$$T: V \to V ext{ is injective}$$
  
 $\langle \, , 
angle_T: V imes V o \mathbb{F}$ 

Show that  $\langle \, , \rangle_T$  is an inner product

Solution:

$$egin{aligned} \langle v+u,w
angle_T = \langle T(v+u),T(w)
angle = \langle T(v)+T(u),T(w)
angle = \langle T(v),T(w)
angle + \langle T(u),T(w)
angle \\ &\langle \alpha v,u
angle_T = \langle T(\alpha v),T(u)
angle = \langle \alpha T(v),T(u)
angle = \alpha \langle T(v),T(u)
angle = \alpha \langle v,u
angle_T \\ &\langle v,u
angle_T = \langle T(v),T(u)
angle = \overline{\langle T(u),T(v)
angle} = \overline{\langle u,v
angle_T} \\ &\langle v,v
angle_T = \langle T(v),T(v)
angle \geq 0 \\ &\langle v,v
angle_T = 0 \iff \langle T(v),T(v)
angle = 0 \iff v \in ker(T) \\ &T \text{ is injective} \iff ker(T) = \{0\} \implies \langle\,,\,\rangle_T \text{ is an inner product} \end{aligned}$$

Let V be an inner product space

Let 
$$\{v_1,\ldots,v_n\}\subseteq V$$

Prove or disprove: 
$$\sum_{i=1}^n \sum_{j=1}^n \langle v_i, v_j 
angle \geq 0$$

Proof:

$$\sum_{i=1}^n \sum_{j=1}^n \langle v_i, v_j 
angle = \sum_{j=1}^n \sum_{i=1}^n \langle v_i, v_j 
angle = \sum_{j=1}^n \left\langle \sum_{i=1}^n v_i, v_j 
ight
angle = \ = \sum_{j=1}^n \overline{\left\langle v_j, \sum_{i=1}^n v_i 
ight
angle} = \overline{\left\langle \sum_{j=1}^n v_j, \sum_{i=1}^n v_i 
ight
angle} = \overline{\left\langle v, v 
ight
angle} \geq 0$$

$$\text{Prove: } v = 0 \iff \forall u \in V : \langle v, u \rangle = 0$$

Let B be a basis of V

$$\text{Prove: } \forall i \in [1,n]: \langle v,v_i \rangle = \langle u,v_i \rangle \iff v=u$$

Proof:

Let 
$$u \in V$$

$$egin{aligned} v = 0 & \Longrightarrow \langle v, u 
angle = \langle 2v, u 
angle = \langle v, u 
angle + \langle v, u 
angle & \Longrightarrow \langle v, u 
angle = 0 \ orall u \in V : \langle v, u 
angle = 0 & \Longrightarrow \langle v, v 
angle = 0 & \Longrightarrow v = 0 \end{aligned}$$

Proof:

$$\langle v, v_i 
angle = \langle u, v_i 
angle \implies \langle v - u, v_i 
angle = 0$$

Let 
$$w \in V$$

$$w = \sum_{i=1}^n lpha_i v_i$$

$$\langle v-u,w
angle = \left\langle v-u,\sum_{i=1}^n lpha_i v_i
ight
angle = \overline{\left\langle \sum_{i=1}^n lpha_i v_i,v-u
ight
angle} = \sum_{i=1}^n \overline{lpha_i} \overline{\langle v-u,v_i
angle} = 0$$

## **Gram-Schmidt matrix**

$$egin{aligned} \operatorname{Let} S &= \{v_1, \dots, v_n\} \subseteq V \ G_S &\in \mathbb{F}^{n imes n} : (G_S)_{ij} = \langle v_i, v_j 
angle \end{aligned}$$

Prove:  $G_S$  is non-invertible  $\iff S$  is a linear dependence

Proof:

Let 
$$G_S$$
 be non-invertible

$$\implies \exists \sum_{j=1}^{n} \alpha_{j} C_{j}(G_{S}) = 0 : \exists j \in [1, n] : \alpha_{j} \neq 0$$

$$\implies 0 = \sum_{j=1}^{n} \alpha_{j} C_{j}(G_{S}) = \sum_{j=1}^{n} \alpha_{j} \left( \begin{array}{c} \langle v_{1}, v_{j} \rangle \\ \langle v_{2}, v_{j} \rangle \end{array} \right)$$

$$\implies 0 = \sum_{j=1}^{n} \alpha_{j} C_{j}(G_{S}) = \sum_{j=1}^{n} \alpha_{j} \left( \begin{array}{c} \langle v_{1}, v_{j} \rangle \\ \langle v_{2}, v_{j} \rangle \end{array} \right)$$

$$\implies 0 = \sum_{j=1}^{n} \alpha_{j} C_{j}(G_{S}) = \sum_{j=1}^{n} \alpha_{j} \left( \begin{array}{c} \langle v_{1}, v_{j} \rangle \\ \langle v_{2}, v_{j} \rangle \end{array} \right)$$

$$\implies 0 = \sum_{j=1}^{n} \alpha_{j} \left( \begin{array}{c} \langle v_{1}, v_{j} \rangle \\ \langle v_{2}, v_{j} \rangle \end{array} \right)$$

$$\implies \sum_{j=1}^{n} \alpha_{j} \langle v_{i}, v_{j} \rangle = 0$$

$$\implies \sum_{j=1}^{n} \langle \overline{\alpha_{j}} v_{j}, v_{i} \rangle = 0$$

$$\implies \sum_{j=1}^{n} \langle \overline{\alpha_{j}} v_{j}, v_{i} \rangle = \sum_{j=1}^{n} \langle \overline{\alpha_{j}} v_{j}, v_{i} \rangle = 0$$

$$\implies \left( \sum_{j=1}^{n} \overline{\alpha_{j}} v_{j}, u \right) = \sum_{j=1}^{n} \overline{\alpha_{j}} \langle v_{j}, u \rangle = 0$$

$$\implies u = 0 \implies \sum_{i=1}^{n} \overline{\alpha_{i}} v_{i} = 0$$

 $\exists j \in [1,n]: lpha_j 
eq 0 \implies \overline{lpha_j} 
eq 0$ 

 $\implies$  S is a linear dependence