Eigenvalues of linear transformation and representation matrix #lemma

$$\text{Let } V \text{ be a finitely generated vector space over } \mathbb{F}$$
$$\text{Let } B \text{ be a basis of } V$$
$$\text{Let } T: V \to V \text{ be a linear transformation}$$
$$\text{Then } \lambda \text{ is an eigenvalue of } T \iff \lambda \text{ is an eigenvalue of } [T]_B^B$$

$$\text{Proof:}$$
$$\lambda \text{ is an eigenvalue of } T \iff \exists v \neq 0 \in V : T(v) = \lambda v$$
$$\iff \exists v \neq 0 \in V : [T]_B^B [v]_B = [\lambda v]_B = \lambda [v]_B \iff \lambda \text{ is an eigenvalue of } [T]_B^B$$

Cayley-Hamilton theorem #theorem

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\text{Then } P_A(A) = 0$$

$$\text{Explanation, not proof:}$$
$$P_A(A) = \det(AI - A) = \det(0) = 0$$

$$\text{Proof:}$$
$$P_A(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i$$
$$(\lambda I - A) \cdot \operatorname{adj}(\lambda I - A) = \det(\lambda I - A) I = P_A(\lambda) I$$
$$\operatorname{adj}(\lambda I - A) \in \mathbb{F}_{n-1}[\lambda]^{n \times n}$$
$$\implies \exists \{B_0, \dots, B_{n-1}\} \subseteq \mathbb{F}^{n \times n} : \operatorname{adj}(\lambda I - A) = \sum_{i=0}^{n-1} \lambda^i B_i$$
$$\implies (\lambda I - A) \cdot \sum_{i=0}^{n-1} \lambda^i B_i = P_A(\lambda) I$$
$$\implies (\lambda I - A) \cdot \sum_{i=0}^{n-1} \lambda^i B_i = \sum_{i=0}^{n} \lambda^i a_i I$$
$$\implies \begin{array}{c|c|c|c|c|c|c} & \lambda^n & \lambda^{n-1} & \lambda^{n-2} & \dots & \lambda & 1 \\ \text{Left side} & B_{n-1} & B_{n-2} - AB_{n-1} & B_{n-3} - AB_{n-2} & \dots & B_0 - AB_1 & -AB_0 \\ \text{Right side} & I & a_{n-1} I & a_{n-2} I & \dots & a_1 I & a_0 I \end{array}$$
$$\implies A^n B_{n-1} + A^{n-1}(B_{n-2} - AB_{n-1}) + \dots + A(B_0 - AB_1) - AB_0 =$$
$$= A^n + a_{n-1} A^{n-1} + \dots + a_0 I = P_A(A)$$
$$\implies \boxed{0 = P_A(A)}$$

Diagonalization and minimal polynomial #theorem

$$\text{Let } A \in \mathbb{F}^{n\times n}$$

$$A \sim D \iff m_A \text{ is factorizable into distinct linear factors}$$

$$\text{Proof:}$$

$$\boxed{\implies} \text{Let } A \sim D$$

$$\implies P_A(\lambda) = \prod_{i=1}^{k}(\lambda - \alpha_i)^{\mu_A(\alpha_i)}$$

$$\implies m_A(\lambda) = \prod_{i=1}^{k}(\lambda - \alpha_i)^{t_i}, t_i \le \mu_A(\alpha_i)$$

$$A_J = D$$

$$\implies \text{Largest Jordan block corresponding to any eigenvalue of } A \text{ is of size } 1$$

$$\implies \forall i \in [1, k] : t_i = 1 \implies \boxed{m_A(\lambda) = \prod_{i=1}^{k}(\lambda - \alpha_i)}$$

$$\boxed{\impliedby} \text{Let } m_A \text{ be factorizable into distinct linear factors}$$

$$\implies m_A(\lambda) = \prod_{i=1}^{k}(\lambda - \alpha_i)$$

$$\implies P_A(\lambda) = \prod_{i=1}^{k}(\lambda - \alpha_i)^{\mu_A(\alpha_i)} \implies A \sim A_J$$

$$\implies \text{Largest Jordan block corresponding to any eigenvalue of } A \text{ is of size } 1$$

$$\implies A_J = D \implies \boxed{A \sim D}$$

Pythagorean theorem #theorem

$$\text{Let } B \text{ be an orthogonal basis of } V$$

$$\text{Let } v \in V$$

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

$$\implies \left\| \sum_{i=1}^{n} \alpha_i v_i \right\|^2 = \sum_{i=1}^{n} \|\alpha_i v_i\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \|v_i\|^2$$

$$\text{Proof:}$$

$$\left\| \sum_{i=1}^{n} \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{i=1}^{n} \alpha_i v_i \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \alpha_i v_i, \alpha_j v_j \rangle =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \langle v_i, v_j \rangle = \sum_{i=1}^{n} \alpha_i \overline{\alpha_i} \langle v_i, v_i \rangle = \sum_{i=1}^{n} |\alpha_i|^2 \|v_i\|^2$$

Cauchy-Schwarz inequality #lemma

$$\text{Let } V \text{ be an inner product space}$$
$$\text{Let } v, u \in V$$
$$\|v\| \cdot \|u\| \geq |\langle v, u\rangle|$$
$$\|v\| \cdot \|u\| = |\langle v, u\rangle| \iff u = \alpha v$$

$$\text{Proof:}$$
$$\text{Case 1. } u = v = 0$$
$$\underbrace{\|v\|}_{0} \cdot \underbrace{\|u\|}_{0} \geq \underbrace{|\langle v, u\rangle|}_{0}$$
$$\text{Case 2. } \begin{bmatrix} v \neq 0 \\ u \neq 0 \end{bmatrix} \quad \text{Let } v \neq 0$$
$$\text{Let } A = \left\{ \frac{v}{\|v\|} \right\}$$
$$A \text{ is an orthonormal set}$$
$$\text{By Bessel's inequality: } \|u\|^2 \geq \left| \left\langle u, \frac{v}{\|v\|} \right\rangle \right|^2 = \frac{1}{\|v\|^2} |\langle u, v\rangle|^2 \underset{|z|=|\bar{z}|}{=} \frac{1}{\|v\|^2} |\langle v, u\rangle|^2$$
$$\implies \|u\|^2 \cdot \|v\|^2 \geq |\langle v, u\rangle|^2$$
$$\implies \boxed{\|v\| \cdot \|u\| \geq |\langle v, u\rangle|}$$

$$\|v\| \cdot \|u\| = |\langle v, u\rangle| \iff \underbrace{\|u\|^2 = \frac{1}{\|v\|^2} |\langle v, u\rangle|^2}_{\text{Bessel's equality case}}$$
$$\iff u \in spA \iff \boxed{u = \alpha v}$$

Riesz Representation theorem #theorem

$$\text{Let } V \text{ be an inner product space over } \mathbb{F}$$
$$\text{Let } T : V \to \mathbb{F} \text{ be a linear transformation}$$
$$\text{Then } \exists! h \in V : \forall v \in V : T(v) = \langle v, h\rangle$$

$$\text{Proof:}$$
$$\text{Let } B \text{ be an orthonormal basis of } V$$
$$\text{Let } C = \{1\} \text{ be a standard basis of } \mathbb{F}$$
$$T(v) = [T(v)]_C = [T]_C^B [v]_B \in \mathbb{F} \implies T(v) = ([T]_C^B [v]_B)^T = [v]_B^T ([T]_C^B)^T$$
$$\langle v, h\rangle = [v]_B^T G_B \overline{[h]^B} = [v]_B^T \overline{[h]_B}$$
$$[\cdot]_B \text{ is surjective} \implies \boxed{\exists h \in V : [h]_B = \overline{([T]_C^B)^T} = (\overline{[T]_C^B})^T}$$
$$\text{Let } \forall v \in V : \langle v, h_1\rangle = T(v) = \langle v, h_2\rangle$$
$$\implies [v]_B^T \overline{[h_1]_B} = [v]_B^T \overline{[h_2]_B}$$
$$\implies [h_1]_B = [h_2]_B \implies \boxed{h_1 = h_2}$$

Existence and uniqueness of adjoint linear transformation #theorem

$$\text{Let } V, U \text{ be inner product spaces over } \mathbb{F}$$
$$\text{Let } T : V \to U \text{ be a linear transformation}$$
$$\text{Let } S : U \to V \text{ be a linear transformation}$$
$$\text{Then } \exists! S : S \text{ is a adjoint linear transformation of } T$$

$$\text{Proof:}$$
$$\text{Let } u \in U$$
$$\text{Let } K_u : V \to \mathbb{F}, K_u(v) = \langle T(v), u \rangle$$
$$\implies \text{By Riesz theorem } \exists h_u \in V : \forall v \in V : K_u(v) = \langle v, h_u \rangle$$
$$\text{Let } S : U \to V, S(u) = h_u$$
$$\implies \forall v \in V, u \in U : \langle T(v), u \rangle = \langle v, h_u \rangle = \langle v, S(u) \rangle$$

$$\text{Let } v \in V, u_1, u_2 \in U, \alpha \in \mathbb{F}$$
$$\langle v, S(u_1 + \alpha u_2) \rangle = \langle T(v), u_1 + \alpha u_2 \rangle = \langle T(v), u_1 \rangle + \overline{\alpha} \langle T(v), u_2 \rangle = K_{u_1}(v) + \overline{\alpha} K_{u_2}(v) =$$
$$= \langle v, h_{u_1} \rangle + \overline{\alpha} \langle v, h_{u_2} \rangle = \langle v, S(u_1) \rangle + \overline{\alpha} \langle v, S(u_2) \rangle = \langle v, S(u_1) + \alpha S(u_2) \rangle$$
$$\forall v \in V : \langle v, S(u_1 + \alpha u_2) \rangle = \langle v, S(u_1) + \alpha S(u_2) \rangle$$
$$\implies S(u_1 + \alpha u_2) = S(u_1) + \alpha S(u_2) \implies \boxed{S \text{ is a linear transformation}}$$
$$\text{Let } \hat{S} : U \to V, \forall v \in V, u \in U : \langle T(v), u \rangle = \langle v, S(u) \rangle$$
$$\implies \forall v \in V, u \in U : \langle v, S(u) \rangle = \langle v, \hat{S}(u) \rangle \implies \forall u \in U : S(u) = \hat{S}(u)$$
$$\implies \boxed{S = \hat{S}}$$

Representation matrix of adjoint linear transformation #lemma

$$B \text{ orthonormal basis of } V$$
$$C \text{ orthonormal basis of } U$$
$$T : V \to U \text{ is a linear transformation}$$
$$[T^*]_B^C = ([T]_C^B)^*$$

$$\text{Proof:}$$
$$\dim V = n$$
$$\dim U = m$$
$$[T]_C^B \in \mathbb{F}^{m \times n}$$
$$[T^*]_B^C \in \mathbb{F}^{n \times m}$$
$$\forall v \in V, u \in U : \langle T(v), u \rangle = \langle v, T*(u) \rangle$$
$$\langle T(v), u \rangle = [T(v)]_C^T G_C \overline{[u]_C} = [T(v)]_C^T \overline{[u]_C} = [v]_B^T ([T]_C^B)^T \overline{[u]_C}$$
$$\langle v, T^*(u) \rangle = [v]_B^T G_B \overline{[T^*(u)]_B} = [v]_B^T \overline{[T^*]_B^C [u]_C} = [v]_B^T \overline{[T^*]_B^C} \overline{[u]_C}$$
$$\implies ([T]_C^B)^T = \overline{[T^*]_B^C} \implies \boxed{([T]_C^B)^* = [T^*]_B^C}$$

Gram-Schmidt matrix of two bases #lemma

$$\text{Let } B, \hat{B} \text{ be bases of } V$$
$$\text{Let } C = [I]_B^{\hat{B}}$$
$$\text{Then } G_{\hat{B}} = C^T G_B \overline{C}$$

$$\text{Proof:}$$
$$\langle v, u \rangle = [v]_B^T G_B \overline{[u]_B} = ([I]_B^{\hat{B}} [v]_{\hat{B}})^T G_B \overline{[I]_B^{\hat{B}} [u]_{\hat{B}}} = [v]_{\hat{B}}^T \cdot C^T G_B \overline{C} \cdot \overline{[u]_{\hat{B}}}$$
$$= \langle v, u \rangle = [v]_{\hat{B}}^T G_{\hat{B}} \overline{[u]_{\hat{B}}}$$
$$\implies [v]_{\hat{B}}^T \cdot C^T G_B \overline{C} \cdot \overline{[u]_{\hat{B}}} = [v]_{\hat{B}}^T \cdot G_{\hat{B}} \cdot \overline{[u]_{\hat{B}}}$$
$$\text{Let } \hat{B} = \{v_1, \ldots, v_n\}$$
$$\forall i, j \in [1, n] : [v_i]_{\hat{B}}^T \cdot C^T G_B \overline{C} \cdot \overline{[v_j]_{\hat{B}}} = [v_i]_{\hat{B}}^T \cdot G_{\hat{B}} \cdot \overline{[v_j]_{\hat{B}}}$$
$$\implies \forall i, j \in [1, n] : e_i^T \cdot C^T G_B \overline{C} \cdot e_j = e_i^T \cdot G_{\hat{B}} \cdot e_j$$
$$\implies \forall i, j \in [1, n] : (C^T G_B \overline{C})_{ij} = (G_{\hat{B}})_{ij}$$
$$\implies \boxed{C^T G_B \overline{C} = G_{\hat{B}}}$$

Polar norm equations #lemma

$$\text{Re}(\langle v, u \rangle) = \frac{1}{2}(\|v + u\|^2 - \|v\|^2 - \|u\|^2)$$
$$\text{Im}(\langle v, u \rangle) = i\frac{1}{2}(\|v + iu\|^2 - \|v\|^2 - \|u\|^2)$$

Normal linear operator criterion #theorem

$$\text{Let } V \text{ be a finitely generated inner product space over } \mathbb{F}$$
$$\text{Let } B \text{ be an orthonormal basis of } V$$
$$\text{Let } T : V \to V \text{ be a linear operator}$$
$$\text{Then } T \text{ is normal } \iff \forall v \in V : \|T(v)\| = \|T^*(v)\|$$

$$\text{Proof:}$$
$$\boxed{\implies} \text{ Let } T \text{ be normal}$$
$$TT^* = T^*T$$
$$\text{Let } v \in V$$
$$\|T(v)\| = \sqrt{\langle Tv, Tv \rangle} = \sqrt{\langle v, T^*Tv \rangle}$$
$$\|T^*(v)\| = \sqrt{\langle T^*v, T^*v \rangle} = \sqrt{\langle v, TT^*v \rangle} = \sqrt{\langle v, T^*Tv \rangle} = \|T(v)\|$$
$$\implies \boxed{\forall v \in V : \|T^*(v)\| = \|T(v)\|}$$

$$\boxed{\impliedby} \text{ Let } \forall v \in V : \|T(v)\| = \|T^*(v)\|$$
$$\text{Let } v, u \in V$$
$$\langle u, TT^*v \rangle = \langle T^*u, T^*v \rangle =$$
$$= \frac{1}{2}(\|T^*u + T^*v\|^2 - \|T^*u\|^2 - \|T^*v\|^2) + i\frac{1}{2}(\|T^*u + iT^*v\|^2 - \|T^*u\|^2 - \|T^*v\|^2)$$
$$\langle u, T^*Tv \rangle = \langle Tu, Tv \rangle =$$
$$= \frac{1}{2}(\|Tu + Tv\|^2 - \|Tu\|^2 - \|Tv\|^2) + i\frac{1}{2}(\|Tu + iTv\|^2 - \|Tu\|^2 - \|Tv\|^2)$$
$$\implies \langle u, TT^*v \rangle - \langle u, T^*Tv \rangle =$$
$$= \frac{1}{2}(\|T^*u + T^*v\|^2 - \|Tu + Tv\|^2) + i\frac{1}{2}(\|T^*u + iT^*v\|^2 - \|Tu + iTv\|^2) =$$
$$= \frac{1}{2}(\|T^*(u + v)\|^2 - \|T(u + v)\|^2) + i\frac{1}{2}(\|T^*(u + iv)\|^2 - \|T(u + iv)\|^2) = 0 - 0i = 0$$
$$\implies \forall v, u \in V : \langle u, TT^*v \rangle = \langle u, T^*Tv \rangle \implies \boxed{TT^* = T^*T}$$