# Functions of multiple variables #definition

$$f : A \to \mathbb{R}$$
$$A \subseteq \mathbb{R}^n$$
$$\text{For exampel: } f : \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = x^2 z - \sin(y) + x$$
$$\text{Note: we are only talking about real scalar functions, that is, about functions with range } \mathbb{R}$$

# Limits and continuity

$$\text{For one variable:}$$
$$\lim_{x \to a} f(x) = L \iff \forall \{x_m\} \to a : \{f(x_m)\} \to L$$
$$\text{For multiple variables:}$$
$$\text{Let } x, a \in A \subseteq \mathbb{R}^n$$
$$\lim_{x \to a} f(x) = L \iff \forall \{x_m\} \to a : \{f(x_m)\} \to L$$
$$\text{Meaning that every "path" of sources leading to } a \text{ leads images to } L$$
$$\text{There is an infinite number of such paths for two or more variables}$$
$$\implies \text{To show that there is no limit, it is enough to find two "paths" with different results}$$

## Examples

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$
$$\text{Let } (x,y)_n = \left(\frac{1}{n}, \frac{1}{n}\right)$$
$$\text{Let } (x,y)_m = \left(\frac{1}{m}, \frac{2}{m}\right)$$
$$\lim_{(x,y)\to(0,0)} = \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} = 0$$
$$\lim_{(x,y)\to(0,0)} = \lim_{m\to\infty} \frac{\frac{1}{m^2} - \frac{4}{m^2}}{\frac{1}{m^2} + \frac{4}{m^2}} = -\frac{3}{5}$$
$$\implies \boxed{\nexists \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}}$$

$$\text{Alternative solution:}$$
$$\text{Let } y = 0$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$
$$\text{Let } x = 0$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} -\frac{y^2}{y^2} = -1$$
$$\implies \boxed{\nexists}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$
$$\text{Let } y = x$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$
$$\text{Let } y = -x$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} -\frac{x^2}{2x^2} = -\frac{1}{2}$$
$$\text{Let } y = 2x$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{5x^2} = \frac{2}{5}$$
$$\implies \boxed{\nexists}$$

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$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$
$$\text{Let } y = 0$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{x\to 0} \frac{x^3\cdot 0}{x^6} = 0$$
$$\text{Let } y = x^3$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$$
$$\text{Let } y = -x^3$$
$$\implies \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{x\to 0} \frac{-x^6}{2x^6} = -\frac{1}{2}$$
$$\implies \boxed{\nexists}$$

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$$\lim_{(x,y)\to(0,0)} \frac{x^ny}{x^{2n}+y^2}$$
$$y = x^n \implies \lim = \frac{1}{2}$$
$$y = -x^n \implies \lim = -\frac{1}{2}$$
$$\implies \boxed{\nexists}$$

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But how do we show that there is a limit?

# Continuity

Continuity is defined the same way as for functions of one variable

$$f \text{ is continuous at } a \iff \lim_{x\to a} f(x) = f(a)$$

# How to find the limit? Special cases

If function is continuous, for example elementary, limit is easy to calculate:

$$\text{For example:} \quad \lim_{(x,y,z)\to(0,1,2)} \frac{x^2+y}{x^2+y^2+z^4} = \frac{0^2+1}{0^2+1^2+2^4} = \frac{1}{17}$$

If we can arrive to the limit of function of one variable, it is also easy to find the limit:

$$\lim_{\vec{x}\to 0} \frac{1}{\|x\|} e^{-1/\|x\|} = \left\{ \begin{array}{c} t = \|x\| \\ x \to 0 \implies \|x\| \to 0^+ \end{array} \right\} = \lim_{t\to 0^+} \frac{1}{t} e^{-1/t} =$$
$$= \left\{ u = \tfrac{1}{t} \right\} = \lim_{u\to\infty^+} \frac{u}{e^u} = 0$$

## Application of sandwich theorem

If, intuitively, we think that the limit is 0
We can show it with Sandwich theorem:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$
$$0 \le \left|\frac{x^3+y^3}{x^2+y^2}\right| = \left|\frac{x^3}{x^2+y^2} + \frac{y^3}{x^2+y^2}\right| \le$$
$$\le \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2}\right| + \left|\frac{y^3}{y^2}\right| = \underbrace{|x|}_{\to 0} + \underbrace{|y|}_{\to 0} \to 0$$
$$\implies \boxed{\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0}$$

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$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^6+z^6}{x^4+z^4}\cdot\frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{x^6+z^6}{x^4+z^4}\cdot \lim_{(x,y,z)\to(0,0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} =$$
$$= \begin{Bmatrix} t = x^2+y^2 \\ t\to 0^+ \end{Bmatrix} = \lim_{(x,y,z)\to(0,0,0)} \frac{x^6+z^6}{x^4+z^4}\cdot \lim_{t\to 0^+}\frac{\sin(t)}{t} =$$
$$= \lim_{(x,y,z)\to(0,0,0)} \frac{x^6+z^6}{x^4+z^4}\cdot 1$$

$$0 \le \left|\frac{x^6+z^6}{x^4+z^4}\right| \le \left|\frac{x^6}{x^4+z^4}\right| + \left|\frac{z^6}{x^4+z^4}\right| \le \left|\frac{x^6}{x^4}\right| + \left|\frac{z^6}{z^4}\right| =$$
$$= \underbrace{x^2}_{\to 0} + \underbrace{z^2}_{\to 0} \to 0$$
$$\implies \lim_{(x,y,z)\to(0,0,0)} \frac{x^6+z^6}{x^4+z^4} = 0$$
$$\implies \boxed{\lim_{(x,y,z)\to(0,0,0)} \frac{x^6+z^6}{x^4+z^4}\cdot\frac{\sin(x^2+y^2)}{x^2+y^2} = 0}$$

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$$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4+y^4}$$
Let $a, b > 0$
$$(a-b)^2 = a^2 - 2ab + b^2 = a^2 + 2ab + b^2 - 4ab = (a+b)^2 - 4ab \ge 0$$
$$\implies 4ab \le (a+b)^2$$
$$\implies \sqrt{ab} \le \frac{a+b}{2} \text{ when } a, b > 0$$
$$0 \le \left|\frac{x^3y^3}{x^4+y^4}\right| = \frac{\sqrt{x^6y^6}}{x^4+y^4} \le \frac{1}{2}\cdot \underbrace{\frac{x^6+y^6}{x^4+y^4}}_{\to 0} \to 0$$
$$\implies \boxed{\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4+y^4} = 0}$$

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## Polar coordinates #definition

Polar coordinates are coordinates derived from distance of $(x, y)$ from zero, denoted $r$

And signed angle $\theta$ between vector $\vec{(x, y)}$ and axis $X$

$$r = \sqrt{x^2 + y^2} = \|(x, y)\|$$

$$\sin\theta = \frac{y}{x} \implies \theta = \arcsin\left(\frac{y}{x}\right)$$

$$\implies \boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}}$$

$$(x, y) \to (0, 0) \implies r \to 0, \theta \text{ can be anything}$$

So, it is enough to show that for all $\theta$ the limit when $r \to 0$ is the same to show existence

$$\frac{x^3y^3}{x^4 + y^4} = \frac{r^3\cos^3\theta r^3\sin^3\theta}{r^4\cos^4\theta + r^4\sin^4\theta} = r^2 \cdot \frac{\sin^3\theta\cos^3\theta}{(\cos^4\theta + \sin^4\theta)}$$

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