Extremums

Local extremums #definition

To find local extremums, we have to go through two steps:

1. Find critical points
2. Determine which kind of extremum they are, local minimum or maximum, if any

Step 1:
For functions of one variable, critical points are points where: $f' = 0$
For functions of multiple variables, we do the same, but with gradient: $\nabla f = 0$

Step 2:
For functions of one variable, we can check the behaviour of the function
If it increases/decreases up to the point and then decreases/increases,
the point is a local maximum/minimum. otherwise it is not an extremum
For function of multiple variables, it is problematic as there are infinite paths to the point

Another option is to check the second derivative
If it is positive/negative, the point is minimum/maximum, if it is zero, there is no result
We can continue to differentiate until we get the result, but we'll stop at the second
For functions of multiple variables, matrix H_f named Hessian, is defined to consist of all
partial second derivatives, e.g. for function of two or three variables:

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}, H_g = \begin{pmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

Leading principal minors of size i are then denoted as for example:

$$M_1 = (g_{xx})$$
$$M_2 = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}$$
$$M_3 = H_g$$
$$\forall i \in [1, n] : \det(M_i) > 0 \implies \text{Point is a local minimum}$$
$$\forall i \in [1, n] : (-1)^i \det(M_i) > 0 \implies \text{Point is a local maximum}$$
$$\exists i \in [1, n] : (-1)^i \det(M_i) < 0 \implies \text{Point is called a saddle}$$

Otherwise, no conclusion

This can also be written down via eigenvalues:

$$\forall i \in [1, n] : \lambda_i > 0 \implies \text{Point is a local minimum}$$
$$\forall i \in [1, n] : \lambda_i < 0 \implies \text{Points is a local maximum}$$

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\text{Note: further reading - Sylvester's criterion} \
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f(x, y) = 3(x^{2} + y^{2}) + x^{3} + 4y \
\text{Find critical points of } f \text{ and categorize them} \
\
\text{Solution:} \
\nabla f = \begin{pmatrix}
f{x} \
f{y} \
\end{pmatrix} = \begin{pmatrix}

6x + 3x^{2} \
6y + 4 \
\end{pmatrix} \
\nabla f = 0 \iff \left{\begin{array}{}
\left[\begin{array}{}
x = 0 \
x = -2 \
\end{array}\right. \
y = -\frac{2}{3} \
\end{array}\right. \iff \left[\begin{array}{}
\left(0, -\frac{2}{3} \right) \
\left(-2, -\frac{2}{3} \right) \
\end{array}\right. \
H{f} = \begin{pmatrix}
f{xx} & f{xy} \
f{yx} & f{yy} \
\end{pmatrix} = \begin{pmatrix}
6x + 6 & 0 \
0 & 6 \
\end{pmatrix} \
H{f}\left(0, -\frac{2}{3} \right) = \begin{pmatrix}
6 & 0 \
0 & 6 \
\end{pmatrix} \implies \boxed{ \left(0, -\frac{2}{3} \right) \text{ is a local minimum} } \
H_{f}\left(-2, -\frac{2}{3} \right) = \begin{pmatrix}
-6 & 0 \
0 & 6 \
\end{pmatrix} \implies \boxed{ \left(-2, -\frac{2}{3} \right) \text{ is a saddle} } \
}

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1. & \text{Saddle is a point where in some paths it is a local maximum and in some a minimum} \
   & \text{It is named after the form of saddle, or Pringles} \
2. & \text{For functions of one variable, there is a connection between the number of maximums} \
   & \text{and minimums, between any two must be an opposite} \
   & \text{For functions of multiple variable there aren't} \
   }
   $$

$$f(x,y,z) = x^2 + y^2 + z^2 - xy + yz - xz - 4x + 6y + 2z$$

$$\nabla f = \begin{pmatrix} 2x - y - z - 4 \\ 2y - x + z + 6 \\ 2z + y - x + 2 \end{pmatrix}$$

$$\nabla f = 0 \iff \begin{cases} 2x - y - z - 4 = 0 \\ -x + 2y + z + 6 = 0 \\ -x + y + 2z + 2 = 0 \end{cases} \iff \begin{cases} x + y + 2 = 0 \\ -x + 2y + z + 6 = 0 \\ -x + y + 2z + 2 = 0 \end{cases}$$

$$\iff \begin{cases} y = -2 - x \\ -x + 2y + z + 6 = 0 \\ -x + y + 2z + 2 = 0 \end{cases} \iff \begin{cases} y = -2 - x \\ x = 1 \\ z = x \end{cases} \iff \begin{cases} x = 1 \\ y = -3 \\ z = 1 \end{cases}$$

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

$$\det(M_1) = 2 > 0$$

$$\det(M_2) = 3 > 0$$

$$\det(M_3) = 4 > 0$$

$$f(x,y,z) = x^3 + y^3 + z^3 - 3xz - 3yz - 3xy$$

$$\nabla f = \begin{pmatrix} 3x^2 - 3z - 3y \\ 3y^2 - 3z - 3x \\ 3z^2 - 3x - 3y \end{pmatrix}$$

$$\nabla = 0 \iff \begin{cases} x^2 - z - y = 0 \\ y^2 - z - x = 0 \\ z^2 - x - y = 0 \end{cases} \iff \begin{cases} x^2 - z - y = 0 \\ y^2 - x^2 + y - x = 0 \\ z^2 - x - y = 0 \end{cases}$$

$$\iff \begin{cases} x^2 - z - y = 0 \\ (y - x)(y + x + 1) = 0 \\ z^2 - x - y = 0 \end{cases}$$

$$y = x \implies \begin{cases} x^2 - z - x = 0 \\ z^2 - x - x = 0 \end{cases} \implies \begin{cases} 2x = z^2 \\ x^2 - z - x = 0 \end{cases}$$

$$\implies \begin{cases} x = \frac{z^2}{2} \\ \frac{z^4}{4} - \frac{z^2}{2} - z = z\left(z^3 - 2z - 4\right) = 0 \end{cases} \implies \begin{cases} x = \frac{z^2}{2} \\ z(z-2)(z^2 + 2z + 2) = 0 \end{cases}$$

$$\implies \boxed{\left[\begin{array}{l} x = 0, y = 0, z = 0 \\ x = 2, y = 2, z = 2 \end{array}\right.}$$

$$y = -x - 1 \implies \begin{cases} x^2 - z + x + 1 = 0 \\ z^2 - x + x + 1 = 0 \end{cases} \implies \begin{cases} x^2 - z + x + 1 = 0 \\ z^2 = -1 \end{cases} \implies \text{No solutions}$$

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 6x & -3 & -3 \\ -3 & 6y & -3 \\ -3 & -3 & 6z \end{pmatrix}$$

$$\det(M_1) = 6x$$

$$\det(M_2) = 36xy - 9$$

$$\det(M_3) = 216xyz + 27 + 27 - 54y - 54z - 54x$$

$$(0,0,0) \implies \det(M_2) < 0 \implies \boxed{(0,0,0) \text{ is a saddle}}$$

$$\boxed{(2,2,2) \text{ is a local minimum}}$$