1a

$$\int \frac{dx}{x\ln(x)\ln(\ln(x))}$$

Solution:

$$\int \frac{dx}{x\ln(x)\ln(\ln(x))} = \begin{Bmatrix} t = \ln(x) \\ dt = \frac{dx}{x} \end{Bmatrix} = \int \frac{dt}{t\ln(t)} = \begin{Bmatrix} u = \ln(t) \\ du = \frac{dt}{t} \end{Bmatrix} = \int \frac{du}{u} = \ln|u| = \ln|\ln t| =$$
$$= \ln|\ln\ln x| + C$$

1b

$$\int \frac{dx}{x(\ln x - \ln^2(x))}$$

Solution:

$$\int \frac{dx}{x(\ln x - \ln^2(x))} = \begin{Bmatrix} t = \ln(x) \\ dt = \frac{dx}{x} \end{Bmatrix} = \int \frac{dt}{t - t^2} = \int \frac{1}{t} + \frac{1}{1-t}\, dt =$$
$$= \ln|t| - \ln|1-t| = \ln|\ln x| - \ln|1 - \ln x| + C$$

2a

$$\text{Prove or disprove: } \forall x \in [0,\infty) : f(x) \geq 0 \text{ and } \int_0^\infty f(x)\,dx \text{ converges}$$
$$\implies f \text{ is bounded on } [1,\infty)$$

Disproof:

$$\text{Let } f \text{ be a function of triangles of height } n \text{ and base } \frac{1}{n^3}$$
$$\implies \int_0^\infty f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$
$$\forall x \in [0,\infty) : f(x) \geq 0$$
$$\text{But } f \text{ is not bounded}$$

2b

$$\text{Prove or disprove: } \int_1^\infty f(x)\,dx \text{ converges and } f \text{ is bounded on } [1,\infty)$$
$$\implies \lim_{x\to\infty} f(x) = 0$$

Proof:

$$\text{Let } \lim_{x\to\infty} f(x) = L > 0$$
$$\implies \forall\varepsilon > 0 : \exists x_0 : \forall x > x_0 : |f(x) - L| < \varepsilon$$
$$\text{Let } \varepsilon = \frac{L}{2}$$
$$\implies \forall x > x_0 : \frac{L}{2} \le f(x) \le \frac{3L}{2}$$
$$\implies \int_1^\infty f(x)\,dx = \int_1^{x_0+1} f(x)\,dx + \int_{x_0+1}^\infty f(x)\,dx \ge \int_1^{x_0+1} f(x)\,dx + \underbrace{\int_{x_0+1}^\infty \frac{L}{2}\,dx}_{=\infty}$$
$$f \text{ is bounded on } [1,\infty) \implies \exists M : \forall x \ge 1 : |f(x)| \le M$$
$$\implies \left|\int_1^{x_0+1} f(x)\,dx\right| \le \int_1^{x_0+1} |f(x)|\,dx \le Mx_0 \in \mathbb{R}$$
$$\implies \int_1^{x_0+1} f(x)\,dx \text{ converges}$$
$$\implies \int_1^\infty f(x)\,dx \text{ diverges} - \text{Contradiction!}$$

$$\text{Let } \lim_{x\to\infty} f(x) = L < 0$$
$$\implies \forall\varepsilon > 0 : \exists x_0 : \forall x > x_0 : |f(x) - L| < \varepsilon$$
$$\text{Let } \varepsilon = -\frac{L}{2}$$
$$\implies \forall x > x_0 : \frac{3L}{2} \le f(x) \le \frac{L}{2}$$
$$\implies \int_1^\infty f(x)\,dx = \int_1^{x_0+1} f(x)\,dx + \int_{x_0+1}^\infty f(x)\,dx \le \int_1^{x_0+1} f(x)\,dx + \underbrace{\int_{x_0+1}^\infty \frac{L}{2}\,dx}_{=-\infty}$$
$$f \text{ is bounded on } [1,\infty) \implies \exists M : \forall x \ge 1 : |f(x)| \le M$$
$$\implies \left|\int_1^{x_0+1} f(x)\,dx\right| \le \int_1^{x_0+1} |f(x)|\,dx \le Mx_0 \in \mathbb{R}$$
$$\implies \int_1^{x_0+1} f(x)\,dx \text{ converges}$$
$$\implies \int_1^\infty f(x)\,dx \text{ diverges} - \text{Contradiction!}$$
$$\implies \boxed{\lim_{x\to\infty} f(x) = 0}$$

3a

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{n}{n^2+k^2}$$

Solution:

$$\text{Let } x_k=\frac{k}{n}$$

$$\implies \Delta x_k=\frac{1}{n}$$

$$\sum_{k=0}^{n}\frac{n}{n^2+k^2}=\sum_{k=0}^{n}\frac{1}{n}\frac{1}{1+\frac{k^2}{n^2}}$$

$$\text{Let } f(x)=\frac{1}{1+x^2}$$

$$\implies \sum_{k=0}^{n}\frac{n}{n^2+k^2}=\sum_{k=0}^{n}f(x_k)\cdot\Delta x_k$$

$$f(x)=\frac{1}{1+x^2}\text{ is continuous and bounded on }[0,1]$$

$$\implies f\text{ is integrable on }[0,1]$$

$$\implies \int_0^1 f(x)\,dx=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{n}{n^2+k^2}=\int_0^1\frac{1}{1+x^2}\,dx=\arctan(1)-\arctan(0)=\frac{\pi}{4}$$

3b

$$\text{Find the length of graph of } f(x)=\ln(\sin x)\text{ on }\left[\frac{\pi}{3},\frac{\pi}{2}\right]$$

$$\text{You can use: } L(f)=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

Solution:

$$f'(x)=(\ln(\sin x))'=\frac{\cos x}{\sin x}$$

$$\implies L(f)=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sqrt{1+\frac{\cos^2 x}{\sin^2 x}}\,dx$$

$$\sqrt{1+\frac{\cos^2 x}{\sin^2 x}}=\sqrt{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}=\sqrt{\frac{1}{\sin^2 x}}=\frac{1}{\sin x}$$

$$\tan^2 x=\frac{\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}-1\implies\cos^2 x=\frac{1}{1+\tan^2 x}$$

$$\sin x=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)=2\tan\left(\frac{x}{2}\right)\cdot\cos^2\left(\frac{x}{2}\right)=\frac{2t}{1+t^2}$$

$$\left(\tan\left(\frac{x}{2}\right)\right)'=\frac{1}{2}\frac{1}{\cos^2\left(\frac{x}{2}\right)}=\frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)$$

$$\implies L(f)=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{\sin x}\,dx=\left\{\begin{array}{c}t=\tan\frac{x}{2}\\ dt=\frac{1+t^2}{2}dx\\ \sin x=\frac{2t}{1+t^2}\end{array}\right\}=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{t}\,dt=$$

$$=\ln\left|\tan\frac{x}{2}\right|\Bigg|_{x=\frac{\pi}{3}}^{x=\frac{\pi}{2}}=\ln\left|\tan\left(\frac{\pi}{4}\right)\right|-\ln\left|\tan\left(\frac{\pi}{6}\right)\right|=-\ln\left|\frac{1}{\sqrt{3}}\right|=\frac{\ln 3}{2}$$

4

$$\sum_{n=0}^{\infty} \frac{1}{16^n(4n+1)}$$

Solution:

$$x \in (-1,1) \implies \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\implies \frac{1}{1-x^4} = \sum_{n=0}^{\infty} x^{4n} \implies \sum_{n=0}^{\infty} \int_0^x t^{4n}\,dt = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1} = \int_0^x \frac{1}{1-t^4}\,dt =$$

$$= \int_0^x \frac{1}{(1-t)(1+t)(1+t^2)}\,dt$$

$$\frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1+x^2}$$

$$A(1+x)(1+x^2) + B(1-x)(1+x^2) + C(1-x)(1+x) = 1$$

$$\implies \begin{cases} A+B+C=1 \\ A-B=0 \\ A+B-C=0 \\ A-B=0 \end{cases} \implies \begin{cases} A=\frac{1}{4} \\ B=\frac{1}{4} \\ C=\frac{1}{2} \end{cases}$$

$$\implies \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1} = \frac{1}{4} \int_0^x \frac{1}{1-t} + \frac{1}{1+t} + \frac{2}{(1+t^2)}\,dt =$$

$$= \frac{1}{4}(-\ln|1-x| + \ln|1+x| + 2\arctan(x))$$

$$\text{Let } x = \frac{1}{2}$$

$$\implies \sum_{n=0}^{\infty} \frac{1}{2 \cdot 16^n(4n+1)} = \frac{1}{4}\left(-\ln\frac{1}{2} + \ln\frac{3}{2} + 2\arctan\left(\frac{1}{2}\right)\right)$$

$$\implies \sum_{n=0}^{\infty} \frac{1}{16^n(4n+1)} = \frac{1}{2}\left(\ln 3 + 2\arctan\frac{1}{2}\right)$$

5

$$\forall n \in \mathbb{N}$$

Find global extremums of $f(x,y) = x^n + y^n$

Limited by $x^2 + y^2 \leq 1$

Solution:

$$f_x = nx^{n-1} = 0 \iff n \neq 1, x = 0$$

$$f_y = ny^{n-1} = 0 \iff n \neq 1, y = 0$$

$\implies$ The only critical point inside the limits is $(0,0)$

Now we will find critical points on the border:

Let $g(x,y) = x^2 + y^2 - 1$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 0 \end{cases} \implies \begin{cases} nx^{n-1} = 2\lambda x \\ ny^{n-1} = 2\lambda y \\ x^2 + y^2 - 1 = 0 \end{cases}$$

$$n = 1 \implies \begin{cases} 2\lambda x = 1 \\ 2\lambda y = 1 \\ x^2 + y^2 - 1 = 0 \end{cases} \implies \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{2\lambda} \\ \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 = 0 \end{cases} \implies \frac{1}{2\lambda^2} = 1$$

$$\implies \lambda = \pm\sqrt{\frac{1}{2}}$$

$$\implies \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) \text{ are critical points}$$

$$n = 1 \implies f(x,y) = x + y$$

$$\implies \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) \text{ is a global maximum and } \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) \text{ is a global minimum}$$

Let $n \neq 1$

$$\implies \begin{cases} nx^{n-1} = 2\lambda x \\ ny^{n-1} = 2\lambda y \\ x^2 + y^2 - 1 = 0 \end{cases} \implies \begin{cases} x(nx^{n-2} - 2\lambda) = 0 \\ y(ny^{n-2} - 2\lambda) = 0 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

$$\implies \begin{cases} \left[\begin{array}{l} x = 0 \\ nx^{n-2} = 2\lambda \end{array}\right. \\ \left[\begin{array}{l} y = 0 \\ ny^{n-2} = 2\lambda \end{array}\right. \\ x^2 + y^2 - 1 = 0 \end{cases} \implies 4 \text{ possibilities}$$

$x = 0, y = 0$ doesn't work

$$x = 0, ny^{n-2} - 2\lambda = 0 \implies y^2 - 1 = 0 \implies y = \pm 1 \implies (0, \pm 1)$$

$$y = 0, nx^{n-2} - 2\lambda = 0 \implies x^2 - 1 = 0 \implies x = \pm 1 \implies (\pm 1, 0)$$

$$nx^{n-2} - 2\lambda = 0, ny^{n-2} - 2\lambda = 0 \implies x^{n-2}, y^{n-2} = \frac{2\lambda}{n}$$

$$n = 2 \implies \frac{2\lambda}{n} = 1 \implies \lambda = 1 \implies \text{All } x, y \text{ work}$$

$$n \neq 2 \implies x, y = \left(\frac{2\lambda}{n}\right)^{1/(n-2)} \text{ for odd } n \text{ and } \pm\left(\frac{2\lambda}{n}\right)^{1/(n-2)} \text{ for even } n$$

$$\implies 2\left(\frac{2\lambda}{n}\right)^{2/n-2} = 1 \implies x^2, y^2 = \frac{1}{2}$$

$$\implies \text{Points } \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right), \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) \text{ are also critical for even } n$$

In total:

$$n \text{ is odd: } \begin{cases} n = 1 \implies \underbrace{\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)}_{\text{global maximum}}, \underbrace{\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)}_{\text{global minimum}} \\ n \neq 1 \implies \left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right), \left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right), (0, \pm 1), (\pm 1, 0), (0,0) \\ f(x,y) = x^n + y^n \implies (0,1), (1,0) \text{ are global maximums} \end{cases}$$

$$(0,-1), (-1,0) \text{ are global minimums}$$

$$n \text{ is even: } \begin{cases} n = 2 \implies \text{All points on the circle are critical and have the same value 1} \\ f(x,y) = x^2 + y^2 \implies \begin{array}{c} \text{All points on } x^2+y^2=1 \text{ are global maximums} \\ (0,0) \text{ is a global minimum} \end{array} \\ \\ n \neq 2 \implies \left(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}}\right), (0,\pm 1), (\pm 1, 0), (0,0) \\ f(x,y) = x^n + y^n \implies \begin{array}{c} (0,\pm 1), (\pm 1, 0) \text{ are global maximums} \\ (0,0) \text{ is a global minimum} \end{array} \end{cases}$$