Dirichlet's convergence test for integrals

$$\text{Let } f \text{ be a continuously differentiable, monotonically decreasing function}$$
$$\lim_{x\to\infty} f(x) = 0$$
$$\text{Let } g \text{ be continuous}$$
$$\text{Let } G(x) = \int_a^x g(t)\,dt \text{ be bounded}$$
$$\text{Then } \int_a^\infty f(x)g(x)\,dx \text{ converges}$$

$$\int_a^\infty \frac{\cos(x)}{x}\,dx$$
$$f(x) = \frac{1}{x}$$
$$g(x) = \cos x = G(x) = \sin(x) - \sin(a)$$
$$\implies \text{Integral converges}$$

$$\int_a^\infty \sin(x^2)\,dx$$
$$t = x^2 \implies dt = 2xdx \implies dx = \frac{dt}{2\sqrt{t}}$$
$$\int_{a^2}^\infty \frac{\sin(t)}{2\sqrt{t}}\,dt \text{ converges}$$
$$\text{But } \lim_{x\to\infty} \sin(x^2) \neq 0$$

$$\int_{-\infty}^\infty \frac{1}{x^2+34}\,dx = \int_{-\infty}^0 \frac{1}{x^2+34}\,dx + \int_0^\infty \frac{1}{x^2+34}\,dx$$
$$= \lim_{b\to\infty} \frac{1}{\sqrt{34}}\arctan\left(\frac{x}{\sqrt{34}}\right)\Bigg|_{x=-b}^{x=0} + \lim_{b\to\infty} \frac{1}{\sqrt{34}}\arctan\left(\frac{x}{\sqrt{34}}\right)\Bigg|_{x=0}^{x=b} = \frac{\pi}{\sqrt{34}}$$

$$\int_0^1 \cot x\,dx = \lim_{a\to 0^+} \int_a^1 \cot x\,dx = \lim_{a\to 0^+} \ln|\sin x|\Bigg|_{x=a}^{x=1} = \lim_{n\to\infty} \ln(\sin(1)) - \ln(\sin(a)) = \infty$$

$$\forall n \geq 0 : \int_0^\infty x^n e^{-x}\,dx = n!$$
$$\text{Proof (a little more complicated than the standard one) in Lecture 8}$$

$$\int_1^\infty \sin\left(\frac{x+20}{x^3-2}\right)dx = \int_1^3 \sin\left(\frac{x+20}{x^3-2}\right)dx + \int_3^\infty \sin\left(\frac{x+20}{x^3-2}\right)dx$$

$$\int_3^\infty \sin\left(\frac{x+20}{x^3-2}\right)dx \text{ converges} \iff \int_3^\infty \frac{x+20}{x^3-2}\,dx \text{ converges}$$
$$\int_3^\infty \frac{x+20}{x^3-2}\,dx \text{ converges} \iff \int_3^\infty \frac{1}{x^2}\,dx \text{ converges}$$

$$\int_0^1 \frac{\sin\left(\frac{1}{x}\right)}{x}\,dx = \left\{\begin{array}{c} t=\frac{1}{x} \\ dt=-\frac{1}{x^2}dx \\ x=0 \implies t=\infty \\ x=1 \implies t=1 \end{array}\right\} = \int_1^\infty \frac{\sin(t)}{t}\,dt$$

$$\implies \text{Converges by the Dirichlet's test}$$

$$\left|\frac{\sin t}{t}\right| \geq \frac{\sin^2 t}{t} = \frac{1-\cos(2t)}{2t} = \underbrace{\frac{1}{2}\left(\frac{1}{t} - \frac{\cos(2t)}{t}\right)}_{\text{Integral diverges}}$$

$$\int_0^\infty (-1)^{\lfloor x^2 \rfloor}\,dx = \sum_{n=0}^{\infty} \int_{\sqrt{n}}^{\sqrt{n+1}} (-1)^n\,dx =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}+\sqrt{n}} \text{ converges by the Leibniz test}$$

$$\int_0^{\pi/2} \frac{1}{\sin^a(x^2)\cos^b(x)}\,dx =$$

$$= \int_0^1 \frac{1}{\sin^a(x^2)\cos^b(x)}\,dx + \int_1^{\pi/2} \frac{1}{\sin^a(x^2)\cos^b(x)}\,dx$$

$$\lim_{x\to 0} \frac{\frac{1}{\cos^b x}\cdot\frac{1}{\sin^a(x^2)}}{\left(\frac{1}{x^2}\right)^a} = \lim_{x\to 0} \frac{1}{\cos^b x}\cdot\left(\frac{\frac{1}{\sin(x^2)}}{\frac{1}{x^2}}\right)^a \underset{x\to 0}{\to} 1^a = 1$$

$$\implies \int_0^1 f(x)\,dx \text{ converges} \iff 2a < 1 \iff a < \frac{1}{2}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{(\pi-x)^b}}{\frac{1}{\sin^a(x^2)\cos^b(x)}} = \frac{1}{\sin^a\left(\frac{\pi}{4}\right)}$$

$$\implies \int_1^{\pi/2} f(x)\,dx \text{ converges} \iff b < 1$$
