

Gradient of function f at point a is defined as

$$\nabla f(a) = (f_{x_1}(a), \ldots, f_{x_n}(a))$$

If partial derivatives are continuous, then directional derivative can be calculated as:

$$f_u(a) = \frac{\langle u, \nabla f(a) \rangle}{\|u\|} = \frac{u \cdot \nabla f(a)}{\|u\|}$$
