Motivation behind diagonalization

$$A \sim D$$
$$A = P^{-1}DP$$
$$\underbrace{A^n}_{\text{Hard}} = (P^{-1}DP)^n = P^{-1}\underbrace{D^n}_{\text{Easy}}P$$

Diagonalizable matrix

$$A \text{ is called diagonalizable iff } \exists D : A \sim D$$

Diagonalizable matrix and eigenvectors

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$A \text{ is diagonalizable } \iff \exists B \text{ basis of } \mathbb{F}^n : \forall i \in [1,n] : Av_i = \lambda_i v_i$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$
$$\text{Determine whether } A \text{ is diagonalizable}$$

$$\text{Solution:}$$
$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 2 & -1 \\ 0 & -2 & \lambda - 3 \end{vmatrix} = (\lambda-1)((\lambda-2)(\lambda-3)-2) = (\lambda-1)^2(\lambda-4)$$
$$\lambda = 1 \implies \left(\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies E_1 = sp\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$
$$\lambda = 4 \implies \begin{pmatrix} 3 & -1 & -1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & -2 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies E_4 = sp\left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$$
$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \implies D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$
$$T(v_1) = 2v_1$$
$$T(v_2) = v_1 + v_2$$
Determine whether T is diagonalizable

Solution:
$$[T]_B^B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$
$$P_T(\lambda) = (\lambda - 2)(\lambda - 1)$$
$$\implies T \text{ is diagonalizable}$$
$$E_2 = sp\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$
$$E_1 = sp\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$
$$C_B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \{[d_1]_B, [d_2]_B\} \implies D = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$
$$[T]_{C_B}^{C_B} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - \text{Not diagonal!}$$
$$[T]_D^D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \text{Diagonal!}$$

$$T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$$
$$B = \{1 + x, x, x^2\}$$
$$T(1 + x) = -1 + x^2$$
$$T(x) = 1 + x^2$$
$$T(x^2) = 1 + 2x - x^2$$
Determine whether T is diagonalizable

Solution:
$$[T]_B^B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$
$$P_T(\lambda) = \begin{vmatrix} \lambda+1 & -1 & -1 \\ -1 & \lambda+1 & -1 \\ -1 & -1 & \lambda+1 \end{vmatrix} = (\lambda - 1)\begin{vmatrix} 1 & 1 & 1 \\ -1 & \lambda+1 & -1 \\ -1 & -1 & \lambda+1 \end{vmatrix} =$$
$$= (\lambda - 1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = (\lambda - 1)(\lambda + 2)^2$$
$$\lambda = 1 \implies \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \implies E_1 = sp\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$
$$\lambda = -2 \implies \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \implies E_{-2} = sp\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$
$$\implies D = \{1 + 2x + x^2, 1 + x - x^2, 1\}$$
$$\implies [T]_D^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \pi & 2 & 0 & 0 \\ -1 & e & 3 & 0 \\ -\frac{1}{2} & \frac{3}{5} & \sqrt{3} & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 1 & \pi & -e \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Determine whether $A \sim B$

Solution:

$$P_A(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$$

$$P_B(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$$

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \sim B \implies A \sim B$$

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & a & 0 & b \end{pmatrix}$$

$$b = 2 \implies [A \text{ is diagonalizable} \iff a = 0]$$

$$b = 3 \implies [A \text{ is diagonalizable} \iff a = 0]$$

$$b \notin \{2, 3\} \implies A \text{ is diagonalizable}$$

$$A \in \mathbb{C}^{3\times 3}$$

$$P_A(\lambda) = \lambda^3$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \dim(N(0I - A)) = \dim(N(-A)) = \dim(N(A)) = 1$$

$$\implies A \text{ is not diagonalizable}$$
