$$\text{Let } A \in \mathbb{R}^{n\times n}$$
$$\text{Prove or disprove: } A \text{ invertible} \iff AA^T \text{ invertible}$$

$$\text{Proof:}$$
$$\det(A) \neq 0 \iff \det(A^T) \neq 0 \iff \det(AA^T) \neq 0$$

$$\text{Let } A \in \mathbb{R}^{n\times n} \text{ be symmetric}$$
$$\text{Prove or disprove: } A \text{ invertible} \iff A + A^T \text{ invertible}$$

$$\text{Proof:}$$
$$A + A^T = 2A$$
$$\det(A) \neq 0 \iff \det(2A) = 2^n \det(A) \neq 0 \iff \det(A + A^T) \neq 0$$

$$\text{Let } \alpha \in \mathbb{R}$$
$$\text{Let } A \in \mathbb{R}^{n\times n}$$
$$A_{ij} = \begin{cases} \alpha & i = j \\ 1 & i \neq j \end{cases}$$
$$\begin{vmatrix} \alpha & 1 & \dots & 1 \\ 1 & \alpha & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & \alpha \end{vmatrix} \overset{\forall i \in [2,n]: R_1 + R_i}{\rightarrow} \begin{vmatrix} \alpha+n-1 & \alpha+n-1 & \dots & \alpha+n-1 \\ 1 & \alpha & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & \alpha \end{vmatrix}$$
$$\overset{\frac{1}{\alpha+n-1} R_1}{\rightarrow} (\alpha+n-1) \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & \alpha & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & \alpha \end{vmatrix} \overset{\forall i \in [2,n]: R_i - R_1}{\rightarrow} (\alpha+n-1) \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & \alpha-1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha-1 \end{vmatrix}$$
$$\implies \det(A) = (\alpha+n-1)(\alpha-1)^{n-1}$$

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Let } \lambda \in \mathbb{F}$$
$$\lambda \text{ is called an eigenvalue of } A \iff \exists v \in \mathbb{F}^n \neq 0 : Av = \lambda A$$
$$v \text{ is then called an eigenvector of } A \text{ in respect to eigenvalue } \lambda$$

$$\lambda \neq 0 \implies A\left(\frac{1}{\lambda} v\right) = v \implies v \in C(A)$$
$$\lambda = 0 \implies Av = 0 \implies v \in N(A)$$

Characteristic polynomial

$$\text{Characteristic polynomial } P_A(\lambda) = |\lambda I - A|$$

$$\begin{vmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{vmatrix} = (\lambda-5)(\lambda+4) + 18 = (\lambda+1)(\lambda-2)$$

$$P_A(\lambda) = 0 \iff \lambda \text{ is an eigenvalue of } A$$
$$\text{Proof:}$$
$$\exists v \neq 0 : Av = \lambda v \iff (\lambda I - A)v = 0 \iff v \in N(\lambda I - A) \iff \det(\lambda I - A) = 0$$
$$\iff P_A(\lambda) = 0$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$P_A(\lambda) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

$$A \in \mathbb{R}^{n\times n} \implies \text{No eigenvalues}$$

$$A \in \mathbb{C}^{n\times n} \implies \pm i \text{ is an eigenvalue}$$

$$A \in \mathbb{Z}_2^{n\times n} \implies 1 \text{ is an eigenvalue}$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

$$\begin{vmatrix} \lambda-3 & -1 & -1 \\ -2 & \lambda-4 & -2 \\ -1 & -1 & \lambda-3 \end{vmatrix} \overset{R_1+R_2+R_3}{\rightarrow} \begin{vmatrix} \lambda-6 & \lambda-6 & \lambda-6 \\ -2 & \lambda-4 & -2 \\ -1 & -1 & \lambda-3 \end{vmatrix} = (\lambda-6)\begin{vmatrix} 1 & 1 & 1 \\ -2 & \lambda-4 & -2 \\ -1 & -1 & \lambda-3 \end{vmatrix}$$

$$\rightarrow (\lambda-6)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda-2 & -2 \\ 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda-6)(\lambda-2)^2$$

$$P_A(\lambda) = 0 \iff \left[\begin{array}{l} \lambda = 6 \\ \lambda = 2 \end{array}\right.$$

$$\lambda = 6 \implies v_\lambda = N\begin{pmatrix} 0 & 0 & 0 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{pmatrix} = N\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} = N\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} =$$

$$= sp\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}$$

$$\lambda = 2 \implies v_\lambda = N\begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -1 & -1 & -1 \end{pmatrix} = N\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = sp\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$\text{Let } A \sim B$$

$$P_A(\lambda) = P_B(\lambda)$$

$$\text{Proof:}$$

$$|\lambda I - B| = \left|\lambda I - P^{-1}AP\right| = \left|P^{-1}(\lambda I - A)P\right| =$$

$$= \left|P^{-1}\right| \cdot |\lambda I - A| \cdot |P| = |\lambda I - A|$$

$$\text{Let } T : V \to V \text{ be a linear transformation}$$

$$\forall B \text{ basis of } V : \lambda \text{ is an eigenvalue of } T \iff \lambda \text{ is an eigenvalue of } [T]_B$$

$$\text{Proof:}$$

$$\text{Let } v \neq 0 \in V$$

$$Tv = \lambda v \iff [Tv]_B = [\lambda v]_B \iff [T]_B[v]_B = \lambda[v]_B$$
