Linear independence of orthogonal sets #lemma

$$S \subseteq V \text{ is orthogonal and } 0 \notin S \implies S \text{ is a linear independence}$$

Coordinates in orthogonal basis #definition

$$\text{Let } V \text{ be an inner product space over } \mathbb{F}$$
$$\text{Let } B \text{ be an orthogonal basis of } V$$
$$\text{Let } v \in V$$
$$\exists\{\alpha_i\}_{i\in[1,n]} : v = \sum_{i=1}^{n} \alpha_i v_i$$
$$\implies \forall i \in [1,n] : \alpha_i = \frac{\langle v, v_i\rangle}{\|v\|^2}$$
$$\text{Or in other words: } [v]_B = \frac{1}{\|v\|^2}\begin{pmatrix}\langle v, v_1\rangle \\ \vdots \\ \langle v, v_n\rangle\end{pmatrix}$$

Proof:

$$\text{Let } \{\alpha_i\}_{i\in[1,n]} : v = \sum_{i=1}^{n} \alpha_i v_i$$
$$\forall i \in [1,n] : \langle v, v_i\rangle = \left\langle \sum_{k=1}^{n} \alpha_k v_k, v_i \right\rangle = \sum_{k=1}^{n} \alpha_k \langle v_k, v_i\rangle = \alpha_i \langle v_i, v_i\rangle = \alpha_i \|v_i\|^2$$
$$\forall i \in [1,n] : v_i \neq 0 \implies \|v_i\| > 0$$
$$\implies \boxed{\forall i \in [1,n] : \alpha_i = \frac{\langle v, v_i\rangle}{\|v_i\|^2}}$$

Pythagorean theorem #theorem

$$\text{Let } B \text{ be an orthogonal basis of } V$$
$$\text{Let } v \in V$$
$$v = \sum_{i=1}^{n} \alpha_i v_i$$
$$\implies \left\| \sum_{i=1}^{n} \alpha_i v_i \right\|^2 = \sum_{i=1}^{n} \|\alpha_i v_i\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \|v_i\|^2$$

Proof:

$$\left\| \sum_{i=1}^{n} \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{i=1}^{n} \alpha_i v_i \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \alpha_i v_i, \alpha_j v_j\rangle =$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \langle v_i, v_j\rangle = \sum_{i=1}^{n} \alpha_i \overline{\alpha_i} \langle v_i, v_i\rangle = \sum_{i=1}^{n} |\alpha_i|^2 \|v_i\|^2$$

Orthogonal complement #definition

Let V be an inner product space over $\mathbb{F}$

Let $S \subseteq V$

Set of vectors that are orthogonal to all vectors in S is then called an orthogonal complement and denoted

$$S^\perp = \{v \in V | \forall s \in S : \langle v, s\rangle = 0\}$$

Properties of orthogonal complements #lemma

Let V be an inner product space over $\mathbb{F}$

$S \subseteq V \implies S^{\perp}$ is a subpspace of V

Proof:

$$\forall s \in S : \langle 0, s \rangle = 0 \implies 0 \in S^{\perp}$$

Let $v, u \in S^{\perp}, \alpha \in \mathbb{F}$

$$\forall s \in S : \langle v + \alpha u, s \rangle = \langle v, s \rangle + \alpha \langle u, s \rangle = 0 + \alpha \cdot 0 = 0$$

$$\implies v + \alpha u \in S^{\perp} \implies \boxed{S^{\perp} \text{ is a subspce of } V}$$

$$S \subseteq (S^{\perp})^{\perp}$$

Proof:

$$(S^{\perp})^{\perp} = \left\{ v \in V \middle| \forall s' \in S^{\perp} : \langle v, s' \rangle = 0 \right\}$$

Let $s \in S$

$$\forall s' \in S^{\perp} : \langle s', s \rangle = 0 \implies \langle s, s' \rangle = 0 \implies s \in (S^{\perp})^{\perp} \implies \boxed{S \subseteq (S^{\perp})^{\perp}}$$

$$A \subseteq B \implies A^{\perp} \supseteq B^{\perp}$$

Proof:

Let $v \in B^{\perp}$

$$\forall b \in B : \langle v, b \rangle = 0 \underset{A \subseteq B}{\implies} \forall a \in A : \langle v, a \rangle = 0 \implies v \in A^{\perp}$$

$$\implies \boxed{B^{\perp} \subseteq A^{\perp}}$$

$$S^{\perp} = (sp(S))^{\perp}$$

Proof:

$$S \subseteq sp(S) \implies (sp(S))^{\perp} \subseteq S^{\perp}$$

Let $v \in S^{\perp}$

Let $u \in sp(S)$

$$u = \sum_{i=1}^{k} \alpha_i s_i$$

$$\langle v, u \rangle = \left\langle v, \sum_{i=1}^{k} \alpha_i s_i \right\rangle = \sum_{i=1}^{k} \overline{\alpha_i} \langle v, s_i \rangle = 0$$

$$\implies v \in (sp(S))^{\perp} \implies S^{\perp} \subseteq (sp(S))^{\perp}$$

$$\implies \boxed{S^{\perp} = (sp(S))^{\perp}}$$

Orthogonal projection #definition

Let V be an inner product space

Let W be a subspace of V

Let $B = \{w_1, \dots, w_k\}$ be an orthogonal basis of W

Let $v \in V$

Then orthogonal projection $P_W(v) = \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i$

Equivalent: Orthogonal projection is a vector such that $\forall w \in W : \|v - w\| \geq \|v - P_W(v)\|$

Properties of orthogonal projection #lemma

$$\forall v \in V : P_W(v) \in W$$

$$v \in W \iff P_W(v) = v$$

Proof:

$\boxed{\implies}$ Let $v \in W$

$$B \text{ is an orthogonal basis} \implies v = \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i$$

$$\implies \boxed{v = P_W(v)}$$

$\boxed{\impliedby}$ Let $P_W(v) = v$

$$P_W(v) \in W \implies \boxed{v \in W}$$

$$P_W(v) = 0 \iff v \in W^\perp$$

Proof:

$\boxed{\implies}$ Let $P_W(v) = 0$

$$B \text{ is a linear independence} \implies \forall w_i \in W : \frac{\langle v, w_i \rangle}{\|w_i\|^2} = 0$$

$$\implies \forall w_i \in B : \langle v, w_i \rangle = 0 \implies \boxed{v \in B^\perp = (sp(B))^\perp = W^\perp}$$

$\boxed{\impliedby}$ Let $v \in W^\perp$

$$\implies \forall w_i \in B : \langle v, w_i \rangle = 0 \implies \boxed{P_W(v) = 0}$$

$$w \in W \iff \forall v \in V : \langle v - P_W(v), w \rangle = 0$$

Or in other words: $v - P_W(v) \in W^\perp$

Proof:

$$w \in W \implies w = \sum_{i=1}^{k} \alpha_i w_i$$

$$\langle v - P_W(v), w \rangle = 0 \iff \langle v, w \rangle = \langle P_W(v), w \rangle$$

$$\langle v, w \rangle = \left\langle v, \sum_{i=1}^{k} \alpha_i w_i \right\rangle = \boxed{\sum_{i=1}^{k} \overline{\alpha_i} \langle v, w_i \rangle}$$

$$\langle P_W(v), w \rangle = \left\langle P_W(v), \sum_{i=1}^{k} \alpha_i w_i \right\rangle = \sum_{i=1}^{k} \beta_i \langle w_i, w \rangle =$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \beta_i \overline{\alpha_j} \langle w_i, w_j \rangle = \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\|w_i\|^2} \overline{\alpha_i} \langle w_i, w_i \rangle = \boxed{\sum_{i=1}^{k} \overline{\alpha_i} \langle v, w_i \rangle}$$

$$\implies \boxed{\langle v, w \rangle = \langle P_W(v), w \rangle}$$