Convergence of integral and series (Integral convergence test) #theorem

$$\text{Let } f \text{ be a continuous, monotonically decreasing and positive function on } [a, \infty)$$

$$\text{Then } \int_a^\infty f(x)\,dx \text{ converges } \iff \sum_{n=a}^{\infty} f(n) \text{ converges}$$

$$\text{Note:}$$

$$\text{Integral and series might not converge to the same limit(!)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ but } \int_1^\infty \frac{1}{x^2}\,dx = 1$$

$$\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-\frac{1}{e}} \text{ but } \int_0^\infty e^{-x}\,dx = 1$$

$$\int_7^\infty \left(1-\frac{1}{x}\right)^{x^2} dx \iff \sum_{n=7}^{\infty} \left(1-\frac{1}{n}\right)^{n^2} \iff \lim_{n\to\infty} \sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} < 1$$

$$\iff \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n} < 1 \iff e^{-1} < 1$$

$$\int_a^\infty f(x)\,dx \text{ converges } \overset{?}{\implies} \lim_{x\to\infty} f(x) = 0$$

$$\text{No}$$

$$\int_a^\infty \sin(x^6)\,dx \text{ converges, but } \lim_{x\to\infty} \sin(x^6) \neq 0$$
