Equivalent definitions of uniform convergence (oscillation) #theorem

$$f_n \rightrightarrows f \iff d_n \to 0$$

Proof:

$$\boxed{\implies} \text{Let } f_n \rightrightarrows f$$
$$\text{Let } \varepsilon > 0$$
$$\forall x \in A : \exists N_\varepsilon : \forall n > N_\varepsilon : |f_n(x) - f(x)| < \frac{\varepsilon}{2}$$
$$\implies \exists N_\varepsilon : \forall n > N_\varepsilon : \sup_{x \in A} |f_n(x) - f(x)| \le \frac{\varepsilon}{2}$$
$$\implies \exists N_\varepsilon : \forall n > N_\varepsilon : |d_n - 0| \le \frac{\varepsilon}{2} < \varepsilon$$
$$\implies d_n \to 0$$

$$\boxed{\impliedby} \text{Let } d_n \to 0$$
$$\text{Let } \varepsilon > 0$$
$$\exists N_\varepsilon : \forall n > N_\varepsilon : |d_n - 0| < \varepsilon$$
$$\implies \exists N_\varepsilon : \forall n > N_\varepsilon : \sup_{x \in A} |f_n(x) - f(x)| < \varepsilon$$
$$\implies \forall x \in A : \exists N_\varepsilon : \forall n > N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$
$$\implies \boxed{f_n \rightrightarrows f}$$

Properties of uniform limit (integral) #theorem

Let f_n be integrable on $[a, b]$

Then f is integrable on $[a, b]$ and $\forall x \in [a, b] : \int_a^x f_n(t)\, dt \rightrightarrows \int_a^x f(t)\, dt$

Proof:

Let $x \in [a, b]$

$$\int_a^x f_n(t)\, dt \to \int_a^x f(t)\, dt \iff \int_a^x f_n(t)\, dt - \int_a^x f(t)\, dt \to 0$$
$$\iff \int_a^x f_n(t) - f(t)\, dt \to 0$$
$$\iff \left| \int_a^x f_n(t) - f(t)\, dt \right| \to 0$$
$$\overset{|\int f| \le \int |f|}{\iff} \int_a^x |f_n(t) - f(t)|\, dt \to 0$$
$$\overset{|f_n(t) - f(t)| \le d_n}{\iff} \int_a^x d_n\, dt \to 0 = \underbrace{d_n}_{\to 0} \cdot (x - a) \to 0$$
$$\left| \int_a^x f_n(t) - f(t)\, dt \right| \le d_n(x - a)$$
$$\implies \sup_{x \in [a,b]} \left| \int_a^x f_n(t) - f(t)\, dt \right| \le d_n(x - a)$$
$$\implies \sup_{x \in [a,b]} \left| \int_a^x f_n(t) - f(t)\, dt \right| \to 0$$
$$\implies \boxed{\int_a^x f_n(t)\, dt \rightrightarrows \int_a^x f(t)\, dt}$$

Function series #definition

$$\sum_{n=1}^{\infty} f_n(x)$$

$$\text{For example: geometric series } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, x \in (-1,1)$$

$$S_N(x) = \sum_{n=1}^{N} f_n(x)$$

$$\lim_{N\to\infty} S_N(x) = S(x) \iff \sum_{n=1}^{\infty} f_n(x) = S(x)$$

$$f_n \text{ is continuous/differentiable/integrable} \implies S_N \text{ is too}$$

$$\text{And even more: } \int S_N = \int \sum_{n=0}^{N} f_n = \sum_{n=0}^{N} \int f_n = \left(\sum_{n=0}^{N} f_n\right)' = \sum_{n=0}^{N} f_n'$$

Properties of series uniform convergence #theorem

$$1. \quad \text{Let } f_n(x) \text{ be continuous}$$

$$\text{Then } S(x) \text{ is also continuous}$$

$$2. \quad \text{Let } f_n(x) \text{ be integrable}$$

$$\text{Then } S(x) \text{ is also integrable and}$$

$$\int S_N(x) \to \int S(x)$$

$$\int \sum_{n=0}^{N} f_n(x) \to \int \sum_{n=0}^{\infty} f_n(x)$$

$$\int_a^x \sum_{n=0}^{\infty} f_n(t)\,dt = \sum_{n=0}^{\infty} \int_a^x f_n(t)\,dt$$

$$3. \quad \text{Let } S_N' \rightrightarrows g(x)$$

$$\text{Let } \exists x_0 \in A : \sum_{n=0}^{\infty} f_n(x) \to M$$

$$\text{Then } S_N'(x) \rightrightarrows S'(x)$$

$$\text{Or } \sum_{n=0}^{\infty} f_n'(x) \rightrightarrows \left(\sum_{n=0}^{\infty} f_n(x)\right)'$$