$$a_n : \mathbb{N} \to \mathbb{R}$$
$$f_n : \mathbb{N} \to \mathbb{R}^D$$

Pointwise convergence

$$f_n \to f$$
$$\forall x \in D : \forall \varepsilon > 0 : \exists N_{x,\varepsilon} : \forall n \geq N_{x,\varepsilon} : |f_n(x) - f(x)| < \varepsilon$$

$$(0, \infty)$$
$$f_n(x) = n(\sqrt[n]{x} - 1)$$
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sqrt[n]{x} - 1}{\frac{1}{n}} \overset{t=\frac{1}{n}}{=} \lim_{t\to 0} \frac{x^t - 1}{t} = \ln(x)$$

$$f_n(x) = \frac{n^2}{1 + n^2x^2}$$
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2} + x^2} - = \frac{1}{x^2}$$

$$\text{Let } \{f_n(x)\} \to f(x) \text{ on } [a, b]$$
$$\text{Let } \forall n, x \in [a, b] : f_n(x) \text{ is bounded}$$
$$\text{Then not necessarily } f \text{ is bounded}$$

Uniform convergence

$$f_n \rightrightarrows f$$
$$\forall \varepsilon > 0 : \exists N_\varepsilon : \forall n \geq N_\varepsilon : \forall x \in D : |f_n(x) - f(x)| < \varepsilon$$

$$\text{Let } d_n = \sup_{x\in D} |f_n(x) - f(x)|$$
$$f_n \rightrightarrows f \iff d_n \to 0$$

$$f_n \rightrightarrows f \text{ and } f_n \text{ is continuous} \implies f \text{ is continuous}$$

$$\text{Let } f_n \rightrightarrows f \text{ and } f_n(x) \text{ be bounded}$$
$$\implies f \text{ is bounded}$$
$$\text{And exists } M \text{ that bounds } f_n \text{ for every } n$$
$$\forall n \geq N : |f_n| \leq |f_n - f| + |f - f_N| + |f_N| < 2 + M_N$$
$$\implies \forall n \in \mathbb{N} : |f_n| \leq \max\{M_1, M_2, \ldots, M_{N-1}, 2 + M_N\}$$

$$[0,1]$$

$$f_n(x) = \frac{x}{1+n^2x^2}$$

$$f_n \to 0$$

$$d_n = \sup_{x\in[0,1]} \frac{x}{1+n^2x^2} = \max_{x\in[0,1]} \frac{x}{1+n^2x^2}$$

$$x = 0 \implies d_n = 0$$

$$x = 1 \implies d_n = \frac{1}{1+n^2}$$

$$0 = \left(\frac{x}{1+n^2x^2}\right)' = \frac{(1+n^2x^2) - 2n^2x^2}{(1+n^2x^2)^2} = \frac{1-n^2x^2}{(1+n^2x^2)^2}$$

$$\implies x^2 = \frac{1}{n^2} \implies x = \frac{1}{n}$$

$$f_n\left(\frac{1}{n}\right) = \frac{1}{2n}$$

$$\implies d_n = \frac{1}{2n} \to 0$$

$$\implies \boxed{f_n \rightrightarrows 0}$$