## **Conjugate linear operator**

$$egin{aligned} T:V &
ightarrow W \ T^*:W &
ightarrow V \ orall v \in V, w \in W: \langle Tv,w 
angle = \langle v,T^*w 
angle \end{aligned}$$

## **Properties**

1. 
$$T^*$$
 is a linear operator and unique

$$2. I^* = I$$

3. 
$$(T+S)^* = T^* + S^*$$

$$4. (T^*)^* = T$$

5. 
$$B, C$$
 orthonormal bases  $\Longrightarrow [T^*]_B^C = ([T]_C^B)^*$ 

6. 
$$(ST)^* = T^*S^*$$

Proofs go as follows:

$$orall v \in V: \langle v,v 
angle = \langle Iv,v 
angle = \langle v,I^*v 
angle \implies I^* = I$$

3.

$$egin{aligned} orall v \in V : \langle v, (T+S)^*w 
angle = \langle (T+S)v, w 
angle = \langle Tv, w 
angle + \langle Sv, w 
angle = \langle v, T^*w 
angle + \langle v, S^*w 
angle = \\ = \langle v, (T^*+S^*)w 
angle \implies (T+S)^* = T^* + S^* \end{aligned}$$

$$egin{aligned} orall w \in W: \langle Tv, w 
angle = \langle v, T^*w 
angle = \overline{\langle T^*w, v 
angle} = \overline{\langle w, (T^*)^*v 
angle} = \langle (T^*)^*v, w 
angle \ \Longrightarrow \ (T^*)^* = T \end{aligned}$$

5. and 6. in lecture 10

 $V=\mathbb{R}^2$  with inner product  $\langle v,u
angle=2v_1u_1+v_2u_2$   $W=\mathbb{R}^2$  with standrad inner product

$$T:V o W \ Tinom{x}{y}=inom{2x+3y}{2y}$$
 Find  $T^*$ 

Solution:

$$\operatorname{Let} B = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\operatorname{Let} C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$[T]_{C}^{B} = \begin{pmatrix} \sqrt{2} & 3 \\ 0 & 2 \end{pmatrix}$$

$$\implies [T^{*}]_{B}^{C} = ([T]_{C}^{B})^{*} = ([T]_{C}^{B})^{T} = \begin{pmatrix} \sqrt{2} & 0 \\ 3 & 2 \end{pmatrix}$$

$$[T^{*} \begin{pmatrix} 1 \\ 0 \end{pmatrix}]_{B} = \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix} \implies T^{*} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$[T^{*} \begin{pmatrix} 0 \\ 1 \end{pmatrix}]_{B} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \implies T^{*} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\implies T^{*} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x + 2y \end{pmatrix}$$

Alternative solution:

$$egin{aligned} [T^*egin{pmatrix} x \ y \end{bmatrix}]_B &= [T^*]_B^C [egin{pmatrix} x \ y \end{bmatrix}]_C &= egin{pmatrix} \sqrt{2} & 0 \ 3 & 2 \end{pmatrix} egin{pmatrix} x \ y \end{pmatrix} &= egin{pmatrix} \sqrt{2} \cdot x \ 3x + 2y \end{pmatrix} \end{aligned} \ \implies egin{pmatrix} T^*egin{pmatrix} x \ y \end{pmatrix} &= egin{pmatrix} x \ 3x + 2y \end{pmatrix} \end{aligned}$$

Let V, W be inner product spaces over  $\mathbb{F}$ 

Let B be an orthonormal basis of V

Let 
$$T:V \to W$$

$$\text{Prove:} \, \forall w \in W : T^*w = \sum_{i=1}^n \overline{\langle Tv_i, w \rangle} v_i$$

**Proof:** 

Let 
$$w \in W$$

Let 
$$v \in V$$

$$egin{aligned} v &= \sum_{i=1}^n lpha_i v_i \left( = \sum_{i=1}^n \langle v, v_i 
angle v_i 
ight) \ \left\langle v, \sum_{i=1}^n \overline{\langle T v_i, w 
angle} v_i 
ight
angle = \left\langle \sum_{i=1}^n lpha_i v_i, \sum_{i=1}^n \overline{\langle T v_i, w 
angle} v_i 
ight
angle = \ &= \sum_{i=1}^n lpha_i \langle T v_i, w 
angle \langle v_i, v_i 
angle = \sum_{i=1}^n lpha_i \langle T v_i, w 
angle = \left\langle \sum_{i=1}^n lpha_i T v_i, w 
ight
angle = \ &= \left\langle \sum_{i=1}^n T (lpha_i v_i), w 
ight
angle = \left\langle T \left( \sum_{i=1}^n lpha_i v_i 
ight), w 
ight
angle = \langle T v, w 
angle \end{aligned}$$

Alternative solution:

$$\left\langle v, \sum_{i=1}^n \overline{\langle Tv_i, w \rangle} v_i \right\rangle = \sum_{i=1}^n \langle Tv_i, w \rangle \langle v, v_i \rangle = \sum_{i=1}^n \langle \langle v, v_i \rangle Tv_i, w \rangle = \\ = \sum_{i=1}^n \langle T(\langle v, v_i \rangle v_i), w \rangle = \left\langle \sum_{i=1}^n T(\langle v, v_i \rangle v_i), w \right\rangle = \left\langle T \left( \sum_{i=1}^n \langle v, v_i \rangle v_i \right), w \right\rangle = \langle Tv, w \rangle \\ \text{Because of: } v = \sum_{i=1}^n \langle v, v_i \rangle v_i \\ \Longrightarrow \left\langle Tv, w \right\rangle = \left\langle v, \sum_{i=1}^n \overline{\langle Tv_i, w \rangle} v_i \right\rangle = \langle v, T^*w \rangle \\ \Longrightarrow \left[ T^*w = \sum_{i=1}^n \overline{\langle Tv_i, w \rangle} v_i \right]$$

Let  $V=\mathbb{R}^{2 imes 2}$  with standard inner product  $\langle A,B
angle=tr(AB^*)$ 

Let 
$$W=\mathbb{R}^2$$
 with inner product  $\langle v,w
angle=v^Tinom{1}{-1}-1 \choose -1$   $2$   $w$  Let  $T:V o W$   $T(A)=C_1(A)+C_2(A)$  Find  $T^*$ 

Solution:

Let B be a standard basis of V

$$T^*(w) = \sum_{i=1}^4 \overline{\langle TE_i,w
angle} E_i \ orall i \in [1,4:] T(E_i) = C_1(E_i) + C_2(E_i) = egin{cases} \left(egin{array}{c} 1 \ 0 \end{pmatrix} & i \leq 2 \ \left(egin{array}{c} 0 \ 1 \end{pmatrix} & i > 2 \end{cases} \ i = 1,2 \implies \langle TE_i,w
angle = \langle inom{1}{0},inom{x}{y}
angle = (1 \quad 0) inom{1 \quad -1}{-1 \quad 2} inom{x}{y} = x-y \ i = 3,4 \implies \langle TE_i,w
angle = (0 \quad 1) inom{1 \quad -1}{-1 \quad 2} inom{x}{y} = -x+2y \ \implies \hline T^*(w) = (x-y)(E_1+E_2) + (-x+2y)(E_3+E_4) = inom{x-y \quad x-y}{-x+2y \quad -x+2y} \ \end{cases}$$

Let W be T-invariant

Prove or disprove:

- 1.  $W^{\perp}$  is T-invariant
- 2.  $W^{\perp}$  is  $T^*$ -invariant

Disproof for 1:

$$Tegin{pmatrix} 0 \ 1 \end{pmatrix} = egin{pmatrix} 0 \ 1 \end{pmatrix}$$

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$W=sp\left\{egin{pmatrix}0\1\end{pmatrix}
ight\}$$

Proof for 2:

$$T[W]\subseteq W$$

Let 
$$v \in W^\perp$$

Let 
$$w \in W$$

$$\langle w,v
angle =0 \ 0 \mathop{=}\limits_{Tw\in W} \langle Tw,v
angle =\langle w,T^*v
angle$$

$$\implies T^*v \in W^\perp \implies T^*[W^\perp] \subseteq W^\perp$$