Absolute and conditional convergence #definition

$$\int f \text{ is called absolutely convergent if } \int |f| \text{ converges}$$
$$\int f \text{ is called conditionally convergent if } \int f \text{ converges but } \int |f| \text{ diverges}$$

Absolute convergence preserves "regular" convergence #lemma

$$\text{If } \int |f| \text{ converges, then } \int f \text{ also converges}$$

$$\text{Proof:}$$
$$\text{Let } \int |f| \text{ converges}$$
$$\text{Let } f_+ = \begin{cases} f & f \geq 0 \\ 0 & f < 0 \end{cases}$$
$$\text{Let } f_- = \begin{cases} 0 & f \geq 0 \\ -f & f < 0 \end{cases}$$

$$0 \leq f_+ \leq |f| \implies \int f_+ \text{ converges}$$
$$0 \leq f_- \leq |f| \implies \int f_- \text{ converges}$$
$$\int f = \int (f_+ - f_-) = \underbrace{\int f_+}_{\text{Converges}} - \underbrace{\int f_-}_{\text{Converges}} \implies \boxed{\int f \text{ converges}}$$

Dirichlet's convergence test for integrals #theorem

Let f be a continuously differentiable, monotonically decreasing function

$$\lim_{x\to\infty} f(x) = 0$$

Let g be continuous

$$\text{Let } G(x) = \int_a^x g(t)\,dt \text{ be bounded}$$

$$\text{Then } \int_a^\infty f(x)g(x)\,dx \text{ converges}$$

Proof:

Let f be a continuously differentiable, monotonically decreasing function

$$\lim_{x\to\infty} f(x) = 0$$

Let g be continuous

$$\text{Let } G(x) = \int_a^x g(t)\,dt \text{ be bounded}$$

$$\int_a^\infty f(x)g(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)g(x)\,dx$$

$$\int_a^b f(x)g(x)\,dx = \int_a^b f(x)G'(x)\,dx = f(x)G(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)G(x)\,dx$$

$$f(x)G(x)\Big|_{x=a}^{x=b} = \underbrace{f(b)}_{\to 0}\ \underbrace{G(b)}_{\text{Bounded}} - f(a)\underbrace{G(a)}_{\int_a^a = 0} \xrightarrow[b\to\infty]{} 0$$

$$\text{It is now enough to show that } \int_a^\infty f'(x)G(x)\,dx \text{ converges}$$

$$\text{Consequently, we can just show that } \int_a^\infty |f'(x)G(x)|\,dx \text{ converges}$$

$$|G(x)| \le M \implies \int_a^\infty |f'(x)G(x)|\,dx \le M\int_a^\infty |f'(x)|\,dx$$

$$f \text{ is monotonically decreasing} \implies f'(x) < 0$$

$$\implies M\int_a^\infty |f'(x)|\,dx = -M\int_a^\infty f'(x)\,dx = -M\lim_{b\to\infty}\int_a^b f'(x)\,dx = -M\lim_{b\to\infty} f(x)\Big|_{x=a}^{x=b} =$$

$$= -M\lim_{b\to\infty}(\underbrace{f(b)}_{\to 0} - f(a)) = Mf(a)$$

$$\implies \int_a^\infty |f'(x)G(x)| \text{ converges}$$

$$\implies \boxed{\int_a^\infty f(x)g(x)\,dx \text{ converges}}$$