Polynomial division #theorem

$$\text{Let } f(x), g(x) \in \mathbb{F}[x] : \deg(f(x)) \geq \deg(g(x))$$
$$\text{Then } \exists q(x), r(x) \in \mathbb{F}[x] : \begin{cases} f(x) = q(x)g(x) + r(x) \\ \deg(r(x)) < \deg(g(x)) \end{cases}$$

Divisibility of polynomial #lemma

$$\text{Let } f(x) \in \mathbb{F}[x]$$
$$\text{Let } \alpha \in \mathbb{F}$$
$$\text{Then } (x - \alpha) \mid f(x) \iff f(\alpha) = 0$$

$$\text{Proof:}$$
$$\boxed{\Longleftarrow} \text{ Let } f(\alpha) = 0$$
$$\deg(f(x)) \geq 1 = deg(x - \alpha)$$
$$\implies \exists q(x), r(x) \in \mathbb{F}[x] : \begin{cases} f(x) = q(x)(x - \alpha) + r(x) \\ \deg(r(x)) < 1 \end{cases} \implies r(x) = \beta$$
$$f(\alpha) = q(\alpha) \cdot 0 + r(\alpha) = 0 \implies r(\alpha) = 0 \implies r(x) = 0$$
$$\implies f(x) = q(x)(x - \alpha) \implies \boxed{(x - \alpha) \mid f(x)}$$

$$\boxed{\Longrightarrow} \text{ Let } (x - \alpha) \mid f(x)$$
$$\implies \exists q(x) \in \mathbb{F}[x] : f(x) = q(x)(x - \alpha)$$
$$\implies \boxed{f(\alpha) = q(\alpha) \cdot 0 = 0}$$

Algebraic and geometric multiplicities limits #lemma

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\text{Let } \alpha \in \mathbb{F} \text{ be an eigenvalue of } A$$
$$\text{Then } 1 \leq \mu_A(\alpha), \gamma_A(\alpha) \leq n$$

$$\text{Proof:}$$
$$\exists v \neq 0 : (\alpha I - A)v = 0 \implies N(\alpha I - A) \neq 0$$
$$\implies \gamma_A(\alpha) = \dim(N(\alpha I - A)) \geq 1$$
$$A \in \mathbb{F}^{n \times n} \implies N(\alpha I - A) \subseteq \mathbb{F}^n \implies \dim(N(\alpha I - A)) \leq n$$
$$\implies \boxed{1 \leq \gamma_A(\alpha) \leq n}$$
$$(\lambda - \alpha)^{\mu_A(\alpha)} \mid P_A(\lambda)$$
$$\implies \deg((\lambda - \alpha)^{\mu_A(\alpha)}) \leq \deg(P_A(\lambda)) = n \implies \mu_A(\alpha) \leq n$$
$$P_A(\alpha) = 0 \implies (\lambda - \alpha) \mid P_A(\lambda)$$
$$\implies \mu_A(\alpha) \geq 1 \implies \boxed{1 \leq \mu_A(\alpha) \leq n}$$

Geometric multiplicity is at most algebraic multiplicity #theorem

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Let } \alpha \in \mathbb{F} \text{ be an eigenvalue of } A$$
$$\text{Then } \gamma_A(\alpha) \le \mu_A(\alpha)$$

Proof:

$$\text{Let } \gamma_A(\alpha) = t \ge 1$$
$$\text{Let } B = \{v_1, \dots, v_t\} \text{ be a basis of eigenspace in respect to } \alpha$$
$$\text{Let } C = B \cup \{u_1, \dots, u_{n-t}\}$$
$$\text{Let } P = \begin{pmatrix} | & & | & | & & | \\ v_1 & \dots & v_t & u_1 & \dots & u_{n-t} \\ | & & | & | & & | \end{pmatrix}$$
$$P \text{ is invertible}$$
$$P^{-1}AP = P^{-1}\begin{pmatrix} | & & | & | & & | \\ Av_1 & \dots & Av_t & Au_1 & \dots & A_{u_{n-t}} \\ | & & | & | & & | \end{pmatrix} =$$
$$= P^{-1}\begin{pmatrix} \alpha v_1 & \dots & \alpha v_n & \overset{|}{\underset{|}{*}} & \dots & \overset{|}{\underset{|}{*}} \end{pmatrix} = \begin{pmatrix} \alpha P^{-1}v_1 & \dots & \alpha P^{-1}v_t & \overset{|}{\underset{|}{*}} & \dots & \overset{|}{\underset{|}{*}} \end{pmatrix} =$$
$$\underset{\substack{P^{-1}v_i=P^{-1}C_i(P) \\ P^{-1}C_i(P)=C_i(P^{-1}P) \\ C_i(P^{-1}P)=e_i}}{=} \begin{pmatrix} \alpha e_1 & \dots & \alpha e_t & \overset{|}{\underset{|}{*}} & \dots & \overset{|}{\underset{|}{*}} \end{pmatrix}$$
$$\implies P^{-1}AP = \begin{pmatrix} \alpha I_t & C \\ 0 & B \end{pmatrix}$$
$$A \sim P^{-1}AP \implies P_A(\lambda) = P_{P^{-1}AP}(\lambda) = |\lambda I - P^{-1}AP| = \begin{vmatrix} (\lambda-\alpha)I_t & C \\ 0 & \lambda I - B \end{vmatrix} =$$
$$= |(\lambda-\alpha)I_t| \cdot |\lambda I - B| = (\lambda-\alpha)^t \cdot |\lambda I - B|$$
$$\implies \boxed{\mu_A(\alpha) \ge t = \gamma_A(\alpha)}$$

Linear independency of eigenvectors in respect to distinct eigenvalues #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Let } \{\lambda_1, \dots, \lambda_t\} \text{ be eigenvalues of } A$$
$$\text{Let } \forall i \in [1,t] : Av_i = \lambda_i v_i$$
$$\text{Then } \{v_1, \dots, v_t\} \text{ is a linear independence}$$

Proof:

$$\text{Base case. } \{v_t\} \text{ is a linear independence}$$
$$\text{Induction step. Let } \{v_2, \dots, v_t\} \text{ be a linear independence}$$
$$\text{Let } \sum_{i=1}^{t} \alpha_i v_i = 0$$
$$\implies \begin{cases} \sum_{i=1}^{t} \alpha_i A v_i = \sum_{i=1}^{t} \alpha_i \lambda_i v_i = 0 \\ \sum_{i=1}^{t} \alpha_i \lambda_1 v_i = 0 \end{cases}$$
$$\implies \sum_{i=2}^{t} \alpha_i(\lambda_i - \lambda_1)v_i = 0$$
$$\forall i \in [2,n] : \lambda_i \ne \lambda_1 \implies \alpha_i = 0$$
$$\implies \text{By induction: } \boxed{\{v_1, \dots, v_t\} \text{ is a linear independence}}$$

Union of Eigenspace bases #theorem

$$\text{Let } A \in \mathbb{F}^{n\times n}$$

$$\text{Let } \{\alpha_i\}_{i\in[1,t]} \subseteq \mathbb{F} \text{ be eigenvalues of } A$$

$$\text{Let } \forall i \in [1,t] : B_i \text{ be a basis of eigenspace in respect to } \alpha_i$$

$$\text{Then } \bigcup_{i=1}^{t} B_i \text{ is a linear independence}$$

Proof:

$$\text{Let } \forall i \in [1,t] : B_i = \left\{v_{i_1}, \dots, v_{i_{k_i}}\right\}$$

$$\text{Let } \left\{\alpha_{i_j}\right\}_{\substack{i\in[1,t] \\ j\in[1,k_i]}} \subseteq \mathbb{F}$$

$$\text{Let } \sum_{i=1}^{t}\sum_{j=1}^{k_i} \alpha_{i_j} v_{i_j} = 0$$

$$\text{Case 1. Let } \forall i \in [1,t] : \sum_{j=1}^{k_i} \alpha_{i_j} v_{i_j} = 0$$

$$\implies \forall i \in [1,t] : B_i \text{ is a linear independence} \implies \forall j \in [1,k_i] : \alpha_{i_j} = 0$$

$$\implies \boxed{\bigcup_{i=1}^{t} B_i \text{ is a linear independence}}$$

$$\text{Case 2. Let } \exists i \in [1,t] : \sum_{j=1}^{k_i} \alpha_{i_j} v_{i_j} \neq 0$$

$$\text{Let } \forall i \in [1,t] : u_i = \sum_{j=1}^{k_i} \alpha_{i_j} v_{i_j}$$

$$\forall i \in [1,t] : u_i \in sp(B_i) \implies \left[\begin{array}{l} u_i = 0 \\ Au_i = \lambda_i u_i \end{array}\right.$$

$$\sum_{i=1}^{t} u_i = 0 \underset{\substack{\{u_1,\dots,u_t\} \\ \text{is a linear independence}}}{\implies} \forall i \in [1,t] : u_i = 0 - \text{Contradiction!}$$

$$\implies \boxed{\text{Case 1.}}$$

Diagonalizability and geometric multiplicities #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$

$$\text{Then } A \text{ is diagonalizable } \iff \sum_{i=1}^{t} \gamma_A(\lambda_i) = n$$

Proof:

$\boxed{\implies}$ Let A be diagonalizable

$$\implies \exists B = \{v_1, \dots, v_n\} \text{ basis of } \mathbb{F}^n : \forall i \in [1,n] : Av_i = \lambda_i v_i$$

$$\implies t = n \implies n = \sum_{i=1}^{n} 1 \le \sum_{i=1}^{n} \gamma_A(\lambda_i) \le n \implies \boxed{\sum_{i=1}^{n} \gamma_A(\lambda_i) = n}$$

$\boxed{\impliedby}$ Let $\sum_{i=1}^{t} \gamma_A(\lambda_i) = n$

$$\text{Let } \forall i \in [1,t] : B_i = \left\{v_{i_1}, \dots, v_{i_{k_i}}\right\}$$

$$\text{Let } P \in \mathbb{F}^{n\times n} : C_{i_j}(P) = v_{i_j}$$

$$P^{-1}AP = P^{-1}\begin{pmatrix} | & & | & & & | & & | \\ \lambda_1 v_{1_1} & \dots & \lambda_1 v_{1_{k_1}} & \dots & \dots & \lambda_t v_{t_1} & \dots & \lambda_t v_{t_{k_t}} \\ | & & | & & & | & & | \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1 I_{k_1} & 0 & \dots & 0 \\ 0 & \lambda_2 I_{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_t I_{k_t} \end{pmatrix} = D$$

$$\implies \boxed{A \text{ is diagonalizable}}$$

Diagonalizability criterion #theorem

$$\text{Let } A \in \mathbb{F}^{n\times n}$$

$$\text{Then } A \text{ is diagonalizable } \iff \begin{cases} \sum_{i=1}^{t} \mu_A(\lambda_i) = n \\ \forall i \in [1,t] : \mu_A(\lambda_i) = \gamma_A(\lambda_i) \end{cases}$$

Proof:

$\boxed{\implies}$ Let A be diagonalizable

$$\implies \sum_{i=1}^{t} \gamma_A(\lambda_i) = n$$

$$\forall i \in [1,t] : \gamma_A(\lambda_i) \le \mu_A(\lambda_i)$$

$$\implies n = \sum_{i=1}^{t} \gamma_A(\lambda_i) \le \sum_{i=1}^{t} \mu_A(\lambda_i) \le n$$

$$\implies \boxed{\sum_{i=1}^{t} \mu_A(\lambda_i) = n}$$

$$\implies \sum_{i=1}^{t} \mu_A(\lambda_i) - \sum_{i=1}^{t} \gamma_A(\lambda_i) = 0 \implies \sum_{i=1}^{t} \underbrace{(\mu_A(\lambda_i) - \gamma_A(\lambda_i))}_{\ge 0} = 0$$

$$\implies \boxed{\forall i \in [1,t] : \mu_A(\lambda_i) = \gamma_A(\lambda_i)}$$

$\boxed{\impliedby}$ Let $\begin{cases} \sum_{i=1}^{t} \mu_A(\lambda_i) = n \\ \forall i \in [1,t] : \mu_A(\lambda_i) = \gamma_A(\lambda_i) \end{cases}$

$$\left\{\begin{matrix} \sum_{i=1}^{t} \mu_A(\lambda_i) = n \\ \forall i \in [1,t] : \mu_A(\lambda_i) = \gamma_A(\lambda_i) \end{matrix}\right\} \implies \sum_{i=1}^{t} \gamma_A(\lambda_i) = n$$

$$\implies \boxed{A \text{ is diagonalizable}}$$