Inner product #definition

Let V be a vector space over $\mathbb{F}(\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\})$

$\langle , \rangle : V \times V \to \mathbb{F}$ is called inner product if

1. Linearity in the first argument: $\langle v + \alpha u, w\rangle = \langle v, w\rangle + \alpha\langle u, w\rangle$
2. Conjugate symmetry (Hermitian): $\langle v, u\rangle = \overline{\langle u, v\rangle}$
3. Positive-definiteness: $\forall v \neq 0 : \langle v, v\rangle > 0$

3.1 $\quad v = 0 \iff \langle v, v\rangle = 0$

V is then called an inner product space

Standard inner product

Let $V = \mathbb{R}^n$

Standard inner product is $\langle v, u\rangle = v^T u = \sum_{i=1}^{n} v_i u_i$

$$\langle v + \alpha u, w\rangle = (v + \alpha u)^T w = v^T w + \alpha u^T w = \langle v, w\rangle + \alpha\langle u, w\rangle$$

$$\langle v, w\rangle = v^T w = (v^T w)^T = w^T v = \overline{w^T v} = \overline{\langle w, v\rangle}$$

$$v \neq 0 \implies \langle v, v\rangle = v^T v = \sum_{i=1}^{n} v_i^2 > 0$$

$$v = 0 \iff \sum_{i=1}^{n} v_i^2 = 0 \iff v^T v = 0 \iff \langle v, v\rangle = 0$$

Let $V = \mathbb{C}^n$

Standard inner product is$\langle v, u\rangle = v^T \overline{u} = \sum_{i=1}^{n} v_i \overline{u_i}$

Let $V \in \mathbb{F}^{n\times n}$

Standard inner product is $\langle A, B\rangle = tr(AB^*) = tr(A\overline{B}^T)$

Properties of inner product #lemma

$$\forall v \in V : \langle 0_V, v\rangle = \langle 0_{\mathbb{F}} \cdot 0_V, v\rangle = 0_{\mathbb{F}\cdot\langle 0_V, v\rangle} = 0_{\mathbb{F}}$$

$$\forall v \in V : \langle v, 0_V\rangle = \overline{\langle 0_V, v\rangle} = \langle 0_V, v\rangle = 0_{\mathbb{F}}$$

$$\langle v, w + \alpha u\rangle = \overline{\langle w + \alpha u, v\rangle} = \overline{\langle w, v\rangle} + \overline{\alpha} \cdot \overline{\langle u, v\rangle} = \overline{\overline{\langle v, w\rangle}} + \overline{\alpha} \cdot \overline{\overline{\langle v, u\rangle}} =$$
$$= \langle v, w\rangle + \overline{\alpha} \cdot \langle v, u\rangle$$

Zero inner product #lemma

Let $v \in V : \forall u \in V : \langle v, u\rangle = 0$

Then $v = 0$

Proof:

$$\forall u \in V : \langle v, u\rangle = 0 \implies \langle v, v\rangle = 0 \implies v = 0$$

Norm #definition

$$\text{Let } V \text{ over } \mathbb{F}$$
$$\|\,\| : V \times V \to \mathbb{F} \text{ is called a norm if}$$
$$1. \quad v \neq 0 \implies \|v\| > 0$$
$$2. \quad v = 0 \iff \|v\| = 0$$
$$3. \quad \|\alpha v\| = |\alpha| \cdot \|v\|$$
$$4. \quad \|v + u\| \leq \|v\| + \|u\|$$

"Root" norm #definition

$$\forall v \in V : \|v\| = \sqrt{\langle v, v\rangle}$$
$$\text{This norm will be used throughout the course}$$

Metric #definition

$$p : V \times V \to \mathbb{R} \text{ is called a metric if}$$
$$1. \quad p(v, u) \geq 0$$
$$2. \quad v = u \iff p(v, u) = 0$$
$$3. \quad p(v, u) = p(u, v)$$
$$4. \quad p(v, u) \leq (p, w) + p(w, u)$$

Standard metric

$$\forall v, u \in V : p(v, u) = \|v - u\|$$

Orthogonal vectors #definition

$$v, u \text{ are called orthogonal iff } \langle v, u\rangle = 0$$

Orthogonal set #definition

$$\text{Let } S \subseteq V$$
$$S \text{ is called orthogonal set iff } \forall v, u \in S : \langle v, u\rangle = 0$$

Normal vector #definition

$$v \text{ is called normal iff } \|v\| = 1$$

Orthonormal set #definition

$$\text{Let } S \subseteq V \text{ be a orthogonal set}$$
$$S \text{ is then called orthonormal iff } \forall v \in S : \|v\| = 1$$