1a

$$\text{Let } V \text{ be a vector space over } \mathbb{R}$$
$$\text{Let } \dim V = n$$
$$\text{Let } T : V \to V \text{ be a linear transformation}$$
$$\text{Prove: } \exists B, C \text{ bases of } V : [T]_C^B = 0 \implies T = 0$$

$$\text{Proof:}$$
$$\text{Let } B, C : [T]_C^B = 0$$
$$\forall v \in V : [T(v)]_C = [T]_C^B [v]_B = 0 \cdot [v]_B = 0$$
$$[T(v)]_C = 0 \iff T(v) = 0$$
$$\implies \forall v \in V : T(v) = 0 \implies \boxed{T = 0}$$

1b

$$\text{Let } V \text{ be a vector space over } \mathbb{R}$$
$$\text{Let } \dim V = n$$
$$\text{Let } T : V \to V \text{ be a linear transformation}$$
$$\text{Prove: } \exists B, C \text{ bases of } V : [T]_C^B = D \text{ where } D \text{ is diagonal}$$

$$\text{Proof:}$$
$$\text{Let } \ker T = sp\{v_1, \ldots, v_k\}$$
$$\text{Let } B = \{v_1, \ldots, v_k, \ldots, v_n\}$$
$$\implies \text{Im} T = sp\{T(v_{k+1}), \ldots, T(v_n)\}$$
$$\text{Let } C = \{u_1, \ldots, u_k, T(v_{k+1}), \ldots, T(v_n)\}$$
$$\implies \begin{cases} \forall i \in [1, k] : [T(v_i)]_C = [0]_C = 0 \\ \forall i \in [k+1, n] : [T(v_i)]_C = e_i \end{cases}$$
$$\implies \boxed{[T]_C^B = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-k-1} \end{pmatrix} = D}$$

1c

$$\text{Let } V \text{ be a vector space over } \mathbb{R}$$
$$\text{Let } \dim V = n$$
$$\text{Let } T : V \to V \text{ be a linear transformation}$$
$$\text{Prove: } \forall B, C \text{ bases of } V : ([T]_C^B)^2 = 0 \implies T = 0$$

$$\text{Proof:}$$
$$\text{Let } \forall B, C \text{ bases of } V : ([T]_C^B)^2 = 0$$
$$\text{As proved in 1b: } \exists B, C \text{ bases of } V : [T]_C^B = D$$
$$([T]_C^B)^2 = 0 \implies D^2 = 0 \implies D = 0 \underset{\text{1a}}{\implies} \boxed{T = 0}$$

2

$$\text{Let } T:\mathbb{R}^2\to\mathbb{R}^2$$
$$\text{Let } n\in\mathbb{N}$$
$$T\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}p+nq\\q+np\end{pmatrix}$$
$$\text{Let } T^{17}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-3^{17}\\3^{17}\end{pmatrix}$$
$$\text{Find } n$$

Solution:

$$T^N\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}1&n\\n&1\end{pmatrix}^N\begin{pmatrix}p\\q\end{pmatrix}$$
$$\text{Let } A=\begin{pmatrix}1&n\\n&1\end{pmatrix}$$
$$P_A(\lambda)=(\lambda-1)^2-n^2=(\lambda-(n+1))(\lambda-(1-n))$$
$$\lambda=n+1\implies\begin{pmatrix}n&-n\\-n&n\end{pmatrix}\implies E_{n+1}=sp\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$$
$$\lambda=1-n\implies\begin{pmatrix}-n&-n\\-n&-n\end{pmatrix}\implies E_{1-n}=sp\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}$$
$$\implies A=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}n+1&0\\0&1-n\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1}$$
$$\implies A^N=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}(n+1)^N&0\\0&(1-n)^N\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}=$$
$$\frac{1}{2}\begin{pmatrix}(n+1)^N+(1-n)^N&(n+1)^N-(1-n)^N\\(n+1)^N-(1-n)^N&(n+1)^N+(1-n)^N\end{pmatrix}$$
$$T^N\begin{pmatrix}1\\-1\end{pmatrix}=A^N\begin{pmatrix}1\\-1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}(n+1)^N+(1-n)^N&(n+1)^N-(1-n)^N\\(n+1)^N-(1-n)^N&(n+1)^N+(1-n)^N\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=$$
$$=\frac{1}{2}\begin{pmatrix}2(1-n)^N\\-2(1-n)^N\end{pmatrix}=\begin{pmatrix}(1-n)^N\\-(1-n)^N\end{pmatrix}$$
$$\implies T^{17}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}(1-n)^{17}\\-(1-n)^{17}\end{pmatrix}=\begin{pmatrix}-3^{17}\\3^{17}\end{pmatrix}$$
$$\implies 1-n=-3\implies\boxed{n=4}$$

3a

$$\text{Let } T:\mathbb{R}^n\to\mathbb{R}^n \text{ be a linear transformation}$$
$$\text{Let } \forall\lambda \text{ eigenvalue of } T: g_\lambda=k_\lambda$$
$$\text{Prove or disprove: } T \text{ is diagonalizable}$$

Disproof:

$$\text{Let } B \text{ be a basis of } V: P_{[T]_B^B}(\lambda)=\prod_{i=1}^{n-2}(\lambda-\lambda_i)(\lambda^2+1)$$
$$\implies\forall i\in[1,n-2]:\lambda_i \text{ is an eigenvalue of } T \text{ and } g_{\lambda_i}=k_{\lambda_i}$$
$$\text{However, } T \text{ is not diagonalizable, as } P_T(\lambda) \text{ is not factorizable into linear factors}$$
$$\text{An example would be } T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}$$

3b

$$\text{Prove or disprove: }\exists A\in\mathbb{C}^{n\times n}:\exists\alpha\in\mathbb{C}:\alpha\notin\{-1,0,1\}:\forall\lambda \text{ eigenvalue of } A:$$
$$\alpha\lambda \text{ is also an eigenvalue of } A$$

Proof:

$$\text{Let } A=0$$
$$\implies P_A(\lambda)=\lambda^n\implies \text{Eigenvalues of } A \text{ are } \{0\}$$
$$\implies\forall\alpha\in\mathbb{C}:\alpha 0=0 \text{ is also an eigenvalue of } A$$

3c

Prove or disprove: $\exists A$ invertible $\in \mathbb{C}^{n\times n} : \exists \alpha \in C : \alpha \notin \{-1, 0, 1\} : \forall \lambda$ eigenvalue of A :
$\alpha\lambda$ is also an eigenvalue of A

Disproof:
A is invertible $\implies \forall \lambda$ eigenvalue of $A : \lambda \neq 0$
Let A, α such that the statement holds
$\lambda \neq 0, \alpha \notin \{-1, 0, 1\} \implies \alpha\lambda \neq \{\lambda, 0, -\lambda\}$ and is an eigenvalue of A
$\implies \alpha^2\lambda$ is also an eigenvalue of A
$\alpha^2\lambda \notin \{\lambda, 0, -\lambda, \alpha\lambda, -\alpha\lambda\}$
We can show by induction that $\forall n \neq m \in \mathbb{N} : \alpha^n\lambda \neq \alpha^m\lambda$
$\implies A$ has an infinite number of eigenvalues $\{\alpha\lambda\}_{n\in\mathbb{N}_0} -$ Contradiction!