Determinant

$$\text{Determinant is a way to calculate area (volume in 3D, etc.) of a figure}$$
$$\text{defined by } n \text{ vectors } v_1, v_2, v_3, \dots, v_n$$

Permutation #definition

$$\text{Permutation } \sigma : [n] \to [n] \text{ is a bijective function}$$
$$\text{is a set of values } \{\sigma(1), \sigma(2), \dots, \sigma(n)\}$$

$$\text{For example:}$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$$
$$\text{Which can in turn be expressed as } (2,5,4)(3,1)$$

$$(1,3,2) \neq (1,2,3) = (2,3,1)$$

Transposition #definition

$$\text{Let } i, j \in [n]$$
$$(i,j) \text{ is called transposition if } i < j \text{ and } \sigma(i) > \sigma(j)$$

$$(1,2,3) \implies \begin{cases} \sigma(1) = 2 \\ \sigma(2) = 3 \\ \sigma(3) = 1 \end{cases} \implies \begin{cases} 1 < 2, \sigma(1) < \sigma(2) \\ \boxed{2 < 3, \sigma(2) > \sigma(3)} \\ \boxed{1 < 3, \sigma(1) > \sigma(3)} \end{cases}$$

Permutation sign #definition

$$\text{Let } k \text{ be the number of transpositions in } \sigma$$
$$\text{Permutation sign is then calculated and denoted as } sgn(\sigma) = (-1)^k$$

Symmetric group #definition

$$\text{Set of all possible permutations of } [n] \text{ is denoted as}$$
$$S_n = \{\{\sigma(1), \sigma(2), \dots, \sigma(n)\} | \sigma : [n] \to [n] \text{ is bijective}\}$$

$$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\} = \begin{cases} (1)(2) \\ (1,2) \end{cases}$$

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\} =$$
$$= \begin{cases} (1)(2)(3) \to sgn(\sigma) = (-1)^0 \\ (1)(2,3) \to sgn(\sigma) = (-1)^1 \\ (1,2)(3) \to sgn(\sigma) = (-1)^1 \\ (1,2,3) \to sgn(\sigma) = (-1)^2 \\ (1,3,2) \to sgn(\sigma) = (-1)^2 \\ (1,3)(2) \to sgn(\sigma) = (-1)^3 \end{cases}$$

Determinant #definition

$$\text{Let } |\,| : \mathbb{F}^{n \times n} \to \mathbb{F}$$
$$\forall A \in \mathbb{F}^{n \times n} : |A| = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \dots \cdot a_{n\sigma(n)}$$
$$\text{Sometimes denoted as } \det(A)$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \implies \det(A) = |A| = \underbrace{a_{11}a_{22}}_{\sigma(1)=1,\sigma(2)=2,sgn(\sigma)=1} - \underbrace{a_{12}a_{21}}_{\sigma(1)=2,\sigma(2)=1,sgn(\sigma)=-1}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \implies \det(A) = |A| =$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} =$$
$$= 45 - 48 - 72 + 84 + 96 - 105 = 0$$

Determinant of triangular matrix #lemma

Let $A \in \mathbb{F}^{n\times n}$ be a triangular matrix

$$\text{Then } |A| = \prod_{i=1}^{n} a_{ii}$$

Proof:

Let $\sigma \neq I$

$$\implies \exists k \geq 2 : (x_1, \dots, x_k) \in \sigma$$

Case 1. Let $\exists j \in [k-1] : x_j > x_{j+1}$

$$\implies x_j > \sigma(x_j) \implies a_{x_j\sigma(x_j)} = 0 \implies sgn(\sigma)a_{1\sigma(1)} \dots a_{n\sigma(n)} = 0$$

Let $x_1 < x_2 < \dots < x_k$

$$\implies x_k > x_1 = \sigma(x_k) \implies a_{x_k\sigma(x_k)} = 0 \implies sgn(\sigma)a_{1\sigma(1)} \dots a_{n\sigma(n)} = 0$$

$$\implies |A| = \prod_{i=1}^{n} a_{ii}$$

Similar proof for lower-triangular matrix

Row-linearity of determinant #lemma

Let$A \in \mathbb{F}^{n\times n}$

Let $i \in [n]$

Let $\exists v, u \in \mathbb{F}^n : R_i(A) = v + \alpha u$

$$\text{Let } A_v = \begin{cases} R_j(A_v) = R_j(A) & j \neq i \\ R_j(A_v) = v & j = i \end{cases}$$

$$\text{Let } A_u = \begin{cases} R_j(A_v) = R_j(A) & j \neq i \\ R_j(A_v) = u & j = i \end{cases}$$

Then $|A| = |A_v| + \alpha|A_u|$

Determinant of a matrix with two equal rows #lemma

Let $A \in \mathbb{F}^{n\times n}$

Let $\exists i \neq j \in [n] : R_i(A) = R_j(A)$

Then $|A| = 0$

Determinant of a matrix with a zero row #lemma

Let $\exists i \in [n] : R_i(A) = 0$

Then $|A| = 0$

Proof:

$$\forall \sigma : a_{i\sigma(i)} = 0 \implies \forall \sigma : sgn(\sigma)a_{1\sigma(1)} \dots a_{n\sigma(n)} = 0$$

$$\implies |A| = 0$$

Determinant after elementary row-operations #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$
$$\text{Let } B = p(A)$$
$$\text{Then } \begin{cases} p : \alpha R_i \implies |B| = \alpha|A| \\ p : R_i \leftrightarrow R_j \implies |B| = -|A| \\ p : R_i + \alpha R_j \implies |B| = |A| \end{cases}$$

$$\text{Proof:}$$
$$\text{Let } A = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$
$$\text{Let } p : \alpha R_i$$
$$\implies B = \begin{pmatrix} v_1 \\ \vdots \\ \alpha v_i \\ \vdots \\ v_n \end{pmatrix} \implies R_i(B) = 0 + \alpha v_i \implies |B| = |B_0| + \alpha|B_{v_i}| = 0 + \alpha|A| = \alpha|A|$$

$$\text{Let } p : R_i \leftrightarrow R_j$$
$$\text{Let } i > j \quad \text{(WLOG)}$$
$$\text{Let } X = \begin{pmatrix} v_1 \\ \vdots \\ v_i + v_j \\ \vdots \\ v_i + v_j \\ \vdots \\ v_n \end{pmatrix} \implies 0 = |X| = \begin{vmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_i + v_j \\ \vdots \\ v_n \end{vmatrix} + \begin{vmatrix} v_1 \\ \vdots \\ v_j \\ \vdots \\ v_i + v_j \\ \vdots \\ v_n \end{vmatrix} =$$
$$= \begin{vmatrix} -v_1- \\ \vdots \\ -v_i- \\ \vdots \\ -v_i- \\ \vdots \\ -v_n- \end{vmatrix} + \begin{vmatrix} -v_1- \\ \vdots \\ -v_i- \\ \vdots \\ -v_j- \\ \vdots \\ -v_n- \end{vmatrix} + \begin{vmatrix} -v_1- \\ \vdots \\ -v_j- \\ \vdots \\ -v_i- \\ \vdots \\ -v_n- \end{vmatrix} + \begin{vmatrix} -v_1- \\ \vdots \\ -v_j- \\ \vdots \\ -v_j- \\ \vdots \\ -v_n- \end{vmatrix} = 0 + |A| + |B| + 0$$
$$\implies |A| + |B| = 0 \implies |B| = -|A|$$

$$\text{Let } p : R_i + \alpha R_j$$
$$\implies B = \begin{pmatrix} v_1 \\ \vdots \\ v_i + \alpha v_j \\ \vdots \\ v_n \end{pmatrix} \implies |B| = |A| + \alpha \begin{vmatrix} -v_1- \\ \vdots \\ -v_j- \\ \vdots \\ -v_j- \\ \vdots \\ -v_n- \end{vmatrix} = |A|$$

Properties of elementary row-operations determinant #lemma

$$\text{Let } A \in \mathbb{F}^{n\times n}$$

$$\begin{gathered}
p : \alpha R_i \implies |p(I)| = \alpha|I| = \alpha \\
p : R_i \leftrightarrow R_j \implies |p(I)| = -|I| = -1 \\
p : R_i + \alpha R_j \implies |p(I)| = |I| = 1 \\
\implies |p(I)A| = |p(I)| \cdot |A| \\
\implies \text{(Simple proof by induction) } \left|\left(\prod_{i=1}^{k} p_i(I)\right) A\right| = \left(\prod_{i=1}^{k} |p_i(I)|\right)|A| \\
\implies \forall A, B \in \mathbb{F}^{n\times n} : A = \left(\prod_{i=1}^{k} p_i\right) B \implies \exists \alpha \neq 0 \in \mathbb{F} : |A| = \alpha|B|
\end{gathered}$$

Invertibility of matrix and determinant #theorem

$$\begin{gathered}
\text{Let } A \in \mathbb{F}^{n\times n} \\
\text{Then } |A| \neq 0 \iff A \text{ is invertible}
\end{gathered}$$

$$\begin{gathered}
\text{Proof:} \\
\text{Let } A \text{ be non-invertible} \\
\implies \exists i \in [n] : R_i(CF(A)) = 0 \implies |CF(A)| = 0 = \alpha|A| \underbrace{\implies}_{\alpha\neq 0} |A| = 0 \\
\text{Let } |A| = 0 \\
\implies |A| = \alpha|CF(A)| = 0 \\
\alpha \neq 0 \implies |CF(A)| = 0 \\
|I| \neq 0 \implies CF(A) \neq I \implies A \text{ is not invertible}
\end{gathered}$$

Determinant of two matrix product #lemma

$$\begin{gathered}
\text{Let } A, B \in \mathbb{F}^{n\times n} \\
\text{Then } |AB| = |A| \cdot |B|
\end{gathered}$$

$$\begin{gathered}
\text{Proof:} \\
\text{Let } A \text{ be non-invertible} \\
\implies |A| = 0 \implies |A| \cdot |B| = 0 \\
rank(A) < n \implies rank(AB) \leq rank(A) < n \implies AB \text{ is non-invertible} \\
\implies |AB| = 0 = |A| \cdot |B|
\end{gathered}$$

$$\begin{gathered}
\text{Let } A \text{ be invertible} \\
\implies A = \left(\prod_{i=1}^{k} p_i(I)\right) CF(A) = \prod_{i=1}^{k} p_i(I) \implies |A| = \left|\prod_{i=1}^{k} p_i(I)\right| = \prod_{i=1}^{k} |p_i(I)| \\
\implies AB = \left(\prod_{i=1}^{k} p_i(I)\right) B \\
\implies |AB| = \left|\left(\prod_{i=1}^{k} p_i(I)\right) B\right| = \left(\prod_{i=1}^{k} |p_i(I)|\right)|B| = |A| \cdot |B|
\end{gathered}$$