Integrals

Primitive function #definition

Let f be a function on $[a, b]$ or any other

Function F is called primitive of f if

$$F' = f$$

For example:

$$f(x) = x^2 \implies \forall C \in \mathbb{R} : F(x) = \frac{x^3}{3} + C$$

Two primitives #lemma

Let f be a function

Let F, G be antiderivatives of f

Then $\exists C \in \mathbb{R} : F = G + C$

Proof:

$$(F - G)' = F' - G' = f - f = 0$$

$$\implies F - G \text{ is constant} \implies \exists C \in \mathbb{R} : F - G = C$$

$$\implies \exists C \in \mathbb{R} : F = G + C$$

Indefinite integral #definition

Let f be a function

Integral is a set of primitives of f

Integral is denoted as $\displaystyle\int f(x)dx$

Note: this integral is an indefinite integral

Let f be a function

We want to find it's primitive F

Let there be four types of functions

1. Functions without a primitive

   By Darboux's theorem, if function has a removable or a jump discontinuity

   it has no primitive

2. Functions with "immediate" (known) integrals

   $$\int x^n dx = \tfrac{x^{n+1}}{n+1} + C$$
   $$\int x^{-1} dx = \ln|x| + C$$
   $$\int a^x dx = \tfrac{a^x}{\ln a} + C$$
   $$\int \sin(x) dx = -\cos(x) + C$$
   $$\int \cos(x) dx = \sin(x) + C$$
   $$\int \tfrac{1}{\cos^2(x)} dx = \tan(x) + C$$
   $$\int \tfrac{1}{x^2+1} dx = \arctan(x) + C$$
   $$\int \tfrac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$
   $$\int \ln(x) dx = x \ln x - x + C$$

3. Functions with no elementary primitive

   For example: $\displaystyle\int e^{x^2} dx = \frac{\sqrt{\pi}}{2} erfi(x) + C$

4. ...

Linearity of integrals #lemma

$$\int (af+g)(x)dx = a\int f(x)dx + \int g(x)dx$$

$$\text{Proof:}$$
$$\text{Let } F' = f$$
$$\text{Let } G' = g$$
$$\implies (aF' + G') = af + g$$
$$\implies \int (af+g)(x)dx = (aF+G) = aF + G = a\int f(x)dx + \int g(x)dx$$
$$\implies \int (af+g)(x)dx = a\int f(x)dx + \int g(x)dx$$

Integration by parts #theorem

$$\text{Let } f, g \text{ be functions}$$
$$\text{Then } \int (fg')(x)dx = fg - \int (f'g)(x)dx$$

$$\text{Proof:}$$
$$(fg)' = f'g + fg'$$
$$\implies \int (fg)'(x)dx = \int (f'g + fg')(x)dx = \int (f'g)(x)dx + \int (fg')(x)dx$$
$$\implies fg = \int (f'g)(x)dx - \int (fg')(x)dx \implies \int (fg')(x)dx = fg - \int (f'g)(x)dx$$

When should we use integration by parts?

1. Product of functions, when one function is an obvious derivative with known primitive

$$\int x\sin x dx = \int x(-\cos x)'dx = -x\cos x - \int -\cos x dx = -x\cos x + \sin x + C$$

Usuall, choosing polynomial as f is profitable

$$\int \ln(x)dx = \int \ln(x)x'dx = x\ln x - \int (\ln x)'xdx = x\ln x - \int 1dx = x\ln x - x + C$$

$$\int \sin(\ln x)dx = \int \sin(\ln x)x'dx = x\sin(\ln x) - \int \cos(\ln x)dx$$
$$\int \cos(\ln x)dx = \int \cos(\ln x)x'dx = x\cos(\ln x) + \int \sin(\ln x)dx$$
$$\implies 2\int \cos(\ln x)dx = x\sin(\ln x) + x\cos(\ln x)$$
$$\implies \int \cos(\ln x)dx = \frac{x\sin(\ln x) + x\cos(\ln x)}{2} + C$$
$$\implies \int \sin(\ln x)dx = \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C$$