Triangularizable matrix

$$A \text{ is called triangularizable iff } \exists T : T \text{ is a triangular matrix} : A \sim T$$

$$A \sim T \iff P_A(\lambda) \text{ is factorizable into linear factors}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 0 & 6 & 0 \end{pmatrix}$$
$$\text{Over } \mathbb{C} - \text{ all matrices are triangularizable}$$
$$\text{Over } \mathbb{R}?$$
$$P_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 & -3 \\ -4 & \lambda & -2 \\ 0 & -6 & \lambda \end{vmatrix} = 6 \cdot \begin{vmatrix} \lambda-1 & -3 \\ -4 & -2 \end{vmatrix} + \lambda \begin{vmatrix} \lambda-1 & -2 \\ -4 & \lambda \end{vmatrix} =$$
$$= 12 - 12\lambda - 72 + \lambda^2(\lambda-1) - 8\lambda =$$
$$= \lambda^3 - \lambda^2 - 20\lambda - 60 = -(\lambda-6)(\lambda^2+5\lambda+10)$$
$$\implies A \nsim T$$

$$\text{Let } A \in \mathbb{R}^{n\times n} \text{ be nilpotent}$$
$$\text{Determine whether } A \text{ is triangularizable}$$
$$\text{Determine whether } A \text{ is diagonalizable}$$

$$\text{Solution:}$$
$$A \sim D \iff A = 0$$
$$P_A(\lambda) = \lambda^n \implies A \sim T$$

Triangularization

1. Find $P_A(\lambda)$ and eigenvalues
2. Find eigenspace for each eigenvalue
3. Add vectors to the union of eigenspaces to get a basis of $\mathbb{F}^n$
4. Assign each vector from the basis to be a column of P

$$\text{Resulting matrix is: } P^{-1}AP = \begin{pmatrix} D & * \\ 0 & B \end{pmatrix}$$
$$\text{Where } D \text{ is a diagonal matrix with eigenvalues on the diagonal}$$
$$\text{Each eigenvalue is featured } \gamma_A(\lambda_i) \text{ times}$$
$$\text{And } B \text{ is a matrix with eigenvalues of } A$$
$$\text{For each eigenvalue: } \gamma_B(\lambda_i) = \mu_A(\lambda_i) - \gamma_A(\lambda_i)$$

5.Repeat for B

6. Repeat until not triangularized

$$A=\begin{pmatrix}-1&-3&-4&-5\\1&1&-1&-3\\2&5&9&12\\-1&-2&-3&-3\end{pmatrix}$$

$$P_A(\lambda)=\begin{vmatrix}\lambda+1&3&4&5\\-1&\lambda-1&1&3\\-2&-5&\lambda-9&-12\\1&2&3&\lambda+3\end{vmatrix}\overset{R_1+R_i}{=}\begin{vmatrix}\lambda-1&\lambda-1&\lambda-1&\lambda-1\\-1&\lambda-1&1&3\\-2&-5&\lambda-9&-12\\1&2&3&\lambda+3\end{vmatrix}=$$

$$=(\lambda-1)\begin{vmatrix}1&1&1&1\\-1&\lambda-1&1&3\\-2&-5&\lambda-9&-12\\1&2&3&\lambda+3\end{vmatrix}=(\lambda-1)\begin{vmatrix}1&0&0&0\\-1&\lambda&2&4\\-2&-3&\lambda-7&-10\\1&1&2&\lambda+2\end{vmatrix}=$$

$$=(\lambda-1)\begin{vmatrix}\lambda&2&4\\-3&\lambda-7&-10\\1&2&\lambda+2\end{vmatrix}=(\lambda-1)\begin{vmatrix}\lambda&2&4\\0&\lambda-1&3\lambda-4\\1&2&\lambda+2\end{vmatrix}=$$

$$=(\lambda-1)(\lambda(\lambda-1)(\lambda+2)-2\lambda(3\lambda-4)+(6\lambda-8-4\lambda+4))=$$

$$=(\lambda-1)(\lambda(\lambda+2)(\lambda-1)-6\lambda(\lambda-1)+4(\lambda-1))=$$

$$=(\lambda-1)^2(\lambda(\lambda+2)-6\lambda+4)=(\lambda-1)^2(\lambda-2)^2$$

$$\lambda=1\implies\begin{pmatrix}2&3&4&5\\-1&0&1&3\\-2&-5&-8&-12\\1&2&3&4\end{pmatrix}\to\begin{pmatrix}1&1&1&1\\0&1&2&4\\0&-3&-6&-10\\0&1&2&3\end{pmatrix}$$

$$\to\begin{pmatrix}1&1&1&1\\0&1&2&4\\0&0&0&1\\0&0&0&0\end{pmatrix}\implies E_1=sp\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}\right\}$$

$$\ldots\implies E_2=sp\left\{\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}\right\}$$

$$\implies P=\begin{pmatrix}1&1&0&0\\-2&0&0&0\\1&-2&1&0\\0&1&0&1\end{pmatrix}$$

$$P^{-1}=\begin{pmatrix}0&-\frac{1}{2}&0&0\\1&\frac{1}{2}&0&0\\2&\frac{3}{2}&1&0\\-1&-\frac{1}{2}&0&1\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}1&0&*&*\\0&2&*&*\\0&0&-\frac{1}{2}&-\frac{1}{2}\\0&0&\frac{3}{2}&\frac{7}{2}\end{pmatrix}$$

$$P_B(\lambda)=(\lambda-1)(\lambda-2)$$

$$\implies\hat{P}^{-1}B\hat{P}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$$

$$\begin{pmatrix}I&0\\0&\hat{P}\end{pmatrix}^{-1}=\begin{pmatrix}I&0\\0&\hat{P}^{-1}\end{pmatrix}$$

$$\implies\begin{pmatrix}I&0\\0&\hat{P}\end{pmatrix}^{-1}P^{-1}AP\begin{pmatrix}I&0\\0&\hat{P}\end{pmatrix}=\begin{pmatrix}D&*\\0&D\end{pmatrix}$$