Taylor series #definition

$$f(x) \text{ is infinitely differentiable} \implies f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$f(a) = a_0$$

$$f'(a) = a_1$$

$$f''(a) = 2a_2 \implies a_2 = \frac{f''(a)}{2}$$

$$f^{(3)}(a) = 6a^3 \implies a_3 = \frac{f^{(3)}(a)}{6} = \frac{f^{(3)}(a)}{3!}$$

$$\dots$$

$$\implies a_n = \frac{f^{(n)}(a)}{n!}$$

$$\implies f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin series #definition

$$\text{Taylor series with } a = 0$$

Examples

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sin x = \int_0^x \cos t \, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^x t^{2n} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Applications

$$\text{Let } g(x) = \frac{1}{1+x^2}$$

$$\text{Find } g^{(69420)}(0)$$

$$\text{Solution:}$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

$$\text{Let } t = -x^2$$

$$\implies g(x) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \implies a_n = \begin{cases} (-1)^k & n = 2k \\ 0 & n = 2k-1 \end{cases}$$

$$g^{(n)}(0) = n! \cdot a_n \implies g^{(69420)}(0) = 69420! \cdot (-1)^{69420/2} = 69420!$$

Approximations

Let $\varepsilon > 0$

Find value of $\dfrac{1}{e}$ with accuracy ε

Solution:

$$f(x) \approx P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Let for example $\varepsilon = \dfrac{1}{1000}$

And $f(x) = e^x = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$$x = -1 \implies \frac{1}{e} = f(-1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

For alternating series $|R_k(x)| \le |a_{k+1}|$

$$\left| f(x) - \sum_{n=0}^{k} \frac{(-1)^n}{n!} \right| = |R_k(x)|$$

$\implies$ We need to find minimal k such that $\left| \dfrac{(-1)^{k+1}}{(k+1)!} \right| < \varepsilon$

$\implies$ We need to find first k such that $(k+1)! > 1000$

$$\implies k = 6 \implies \frac{1}{e} = f(-1) \approx \sum_{n=0}^{6} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$