Primitive function and indefinite integral

$$\text{Let } f \text{ be a function}$$
$$\text{Let } F \text{ be a differentiable function such that } F' = f \left(\frac{dF}{dx} = f\right)$$
$$F \text{ is then called a primitive function of } f$$
$$\text{Indefinite integral of } f \text{ is a set of its primitive functions}$$
$$\text{It is denoted as } \int f(x)dx = F + C$$

Known integrals

$$n \neq 1, \int x^n dx = \frac{x^{n+1}}{n+1} + C$$
$$\int \frac{1}{x} dx = \ln|x| + C$$
$$\int a^x dx = \frac{a^x}{\ln(a)} + c$$
$$\int e^x dx = e^x + C$$
$$\int \cos(x) dx = \sin(x) + C$$
$$\int \sin(x) dx = -\cos(x) + C$$
$$\int \frac{1}{1+x^2} dx = \arctan(x) + C$$
$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$
$$\int \frac{1}{\cos^2(x)} dx = \tan(x) + C$$

Linearity of integral

$$\int (af+g)(x)dx = a\int f(x)dx + \int g(x)dx$$

Examples

$$\int \sqrt[6]{x} dx = \int x^{1/6} dx = \frac{6}{7}\sqrt[6]{x^7} + C$$

$$\int \frac{7\cos^2(x) + 2\sin^2(x)}{\cos^2(x)} dx = \int \frac{5\cos^2(x)}{\cos^2(x)} + \frac{2(\cos^2(x) + \sin^2(x))}{\cos^2(x)} dx =$$
$$= \int 5 dx + \int \frac{2}{\cos^2 x} dx = 5x + 2\tan(x) + C$$

Linear composition

$$\int f(x)dx = F(x) + C$$
$$\implies \int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

Examples

$$\int \cos(3x+5)dx = \frac{1}{3}\sin(3x+5) + C$$

$$\int e^{3x}dx = \frac{1}{3}e^{3x} + C$$

Trigonometric equalities

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \\
\sin^2(x) &= \frac{1-\cos(2x)}{2} \\
\cos^2(x) &= \frac{1+\cos(2x)}{2} \\
\sin(x \pm y) &= \sin(x)\cos(y) \pm \sin(y)\cos(x) \\
\cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y) \\
\cos(x)\cos(y) &= \frac{\cos(x-y)+\cos(x+y)}{2} \\
\sin(x)\sin(y) &= \frac{\cos(x-y)-\cos(x+y)}{2} \\
\sin(x)\cos(y) &= \frac{\sin(x+y)+\sin(x-y)}{2}
\end{aligned}$$

Examples

$$\int \sin(5x)\cos(2x)dx = \frac{1}{2}\int(\sin(7x)+\sin(3x))dx = \frac{1}{2}\left(-\frac{1}{7}\cos(7x) - \frac{1}{3}\cos(3x)\right) + C$$

$$\begin{aligned}
\int \sin^4(x)dx = \int(\sin^2(x))^2dx = \int\left(\frac{1-\cos(2x)}{2}\right)^2 dx = \\
= \frac{1}{4}\int(1-2\cos(2x)+\cos^2(2x))dx = \frac{1}{4}\int\left(1-2\cos(2x)+\frac{1}{2}+\frac{\cos(4x)}{2}\right)dx = \\
= \frac{1}{4}\int\left(\frac{3}{2}-2\cos(2x)+\frac{\cos(4x)}{2}\right)dx = \frac{1}{4}\left(\frac{3x}{2}-\sin(2x)+\frac{1}{8}\sin(4x)\right) + C
\end{aligned}$$

Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Examples

$$\begin{aligned}
\int x^4\ln(x)dx = \int \ln(x)\left(\frac{x^5}{5}\right)' dx = \frac{x^5\ln(x)}{5} - \int\frac{x^5}{5x}dx = \\
= \frac{x^5\ln(x)}{5} + \frac{x^5}{25} + C
\end{aligned}$$

LIATE

Order of functions to choose as f, g in integration by parts

The higher the place, the better choice it is for f and not g

1. Logarithmic
2. Inverse trigonometric
3. Algebraic
4. Trigonometric
5. Exponential

Examples

$$\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$$
$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$$
$$\implies \int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx$$
$$\implies \int \cos(\ln x)dx = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C$$
$$\implies \int \sin(\ln x)dx = \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C$$

Variable substitution

$$(f(g(x)))' = f'(g(x))g'(x)$$
$$\implies \int f'(g(x))g'(x)dx = f(g(x)) + C$$
$$t = g(x) \implies \int f'(t)g'(x)dx = f(t) + C = \int f'(t)dt$$
$$\implies f'(t)g'(x)dx = f'(t)dt \implies dt = g'(x)dx$$

Examples

$$\int \frac{2x}{1+x^2}dx$$
$$t = 1 + x^2 \implies dt = (1+x^2)'dx = 2xdx \implies dx = \frac{dt}{2x}$$
$$\implies \int \frac{2x}{t}\frac{dt}{2x} = \int \frac{1}{t}dt = \ln|t| + C = \ln|1+x^2| + C$$

$$\int \sin^4(x)\cos(x)dx$$
$$t = \sin(x) \implies dt = \cos(x)dx$$
$$\implies \int t^4 dt = \frac{t^5}{5} + C = \frac{\sin^5(x)}{5} + C$$

$$\int \arctan(x)dx = x\arctan(x) - \int \frac{x}{1+x^2}dx =$$
$$= x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$$

$$\int \sin^m(x)\cos^n(x)dx$$
Choose function with even power
$$\int \cos^3(x)\sin^2(x)dx$$
$$t = \sin(x) \implies dt = \cos(x)dx$$
$$\implies \int (1-t^2)t^2 dt = \int (t^2 - t^4)dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$$

$$\int xe^{x^2}dx \underbrace{=}_{t=x^2} \frac{1}{2}\int e^t dt = \frac{e^{x^2}}{2} + C$$

$$\int x^3 e^{x^2}dx = \frac{x^2 e^{x^2}}{2} - \int xe^{x^2}dx = \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$$