Diagonalizable matrix and eigenvectors #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$A \text{ is diagonalizable} \iff \exists B \text{ basis of } \mathbb{F}^n : \forall i \in [1,n] : Av_i = \lambda_i v_i$$

$$\text{Proof:}$$
$$\boxed{\Longleftarrow} \text{ Let } \exists B \text{ basis of } \mathbb{F}^n : \forall i \in [1,n] : Av_i = \lambda_i v_i$$
$$\text{Let } B = \{v_1, \dots, v_n\}$$
$$\text{Let } \{\lambda_i\}_{i\in[1,n]} : \forall i \in [1,n] : Av_i = \lambda_i v_i$$
$$\text{Let } P = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} \in \mathbb{F}^{n\times n}$$
$$B \text{ is a linear independence} \implies rank(P) = n \implies P \text{ is invertible}$$
$$P^{-1}AP = P^{-1}\begin{pmatrix} | & & | \\ Av_1 & \dots & Av_n \\ | & & | \end{pmatrix} = P^{-1}\begin{pmatrix} | & & | \\ \lambda_1 v_1 & \dots & \lambda_n v_n \\ | & & | \end{pmatrix} =$$
$$= \begin{pmatrix} | & & | \\ \lambda_1 P^{-1} v_1 & \dots & \lambda_n P^{-1} v_n \\ | & & | \end{pmatrix}$$
$$\lambda_i P^{-1} v_i = \lambda_i P^{-1} C_i(P) = \lambda_i P^{-1} P e_i = \lambda_i e_i$$
$$\implies P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} = D$$

$$\boxed{\Longrightarrow} \text{ Let } A \text{ be diagonalizable}$$
$$\text{Let } D = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha_n \end{pmatrix}$$
$$\text{Let } P = \begin{pmatrix} | & & | \\ p_1 & \dots & p_n \\ | & & | \end{pmatrix} \in \mathbb{F}^{n\times n} \text{ be invertible}$$
$$D = P^{-1}AP = P^{-1}\begin{pmatrix} | & & | \\ Ap_1 & \dots & Ap_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ P^{-1}Ap_1 & \dots & P^{-1}Ap_n \\ | & & | \end{pmatrix}$$
$$\implies \forall i \in [1,n] : C_i(D) = P^{-1}Ap_i \implies \alpha_i e_i = P^{-1}Ap_i$$
$$\implies \alpha_i P e_i = Ap_i \implies \alpha_i p_i = Ap_i$$
$$\implies \forall i \in [1,n] : Ap_i = \alpha_i p_i$$
$$rank(P) = n \implies \{p_1, \dots, p_n\} \text{ is a linear independence of size } n$$
$$\implies \{p_1, \dots, p_n\} \text{ is a basis of } \mathbb{F}^n$$

Eigenvalues and eigenvectors of a linear transformation #definition

$$\text{Let } V \text{ be a finitely generated vector space over } \mathbb{F}$$
$$\text{Let } T : V \to V \text{ be a linear transformation}$$
$$\lambda \in \mathbb{F} \text{ is called an Eigenvalue of } T \text{ if}$$
$$\exists v \neq 0 \in \mathbb{F}^n : T(v) = \lambda v$$
$$v \text{ is then called an Eigenvector of } T \text{ in respect to Eigenvalue } \lambda$$

Eigenvalues of linear transformation and representation matrix #lemma

$$\text{Let } V \text{ be a finitely generated vector space over } \mathbb{F}$$
$$\text{Let } B \text{ be a basis of } V$$
$$\text{Let } T : V \to V \text{ be a linear transformation}$$
$$\text{Then } \lambda \text{ is an eigenvalue of } T \iff \lambda \text{ is an eigenvalue of } [T]_B^B$$

$$\text{Proof:}$$
$$\lambda \text{ is an eigenvalue of } T \iff \exists v \neq 0 \in V : T(v) = \lambda v$$
$$\iff \exists v \neq 0 \in V : [T]_B^B[v]_B = [\lambda v]_B = \lambda[v]_B \iff \lambda \text{ is an eigenvalue of } [T]_B^B$$

Corollary

$$v \text{ is an eigenvector of } T \iff [v]_B \text{ is an eigenvector of } [T]_B^B$$

Diagonalizable linear operator #definition

$$\text{Let } V \text{ be a finitely generated vector space over } \mathbb{F}$$
$$\text{Let } T : V \to V \text{ be a linear operator}$$
$$T \text{ is then called diagonalizable iff } \exists B \text{ basis of } V : [T]_B^B \text{ is diagonalizable}$$

Diagonalizable linear operator and representation matrix #lemma

$$\text{Let } V \text{ be a finitely generated vector space over } \mathbb{F}$$
$$\text{Let } T : V \to V \text{ be a linear operator}$$
$$\text{Let } C \text{ be a basis of } V$$
$$\text{Then } T \text{ is diagonalizable} \iff [T]_C^C \text{ is diagonalizable}$$

$$\text{Proof:}$$
$$\boxed{\implies} \text{ Let } T \text{ be diagonalizable}$$
$$\implies \exists B : [T]_B^B \text{ is diagonal}$$
$$\implies ([I]_C^B)^{-1}[T]_C^C[I]_C^B = [T]_B^B$$
$$\implies [T]_C^C \text{ is diagonalizable}$$

$$\boxed{\impliedby} \text{ Let } [T]_C^C \text{ be diagonalizable}$$
$$\text{Let } P \in \mathbb{F}^{n \times n} \text{ be invertible}$$
$$\implies \exists B : P = [I]_C^B$$
$$\implies P^{-1}[T]_C^C P = [I]_B^C[T]_C^C[I]_C^B = [T]_B^B = D$$
$$\implies T \text{ is diagonalizable}$$

Properties of characteristic polynomial #lemma

1. Eigenvalues of A are roots of its characteristic polynomial
2. Characteristic polynomial is a monic polynomial of degree n
   That is, its leading coefficient is 1
3. $P_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 \cdot 1 \implies \begin{cases} a_{n-1} = -tr(A) \\ a_0 = (-1)^n \, |A| \end{cases}$

Proof for 2.

$$P_A(\lambda) = |\lambda I - A| = \prod_{i=1}^{n}(\lambda - a_{ii}) + \underbrace{p(\lambda)}_{\text{of degree} \leq n-2} =$$
$$= \lambda^n - a_{11}\lambda^{n-1} - \cdots - a_{nn}\lambda^{n-1} + \underbrace{p_1(\lambda)}_{\text{of degree} \leq n-2} =$$
$$= \lambda^n + (-tr(A))\lambda^{n-1} + p_1(\lambda)$$

Proof for 3.

$$a_0 = P_A(0) = |0I - A| = |-A| = (-1)^n \, |A|$$

Characteristic polynomial of linear operator #definition

Let V be a finitely generated vector space over $\mathbb{F}$
Let B be a basis of V
Let $T : V \to V$ be a linear operator
$P_T(\lambda)$ is called a characteristic polynomial of T
And is equal to $P_T(\lambda) = P_{[T]_B^B}(\lambda)$

Note:
Choice of basis does not mattter
$\forall B, C : [T]_B^B \sim [T]_C^C \implies P_{[T]_B^B}(\lambda) = P_{[T]_C^C}(\lambda)$

Algebraic multiplicity #definition

Let $A \in \mathbb{F}^{n \times n}$
Let λ_i be an eigenvalue of A
Maximal degree k such that $(\lambda - \lambda_i)^k \mid P_A(\lambda)$ is then called
an algebraic multiplicity of λ_i and denoted $g_A = \mu_A(\lambda_i) = k$

Geometric multiplicity #definition

Let $A \in \mathbb{F}^{n \times n}$
Let λ_i be an eigenvalue of A
$\dim(N(\lambda_i I - A))$ is then called a geometric multiplicity of λ_i
And is denoted $k_\lambda = \gamma_A(\lambda_i) = \dim(N(\lambda_i I - A))$