Power series #definition

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

Where a_n does not depend on x

is called a power series in the neighborhood of a

Convergence domain(interval) #definition

Convergence domain is a set X such that:

$$\forall I \subseteq X : \sum_{n=0}^{\infty} f_n(x) \text{ converges pointwise on } I$$

Convergence domain of a power series #lemma

Convergence domain of $\sum_{n=0}^{\infty} a_n(x-a)^n$ can be one of three options:

1. $\mathbb{R}$, for example $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\{a\}$, for example $\sum_{n=0}^{\infty} n!x^n$
3. Interval, that is symmetric around a

Convergence radius #lemma

$$\text{Let } R \in \mathbb{R} : \sum_{n=0}^{\infty} a_n (x-a)^n \text{ converges on } (a-R, a+R)$$

$$\text{and diverges if } x \in (-\infty, a-R) \cup (a+R, \infty)$$

R is then called a convergence radius of power series

And there are no "holes" in the convergence domain

Note: endpoints of convergence might be included or excluded

Proof:

$$\text{Let } b \in (a-R, a+R)$$

$$\text{Let } |c-a| < |b-a|$$

$$\text{Let } \sum_{n=0}^{\infty} a_n (b-a)^n \text{ converges}$$

$$\sum_{n=0}^{\infty} |a_n (c-a)^n| = \sum_{n=0}^{\infty} \left| a_n (c-a)^n \frac{(b-a)^n}{(b-a)^n} \right| =$$

$$= \sum_{n=0}^{\infty} |a_n (b-a)^n| \cdot \left| \frac{c-a}{b-a} \right|^n$$

$$a_n (b-a)^n \underset{n\to\infty}{\to} 0$$

$$\implies |a_n (b-a)^n| \underset{n\to\infty}{\to} 0 \implies \forall n > N : |a_n (b-a)^n| < 1$$

$$\forall n > N : |a_n (b-a)^n| \cdot \left| \frac{c-a}{b-a} \right|^n < \left| \frac{c-a}{b-a} \right|^n$$

$$\left| \frac{c-a}{b-a} \right| < 1 \implies \sum_{n=0}^{\infty} \left| \frac{c-a}{b-a} \right|^n \text{ converges}$$

$$\implies \sum_{n=0}^{\infty} |a_n (b-a)^n| \cdot \left| \frac{c-a}{b-a} \right|^n \text{ converges}$$

$$\implies \sum_{n=0}^{\infty} |a_n (c-a)^n| \text{ converges} \implies$$

Determining convergence radius

$$\text{Let } \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$\text{Let } L = \lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n (x-a)^n} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x-a| = |x-a| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\text{Let } t = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$t \cdot |x-a| < 1 \implies \text{Series converges}$$

$$t \cdot |x-a| > 1 \implies \text{Series diverges}$$

$$\implies \text{Series converges} \iff |x-a| < \frac{1}{t}$$

$$\implies R = \frac{1}{t} = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

Note:

$$\text{We can also write } R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \implies R = \infty$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \implies R = 0$$

$$\nexists \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \implies ?$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$a_n = \begin{cases} 0 & n = 2k - 1 \\ \frac{(-1)^n}{(2n)!} & n = 2k \end{cases}$$

???

Let $t = x^2$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^n$$

$$R = \lim_{n \to \infty} \frac{1}{2n+1} = 0$$