1

Formulate and prove Cauchy-Schwarz inequality

2

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Find SVD decomposition of A

Solution:

$$A \in \mathbb{R}^{3\times 2}$$

$$A^* = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A^*A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}$$

$$P_{A^*A}(x) = \begin{vmatrix} x-6 & -5 \\ -5 & x-6 \end{vmatrix} = (x-6)^2 - 25 = (x-11)(x-1)$$

$$x = 1 \implies E_1 = sp\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = sp\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right\}$$

$$x = 11 \implies E_{11} = sp\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = sp\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right\}$$

$$\implies \boxed{V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

$$\implies A^*A = V\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}V^*$$

$$\boxed{\Sigma = \begin{pmatrix} \sqrt{11} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{pmatrix}}$$

$$w_1 = Av_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{pmatrix} \implies \tilde{w_1} = \frac{w_1}{\|w_1\|} = \begin{pmatrix} \frac{3}{\sqrt{22}} \\ \frac{3}{\sqrt{22}} \\ \frac{2}{\sqrt{22}} \end{pmatrix}$$

$$w_2 = Av_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \implies \tilde{w_2} = \frac{w_2}{\|w_2\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$w_3 = 11e_3 - \frac{\langle 11e_3, w_1\rangle}{\|w_1\|^2}w_1 - \frac{\langle 11e_3, w_2\rangle}{\|w_2\|^2}w_2 = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 9 \end{pmatrix}$$

$$\implies \tilde{w_3} = \frac{w_3}{\|w_3\|} = \begin{pmatrix} -\frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix}$$

$$\implies \boxed{U = \begin{pmatrix} \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{22}} & 0 & \frac{3}{\sqrt{11}} \end{pmatrix}}$$

$$\implies A = \underbrace{\begin{pmatrix} \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{22}} & 0 & \frac{3}{\sqrt{11}} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{11} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{V^*}$$

3a

$$\text{Let } A \in \mathbb{R}^{4\times 4}$$
$$\text{Let } (A^2+1I)(A^2-9I)=0$$
$$\text{Let } rank(iI-A)=3$$
$$\text{Find all possible Jordan forms of } A$$

Solution:

$$(A^2+1)(A^2-9)=0$$
$$\implies (A-iI)(A+iI)(A-3I)(A+3I)=0$$
$$\implies m_A(x) \mid (x^2+1)(x^2-9)$$
$$\implies \text{Eigenvalues of } A \text{ can only be } \{i,-i,3,-3\}$$
$$\text{And all their geometric multiplicities can only be 1}$$
$$rank(iI-A)=3 \implies i \text{ is an eigenvalue of } A \text{ and } g_i=1$$
$$A \in \mathbb{R}^{4\times 4} \implies -i \text{ is also an eigenvalue of } A, \text{ with the same geometric multiplicity}$$
$$\implies g_i = g_{-i} = 1$$
$$\implies A \text{ has two more eigenvalues which are either 3 or -3}$$
$$\implies J_A = J_1(i)\oplus J_1(-i)\oplus \begin{bmatrix} J_1(3) \\ J_1(-3)\end{bmatrix} \oplus \begin{bmatrix} J_1(3) \\ J_1(-3)\end{bmatrix}$$
$$\text{In other words: } J_A = \begin{pmatrix} i & & & \\ & -i & & \\ & & 3 & \\ & & & 3\end{pmatrix} \text{ or } \underbrace{\begin{pmatrix} i & & & \\ & -i & & \\ & & 3 & \\ & & & -3\end{pmatrix}}_{\text{Order of 3, -3 doesn't matter}} \text{ or } \begin{pmatrix} i & & & \\ & -i & & \\ & & -3 & \\ & & & -3\end{pmatrix}$$

3b

$$\text{Let } \left\langle \begin{pmatrix} x_1 \\ y_1\end{pmatrix}, \begin{pmatrix} x_2 \\ y_2\end{pmatrix}\right\rangle = 2x_1x_2+4y_1y_2 \text{ be an inner product on } \mathbb{R}^2$$
$$\text{Let } T:\mathbb{R}^2\to\mathbb{R}^2 \text{ be a linear operator}$$
$$T\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} 3x+4y \\ 2x-y\end{pmatrix}$$
$$\text{Find explicitly } T^*$$

Solution:

$$\langle e_1,e_1\rangle = 2$$
$$\langle e_2,e_2\rangle = 4$$
$$\langle e_1,e_2\rangle = \langle \begin{pmatrix} 1 \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ 1\end{pmatrix}\rangle = 0$$
$$\implies B=\left\{\frac{e_1}{\sqrt{2}}, \frac{e_2}{2}\right\} \text{ is an orhonormal basis of } \mathbb{R}^2$$
$$T\left(\frac{e_1}{\sqrt{2}}\right) = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{2}{\sqrt{2}}\end{pmatrix} \implies [T(v_1)]_B = \begin{pmatrix} 3 \\ 2\sqrt{2}\end{pmatrix}$$
$$T\left(\frac{e_2}{2}\right) = \begin{pmatrix} 2 \\ -\frac{1}{2}\end{pmatrix} \implies [T(v_2)]_B = \begin{pmatrix} 2\sqrt{2} \\ -1\end{pmatrix}$$
$$\implies [T]_B^B = \begin{pmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & -1\end{pmatrix} \implies [T^*]_B^B = ([T]_B^B)^* = \begin{pmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & -1\end{pmatrix} = [T]_B^B$$
$$\implies T^* = T \implies T^*\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} 3x+4y \\ 2x-y\end{pmatrix}$$

3c

Let $B \in \mathbb{R}^{n\times n}$ be symmetric and all its eigenvalues are real and non-negative

Prove: $\exists C$ symmetric: $B = C^2$

Solution:

B is symmetric $\implies$ B is orthogonal diagonalizable

$$\implies \exists P : B = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^T$$

$$\text{Let } C = P \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} P^T$$

$$C^T = \left(P \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} P^T \right)^T = P \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} P^T = C$$

$$\implies C^2 = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^T = B$$

4a

Let $A \in \mathbb{R}^{n\times n}$

Let $A^3 = 0, A^2 \neq 0$

Prove or disprove: $rank(A) \leq \dfrac{2n}{3}$

Proof:

$A^3 = 0 \implies A$ is nilpotent $\implies$ All eigenvalues of A are 0

$A^2 \neq 0 \implies m_A(x) \neq x^2 \implies m_A(x) = x^3$

$\implies$ Maximal size of one Jordan block is 3

$\implies$ In the Jordan form, there are at least $\dfrac{n}{3}$ Jordan blocks

$$\implies \dim N(A) = k_0 \geq \frac{n}{3} \implies n - rank(A) \geq \frac{n}{3} \implies \boxed{rank(A) \leq \frac{2n}{3}}$$

4b

Let $A \in \mathbb{F}^{n\times n}$

Prove or disprove: A is unitary $\iff \forall\lambda : |\lambda| = 1$

Disproof:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$\lambda = 1 \implies \forall\lambda : |\lambda| = 1$

Columns of A do not form an orthonormal basis $\implies A$ is not unitary

4c

Let $A \in \mathbb{F}^{n\times n}$

Prove or disprove: A is unitary $\iff$ A is normal and $\forall \lambda : |\lambda| = 1$

Proof:

$\boxed{\implies}$ This direction is trivial

$A^*A = I = AA^* \implies A$ is normal

Let λ be an eigenvalue of A with eigenvector v

$$\|Av\|^2 = \langle Av, Av\rangle = |\lambda|^2\langle v, v\rangle = |\lambda|^2 \cdot \|v\|^2$$

$$\|Av\|^2 = \|v\|^2 \implies |\lambda|^2 = 1 \implies |\lambda| = 1$$

$\boxed{\impliedby}$ Let A be normal and $\forall \lambda : |\lambda| = 1$

A is normal and $P_A(x)$ is factorizable into linear factors over $\mathbb{C}$

$\implies A$ is unitary diagonalizable

$$\implies \exists P : A = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^*$$

$$\implies A^* = P \begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix} P^*$$

$$\implies AA^* = P \begin{pmatrix} \lambda_1\overline{\lambda_1} & & \\ & \ddots & \\ & & \lambda_n\overline{\lambda_n} \end{pmatrix} P^* = PIP^* = I \implies \boxed{A \text{ is unitary}}$$

5a

Let $A \in \mathbb{R}^{n\times m}$

Prove: $N(AA^T) = N(A^T)$

Proof:

$$v \in N(A^T) \implies A^T v = 0 \implies AA^T v = 0 \implies N(A^T) \subseteq N(AA^T)$$

Let $v \in N(AA^T)$

$$\implies AA^T v = 0 \implies \langle AA^T v, v\rangle = 0$$

$$\langle AA^T v, v\rangle = [(AA^T v)^T]_S G_S [v]_S = (AA^T v)^T v = v^T AA^T v = (A^T v)^T A^T v = \langle A^T v, A^T v\rangle$$

$$\implies \langle A^T v, A^T v\rangle = 0$$

$$\implies A^T v = 0 \implies v \in N(A^T) \implies N(AA^T) \subseteq N(A^T)$$

$$\implies \boxed{N(AA^T) = N(A^T)}$$

5b

$$\text{Let } A \in \mathbb{R}^{n\times m}$$
$$\text{Prove: } N(AA^T) = (C(A))^\perp$$

$$\text{Proof:}$$
$$\boxed{\subseteq} N(AA^T) = N(A^T)$$
$$\text{Let } v \in N(A^T) \implies A^T v = 0$$
$$\text{Let } u \in \mathbb{R}^m \implies Au \in C(A)$$
$$\langle Au, v\rangle = (Au)^T v = u^T A^T v = u^T 0 = 0$$
$$\implies v \in (C(A))^\perp \implies N(AA^T) = N(A^T) \subseteq (C(A))^\perp$$

$$\boxed{\supseteq} \text{ Let } v \in (C(A))^\perp$$
$$\text{Let } u \in \mathbb{R}^m$$
$$Au \in C(A) \implies \langle Au, v\rangle = (Au)^T v = u^T A^T v = 0$$
$$\forall u \in \mathbb{R}^m : u^T A^T v = 0 \implies \forall i \in [1,m] : e_i^T A^T v = 0 \implies \forall i \in [1,m] : (A^T v)_i = 0$$
$$\implies A^T v = 0 \implies v \in N(A^T)$$
$$\implies (C(A))^\perp \subseteq N(A^T) = N(AA^T)$$
$$\implies \boxed{N(AA^T) = (C(A))^\perp}$$

5c

$$\text{Let } A \in \mathbb{R}^{n\times m}$$
$$\text{Let } n > m$$
$$\text{Prove: 0 is an eigenvalue of } AA^T \text{ with } \gamma_{AA^T}(0) = n - rankA$$

$$\text{Proof:}$$
$$n > m \implies rankA \leq m < n$$
$$AA^T \in \mathbb{R}^{n\times n}$$
$$\implies rank(AA^T) \leq rank(A) < n \implies \boxed{\text{0 is an eigenvalue of } AA^T}$$
$$\boxed{\gamma_{AA^T}(0) = \dim N(AA^T) = \dim N(A^T) = n - rankA^T = n - rankA}$$

5d

$$\text{Let } A \in \mathbb{R}^{n\times m}$$
$$\text{Let } \{C_1(A), \dots, C_m(A)\} \text{ be an orthonormal set}$$

1

$$\text{What can we say about } A^T A?$$

$$\text{Solution:}$$
$$A^T A = \begin{pmatrix} | & | & & | \\ A^T C_1(A) & A^T C_2(A) & \dots & A^T C_m(A) \\ | & | & & | \end{pmatrix}$$
$$(A^T A)_{ij} = R_i(A^T) \cdot C_j(A) = (C_i(A))^T \cdot C_j(A) = \langle C_i(A), C_j(A)\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$
$$\implies \boxed{A^T A = I}$$

2

$$\text{Prove: } \forall v \in \mathbb{R}^n : AA^T v - v \in N(AA^T)$$

$$\text{Proof:}$$
$$AA^T(AA^T v - v) = AA^T AA^T v - AA^T v = AIA^T v - AA^T v = AA^T v - AA^T v = 0$$
$$\implies \boxed{AA^T v - v \in N(AA^T)}$$

3

$$\text{Prove: } \forall v \in \mathbb{R}^n : AA^Tv \in C(A)$$

$$\text{Prove: } \forall u \in C(A) : \|AA^Tv - v\| \leq \|u - v\|$$

Proof:

$$\text{Let } v \in \mathbb{R}^n$$

$$AA^Tv = A\underbrace{(A^Tv)}_{w\in\mathbb{R}^m} = Aw \in C(A)$$

$$\text{Let } u \in C(A)$$

$$AA^Tv - v \in N(AA^T) = (C(A))^\perp$$

$$AA^Tv \in C(A) \implies u - AA^Tv \in C(A)$$

$$\implies u - AA^Tv \perp AA^Tv - v$$

$$\implies \|u - v\|^2 = \|u - AA^Tv + AA^Tv - v\|^2 \overset{\text{By Pythagorean theorem}}{=}$$

$$= \|u - AA^Tv\|^2 + \|AA^Tv - v\|^2 \geq \|AA^Tv - v\|^2$$

$$\implies \boxed{\|u - v\| \geq \|AA^Tv - v\|}$$