1

Formulate and prove unitarity criterion

2a

Let $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ be a linear operator

$T(p(x)) = p(x) + p(x+1)$

Find Jordan form of T

Solution:

$$p(x) = a + bx + cx^2$$

$$T(p(x)) = p(x) + p(x+1) = a + bx + cx^2 + a + b(x+1) + c(x^2 + 2x + 1) =$$

$$= (2a + b + c) + (2b + 2c)x + 2cx^2$$

$$\implies T(1) = 2, T(x) = 2x + 1, T(x^2) = 2x^2 + 2x + 1$$

$$\implies A = [T]_S^S = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\implies P_T(x) = (x-2)^3$$

$$A - 2I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, (A - 2I)^2 \neq 0 \implies m_T(x) = (x-2)^3$$

$$\implies \text{Maximal size of Jordan block is 3}$$

$$\implies \boxed{J_T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}$$

2b

Let $\langle a + bx + cx^2, \alpha + \beta x + \gamma x^2 \rangle = a\alpha + b\beta x + c\gamma x^2$

Find vector $v \in \text{Im}T$ which is the closest to $1 - x^2$

Solution:

$$\text{Im}T = \left\{ (2a + b + c) + (2b + 2c)x + 2cx^2 \middle| a, b, c \in \mathbb{R} \right\}$$

$$\text{Let } \begin{cases} a = \frac{1}{2} \\ b = \frac{1}{2} \\ c = -\frac{1}{2} \end{cases}$$

$$\implies T(a + bx + cx^2) = T\left(\frac{1}{2} + \frac{1}{2}x - \frac{1}{2}x^2\right) = 1 - x^2$$

$$\implies 1 - x^2 \in \text{Im}T \implies \boxed{\text{The closest vector to } 1 - x^2 \text{ in Im}T \text{ is } 1 - x^2 \text{ itself}}$$

3a

Let $T : V \to V$ be a linear operator such that$T = -T^*$

Let λ be an eigenvalue of T

Prove: $\exists b \in \mathbb{R} : \lambda = bi$

Proof:

Let v be an eigenvector of eigenvalue λ

Let $a, b \in \mathbb{R} : \lambda = a + bi$

$$Tv = \lambda v \implies -T^*v = \lambda v \implies T^*v = -\lambda v$$

$$\implies -\lambda \text{ is an eigenvalue of } T^*$$

$$\implies \overline{\lambda} = -\lambda \implies a - bi = -a - bi$$

$$\implies a = -a \implies a = 0 \implies \boxed{\lambda = bi}$$

3b

Let V be an inner product space over $\mathbb{C}$

Let U, W be subspaces of V

Let P_U, P_W be orthogonal projections on U and W accordingly

Prove: $P_U P_W = 0 \iff \forall u \in U, \forall w \in W : \langle u, w \rangle = 0$

Proof:

$\boxed{\Longleftarrow}$ Let $\forall u \in U, \forall w \in W : \langle u, w \rangle = 0$

$\implies W = U^\perp$

Let $v \in V$

$\implies$ Let $u^\perp = P_W(v)$

$W = U^\perp \implies u^\perp \in U^\perp$

$\implies \forall v \in V : P_U P_W(v) = P_U P_{U^\perp}(v) = P_U(u^\perp) = 0 \implies \boxed{P_U P_W = 0}$

$\boxed{\implies}$ Let $P_U P_W = 0$

$\implies \forall v \in V : P_U P_W(v) = 0$

$\implies \forall v \in V : P_W(v) \in U^\perp$

$\implies \forall w \in W : w = P_W(w) \in U^\perp$

$\implies W \subseteq U^\perp \implies \boxed{\forall u \in U, \forall w \in W : \langle u, w \rangle = 0}$

4a

Let $A, B \in \mathbb{R}^{3\times 3}$

Let $v_1, v_2, v_3 \in \mathbb{R}^3$ linearly independant vectors which are all both eigenvector of A and B

Prove or disprove: $\forall P : A = PD_AP^{-1} \implies B = PD_BP^{-1}$

Disproof:

Let $A = I$

Let $v_1, v_2, v_3 = e_1, e_2, e_3$

$$\text{Let } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

v_1, v_2, v_3 are both eigenvectors of A and B

Let $v = v_1 + v_2 + v_3$

$Av = v_1 + v_2 + v_3 = v \implies v$ is an eigenvector of A

$Bv = v_1 + v_2 \neq v \implies v$ is not an eigenvector of B

$$\text{Let } P = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v \\ | & | & | \end{pmatrix}$$

$$A = P \underbrace{D_A}_{=I} P^{-1}$$

$$B \neq P \underbrace{D_B}_{=B} P^{-1}$$

4b

Let $A \in \mathbb{C}^{n\times n}$

Let $A^2 = I$

Prove or disprove: A is unitary

Disproof:

$A^2 = I \implies A^{-1} = A$

A is not necessarily Hermitian $\implies$ Not necessarily $A^{-1} = A^*$

Let us find an example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$A^* = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \neq A^{-1}$$

4c

Let $A \in \mathbb{F}^{n\times n}$

Prove or disprove: A is diagonalizable $\implies$ A^T is diagonalizable

Proof:

Let A be diagonalizable by matrix P

$$\implies A = PDP^{-1}$$

$$\implies A^T = (PDP^{-1})^T = (P^{-1})^T DP^T$$

$$PP^{-1} = I \implies (PP^{-1})^T = I \implies (P^{-1})^T P^T = I$$

$$\implies \boxed{A^T \text{ is diagonalizable by matrix } P^T}$$

5

Let V be a vector space over $\mathbb{F}$

Let $T : V \to V$ be a linear operator

Let α be an eigenvalue of T

Let $K_\alpha = \ker((T - \alpha I)^n) = \{v \in V | (T - \alpha I)^n(v) = 0\}$

5a

Let T be nilpotent over $\mathbb{C}$

Prove: $K_0 = V$

Proof:

T is nilpotent $\implies$ Its eigenvalues are all 0

$$\implies \alpha = 0 \implies K_\alpha = K_0 = \{v \in V | T^n(v) = 0\}$$

Let B be a basis of V

$$P_T(x) = P_{[T]_B^B}(x) = x^n$$

$\implies$ By Cayley-Hamilton theorem: $([T]_B^B)^n = 0$

$$\implies T^n = 0 \implies \boxed{K_0 = \ker T^n = V}$$

5b

Let T be a linear operator over $\mathbb{C}$ with a single eigenvalue α
Prove: $K_\alpha = V$

Proof:
Let B be a basis of V

$$\alpha \text{ is the only eigenvalue of } T \implies P_T(x) = (x-\alpha)^n$$
$$\implies \text{By Cayley-Hamilton theorem, } ([T]_B^B - \alpha I)^n = 0$$
$$[\alpha I]_B^B = \alpha I \implies [T]_B^B - \alpha I = [T]_B^B - [\alpha I]_B^B = [T-\alpha I]_B^B$$
$$\implies ([T-\alpha I]_B^B)^n = 0 \implies (T-\alpha I)^n = 0 \implies \boxed{K_\alpha = \ker((T-\alpha I)^n) = V}$$

5c

Let T be a linear operator over $\mathbb{C}$
Let α, μ be two different eigenvalues of T

1

Let $v \in K_\alpha$
Prove: $(T-\mu I)(v) \in K_\alpha$

Proof:

$$(T-\mu I)(T-\alpha I) = T^2 - \mu T - \alpha T + \alpha\mu I$$
$$(T-\alpha I)(T-\mu I) = T^2 - \alpha T - \mu T + \alpha\mu I$$
$$\implies (T-\mu I)(T-\alpha I) = (T-\alpha I)(T-\mu I)$$
$$(T-\alpha I)^n((T-\mu I)(v)) = (T-\alpha I)^n(T-\mu I)(v) =$$
$$= (T-\mu I)(T-\alpha I)^n(v) = (T-\mu I)(0) = 0$$
$$\implies \boxed{(T-\mu I)(v) \in \ker((T-\alpha I)^n) = K_\alpha}$$

2

Let $v \neq 0 \in K_\alpha$
Prove: $(T-\mu I)(v) \neq 0$

Proof:

$$\text{Let } (T-\mu I)(v) = 0 \implies T(v) = \mu v$$
$$v \in K_\alpha \implies (T-\alpha I)^n(v) = 0$$
$$TI = IT$$
$$\implies (T-\alpha I)^n(v) = \left(\sum_{k=0}^{n} \binom{n}{k} T^k (-\alpha I)^{n-k}\right)(v) = \left(\sum_{k=0}^{n} \binom{n}{k} (-\alpha)^{n-k} T^k\right)(v) =$$
$$= \sum_{k=0}^{n} \binom{n}{k} (-\alpha)^{n-k} T^k(v) = \sum_{k=0}^{n} \binom{n}{k} (-\alpha)^{n-k} \mu^k v = (\mu-\alpha)^n v = 0$$
$$v \neq 0 \implies (\mu-\alpha)^n = 0 \implies \mu = \alpha - \text{Contradiction!}$$
$$\implies \boxed{(T-\mu I)(v) \neq 0}$$

3

Prove: $K_\alpha \cap K_\mu = \{0\}$

Proof:

Let $v \in K_\alpha \cap K_\mu$

$\implies (T - \alpha I)^n(v) = 0 = (T - \mu I)^n(v)$

Let $v \neq 0$

$\implies (T - \mu I)(v) \neq 0 \in K_\alpha$

$\implies$ By Induction: $(T - \mu I)^n(v) \neq 0 \in K_\alpha$ – Contradiction!

$\implies v = 0 \implies \boxed{K_\alpha \cap K_\mu = \{0\}}$