Definite integrals

Riemann sum #definition

$$\text{Let } [a,b] \subseteq \mathbb{R}$$
$$\text{Let } P = \{x_0, x_1, \ldots, x_n\}$$
$$a = x_0 < x_1 < \cdots < x_n = b$$
$$\text{Let } C = \{c_i | \forall i \in [1,n] : c_i \in [x_{i-1} + x_i]\}$$
$$\text{Let } \Delta x_i = x_{i-1} - x_i$$
$$S_R(f,P,C) = \sum_{i=1}^{n} f(c_i) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} f(c_i) \Delta x_i$$

$S_R(f,P,C)$ is called a Riemann sum and is an approximation of area under function f

This sum is the best approximation when all chosen rectangles' areas tend to 0

$$\text{Let } \lambda(P) = \max\{\Delta x_i\}$$

When $\lambda(P)$ tends to $0,$ all Δx tend to 0 and thus all rectangles' areas tend to 0

Riemann-Integrable function #definition

Let f be a function on $[a,b]$

f is called integrable by Riemann if:

1. f is bounded
2. $\forall P, C : S_R(f,P,C) \underset{\lambda(P)\to 0}{\to} L$

Same definition via sequences:

f is integrable on$[a,b]$ if

$$\forall \{P_n\} : \lambda(P_n) \underbrace{\to}_{n\to\infty} 0 : \forall \{C_n\} : S_R(f,P_n,C_n) \to L$$

Definite integral #definition

If such limit L exists, it is called definite integral of f on $[a,b]$

$$\text{And denoted as } \int_a^b f(x)dx = L$$

Example

$$\text{Let } f(x) = C \text{ on } [a,b]$$
$$S_R(f,P,C) = \sum_{i=1}^{n} C\Delta x_i = C\sum_{i=1}^{n} \Delta x_i$$
$$\forall P, C : \sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} [x_i - x_{i-1}] = x_n - x_0 = b - a$$
$$\implies \int_a^b f(x)dx = C(b-a)$$

Oscillation of a function #definition

Let f be a bounded function on I (for example on $[a,b]$)

Oscillation of f is a measure of how function varies between its extreme values

And is denoted as $\omega(I) = \sup f - \inf f$

Lebesgue criterion of Riemann-Integrability #theorem

Let f be a bounded function on $[a,b]$

Then f is integrable on $[a,b]$ iff

$$\forall \{P_n\} : \lambda(P_n) \underset{n\to\infty}{\to} 0 : \sum_{i=1}^{n} \omega_i \Delta x_i \to 0$$

Where $\forall i \in [1,n] : w_i = \omega([x_i, x_{i-1}])$

Exaplanation (not proof):

$\boxed{\implies}$ Let f be integrable on $[a,b]$

$\implies$ All Riemann sums tend to L

$$\sum_{i=1}^{n} \omega_i \Delta x_i = \sum_{i=1}^{n} (\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f) \Delta x_i =$$

$$= \underbrace{\sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \Delta x_i}_{\text{Upper Riemann sum}} - \underbrace{\sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \Delta x_i}_{\text{Lower Riemann sum}} \to L - L = 0$$

Why is it not formal proof:

There might be no point c_i such that $f(c_i) = \sup\limits_{[x_{i-1},x_i]} f$

$\implies$ These sums might not be Riemann sums at all

$\boxed{\impliedby}$ Let $\forall \{P_n\} : \lambda(P_n) \underset{n\to\infty}{\to} 0 : \sum_{i=1}^{n} \omega_i \Delta x_i \to 0$

$$\sum_{i=1}^{n} \omega_i \Delta x_i = \sum_{i=1}^{n} (\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f) \Delta x_i = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \Delta x_i - \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \Delta x_i$$

$$\sum_{i=1}^{n} \omega_i \Delta x_i \to 0 \implies \begin{cases} \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \Delta x_i \to L \\ \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \Delta x_i \to L \end{cases}$$

$$c_i \in [x_{i-1}, x_i] \implies \inf_{[x_{i-1},x_i]} f \le f(c_i) \le \sup_{[x_{i-1},x_i]} f$$

$$\implies \underbrace{\sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \Delta x_i}_{\to L} \le S_R(f,P,C) \le \underbrace{\sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \Delta x_i}_{\to L}$$

$\implies S_R(f,P,C) \to L \implies f$ is Riemann-integrable

Why is it not formal proof:

$$x_n - y_n \to 0 \not\Longrightarrow \begin{cases} x_n \to L \\ y_n \to L \end{cases}$$

x_n and y_n might not have a limit at all