Conjugate linear operator

$$T : V \to W$$
$$T^* : W \to V$$
$$\forall v \in V, w \in W : \langle Tv, w \rangle = \langle v, T^* w \rangle$$

Properties

$$\begin{array}{lc} 1. & T^* \text{ is a linear operator and unique} \\ 2. & I^* = I \\ 3. & (T+S)^* = T^* + S^* \\ 4. & (T^*)^* = T \\ 5. & B, C \text{ orthonormal bases} \implies [T^*]_B^C = ([T]_C^B)^* \\ 6. & (ST)^* = T^* S^* \end{array}$$

$$\text{Proofs go as follows:}$$
$$2.$$
$$\forall v \in V : \langle v, v \rangle = \langle Iv, v \rangle = \langle v, I^* v \rangle \implies I^* = I$$

$$3.$$
$$\forall v \in V : \langle v, (T+S)^* w \rangle = \langle (T+S)v, w \rangle = \langle Tv, w \rangle + \langle Sv, w \rangle = \langle v, T^* w \rangle + \langle v, S^* w \rangle =$$
$$= \langle v, (T^* + S^*)w \rangle \implies (T+S)^* = T^* + S^*$$

$$4.$$
$$\forall w \in W : \langle Tv, w \rangle = \langle v, T^* w \rangle = \overline{\langle T^* w, v \rangle} = \overline{\langle w, (T^*)^* v \rangle} = \langle (T^*)^* v, w \rangle$$
$$\implies (T^*)^* = T$$
$$\text{5. and 6. in lecture 10}$$

$$V = \mathbb{R}^2 \text{ with inner product } \langle v, u \rangle = 2v_1u_1 + v_2u_2$$
$$W = \mathbb{R}^2 \text{ with standrad inner product}$$

$$T : V \to W$$
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 2y \end{pmatrix}$$
$$\text{Find } T^*$$

$$\text{Solution:}$$
$$\text{Let } B = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$
$$\text{Let } C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$
$$[T]_C^B = \begin{pmatrix} \sqrt{2} & 3 \\ 0 & 2 \end{pmatrix}$$
$$\implies [T^*]_B^C = ([T]_C^B)^* = ([T]_C^B)^T = \begin{pmatrix} \sqrt{2} & 0 \\ 3 & 2 \end{pmatrix}$$
$$[T^*\begin{pmatrix} 1 \\ 0 \end{pmatrix}]_B = \begin{pmatrix} \sqrt{2} \\ 3 \end{pmatrix} \implies T^*\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$[T^*\begin{pmatrix} 0 \\ 1 \end{pmatrix}]_B = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \implies T^*\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$
$$\implies \boxed{T^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x + 2y \end{pmatrix}}$$
$$\text{Alternative solution:}$$
$$[T^*\begin{pmatrix} x \\ y \end{pmatrix}]_B = [T^*]_B^C[\begin{pmatrix} x \\ y \end{pmatrix}]_C = \begin{pmatrix} \sqrt{2} & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{2} \cdot x \\ 3x + 2y \end{pmatrix}$$
$$\implies \boxed{T^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x + 2y \end{pmatrix}}$$

Let V, W be inner product spaces over $\mathbb{F}$

Let B be an orthonormal basis of V

Let $T : V \to W$

$$\text{Prove: } \forall w \in W : T^* w = \sum_{i=1}^{n} \overline{\langle Tv_i, w\rangle} v_i$$

Proof:

Let $w \in W$

Let $v \in V$

$$v = \sum_{i=1}^{n} \alpha_i v_i \left(= \sum_{i=1}^{n} \langle v, v_i\rangle v_i \right)$$

$$\left\langle v, \sum_{i=1}^{n} \overline{\langle Tv_i, w\rangle} v_i \right\rangle = \left\langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{i=1}^{n} \overline{\langle Tv_i, w\rangle} v_i \right\rangle =$$

$$= \sum_{i=1}^{n} \alpha_i \langle Tv_i, w\rangle \langle v_i, v_i\rangle = \sum_{i=1}^{n} \alpha_i \langle Tv_i, w\rangle = \left\langle \sum_{i=1}^{n} \alpha_i T v_i, w \right\rangle =$$

$$= \left\langle \sum_{i=1}^{n} T(\alpha_i v_i), w \right\rangle = \left\langle T\left(\sum_{i=1}^{n} \alpha_i v_i \right), w \right\rangle = \langle Tv, w\rangle$$

Alternative solution:

$$\left\langle v, \sum_{i=1}^{n} \overline{\langle Tv_i, w\rangle} v_i \right\rangle = \sum_{i=1}^{n} \langle Tv_i, w\rangle \langle v, v_i\rangle = \sum_{i=1}^{n} \langle \langle v, v_i\rangle Tv_i, w\rangle =$$

$$= \sum_{i=1}^{n} \langle T(\langle v, v_i\rangle v_i), w\rangle = \left\langle \sum_{i=1}^{n} T(\langle v, v_i\rangle v_i), w \right\rangle = \left\langle T\left(\sum_{i=1}^{n} \langle v, v_i\rangle v_i \right), w \right\rangle = \langle Tv, w\rangle$$

$$\text{Because of: } v = \sum_{i=1}^{n} \langle v, v_i\rangle v_i$$

$$\implies \langle Tv, w\rangle = \left\langle v, \sum_{i=1}^{n} \overline{\langle Tv_i, w\rangle} v_i \right\rangle = \langle v, T^* w\rangle$$

$$\implies \boxed{T^* w = \sum_{i=1}^{n} \overline{\langle Tv_i, w\rangle} v_i}$$

Let $V = \mathbb{R}^{2\times 2}$ with standard inner product $\langle A, B\rangle = tr(AB^*)$

Let $W = \mathbb{R}^2$ with inner product $\langle v, w\rangle = v^T \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} w$

Let $T : V \to W$

$T(A) = C_1(A) + C_2(A)$

Find T^*

Solution:

Let B be a standard basis of V

$$T^*(w) = \sum_{i=1}^{4} \overline{\langle TE_i, w\rangle} E_i$$

$$\forall i \in [1, 4 :] T(E_i) = C_1(E_i) + C_2(E_i) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & i \le 2 \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & i > 2 \end{cases}$$

$$i = 1, 2 \implies \langle TE_i, w\rangle = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = (1 \quad 0) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x - y$$

$$i = 3, 4 \implies \langle TE_i, w\rangle = (0 \quad 1) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -x + 2y$$

$$\implies \boxed{T^*(w) = (x - y)(E_1 + E_2) + (-x + 2y)(E_3 + E_4) = \begin{pmatrix} x - y & x - y \\ -x + 2y & -x + 2y \end{pmatrix}}$$

Let W be T-invariant

Prove or disprove:

1. $W^\perp$ is T-invariant
2. $W^\perp$ is T^*-invariant

Disproof for 1:

$$T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$W=sp\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

Proof for 2:

$$T:V\to V$$

$$T[W]\subseteq W$$

Let $v\in W^\perp$

Let $w\in W$

$$\langle w,v\rangle=0$$

$$0\underset{Tw\in W}{=}\langle Tw,v\rangle=\langle w,T^*v\rangle$$

$$\implies T^*v\in W^\perp\implies T^*[W^\perp]\subseteq W^\perp$$