1

Formulate and prove Riesz theorem

2a

$$\text{Let } V = \mathbb{R}^3$$
$$\text{Let } \langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rangle = xx' + 2yy' + 3zz'$$
$$\text{Let } W = sp \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}, v = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$
$$\text{Find } P_W(v)$$

Solution:

Let us orthogonalize W by using Gram-Schmidt process:

$$u_1 = v_1 \implies \|u_1\|^2 = 3$$
$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} | \cdot 3 \implies u_2 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \implies \|u_2\|^2 = 51$$
$$\implies P_W(v) = \frac{\langle v, u_1 \rangle}{\|u_1\|^2} u_1 + \frac{\langle v, u_2 \rangle}{\|u_2\|^2} u_2 = -\frac{2}{3} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \frac{44}{51} \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{34}{51} + \frac{176}{51} \\ -\frac{34}{51} + \frac{88}{51} \\ 0 + \frac{132}{51} \end{pmatrix} =$$
$$= \frac{1}{51} \begin{pmatrix} 210 \\ 54 \\ 132 \end{pmatrix} = \boxed{\frac{1}{17} \begin{pmatrix} 70 \\ 18 \\ 44 \end{pmatrix}}$$

2b

Let V be an inner product space

Let W be a subspace of V

Let $v \in V$

Let $u = P_W(v)$

Find $P_{W^\perp}(u)$

Solution:

$$u = P_W(v) \implies u \in W$$
$$\implies \boxed{P_{W^\perp}(u) = 0}$$

3

Let $x, y, z, w \geq 0 \in \mathbb{R}$

Let $x + y + z + w = 4$

Find $\max\{\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w}\}$

Solution:

$$(\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w})^2 \leq 4 \cdot (\sqrt{x}^2 + \sqrt{y}^2 + \sqrt{z}^2 + \sqrt{w}^2)$$
$$\implies (\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w})^2 \leq 4 \cdot 4 = 16$$
$$\implies (\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w}) \leq 4$$
$$x = y = z = w = 1 \implies \sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w} = 4$$
$$\implies \boxed{\max\{\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w}\} = 4}$$

3b

$$\text{Determine whether } A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{10\times 10} \text{ is diagonalizable}$$

Solution:

$$rank(A) = 2 \implies \dim E_0 = 8$$

$$A \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ \vdots \\ 9 \end{pmatrix} = 9 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \implies \dim E_9 = 1$$

$$\text{By looking at the first row we can see that } \det(I - A) = 0 \implies 1 \text{ is an eigenvalue of } A$$

$$\implies \dim E_1 = 1$$

$$\implies \dim E_1 + \dim E_9 + \dim E_0 = 10 = n \implies \boxed{A \text{ is diagonalizable}}$$

3c

$$\text{Prove or disprove: } \lambda \neq 0 \text{ is an eigenvalue of } A \in \mathbb{F}^{n\times n} \implies \lambda^2 \text{ is an eigenvalue of } AA^T$$

Disproof:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \implies P_A(x) = x^2 + 1 = (x-i)(x+i)$$

$$AA^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \text{Eigenvalues of } AA^T \text{ are } \{1\}$$

$$\text{Let } \lambda = i$$

$$\lambda^2 = -1 \text{ is not an eigenvalue of } AA^T$$

4

$$\text{Let } A \in \mathbb{C}^{5\times 5}$$

$$\text{Let } \forall i, j : A_{ij} \in \mathbb{R}$$

$$\text{Let } rank(A) = 3$$

$$\text{Let } A - (1+i)I \text{ be non-invertible}$$

$$\text{Let } tr(A) = 0$$

4a

Find all possible Jordan forms of A

Solution:

$$rank(A) = 3 \implies g_0 = n - 3 = 2 \implies \text{There are two Jordan blocks with eigenvalue } 0$$

$$\det(A - (1+i)I) = 0 \implies 1+i \text{ is an eigenvalue of } A$$

$$\underbrace{\implies}_{\forall i,j: A_{ij} \in \mathbb{R}} 1 - i \text{ is also an eigenvalue, with the same algebraic multiplicity}$$

$$\implies P_A(x) = x^{2+t}(x - (1+i))^k (x - (1-i))^k \cdot f(x)$$

$$k \geq 1 \implies \begin{cases} 2 + t + 2k \geq 4 + t \\ 2 + t + 2k \leq 5 \end{cases} \implies \begin{cases} 0 \leq t \leq 1 \\ 2k \leq 3 \implies k = 1 \end{cases}$$

$$tr(A) = tr(J_A) = \sum_{i=1}^{5} \lambda_i = 0 + 0 + (1-i) + (1+i) + \lambda = 0$$

$$\implies \lambda = -2 \implies P_A(x) = x^2(x - (1+i))(x - (1-i))(x+2)$$

$$\implies A \text{ is diagonalizable and its Jordan form is}$$

$$J_A = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & -2 & & \\ & & & 1-i & \\ & & & & 1+i \end{pmatrix}$$

4b

Let $f(x) = x^2 - 9x + 20$

Determine whether $f(A)$ is invertible

Solution:

$$f(A) = A^2 - 9A + 20I$$

$$A \sim J_A \implies A^2 - 9A + 20I = P(J_A^2 - 9J_A + 20I)P^{-1}$$

$$J_A^2 - 9J_A + 20I = \begin{pmatrix} 20 & & & & \\ & 20 & & & \\ & & 4 - 9(-2) + 20 & & \\ & & & -2i - 9(1-i) + 20 & \\ & & & & 2i - 9(1+i) + 20 \end{pmatrix} =$$

$$= \begin{pmatrix} 20 & & & & \\ & 20 & & & \\ & & 42 & & \\ & & & 11 + 7i & \\ & & & & -7i + 11 \end{pmatrix}$$

$$\implies \det(J_A^2 - 9J_A + 20I) = 20 \cdot 20 \cdot 42 \cdot (11 + 7i) \cdot (11 - 7i) \neq 0$$

$$\implies f(A) = A^2 - 9A + 20I \text{ is a product of invertible matrices and is itself invertible}$$

5a

Let $A \in \mathbb{C}^{n \times n}$

Prove: $A + A^*$ is Hermitian

Proof:

$$(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$$

5b

$$\text{Let } A \in \mathbb{C}^{n\times n} : \forall v \in \mathbb{C}^n : \langle Av, v\rangle = 0$$
$$\text{Prove: } A \text{ is nilpotent}$$

Proof:

$$\text{Let } \lambda \text{ be an eigenvalue of } A \text{ with eigenvector } v$$
$$\langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda\|v\| = 0$$
$$v \neq 0 \implies \|v\| \neq 0 \implies \lambda = 0 \implies \boxed{A \text{ is nilpotent}}$$

5c

$$\text{Let } M \in \mathbb{C}^{n\times n} \text{ be Hermitian and } \forall v \in \mathbb{C}^n : \langle Mv, v\rangle = 0$$
$$\text{Prove: } M = 0$$

Proof:

$$M \text{ is Hermitian, nilpotent, normal}$$
$$P_M(x) = x^n \implies \text{Characteristic polynomial is factorizable into linear factors}$$
$$\text{and } M \text{ is normal} \implies M \text{ is unitary diagonalizable}$$
$$\implies \exists P : M = PDP^* \text{ where } D = 0$$
$$\implies \boxed{M = 0}$$

5d

$$\text{Let } A \in \mathbb{C}^{n\times n} \text{ and } \forall v \in \mathbb{C}^n : \langle Av, v\rangle = 0$$
$$\text{Prove: } A = 0$$

Proof:

$$A + A^* \text{ is Hermitian}$$
$$\forall v \in \mathbb{C}^n : \langle (A + A^*)v, v\rangle = \langle Av, v\rangle + \langle A^*v, v\rangle = 0 + \langle v, Av\rangle = 0 + 0 = 0$$
$$\implies \text{By 5c: } A + A^* = 0$$
$$\implies A = -A^* \implies AA^* = A^*A = -A^2$$
$$\implies A \text{ is normal} \implies A \text{ is unitary diagonalizable and nilpotent} \implies \boxed{A = 0}$$