Extremums

Global extremums #definition

For functions of one variable:

Weierstrass theorem: continuous function on a closed interval has a global extremum

How do we find it?

We compare all critical points and ends of the interval

For functions of multiple variables, we know what continuity is

Instead of a closed interval, for multiple variables the definition of a compact is necessary

$A \subseteq \mathbb{R}^n$ is called a compact if it is bounded and closed

Bounded $\iff \exists B_r(a) : A \subseteq B_r(a)$ where $B_r(a) = \{x | \|x - a\| < r\}$

Closed $\iff \forall$ convergent $\{x_n\} \subseteq A : x_n \to L$

With these definitions, Weierstrass theorem is applicable to $\mathbb{R}^n$

How do we find a global extremum for functions of multiple variables?

We compare all critical points

But what about the ends(borders) of the compact? There are infinite

Generally there are two ways:

First one is:

Let $f(x, y) = 4x^3 - 2x^2y + y^2$

Bounded by $y = 9, y = x^2$

Intersections of two boundaries are $x = \pm 3, y = 9$

$\implies$ Our domain is $-3 \leq x \leq 3, 0 \leq y \leq 9$

$y = 9 \implies f(x, y) = 4x^3 - 18x^2 + 81 \implies f' = 12x^2 - 36x$

$f' = 0 \implies x = 0$ or $x = 3$

Ends of interval are $x = \pm 3 \implies$ Candidates on the borders of compact are

$(-3, 9), \quad (0, 9), \quad (3, 9)$

We reduced the function to one variable by substituting borders, this way is suitable for functions of two variables and compacts bounded by some lines

Second one is Lagrange's multipliers

Let f be a function of n variables

Let there be m constraints, $\forall i \in [1, m] : g_i = 0$

We then define variables $\lambda_1, \ldots, \lambda_m$

And solve the following equations system:

$$\begin{cases} \nabla f = \sum_{i=1}^{m} \lambda_i \nabla g_i \\ \forall i \in [1, n] : g_i = 0 \end{cases}$$

For example:

Let $f(x, y, z) = x^2 + yz + 1$

On $x^2 + y^2 = 1$ and $x + 2y + 3z = 4$

$$g_1 = x^2 + y^2 - 1$$

$$g_2 = x + 2y + 3z - 4$$

$$\begin{cases} \nabla f = \sum_{i=1}^{2} \lambda_i g_i \\ g_1 = 0 \\ g_2 = 0 \end{cases} \implies \begin{cases} f_x = \lambda_1 {g_1}_x + \lambda_2 {g_2}_x \\ f_y = \lambda_1 {g_1}_y + \lambda_2 {g_2}_y \\ f_z = \lambda_1 {g_1}_z + \lambda_2 {g_2}_z \\ x^2 + y^2 - 1 = 0 \\ x + 2y + 3z - 4 = 0 \end{cases} \implies \begin{cases} 2x = 2x\lambda_1 + \lambda_2 \\ z = 2y\lambda_1 + 2\lambda_2 \\ y = 0 + 3\lambda_2 \\ x^2 + y^2 - 1 = 0 \\ x + 2y + 3z - 4 = 0 \end{cases}$$

$$\implies \begin{cases} x = \frac{\lambda_2}{2(1-\lambda_1)} \\ y = 3\lambda_2 \\ z = 6\lambda_1\lambda_2 + 2\lambda_2 \\ x^2 + y^2 - 1 = 0 \\ x + 2y + 3z - 4 = 0 \end{cases} \implies \cdots$$