Approximations

We know what to do with alternating power series to determine
how many terms we need or accuracy ε
But what do we do with series that do not alternate?

Lagrange's remainder #theorem

$$\text{Let } f \text{ be a function}$$
$$\text{Let } f \text{ be } N+1 \text{ times differentiable at } a$$
$$\implies P_N(a) \text{ exists}$$
$$\implies R_N(a) = f(a) - P_N(a)$$
$$\implies \forall x : \exists c \in (x,a) : R_N(x) = \frac{\overbrace{f^{(N+1)}(c)}^{c \text{ instead of } a}}{(N+1)!}(x-a)^{N+1}$$

Examples

$$\text{Find } \sqrt{2}$$

$$\text{Solution:}$$
$$\text{Let } a = 0$$
$$f(x) = \sqrt{x+1} \to f(0) = 1$$
$$f'(x) = \frac{1}{2}(x+1)^{-1/2} \to f'(0) = \frac{1}{2}$$
$$f''(x) = -\frac{1}{4}(x+1)^{-3/2} \to f''(0) = -\frac{1}{4}$$
$$f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} \to f^{(3)}(0) = \frac{3}{8}$$
$$f(x) \approx P_3(x)$$
$$\implies \sqrt{x+1} \approx \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 =$$
$$= 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{8}x^3$$
$$\implies \sqrt{2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = \frac{23}{16} = 1.4375$$

$$\text{By Lagrange's remainder theorem } \exists c \in (x,a) : R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$
$$\text{In this case: } a = 0, x = 1, N = 3$$
$$\implies \exists c \in (0,1) : R_3(1) = \frac{f^{(4)}(c)}{4!}(1-0)^4 = \frac{f^{(4)}(c)}{4!}$$
$$f^{(4)}(c) = -\frac{15}{16}(c+1)^{-7/2}$$
$$\implies |R_3(1)| = \left|\frac{-\frac{15}{16}(c+1)^{-7/2}}{24}\right| = \frac{5}{128(c+1)^{7/2}}$$
$$\implies |R_3(1)| \text{ is monotonically decreasing on } c \in (0,1)$$
$$\implies c = 0 \text{ is a maximum(supremum) of } |R_3(1)| \text{ which is } \boxed{\frac{5}{128}}$$
$$\text{Or approximately } 0.04$$
$$\left|\sqrt{2} - \frac{23}{16}\right| \approx 0.023 \leq 0.04$$

Analytic function #definition

Let f be a function on I
Let f be infinitely differentiable
If $f(x)$ is equal to a power series, it is necessarily Taylor series

$$\forall x_0 \in I : f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

And f is then called analytic (real analytic)

$$f(x) = P_N(x) + R_N(x)$$
$$N \to \infty \implies P_N(x) \to f(x)$$
$$N \to \infty \implies R_N(x) \to 0$$

By Lagrange's remainder theorem: $R_N(x) = \dfrac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{N+1}$

$$\overset{?}{\implies} \frac{(x-x_0)^{N+1}}{(N+1)!} \underset{N\to\infty}{\to} 0$$

From this we can conclude:

Taylor's theorem #theorem

If $\left\{f^{(n+1)}(c)\right\}$ is bounded on I
Then f is equal to its Taylor series on I

Examples

$$f(x) = \sin x \implies \forall c \in \mathbb{R} : \left|f^{(N+1)}(c)\right| \leq 1$$
$$\implies \sin x \text{ is equal to its Taylor series}$$

$$f(x) = e^x$$
$$\implies \forall N : f^{(N+1)}(c) = e^c$$
$$\forall c \in [a,b] : \left|f^{(N+1)}(c)\right| \leq e^b$$
$$\implies \forall a,b \in \mathbb{R} : e^x \text{ is equal to its Taylor series on } [a,b]$$

An off-topic note:

$$f(x) = \sqrt{x+1}$$
$$\implies f^{(n)}(c) = (-1)^{n+1}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n}(c+1)^{-(2n-1)/2}$$

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
$$\ldots$$
