Reminder

$$\text{Let } A \text{ be diagonalizable}$$
$$\text{Let } B = \{v_1, \dots, v_n\} \text{ a set of eigenvectors of } A$$
$$B \text{ is then a basis of } \mathbb{F}^n$$
$$P^{-1}AP = D$$
$$P = [I]_S^B$$

Triangularizable matrix #definition

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\text{Let } T \in \mathbb{F}^{n \times n} \text{ be a triangular matrix}$$
$$A \sim T \iff A \text{ is triangularizable}$$

Upper and lower triangularizable matrix #lemma

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\text{Let } U \in \mathbb{F}^{n \times n} \text{ be an upper-triangular matrix}$$
$$\text{Let } L \in \mathbb{F}^{n \times n} \text{ be a lower-triangular matrix}$$
$$A \sim U \iff A \sim L$$

Triangularizable matrix #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$A \text{ is triangularizable } \iff P_A(\lambda) \text{ is factorizable into linear factors}$$
$$\text{Corollary: every matrix is triangularizable over } \mathbb{C}$$

Proof:

$$\boxed{\Longrightarrow} \text{ Let } U \in \mathbb{F}^{n\times n} \text{ be a triangular matrix}$$
$$\text{Let } A \sim U$$
$$\text{Let } \forall i \in [1,n] : \alpha_i = U_{ii}$$
$$P_A(\lambda) = P_U(\lambda) = \prod_{i=1}^{n}(\lambda - \alpha_i)$$
$$\Longrightarrow \boxed{P_A(\lambda) \text{ is factorized into linear factors}}$$
$$\boxed{\Longleftarrow} \text{ Let } P_A(\lambda) \text{ be factorized into linear factors}$$
$$\text{Base case. Let } n = 1, P_A(\lambda) = \lambda - A_{11} \text{ and } A \text{ is upper-triangular, } A \sim A$$
$$\text{Induction step.}$$
$$\text{Let } \forall n' \le n : A \in \mathbb{F}^{n'\times n'} : P_A(\lambda) \text{ is factorizable into linear factors } \Longrightarrow A \sim U$$
$$\text{Let } A \in \mathbb{F}^{n+1\times n+1}, P_A(\lambda) = \prod_{i=1}^{n+1}(\lambda - \alpha_i)$$
$$\Longrightarrow A \text{ definitely has eigenvalues in } \mathbb{F}$$
$$\text{Let } \alpha \in \mathbb{F} \text{ be an eigenvalue of } A$$
$$\text{Let } E_\alpha = sp\,\{v_1, \dots, v_t\}, t \ge 1$$
$$\text{Let } B = \{v_1, \dots, v_t\} \cup \{u_{t+1}, \dots, u_{n+1}\} \text{ be a basis of } \mathbb{F}^{n+1}$$
$$\text{Let } P = \begin{pmatrix} | & & | & | & & | \\ v_1 & \dots & v_t & u_{t+1} & \dots & u_{n+1} \\ | & & | & | & & | \end{pmatrix}$$
$$P^{-1}AP = P^{-1}\begin{pmatrix} | & & | & | & & | \\ \alpha v_1 & \dots & \alpha v_t & Au_{t+1} & \dots & Au_{n+1} \\ | & & | & | & & | \end{pmatrix} =$$
$$= \begin{pmatrix} \alpha I_t & B \\ 0 & C \end{pmatrix}$$
$$P_{P^{-1}AP}(\lambda) = (\lambda - \alpha)^t \cdot P_C(\lambda)$$
$$\Longrightarrow P_A(\lambda) = (\lambda - \alpha)^t \cdot P_C(\lambda)$$
$$\Longrightarrow P_C(\lambda) \text{ is factorizable into linear factors}$$
$$n + 1 - t \le n$$
$$\Longrightarrow C \in \mathbb{F}^{n+1-t\times n+1-t} \text{ is triangularizable}$$
$$\Longrightarrow \hat{P^{-1}}C\hat{P} = \hat{U}, \hat{P} \in \mathbb{F}^{n+1-t\times n+1-t}$$
$$\text{Let } Q = P\begin{pmatrix} I & 0 \\ 0 & \hat{P} \end{pmatrix}$$
$$Q^{-1}AQ = \begin{pmatrix} I & 0 \\ 0 & \hat{P} \end{pmatrix}^{-1} P^{-1}AP \begin{pmatrix} I & 0 \\ 0 & \hat{P} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \hat{P} \end{pmatrix}^{-1} \begin{pmatrix} \alpha I & B \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \hat{P} \end{pmatrix} =$$
$$= \begin{pmatrix} I & 0 \\ 0 & \hat{P^{-1}} \end{pmatrix} \begin{pmatrix} \alpha I & B\hat{P} \\ 0 & C\hat{P} \end{pmatrix} = \begin{pmatrix} \alpha I & B\hat{P} \\ 0 & \hat{U} \end{pmatrix} = U$$
$$\Longrightarrow \boxed{A \sim U}$$

Properties of triangularizable matrix #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n} \text{ be triangularizable}$$
$$\Longrightarrow P_A(\lambda) \text{ is factorizable into linear factors}$$
$$\text{Let } \{\alpha_1, \dots, \alpha_t\} \text{ be eigenvalues of } A$$
$$tr(A) = \sum_{i=1}^{t} \alpha_i \cdot \mu_A(\alpha_i)$$
$$|A| = \prod_{i=1}^{t} \alpha_i^{\mu_A(\alpha_i)}$$

Matrix polynomial expression #definition

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Let } P(x) \in \mathbb{F}_t[x]$$
$$P(A) = a_t A^t + \cdots + a_1 A + a_0 I \text{ is then called a matrix polynomial expression}$$

Existence of polynomial with the given matrix as a root #lemma

$$A \in \mathbb{F}^{n\times n}$$
$$\left\{I, A, A^2, \ldots, A^{n^2}\right\} \subseteq \mathbb{F}^{n\times n} \text{ is a linear dependence}$$
$$\sum_{i=0}^{n^2} \alpha_i A^i = 0$$
$$\implies P(A) = \sum_{i=0}^{n^2} \alpha_i A^i = 0$$
$$\implies P(x) = \sum_{i=0}^{n^2} \alpha_i x^i$$

Adjoint matrix #definition

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Adjoint matrix of } A \text{ is denoted as: } \mathrm{adj}A \in \mathbb{F}^{n\times n}$$
$$\forall i, j \in [1, n] : (\mathrm{adj}A)_{ij} = (-1)^{i+j} |M_{ji}|$$

Adjoint of a transpose #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Then } (\mathrm{adj}A)^T = \mathrm{adj}(A^T)$$

Proof:

$$(\mathrm{adj}A)^T_{ij} = (\mathrm{adj}A)_{ji} = (-1)^{i+j} |M_{ij}(A)| = (-1)^{i+j} \left|(M_{ij}(A))^T\right| =$$
$$= (-1)^{i+j} \left|M_{ji}(A^T)\right| = \mathrm{adj}(A^T)_{ij}$$

Product of matrix and its adjoint #theorem

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$A \cdot \mathrm{adj}A = \mathrm{adj}A \cdot A = \det(A) I$$

Proof:

$$\text{Let } i \in [1, n]$$
$$(A \cdot \mathrm{adj}A)_{ii} = \sum_{k=1}^{n} A_{ik} \cdot (\mathrm{adj}A)_{ki} = \sum_{k=1}^{n} A_{ik} \cdot (-1)^{k+i} |M_{ik}| = \det(A)$$
$$\text{Let } j \neq i \in [1, n]$$
$$\text{Let } \hat{A} : R_k(\hat{A}) = \begin{cases} R_k(A) & k \neq j \\ R_i(A) & k = j \end{cases} \implies \det(\hat{A}) = 0$$
$$(A \cdot \mathrm{adj}A)_{ij} = \sum_{k=1}^{n} A_{ik} \cdot (\mathrm{adj}A)_{kj} = \sum_{k=1}^{n} A_{ik} \cdot (-1)^{k+j} |M_{jk}(A)| =$$
$$= \sum_{i=1}^{n} \hat{A}_{jk} \cdot (-1)^{k+j} \left|M_{jk}(\hat{A})\right| = \det(\hat{A}) = 0$$
$$\implies \boxed{A \cdot \mathrm{adj}A = \det(A) I}$$

$$(A^T \cdot \mathrm{adj}(A^T))^T = (\det(A^T) I)^T$$
$$\implies (\mathrm{adj}(A^T))^T \cdot A = \det(A) I \implies \boxed{\mathrm{adj}A \cdot A = \det(A) I}$$

Cayley-Hamilton theorem #theorem

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Then } P_A(A) = 0$$

$$\text{Explanation, not proof:}$$
$$P_A(A) = \det(AI - A) = \det(0) = 0$$

$$\text{Proof:}$$
$$P_A(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i$$
$$(\lambda I - A)\cdot \operatorname{adj}(\lambda I - A) = \det(\lambda I - A) I = P_A(\lambda) I$$
$$\operatorname{adj}(\lambda I - A) \in \mathbb{F}_{n-1}[\lambda]^{n\times n}$$
$$\implies \exists \{B_0, \dots, B_{n-1}\} \subseteq \mathbb{F}^{n\times n} : \operatorname{adj}(\lambda I - A) = \sum_{i=0}^{n-1} \lambda^i B_i$$
$$\implies (\lambda I - A)\cdot \sum_{i=0}^{n-1} \lambda^i B_i = P_A(\lambda) I$$
$$\implies (\lambda I - A)\cdot \sum_{i=0}^{n-1} \lambda^i B_i = \sum_{i=0}^{n} \lambda^i a_i I$$
$$\implies \begin{array}{c|c|c|c|c|c|c} & \lambda^n & \lambda^{n-1} & \lambda^{n-2} & \dots & \lambda & 1 \\ \text{Left side} & B_{n-1} & B_{n-2} - AB_{n-1} & B_{n-3} - AB_{n-2} & \dots & B_0 - AB_1 & -AB_0 \\ \text{Right side} & I & a_{n-1} I & a_{n-2} I & \dots & a_1 I & a_0 I \end{array}$$
$$\implies A^n B_{n-1} + A^{n-1}(B_{n-2} - AB_{n-1}) + \dots + A(B_0 - AB_1) - AB_0 =$$
$$= A^n + a_{n-1}A^{n-1} + \dots + a_0 I = P_A(A)$$
$$\implies \boxed{0 = P_A(A)}$$