$$T : V \to V \text{ is injective}$$
$$\langle\,,\rangle_T : V \times V \to \mathbb{F}$$

$$\text{Show that } \langle\,,\rangle_T \text{ is an inner product}$$

Solution:

$$\langle v+u, w\rangle_T = \langle T(v+u), T(w)\rangle = \langle T(v)+T(u), T(w)\rangle = \langle T(v), T(w)\rangle + \langle T(u), T(w)\rangle$$
$$\langle \alpha v, u\rangle_T = \langle T(\alpha v), T(u)\rangle = \langle \alpha T(v), T(u)\rangle = \alpha\langle T(v), T(u)\rangle = \alpha\langle v, u\rangle_T$$
$$\langle v, u\rangle_T = \langle T(v), T(u)\rangle = \overline{\langle T(u), T(v)\rangle} = \overline{\langle u, v\rangle_T}$$
$$\langle v, v\rangle_T = \langle T(v), T(v)\rangle \geq 0$$
$$\langle v, v\rangle_T = 0 \iff \langle T(v), T(v)\rangle = 0 \iff T(v) = 0 \iff v \in ker(T)$$
$$\underline{T \text{ is injective}} \iff ker(T) = \{0\} \implies \langle\,,\rangle_T \text{ is an inner product}$$

$$\text{Let } V \text{ be an inner product space}$$
$$\text{Let } \{v_1, \ldots, v_n\} \subseteq V$$

$$\text{Prove or disprove: } \sum_{i=1}^{n}\sum_{j=1}^{n}\langle v_i, v_j\rangle \geq 0$$

Proof:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle v_i, v_j\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n}\langle v_i, v_j\rangle = \sum_{j=1}^{n}\left\langle \sum_{i=1}^{n} v_i, v_j\right\rangle =$$
$$= \sum_{j=1}^{n}\overline{\left\langle v_j, \sum_{i=1}^{n} v_i\right\rangle} = \overline{\sum_{j=1}^{n}\left\langle v_j, \sum_{i=1}^{n} v_i\right\rangle} = \overline{\left\langle \sum_{j=1}^{n} v_j, \sum_{i=1}^{n} v_i\right\rangle} = \langle v, v\rangle \geq 0$$

$$\text{Prove: } v = 0 \iff \forall u \in V : \langle v, u\rangle = 0$$

$$\text{Let } B \text{ be a basis of } V$$
$$\text{Prove: } \forall i \in [1, n] : \langle v, v_i\rangle = \langle u, v_i\rangle \iff v = u$$

Proof:

$$\text{Let } u \in V$$
$$v = 0 \implies \langle v, u\rangle = \langle 2v, u\rangle = \langle v, u\rangle + \langle v, u\rangle \implies \langle v, u\rangle = 0$$
$$\forall u \in V : \langle v, u\rangle = 0 \underset{u=v}{\implies} \langle v, v\rangle = 0 \implies v = 0$$

Proof:

$$\langle v, v_i\rangle = \langle u, v_i\rangle \implies \langle v-u, v_i\rangle = 0$$
$$\text{Let } w \in V$$
$$w = \sum_{i=1}^{n}\alpha_i v_i$$
$$\langle v-u, w\rangle = \left\langle v-u, \sum_{i=1}^{n}\alpha_i v_i\right\rangle = \overline{\left\langle \sum_{i=1}^{n}\alpha_i v_i, v-u\right\rangle} = \sum_{i=1}^{n}\overline{\alpha_i}\overline{\langle v-u, v_i\rangle} = 0$$

Gram-Schmidt matrix

$$\text{Let } S = \{v_1, \ldots, v_n\} \subseteq V$$
$$G_S \in \mathbb{F}^{n\times n} : (G_S)_{ij} = \langle v_i, v_j\rangle$$

Prove: G_S is non-invertible $\iff$ S is a linear dependence

Proof:

Let G_S be non-invertible

$$\implies \exists \sum_{j=1}^{n} \alpha_j C_j(G_S) = 0 : \exists j \in [1,n] : \alpha_j \neq 0$$

$$\implies 0 = \sum_{j=1}^{n} \alpha_j C_j(G_S) = \sum_{j=1}^{n} \alpha_j \begin{pmatrix} \langle v_1, v_j \rangle \\ \langle v_2, v_j \rangle \\ \vdots \\ \langle v_n, v_j \rangle \end{pmatrix}$$

$$\forall i \in [1,n] : \implies \sum_{j=1}^{n} \alpha_j \langle v_i, v_j \rangle = 0$$

$$\implies \sum_{j=1}^{n} \langle v_i, \overline{\alpha_j} v_j \rangle = 0$$

$$\implies \sum_{j=1}^{n} \overline{\langle \overline{\alpha_j} v_j, v_i \rangle} = \overline{\sum_{j=1}^{n} \langle \overline{\alpha_j} v_j, v_i \rangle} = 0$$

$$\implies \left\langle \underbrace{\sum_{j=1}^{n} \overline{\alpha_j} v_j}_{u}, v_i \right\rangle = 0$$

$$\langle u, u \rangle = \left\langle \sum_{j=1}^{n} \overline{\alpha_j} v_j, u \right\rangle = \sum_{j=1}^{n} \overline{\alpha_j} \left\langle v_j, u \right\rangle = 0$$

$$\implies u = 0 \implies \sum_{i=1}^{n} \overline{\alpha_i} v_i = 0$$

$$\exists j \in [1,n] : \alpha_j \neq 0 \implies \overline{\alpha_j} \neq 0$$

$$\implies \boxed{S \text{ is a linear dependence}}$$