1a

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

$$\text{Prove: } A^2 = A^{2024}$$

Proof:

$$A^2 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} = A$$

$$\implies A^4 = A^2$$

$$\implies A^5 = A^3 = A$$

$$\implies \dots \implies A^{2k} = A^2 \implies A^{2024} = A^2$$

$$P_A(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 2 & \lambda + 1 & 0 \\ -2 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda - 1 & \lambda - 1 & \lambda - 1 \\ 2 & \lambda + 1 & 0 \\ -2 & -1 & \lambda \end{vmatrix} = (\lambda - 1) \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda - 1 & -2 \\ 0 & 1 & \lambda + 2 \end{vmatrix} =$$

$$(\lambda - 1)(\lambda^2 + \lambda - 2 + 2) = (\lambda - 1)(\lambda + 1)\lambda$$

$$\implies A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}$$

$$\implies A^{2024} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} = A^2$$

1b

$$\text{Let } A \in \mathbb{R}^{n \times n} \text{ nilpotent and diagonalizable}$$

$$\text{Prove: } N(A) = \mathbb{R}^n$$

Proof:

$$P_A(\lambda) = \lambda^n$$

$$\implies g_0 = n \implies N(0I - A) = N(-A) = N(A) = sp\{v_1, \dots, v_n\}$$

$$\dim sp\{v_1, \dots, v_n\} = n \text{ and } sp\{v_1, \dots, v_n\} \subseteq \mathbb{R}^n$$

$$\implies \boxed{N(A) = sp\{v_1, \dots, v_n\} = \mathbb{R}^n}$$

2a

$$\text{Let } T : V \to V \text{ be an invertible linear transformation}$$

$$\text{Prove or disprove: } \exists B : [T]_B^B = [T^{-1}]_B^B$$

Disproof:

$$\text{Let } T = 2I$$

$$\implies T^{-1} = \frac{1}{2} I$$

$$\forall B : \forall v_i \in B : [T(v)]_B = [2v]_B = 2e_i \implies [T]_B^B = 2I$$

$$\forall B : \forall v_i \in B : [T^{-1}(v)]_B = \left[\frac{1}{2} v\right]_B = \frac{1}{2} e_i \implies [T]_B^B = \frac{1}{2} I$$

2b

Let $T : V \to V$ be an invertible linear transformation

Prove or disprove: $\exists B, C : [T]_C^B = [T^{-1}]_B^C$

Proof:

Let $B = \{v_1, \ldots, v_n\}$

T is invertible $\implies \text{Im}T = sp\,\{T(v_1), \ldots, T(v_n)\}$

Let $C = \{T(v_1), \ldots, T(v_n)\}$

$\forall v_i \in B : [T(v_i)]_C = e_i \implies [T]_C^B = I$

$\forall v_i \in B : [T^{-1}(T(v_i))]_B = [v_i]_B = e_i \implies [T^{-1}]_B^C = I$

$\implies \boxed{[T]_C^B = [T^{-1}]_B^C}$

3

Let $V = \mathbb{R}_n[x]$

Let $T : V \to V$

Let $T(p(x)) = p'(x) + p(0) \cdot x^n$

Prove: T is a linear transformation

Proof:

Let $p(x), q(x) \in V$

Let $\alpha \in \mathbb{R}$

$$T((p + \alpha q)(x)) = (p + \alpha q)'(x) + (p + \alpha q)(0) \cdot x^n =$$
$$= p'(x) + p(0) \cdot x^n + \alpha(q'(x) + q(0) \cdot x^n) = T(p(x)) + \alpha T(q(x))$$

3a

Let $S = \{1, x, \ldots, x^n\}$

Find: $[T]_S^S$

Solution:

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$

$p(0) = a_0$

$$\implies T(p(x)) = a_0 x^n + n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

$$\implies [T(p(x))]_S = \begin{pmatrix} a_1 \\ 2a_2 \\ \vdots \\ (n-1)a_{n-1} \\ n a_n \\ a_0 \end{pmatrix}$$

$$\implies [T(x^i)]_S = \begin{cases} e_n & i = 0 \\ i \cdot e^{i-1} & i \in [1, n] \end{cases}$$

$$\implies [T]_S^S = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & n \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

3b

Let $A = [T]_S^S$

Find: $P_A(\lambda)$

Solution:

$$P_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & & \\ & \lambda & -2 & \\ & & \ddots & -n \\ -1 & & & \lambda \end{vmatrix} =$$

$$= \lambda \begin{vmatrix} \lambda & -1 & & \\ & \lambda & -2 & \\ & & \ddots & -n \\ & & & \lambda \end{vmatrix} - (-1)^{n+2} \begin{vmatrix} -1 & & & \\ \lambda & -2 & & \\ & \ddots & \ddots & \\ & & \lambda & -n \end{vmatrix} =$$

$$= \lambda^{n+1} - (-1)^{n+2} \cdot ((-1)^n n!) =$$

$$= \lambda^{n+1} - n!$$

3c

Find: $m_A(\lambda)$

Solution:

$$P_A(\lambda) = \lambda^{n+1} - n!$$

$$P_A(\lambda) = 0 \iff \lambda^{n+1} = n! \iff \lambda = \sqrt[n+1]{n} \cdot e^{(2\pi k)i/n+1}, k \in [0, n]$$

$P_A(\lambda)$ has $n+1$ roots

$$\implies m_A(\lambda) = P_A(\lambda) = \lambda^{n+1} - n!$$