Improper integrals

$$\int_a^\infty f(x)\,dx = ???$$
$$\int_{-\infty}^b f(x)\,dx = ???$$
$$\int_{-\infty}^\infty f(x)\,dx = ???$$

Improper integral of the first type #definition

$$\text{Function } f \text{ is called Riemann-integrable on } [a,\infty) \text{ iff}$$
$$\forall b > a : f \text{ is Riemann-integrable on } [a,b]$$

$$\text{If function } f \text{ is Riemann-integrable on } [a,\infty)$$
$$\text{It's integral is called improper and is equal to}$$
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$
$$\text{Similarly, } \int_{-\infty}^a f(x)\,dx = \lim_{b\to-\infty}\int_b^a f(x)\,dx = L$$
$$\text{If } L \in \mathbb{R}\text{, improper integral is said to converge}$$

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx$$
$$\text{Let } t = \ln x \implies dt = \frac{dx}{x}$$
$$\implies \int \frac{1}{x\ln x}\,dx = \int \frac{1}{t}\,dt = \ln|t| = \ln|\ln x| + C$$
$$\implies \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\ln|\ln b| - \ln|\ln 2| = \lim_{b\to\infty}\ln(\ln b) = \infty$$
$$\implies \text{This integral diverges}$$

$$\sum_{n=2}^\infty \frac{1}{n\ln(n)} \text{ also diverges, is there a connection?}$$

$$\int_0^\infty xe^{-x}\,dx$$
$$f(x)=x, g'(x)=e^{-x}$$
$$f'(x)=1, g(x)=-e^{-x}$$
$$\int xe^{-x}\,dx=-xe^{-x}+\int e^{-x}\,dx=-xe^{-x}-e^{-x}+C$$
$$\implies \int_0^\infty xe^{-x}\,dx=\lim_{b\to\infty}\int_0^b xe^{-x}\,dx=\lim_{b\to\infty}(-be^{-b}-e^{-b}+e^{-0})=\lim_{b\to\infty}\frac{-b}{e^b}+1=$$
$$=-0+1=1$$
$$\implies \text{This integral converges to 1}$$

$$\text{Note: }\forall n\in\mathbb{N}_0:\int_0^\infty x^ne^{-x}\,dx=n!$$
$$\text{Proof:}$$
$$\text{Let }\Gamma(n)=\int x^ne^{-x}\,dx$$
$$\Gamma(n+1)=\int x^{n+1}e^{-x}\,dx=-x^{n+1}e^{-x}+(n+1)\int x^ne^{-x}\,dx=$$
$$=-x^{n+1}e^{-x}+(n+1)\Gamma(n)$$
$$\Gamma(0)=-e^{-x}$$
$$\implies \Gamma(n)=\sum_{i=0}^n\frac{n!}{(n-i)!}x^{n-i}(-e^{-x})$$

$$\implies \int_0^\infty x^ne^{-x}\,dx=\lim_{b\to\infty}\Gamma(n)\Big|_{x=0}^{x=b}=$$
$$=\lim_{b\to\infty}\underbrace{\sum_{i=0}^n\frac{(n)!}{(n-i)!}b^{n-i}(-e^{-b})}_{\to 0}-n!(-e^{-0})=n!$$

Improper integral from -inf to +inf #definition

$$\int_{-\infty}^\infty f(x)\,dx=\int_{-\infty}^a f(x)\,dx+\int_a^\infty f(x)\,dx$$
$$\int_{-\infty}^\infty f(x)\,dx\text{ converges}\iff\text{Both }\int_{-\infty}^a f(x)\,dx,\int_a^\infty f(x)\,dx\text{ converge}$$

$$\text{Note: }\int_{-\infty}^\infty f(x)\,dx\neq\lim_{b\to\infty}\int_{-b}^b f(x)\,dx$$
$$\text{Example:}$$
$$\int_{-\infty}^\infty x^7\,dx=\int_{-\infty}^a x^7\,dx+\lim_{b\to\infty}\int_a^b x^7\,dx=\int_{-\infty}^a x^7\,dx+\underbrace{\lim_{b\to\infty}\left(\frac{b^8}{8}-\frac{a^8}{8}\right)}_{\infty}=\infty$$
$$\lim_{b\to\infty}\int_{-b}^b x^7\,dx=\lim_{b\to\infty}\left(\frac{b^8}{8}-\frac{(-b)^8}{8}\right)=0$$

Convergence tests

p-Integral test for improper integrals of the first type #lemma

$$\int_a^\infty \frac{1}{x^p}\,dx \text{ converges} \iff p > 1$$

Proof:

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_a^b \frac{1}{x^p}\,dx$$

Let $p = 1$

$$\int_a^b \frac{1}{x^p}\,dx = \ln|b| - \ln|a| \implies \int_a^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}(\ln|b| - \ln|a|) = \infty$$

Let $p \neq 1$

$$\int_a^b \frac{1}{x^p}\,dx = \int_a^b x^{-p}\,dx = \frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}$$

$$\implies \lim_{b\to\infty}\int_a^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\frac{1}{1-p}(b^{1-p} - a^{1-p})$$

$$p < 1 \iff b^{1-p} \to \infty \iff \int_a^\infty \frac{1}{x^p}\,dx \text{ diverges}$$

$$p > 1 \iff b^{1-p} \to 0 \iff \int_a^\infty \frac{1}{x^p}\,dx \text{ converges}$$

Comparison test for improper integrals of the first type #lemma

Let f, g be Riemann-integrable on $[a, \infty)$

Let $0 \leq f \leq g$

$$\text{Then } \int_a^\infty g(x)\,dx \text{ converges} \implies \int_a^\infty f(x)\,dx \text{ converges}$$

Limit comparison text for improper integrals of the first type #lemma

Let f, g be Riemann-integrable on $[a, \infty)$

Let $0 \leq f, g$

$$L = \lim_{x\to\infty}\frac{f(x)}{g(x)}$$

$$\begin{aligned}
L = \infty \implies & \left[\textstyle\int_a^\infty f(x)\,dx \text{ converges} \implies \int_a^\infty g(x)\,dx \text{ converges}\right] \\
L = 0 \implies & \left[\textstyle\int_a^\infty f(x)\,dx \text{ converges} \impliedby \int_a^\infty g(x)\,dx \text{ converges}\right] \\
0 < L < \infty \implies & \left[\textstyle\int_a^\infty f(x)\,dx \text{ converges} \iff \int_a^\infty g(x)\,dx \text{ converges}\right]
\end{aligned}$$