$$\text{Let } S \subseteq V$$
$$S^{\perp} = \{v \in V | \forall u \in S : \langle v, u \rangle = 0\}$$

$$S \subseteq (S^{\perp})^{\perp}$$
$$S = (S^{\perp})^{\perp} \iff S \text{ is a subspace of } V$$

$$W \text{ is a subspace of } V$$
$$B \text{ is a basis of } W$$
$$v \in V$$
$$\forall u \in B : \langle v, u \rangle = 0 \implies v \in W^{\perp}$$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$
$$S^{\perp} = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \middle| \begin{cases} v_1 + v_3 = 0 \\ v_1 + v_2 = 0 \end{cases} \right\} = \left\{ \begin{pmatrix} v_1 \\ -v_1 \\ -v_1 \end{pmatrix} \right\} = sp \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

$$A \in \mathbb{F}^{m \times n}$$
$$\text{Find } C(A)^{\perp}, C(A^T)^{\perp}$$

$$\text{Solution:}$$
$$v \in C(A^T)^{\perp} \iff \forall i \in [1, m] : \langle C_i(A^T), v \rangle = 0$$
$$\iff \forall i \in [1, m] : \langle R_i(A), v \rangle = 0$$
$$\iff Av = 0$$
$$v \in C(A^T)^{\perp} \iff v \in N(A)$$
$$v \in C(A)^{\perp} \iff v \in N(A^T)$$

$$\text{Prove: } (U + W)^{\perp} = U^{\perp} \cap W^{\perp}$$

$$\text{Proof:}$$
$$\boxed{\supseteq} \text{ Let } v \in U^{\perp} \cap W^{\perp}$$
$$v \in U^{\perp}$$
$$v \in W^{\perp}$$
$$\implies \forall u + w \in U + W : \langle v, u + w \rangle = \langle v, u \rangle + \langle v, w \rangle = 0 + 0 = 0$$
$$\implies v \in (U + W)^{\perp} \implies U^{\perp} \cap W^{\perp} = (U + W)^{\perp}$$
$$\boxed{\subseteq} \text{ Let } v \in (U + W)^{\perp}$$
$$u \in U \implies u = u + 0 \in U + W \implies \langle v, u \rangle = 0 \implies v \in U^{\perp}$$
$$w \in W \implies w = 0 + w \in U + W \implies \langle v, w \rangle = 0 \implies v \in W^{\perp}$$

Let B be a basis of V

$$\forall v, u \in V : \langle v, u \rangle = [v]_B^T G_B \overline{[u]_B}$$

Proof:

Let $B = \{v_1, \ldots, v_n\}$

Let $v, u \in V$

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

$$u = \sum_{i=1}^{n} \beta_i v_i$$

$$\langle v, u \rangle = \left\langle \sum_{i=1}^{n} \alpha_i v_i, u \right\rangle = \sum_{i=1}^{n} \alpha_i \langle v_i, u \rangle =$$

$$= \sum_{i=1}^{n} \alpha_i \left\langle v_i, \sum_{j=1}^{n} \beta_j v_j \right\rangle = \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i, v_j \rangle}$$

$$[v]_B^T G_B \overline{[u]_B} = (\alpha_1 \quad \ldots \quad \alpha_n) G_B \begin{pmatrix} \overline{\beta_1} \\ \vdots \\ \overline{\beta_n} \end{pmatrix} =$$

$$= (\alpha_1 \quad \ldots \quad \alpha_n) \begin{pmatrix} \sum_{i=1}^{n} \overline{\beta_i} (G_B)_{1i} \\ \vdots \\ \sum_{i=1}^{n} \overline{\beta_i} (G_B)_{ni} \end{pmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} (G_B)_{ij} =$$

$$= \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i, v_j \rangle}$$

Let $a_1, \ldots, a_n \in \mathbb{R}$

Prove: $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$

Proof:

$$\langle v, 1 \rangle = a_1 + \cdots + a_n$$

$$\implies |\langle v, 1 \rangle| = |a_1 + \cdots + a_n|$$

$$|\langle v, 1 \rangle| \leq \|v\| \cdot \|1\| = \sqrt{a_1^2 + \cdots + a_n^2} \cdot \sqrt{n}$$

$$\implies |\langle v, 1 \rangle|^2 = (a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$$

$$\text{Let } B_1 = \{v \in V : \|v\| = 1\}$$
$$\text{Let } v \neq 0 \in V$$
$$\text{Prove: } \forall u \in B_1 : \|v-u\| \geq \left\| v - \frac{v}{\|v\|} \right\|$$

$$\text{Proof:}$$
$$\text{Let } v \neq 0 \in V$$
$$\|v-u\|^2 = \langle v-u, v-u \rangle = \|v\|^2 - \langle v, u \rangle - \overline{\langle v, u \rangle} + \|u\|^2 = \|v\|^2 - 2\mathrm{Re}(\langle v, u \rangle) + 1$$
$$\left\| v - \frac{v}{\|v\|} \right\|^2 = \left| 1 - \frac{1}{\|v\|} \right|^2 \cdot \|v\|^2 = \left(1 - \frac{2}{\|v\|} + \frac{1}{\|v\|^2} \right) \cdot \|v\|^2 = \|v\|^2 - 2\|v\| + 1$$

$$\|v-u\|^2 - \left\| v - \frac{v}{\|v\|} \right\|^2 = 2(\|v\| - \mathrm{Re}(\langle v, u \rangle))$$
$$\mathrm{Re}(\langle v, u \rangle) \leq |\langle v, u \rangle| \leq \|v\| \cdot \|u\| = \|v\|$$
$$\implies \|v\| - \mathrm{Re}(\langle v, u \rangle) \geq 0 \implies \|v-u\|^2 - \left\| v - \frac{v}{\|v\|} \right\|^2 \geq 0$$
$$\implies \|v-u\|^2 \geq \left\| v - \frac{v}{\|v\|} \right\|^2 \implies \boxed{\|v-u\| \geq \left\| v - \frac{v}{\|v\|} \right\|}$$

$$W \text{ is a subspace of } V$$
$$\implies W \oplus W^\perp = V$$
$$\implies \dim W + \dim W^\perp = V$$
$$\implies dim W^\perp + \dim(W^\perp)^\perp = \dim V$$
$$\implies \dim W = \dim(W^\perp)^\perp \implies W = (W^\perp)^\perp$$

$$P_W : V \to V \text{ is a linear transformation}$$
$$\mathrm{Im} P_W = W$$
$$\ker P_W = W^\perp$$
$$v - P_W(v) \in W^\perp$$
$$P_W^2 = P_W$$
$$B = \{w_1, \ldots, w_k\} \text{ is an orthogonal basis of } W$$
$$\implies P_W(v) = \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i$$

$$\text{Let } W \text{ be a subspace of } V$$
$$\text{Let } v \in V$$
$$\forall w \in W : \|v-w\| \geq \|v - P_W(v)\|$$
