$$T \text{ is normal} \implies S = aT + bT^* \text{ is normal}$$

Proof:

$$SS^* = (aT + bT^*)(aT + bT^*)^* = (aT + bT^*)(\overline{a}T^* + \overline{b}T) =$$
$$= |a|^2 TT^* + a\overline{b}TT + b\overline{a}T^*T^* + |b|^2 T^*T$$
$$S^*S = (\overline{a}T^* + \overline{b}T)(aT + bT^*) =$$
$$= |a|^2 T^*T + \overline{b}aTT + \overline{a}bT^*T^* + |b|^2 TT^*$$
$$T^*T = TT^* \implies S^*S = SS^*$$

Note:

$$\forall T : S \text{ is normal} \iff |a| = |b|$$

$$A = \begin{pmatrix} a & 0 & b \\ 0 & 2a & a \\ i & 1 & a \end{pmatrix}$$

Find all values of a, b such that A is normal/unitary/hermitian

Solution:

$$A^* = \begin{pmatrix} \overline{a} & 0 & -i \\ 0 & 2\overline{a} & 1 \\ \overline{b} & \overline{a} & \overline{a} \end{pmatrix}$$

$$AA^* = \begin{pmatrix} |a|^2 + |b|^2 & b\overline{a} & \overline{a}b - ai \\ a\overline{b} & 5|a|^2 & 2a + |a|^2 \\ \overline{a}i + a\overline{b} & 2\overline{a} + |a|^2 & 2 + |a|^2 \end{pmatrix}$$

$$A^*A = \begin{pmatrix} 2 & -i & -2i \\ i & 4|a|^2 + 1 & 2|a|^2 + a \\ a\overline{b} + \overline{a}i & 2|a|^2 + \overline{a} & |b|^2 + 2|a|^2 \end{pmatrix}$$

$$AA^* = A^*A \iff \begin{cases} |a|^2 + |b|^2 = 2 \\ b\overline{a} = -i \\ a\overline{b} = i \\ |a|^2 = 1 \\ |a|^2 = a \end{cases} \implies \boxed{\begin{cases} a = 1 \\ b = -i \end{cases}}$$

Now let us check unitarity and hermitianity:

$$(AA^*)_{11} \neq 1 \implies \boxed{A \text{ is not unitary}}$$

$$\text{Let } a = 1, b = -i \implies \boxed{A \text{ is hermitian}}$$

$$A \text{ is hermitian and untary} \overset{?}{\implies} A = I$$

$$A \text{ is hermitian and untary} \overset{?}{\implies} A^2 = I$$

$$A \text{ is hermitian and } A^2 = I \overset{?}{\implies} A \text{ is unitary}$$

$$A \text{ is unitary and } A^2 = I \overset{?}{\implies} A \text{ is hermitian}$$

Solution:

First is wrong: $\boxed{A = -I} \implies A^* = -I \implies AA^* = A^*A = I$

Second is correct: $A = A^* \implies AA^* = I \implies \boxed{A^2 = I}$

Third is correct: $A = A^* \implies \boxed{I = AA = AA^* = A^*A}$

Fourth is correct: $A^2 = I, AA^* = I \implies \boxed{A^* = A^{-1} = A}$