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Let $V=\mathbb{R}^{2\times 2}$ with a standrad inner product

Let $W=\mathbb{R}^2$ with inner product:

$$\langle\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x'\\y'\end{pmatrix}\rangle = xx'-xy'-x'y+2yy'$$

Let $T:V\to W$ be a linear operator

$$\forall A\in V: T(A)=C_1(A)+C_2(A)$$

2a

Find T^*

Solution:

Standard basis of $\mathbb{R}^{2\times 2}$ is an orthonormal basis of V

$$\forall w\in W: T^*(w)=\sum_{i=1}^{4}\langle w,T(e_i)\rangle e_i$$

$$T(e_1)=T(e_2)=\begin{pmatrix}1\\0\end{pmatrix}$$

$$T(e_3)=T(e_4)=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\implies T^*\begin{pmatrix}x\\y\end{pmatrix}=\langle\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\rangle(e_1+e_2)+\langle\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\rangle(e_3+e_4)=$$

$$=(x-y)(e_1+e_2)+(2y-x)(e_3+e_4)=\begin{pmatrix}x-y & x-y\\2y-x & 2y-x\end{pmatrix}$$

2b

Find an orthonormal basis of $\ker T$

Solution:

Let $S=\{e_1,e_2,e_3,e_4\}$ be a standard basis of V

$$\ker T\subseteq V$$

$$T(e_1)=T(e_2)=\begin{pmatrix}1\\0\end{pmatrix}$$

$$T(e_3)=T(e_4)=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\implies \forall v\in V: T(v)=T(\alpha e_1+\beta e_2+\gamma e_3+\delta e_4)=(\alpha+\beta)\begin{pmatrix}1\\0\end{pmatrix}+(\gamma+\delta)\begin{pmatrix}0\\1\end{pmatrix}$$

$$\implies T(v)=0 \iff \begin{cases}\alpha=-\beta\\ \gamma=-\delta\end{cases} \iff v=\alpha(e_1-e_2)+\gamma(e_3-e_4)$$

$$\iff v\in sp\left\{\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix}\right\}$$

$$\langle\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix}\rangle = tr\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & -1\end{pmatrix}=tr\begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}=0$$

$$\implies \text{This is an orthogonal basis of } \ker T$$

$$\left\|\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\right\|^2=tr\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix}=tr\begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}=2$$

$$\left\|\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix}\right\|^2=tr\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\0 & -1\end{pmatrix}=tr\begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}=2$$

$$\implies \text{An orthonormal basis of } \ker T \text{ is } \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix}\right\}$$