# Linear independence of orthogonal sets #lemma

 $S\subseteq V$  is orthogonal and  $0\not\in S\implies S$  is a linear independence

# Coordinates in orthogonal basis #definition

Let V be an inner product space over  $\mathbb F$ 

Let B be an orthogonal basis of V

Let  $v \in V$ 

$$\exists \{\alpha_i\}_{i \in [1,n]} : v = \sum_{i=1}^n \alpha_i v_i$$

$$\implies orall i \in [1,n]: lpha_i = rac{\langle v, v_i 
angle}{\|v\|^2}$$

Or in other words:  $[v]_B = rac{1}{\|v\|^2} egin{pmatrix} \langle v, v_1 
angle \ dots \ \langle v, v_n 
angle \end{pmatrix}$ 

Proof:

$$egin{aligned} \operatorname{Let} \left\{lpha_i
ight\}_{i\in[1,n]} : v = \sum_{i=1}^n lpha_i v_i \ orall i \in [1,n] : \langle v,v_i
angle = \left\langle \sum_{k=1}^n lpha_k v_k,v_i
ight
angle = \sum_{k=1}^n lpha_k \langle v_k,v_i
angle = lpha_i \langle v_i,v_i
angle = lpha_i \|v_i\|^2 \ orall i \in [1,n] : v_i 
eq 0 \implies \|v_i\| > 0 \ \implies \left| orall i \in [1,n] : lpha_i = rac{\langle v,v_i
angle}{\|v_i\|^2} 
ight| \end{aligned}$$

# Pythagorean theorem #theorem

Let B be an orthogonal basis of V

Let 
$$v \in V$$

$$v = \sum_{i=1}^n \alpha_i v_i$$

$$\implies \sum_{i=1}^n lpha_i v_i^{-2} = \sum_{i=1}^n \|lpha_i v_i\|^2 = \sum_{i=1}^n |lpha_i|^2 \|v_i\|^2$$

Proof

$$egin{aligned} \sum_{i=1}^n lpha_i v_i & = \left\langle \sum_{i=1}^n lpha_i v_i, \sum_{i=1}^n lpha_i v_i 
ight
angle = \sum_{i=1}^n \sum_{j=1}^n \left\langle lpha_i v_i, lpha_j v_j 
ight
angle = \ & = \sum_{i=1}^n \sum_{j=1}^n lpha_i \overline{lpha_j} \langle v_i, v_j 
angle = \sum_{i=1}^n lpha_i \overline{lpha_i} \langle v_i, v_i 
angle = \sum_{i=1}^n |lpha_i|^2 \|v_i\|^2 \end{aligned}$$

# Orthogonal complement #definition

Let V be an inner product space over  $\mathbb F$ 

Let 
$$S \subseteq V$$

Set of vectors that are orthogonal to all vectors in S is then called an orthogonal complement and denoted

$$S^\perp = \{v \in V | orall s \in S : \langle v,s 
angle = 0\}$$

### **Properties of orthogonal complements**

Let V be an inner product space over  $\mathbb{F}$  $S \subseteq V \implies S^{\perp}$  is a subpspace of V

Proof:

$$egin{aligned} orall s \in S : \langle 0,s 
angle = 0 \implies 0 \in S^\perp \ & ext{Let } v,u \in S^\perp, lpha \in \mathbb{F} \ & orall s \in S : \langle v + lpha u,s 
angle = \langle v,s 
angle + lpha \langle u,s 
angle = 0 + lpha \cdot 0 = 0 \ & \implies v + lpha u \in S^\perp \implies \boxed{S^\perp ext{ is a subspec of } V} \end{aligned}$$

$$S\subseteq (S^\perp)^\perp$$

Proof:

$$(S^\perp)^\perp = \left\{v \in V \; orall s' \in S^\perp : \langle v, s' 
angle = 0
ight\}$$

 $\mathrm{Let}\ s\in S$ 

$$orall s' \in S^\perp : \langle s', s 
angle = 0 \implies \langle s, s' 
angle = 0 \implies s \in (S^\perp)^\perp \implies \boxed{S \subseteq (S^\perp)^\perp}$$

$$A\subseteq B \implies A^{\perp}\supseteq B^{\perp}$$

Proof:

Let 
$$v \in B^\perp$$

$$egin{aligned} orall b \in B: \langle v, b 
angle = 0 & \Longrightarrow \ orall a \in A: \langle v, a 
angle = 0 & \Longrightarrow \ v \in A^\perp \ & \Longrightarrow egin{aligned} B^\perp \subseteq A^\perp \end{aligned}$$

$$S^\perp = (sp(S))^\perp$$

Proof:

$$S\subseteq sp(S) \implies (sp(S))^{\perp} \subseteq S^{\perp} \ ext{Let } v \in S^{\perp} \ ext{Let } u \in sp(S) \ u = \sum_{i=1}^k lpha_i s_i \ \langle v,u 
angle = \left\langle v, \sum_{i=1}^k lpha_i s_i 
ight
angle = \sum_{i=1}^k \overline{lpha_i} \langle v,s_i 
angle = 0 \ \implies v \in (sp(S))^{\perp} \implies S^{\perp} \subseteq (sp(S))^{\perp} \ \implies \left[ S^{\perp} = (sp(S))^{\perp} 
ight]$$

# Orthogonal projection (#definition

Let V be an inner product space

Let W be a subspace of V

Let  $B = \{w_1, \dots, w_k\}$  be an orthogonal basis of W

Let  $v \in V$ 

Then orthogonal projection  $P_W(v) = \sum_{i=1}^k rac{\langle v, w_i 
angle}{\|w_i\|^2} w_i$ 

Equivalent: Orthogonal projection is a vector such that  $orall w \in W: \|v-w\| \geq \|v-P_W(v)\|$ 

# Properties of orthogonal projection #lemma

$$orall v \in V: P_W(v) \in W$$

$$v \in W \iff P_W(v) = v$$
  $ext{Proof:}$   $ext{} ext{} ext{}$ 

$$B$$
 is an orthogonal basis  $\implies v = \sum_{i=1}^k rac{\langle v, w_i 
angle}{\|w_i\|^2} w_i$   $\implies \boxed{v = P_W(v)}$ 

$$igsquare$$
 Let  $P_W(v) = v$   $P_W(v) \in W \implies igsquare$ 

$$P_W(v) = 0 \iff v \in W^\perp$$

$$oxed{\Longrightarrow} \operatorname{Let} P_W(v) = 0$$

$$B ext{ is a linear independence } \implies orall w_i \in W: rac{\langle v, w_i 
angle}{\|w_i\|^2} = 0$$

$$\implies orall w_i \in B: \langle v, w_i 
angle = 0 \implies \boxed{v \in B^\perp = (sp(B))^\perp = W^\perp}$$

$$igstack igstack iggtack igstack iggtack igstack igstack igstack iggtack$$

$$w \in W \iff orall v \in V: \langle v - P_W(v), w 
angle = 0$$
 Or in other words:  $v - P_W(v) \in W^\perp$ 

#### Proof:

$$egin{aligned} w \in W \implies w = \sum_{i=1}^k lpha_i w_i \ & \langle v - P_W(v), w 
angle = 0 \iff \langle v, w 
angle = \langle P_W(v), w 
angle \ & \langle v, w 
angle = \left\langle v, \sum_{i=1}^k lpha_i w_i 
ight
angle = \left\langle v, w_i 
ight
a$$