Linearity of definite integral #lemma

$$\int_a^b (\alpha f + g)(x)dx = \alpha \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\text{Proof:}$$
$$\sum(\alpha f + g)(c_i)\Delta x_i = \alpha \sum f(c_i)\Delta x_i + \sum g(c_i)\Delta x_i$$

Monotonicity of definite integral #lemma

$$f(x) \leq g(x) \implies \int_a^b f(x)dx \leq \int_a^b g(x)dx$$
$$0 \leq f(x) \implies 0 \leq \int_a^b f(x)dx$$
$$m \leq f(x) \leq M \implies m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Summation of definite integrals #lemma

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Absolute value of definite integral #lemma

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

$$\int_0^{2\pi} \sin(x)dx = 0$$
$$\int_0^{2\pi} |\sin(x)|dx = 4$$

Continuous function is Riemann-integrable #theorem

$$\text{Let } f \text{ be a continuous function on } [a,b]$$
$$\text{Then } f \text{ is Riemann-integrable}$$

$$\text{Proof:}$$
$$\text{Let } \{a_n\},\{b_n\} \subseteq [a,b] : a_n - b_n \to 0$$
$$f \text{ is continuous } \implies f(a_n) - f(b_n) \to 0$$

$$\text{Let } \{P_n\} : \lambda(P_n) \to 0$$

$$\text{Let } \omega_k \text{ be a maximal oscilate in } P_n$$
$$\implies \sum \omega_i \Delta x_i \le \sum \omega_k \Delta x_i = \omega_k \sum \Delta x_i = \omega_k(b-a)$$
$$0 \le \sum \omega_i \Delta x_i \le \omega_k(b-a)$$
$$\omega_k = \sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f$$
$$\text{By Weierstrass theorem: } \begin{cases} \sup_{[x_{k-1},x_k]} f = \max_{[x_{k-1},x_k]} f \\ \inf_{[x_{k-1},x_k]} f = \min_{[x_{k-1},x_k]} f \end{cases}$$
$$\implies \exists m_k, M_k \in [x_{k-1}, x_k] : \omega_k = \sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f = f(M_k) - f(m_k)$$
$$\lambda(P_n) \to 0$$
$$0 \le x_k - x_{k-1} \le \lambda(P_n) \implies x_k - x_{k-1} \to 0$$
$$0 \le |M_k - m_k| \le x_k - x_{k-1} \implies |M_k - m_k| \to 0 \implies M_k - m_k \to 0$$
$$\implies f(M_k) - f(m_k) \to 0 \implies \omega_k \to 0$$
$$\implies \omega_k(b-a) \to 0 \implies \sum \omega_i \Delta x_i \to 0$$
$$\implies \text{By Lebesgue criterion: } \boxed{f \text{ is Riemann-integrable}}$$

Bounded function with a finite number of discontinuities is Riemann-integrable #theorem

f is Riemann-integrable $\iff$ f is bounded and has a finite number of discontinuities

Explanation (not proof):

$\boxed{\implies}$ Let f be bounded

Let there be one discontinuity C

Let $\{P_n\} : \lambda(P_n) \to 0$

Let $C \in [x_{k-1}, x_k]$

$$\sum \omega_i \Delta x_i = \sum_{i<k} \omega_i \Delta x_i + \omega_k \Delta x_k + \sum_{i>k} \omega_i \Delta x_i$$

f is bounded $\implies$ ω_k is finite

$\Delta x_k \to 0 \implies \omega_k \Delta x_k \to 0$

$\sum_{i<k} \omega_i \Delta x_i \to 0$ (see previous theorem)

$\sum_{i>k} \omega_i \Delta x_i \to 0$ (see previous theorem)

$$\implies \sum \omega_i \Delta x_i \to 0$$

If number of discontinuities is finite, there is a finite number of such $\omega_{k_j} \Delta x_{k_j}$

that all tend to $0 \implies$ Their sum also tends to 0

$\implies$ f is Riemann-integrable

$\boxed{\impliedby}$ Let f be Riemann-integrable

Let $\{P_n\} : \lambda(P_n) \to 0$

$$\implies \sum \omega_i \Delta x_i \to 0$$

Let $D = \{k\} \subseteq [1, n]$ be a set of intervals with discontinuities

$$\sum \omega_i \Delta x_i = \sum_{i \notin D} \omega_i \Delta x_i + \sum_{k \in D} \omega_k \Delta x_k$$

$$\sum_{i \notin D} \omega_i \Delta x_i \to 0 \implies \sum_{k \in D} \omega_k \Delta x_k \to 0$$

And this is only possible when number of discontinuitites is finite