Uniform convergence #definition

$$\{f_n(x)\} \text{ is called point-convergent on set} A \text{ if}$$
$$\forall \varepsilon > 0 : \exists N_\varepsilon : \forall n > N_\varepsilon : \forall x_0 \in A : |f_n(x_0) - f(x_0)| < \varepsilon$$
$$f(x) \text{ is then called a uniform limit of } f_n$$
$$f_n \rightrightarrows f$$

Note: N_ε only depends on ε and works for all $x \in A$

Hence uniform convergence which is much stronger than pointwise convergence

Equivalent definitions of uniform convergence #theorem

The following are equivalent:

1. $f_n \rightrightarrows f$ on A
2. Equivalent definition via sequences
3. Equivalent definition by Cauchy
4. $d_n = \sup\limits_{x \in A} |f_n(x) - f(x)|, d_n \to 0$

$$f_n(x) = x^n - x^{2n} \text{ on } [0,1]$$
$$f_n(x) \to 0$$
$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|x^n - x^{2n}\right| = \sup_{x\in[0,1]} (x^n - x^{2n}) = \max_{x\in[0,1]} (x^n - x^{2n})$$
$$0^n - 0^{2n} = 0$$
$$1^n - 1^{2n} = 0$$
$$(x^n - x^{2n})' = nx^{n-1} - 2nx^{2n-1} = nx^{n-1}(1 - 2x^n)$$
$$(x^n - x^{2n})' = 0 \iff \begin{cases} x = 0 \\ x = \sqrt[n]{\frac{1}{2}} \end{cases} \implies d_n = \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4} \neq 0 \implies f_n \not\rightrightarrows f$$

Properties of uniform limit #theorem

Let $f_n \rightrightarrows f$

1. Continuity

Let $\forall n : f_n$ is continuous

Then f is continuous

2. Integral

Let f_n be integrable on $[a,b]$

Then f is integrable on $[a,b]$ and $\forall x \in [a,b] : \int_a^x f_n(t)\,dt \to \int_a^x f(t)\,dt$

3. Differentiability

Let f_n, f be differentiable

Then not necessarily $f_n' \to f'$

Let $\exists x_0 \in A : f_n(x_0)$ converges

Let $\exists g : f_n' \rightrightarrows g$

Then $\exists f : f_n \rightrightarrows f$ and $f' = g$

Proof for 1.

Let $f_n \rightrightarrows f$

Let f_n be continuous

Let $x_0 \in A$

Let $x_n : x_n \to x_0$

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 : \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon > 0$

$$f_n \rightrightarrows f \implies \exists N_1 : \forall n > N_1 : \forall x \in A : |f(x_n) - f_n(x_n)| < \varepsilon_1$$

$$f_n \text{ is continuous} \implies \exists N_2 : \forall n > N_2 : \forall x \in A : |f_n(x_n) - f_n(x_0)| < \varepsilon_2$$

$$f_n \to f \implies \exists N_3 : \forall n > N_3 : \forall x \in A : |f_n(x_0) - f(x_0)| < \varepsilon_3$$

$$\implies |f(x_n) - f(x_0)| = |f(x_n) - f_n(x_n) + f_n(x_n) - f_n(x_0) + f_n(x_0) - f(x_0)| \leq$$
$$\leq |f(x_n) - f_n(x_n)| + |f_n(x_n) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \varepsilon$$
$$\implies \boxed{f \text{ is continuous}}$$