Function sequences #definition

Function sequence is a sequence in which each element is a function

For example:

$$f_n(x) = x^n$$

$$f_1(x) = x, f_2(x) = x^2, f_3(x) = x^3, \dots$$

Pointwise convergence #definition

$\{f_n(x)\}$ is called point-convergent on set A if

$$\forall x_0 \in A : \forall \varepsilon > 0 : \exists N_\varepsilon : \forall n > N_\varepsilon : |f_n(x_0) - f(x_0)| < \varepsilon$$

$f(x)$ is then called a pointwise limit of f_n

$$f_n \to f$$

Note: N_ε depends both on ε and $x_0 \in A$

Hence pointwise convergence

$$f_n(x) = x^n \text{ on } [0, 1]$$

$$x < 1 \implies f_n(x) = x^n \to 0$$

$$x = 1 \implies f_n(x) = 1^n = 1 \to 1$$

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

$$f_n \to f$$

$$f_n(x) = x^2 + \frac{x}{n} + \frac{7}{n^2}$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(x^2 + \underbrace{\frac{x}{n}}_{\to 0} + \underbrace{\frac{7}{n^2}}_{\to 0} \right) = x^2$$

$$f(x) = x^2$$

$$f_n \to f$$

$$\lim_{n\to\infty} \frac{n^2x^6}{n^2 + x^6} = \lim_{n\to\infty} \frac{x^6}{1 + \underbrace{\frac{x^6}{n^2}}_{\to 0}} = x^6$$

$$\lim_{n\to\infty} \sqrt{n^2x^2 + x^4} - nx = \lim_{n\to\infty} \frac{x^4}{\sqrt{n^2x^2 + x^4} + nx}$$

$$x > 0 \implies f_n \to 0$$

$$x = 0 \implies f_n(x) = 0$$

$$x < 0 \implies f_n(x) = \underbrace{\sqrt{n^2x^2 + x^4}}_{\to\infty} - \underbrace{nx}_{\to-\infty} \to \infty$$

$$f(x) = 0 \text{ on } [0, \infty)$$

$$f_n \to f$$

$$f_n(x) = n\arctan\left(\frac{x}{n}\right)$$

$$\lim_{n\to\infty} \frac{\arctan\left(\frac{x}{n}\right)}{\frac{1}{n}} \overset{t=\frac{1}{n}}{=} \lim_{t\to 0} \frac{\arctan(tx)}{t} \overset{L}{=} \lim_{t\to 0} \frac{x}{1+(tx)^2} = x$$

$$f_n(x) = n^2 \ln\left(1+\sin\left(\frac{x^9}{n^2}\right)\right)$$

$$\lim_{n\to\infty} n^2 \ln\left(1+\sin\left(\frac{x^9}{n^2}\right)\right) = \lim_{n\to\infty} \frac{x^9 \cdot \ln\left(1+\sin\left(\frac{x^9}{n^2}\right)\right)}{\frac{\frac{x^9}{n^2}}{\sin\left(\frac{x^9}{n^2}\right)} \cdot \sin\left(\frac{x^9}{n^2}\right)} =$$

$$= \lim_{t\to 0} x^9 \cdot \frac{\ln(1+\sin(t))}{\sin(t)} \cdot \frac{\sin(t)}{t} = x^9$$

$$x = 0 \implies f_n(x) = 0 = x^9$$

$$f_n(x) = \sin^{4n}(x)$$

$$\lim_{n\to\infty} \sin^{4n}(x) = \lim_{n\to\infty} (\sin^4(x))^n = \begin{cases} 1 & \sin^4(x) = 1 \\ 0 & \sin^4(x) \neq 1 \end{cases} = \begin{cases} 1 & x = \frac{\pi}{2} + \pi k \\ 0 & \text{otherwise} \end{cases}$$

$$f_n(x) = \begin{cases} 1 & x \in \left[0, \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases} \text{ on } [0,1]$$

$$x = 0 \implies f_n(x) = 1 \to 1$$

$$x > 0 \implies \forall n > \frac{1}{x} : f_n(x) = 0 \to 0$$

$$\implies f = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} \text{ on } [0,1]$$

Let $f_n \to f$

Let f_n be continuous

Is f necessarily continuous?

No: $f_n(x) = x^n$, $f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in [0,1) \end{cases}$

Let f_n, f be differentiable

Does necessarily $f_n' \to f'$?

No: $f_n(x) = \dfrac{\sin(n^8 x)}{n} \to 0$

$f_n'(x) = n^7 \cos(n^8 x)$ doesn't converge

Let f_n, f be Riemann-integrable on $[a,b]$

Does necessarily $\displaystyle\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$?

No: $f_n(x) = \begin{cases} n & x \in \left[0, \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases} \to 0$

$$\int_0^1 f_n(t)\,dt = \int_0^{1/n} f_n(t)\,dt + \int_{\frac{1}{n}}^1 f_n(t)\,dt = nt\Big|_{t=0}^{t=1/n} + C\Big|_{t=\frac{1}{n}}^{t=1} = 1$$