1a

$$\int e^x \sin(3x)dx$$

Solution:

$$f(x) = \sin(3x) \implies f'(x) = 3\cos(3x)$$
$$g'(x) = e^x \implies g(x) = e^x$$

$$\implies \int e^x \sin(3x)dx = e^x \sin(3x) - 3\int e^x \cos(3x)dx$$
$$h(x) = \cos(3x) \implies h'(x) = -3\sin(3x)$$
$$\int e^x \cos(3x)dx = e^x \cos(3x) + 3\int e^x \sin(3x)dx = e^x \cos(3x) + 3e^x \sin(3x) - 9\int e^x \cos(3x)dx$$
$$\implies \int e^x \cos(3x)dx = \frac{e^x(3\sin(3x) + \cos(3x))}{10} + C$$
$$\implies \int e^x \sin(3x)dx = \frac{e^x}{10}(\sin(3x) - 3\cos(3x)) + C$$

1b

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

Solution:

$$\text{Let } t = \frac{\pi}{2} - x$$
$$\implies \cos(t) = \sin(x), \sin(t) = \cos(x)$$
$$\implies \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \left\{ \begin{array}{c} t = \frac{\pi}{2} - x \\ dt = -dx \\ x = 0 \implies t = \frac{\pi}{2} \\ x = \frac{\pi}{2} \implies t = 0 \end{array} \right\} = \int_0^{\pi/2} \frac{\sqrt{\cos t}}{\sqrt{\cos t} + \sqrt{\sin t}}\, dt$$
$$\implies 2\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx =$$
$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$$
$$\implies \boxed{\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \frac{\pi}{4}}$$

2

Let f be a continuous function defined on $[0, \infty)$

2a

$$\text{Prove or disprove: } \int_0^\infty f(x)\,dx \text{ converges} \implies f \text{ is bounded}$$

Disproof:

Let us define a function of "triangles" (starting from n=2):

$$\forall n \in \mathbb{N}\ \{1\} : f(x) = \begin{cases} n^3\left(x - n + \frac{1}{n^2}\right) & x \in \left[n - \frac{1}{n^2}, n\right] \\ n^3\left(n + \frac{1}{n^2} - x\right) & x \in \left[n, n + \frac{1}{n^2}\right] \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Area of each triangle is } \frac{1}{n^2}$$

$$f \text{ is continuous on } [0, \infty)$$

$$f \text{ is unbounded on } [0, \infty)$$

$$\int_0^\infty f(x)\,dx = \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ converges}$$

2b

$$\text{Prove or disprove: } f \text{ is monotonically decreasing and } \int_0^\infty f(x)\,dx \text{ converges}$$

$$\implies \lim_{x\to\infty} f(x) = 0$$

Proof:

$$\text{Let } \lim_{x\to\infty} f(x) = L > 0$$

$$\implies \forall \varepsilon > 0 : \exists x_0 \in \mathbb{R} : \forall x > x_0 : |f(x) - L| < \varepsilon$$

$$\text{Let } \varepsilon = \frac{L}{2}$$

$$\text{Let } x_0 \in \mathbb{R} : \forall x > x_0 : |f(x) - L| < \varepsilon$$

$$\implies \forall x > x_0 : \frac{L}{2} = L - \varepsilon \le f(x) \le L + \varepsilon = \frac{3}{2}L$$

$$\implies \int_0^\infty f(x)\,dx = \int_0^{x_0+1} f(x)\,dx + \int_{x_0+1}^\infty f(x)\,dx \ge \int_0^{x_0+1} f(x)\,dx + \underbrace{\int_{x_0+1}^\infty \frac{L}{2}\,dx}_{=\infty}$$

$$\implies \int_0^\infty f(x)\,dx \text{ diverges}$$

$$\text{Let } \lim_{x\to\infty} f(x) = L < 0$$

$$\implies \forall \varepsilon > 0 : \exists x_0 \in \mathbb{R} : \forall x > x_0 : |f(x) - L| < \varepsilon$$

$$\text{Let } \varepsilon = -\frac{L}{2}$$

$$\text{Let } x_0 \in \mathbb{R} : \forall x > x_0 : |f(x) - L| < \varepsilon$$

$$\implies \forall x > x_0 : \frac{3L}{2} = L - \varepsilon \le f(x) \le L + \varepsilon = \frac{L}{2}$$

$$\implies \int_0^\infty f(x)\,dx = \int_0^{x_0+1} f(x)\,dx + \int_{x_0+1}^\infty f(x)\,dx \le \int_0^{x_0+1} f(x)\,dx + \underbrace{\int_{x_0+1}^\infty \frac{L}{2}\,dx}_{=-\infty}$$

$$\implies \int_0^\infty f(x)\,dx \text{ diverges}$$

$$\implies \boxed{\lim_{x\to\infty} f(x) = 0}$$

3a

$$\text{Prove or disprove: } \sum f_n(x) \text{ converges absolutely on } I$$
$$\implies \exists \sum a_n \text{ convergent} : \forall x \in I : \forall n \in \mathbb{N} : |f_n(x)| \leq a_n$$

$$\text{Disproof:}$$
$$\text{Let } I = [0, 2\pi]$$
$$\text{Let } f_n(x) = (-1)^n$$
$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} (-1)^n = 0$$
$$\forall x \in I : \forall n \in \mathbb{N} : |f_n(x)| = 1$$
$$\implies \forall a_n : \forall x \in I : \forall n \in \mathbb{N} : |f_n(x)| \leq a_n \implies a_n \geq 1$$
$$\implies \lim_{n\to\infty} a_n \neq 0 \implies \sum a_n \text{ diverges}$$

3b

$$\text{Determine whether } \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}}(x^n + x^{-n}) \text{ converges uniformly on } \left[\frac{1}{2}, 2\right]$$

$$\text{Solution:}$$
$$\frac{1}{2} \leq |x| \leq 2 \implies \begin{cases} |x^n| \leq 2^n \\ |x^{-n}| \leq 2^n \end{cases}$$
$$\left|x^n + x^{-n}\right| \leq |x^n| + \left|x^{-n}\right| \leq 2^{n+1}$$
$$\implies \sum_{n=1}^{\infty} \left| \frac{n^2}{\sqrt{n!}}(x^n + x^{-n}) \right| \leq \sum_{n=1}^{\infty} \frac{2^{n+1}n^2}{\sqrt{n!}}$$
$$\lim_{n\to\infty} \frac{2^{n+1}(n+1)^2 \cdot \sqrt{n!}}{\sqrt{(n+1)!} \cdot 2^n \cdot n^2} = \lim_{n\to\infty} 2e^2 \cdot \sqrt{\frac{1}{n+1}} = 0$$
$$\implies \text{Series converges}$$
$$\implies \text{By Weierstrass M-test: } \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}}(x^n + x^{-n}) \text{ converges absolutely on } \left[\frac{1}{2}, 2\right]$$

4

$$\text{Calculate: } \sum_{n=1}^{\infty} \frac{1}{2^n n(n+1)}$$

Solution:

$$\text{Let } x = \frac{1}{2}$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{2^n n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^n$$

$$\lim_{n\to\infty} \frac{n(n+1)}{(n+1)(n+2)} = 1 \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^n \text{ converges absolutely on } (-1,1)$$

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \implies \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{(1-x)}$$

$$\implies \int_0^x \sum_{n=1}^{\infty} t^{n-1}\, dt = \sum_{n=1}^{\infty} \int_0^x t^{n-1}\, dt = \sum_{n=1}^{\infty} \frac{x^n}{n} = \int_0^x \frac{1}{1-t}\, dt = -\ln|x-1| = \ln(1-x)$$

$$\int_0^x \sum_{n=1}^{\infty} \frac{t^n}{n}\, dt = \sum_{n=1}^{\infty} \int_0^t \frac{t^n}{n}\, dt = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = \int_0^x -\ln(1-t)\, dt$$

$$\int -\ln(1-t)\, dt \underbrace{=}_{u=1-t} \int \ln(u)\, du = u\ln(u) - u = (1-t)\ln(1-t) - (1-t)$$

$$\implies \int_0^x -\ln(1-t)\, dt = (1-x)\ln(1-x) - (1-x) + 1 =$$

$$= x + (1-x)\ln(1-x)$$

$$\implies \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = 1 + \frac{1-x}{x}\ln(1-x)$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{2^n n(n+1)} = \boxed{1 + \ln\left(\frac{1}{2}\right)}$$

5a

Find critical points of function:

$$f(x,y) = xy^2 - 2x^2y - 4xy$$

Solution:

$$f_x = y^2 - 4xy - 4y = 0 \implies y(y-4x-4) = 0 \implies \left[\begin{array}{c} y=0 \\ y=4(x+1) \end{array}\right.$$

$$f_y = 2xy - 2x^2 - 4x \implies x(y-x-2) = 0 \implies \left[\begin{array}{c} x=0 \\ y=x+2 \end{array}\right.$$

$$y = 0 \implies \left[\begin{array}{c} x=0 \\ x=-2 \end{array}\right.$$

$$y = 4(x+1) \implies \left[\begin{array}{l} x=0 \implies y=4 \\ y=x+2 \implies 4x+2=x \implies x=-\frac{2}{3}, y=\frac{4}{3} \end{array}\right.$$

$$\implies \text{Critical points are:}$$

$$(0,0), (-2,0), (0,4), \left(-\frac{2}{3}, \frac{4}{3}\right)$$

$$f_{xx} = -4y$$
$$f_{xy} = 2y - 4x - 4$$
$$f_{yx} = 2y - 4x - 4$$
$$f_{yy} = 2x$$

$$\implies H_f = \begin{pmatrix} -4y & 2y-4x-4 \\ 2y-4x-4 & 2x \end{pmatrix}$$

$$M_1 = -4y$$
$$M_2 = -8xy - (2y-4x-4)^2$$

$$(0,0) \to M_1 = 0, M_2 = -16 \implies \text{Saddle}$$
$$(-2,0) \to M_1 = 0, M_2 = -16 \implies \text{Saddle}$$
$$(0,4) \to M_1 = -16, M_2 = -16 \implies \text{Saddle}$$
$$\left(-\frac{2}{3}, \frac{4}{3}\right) \to M_1 = -\frac{16}{3}, M_2 = \frac{64}{9} - \left(\frac{8}{3} + \frac{8}{3} - 4\right)^2 = \frac{16}{3} > 0 \implies \text{Local maximum}$$