Cayley-Hamilton theorem

$$A \in \mathbb{F}^{n \times n} \implies P_A(A) = 0$$

$$\text{Let } A \in \mathbb{F}^{n \times n}$$
$$\text{Prove: } \dim(sp\left\{I, A, A^2, \ldots\right\}) \leq n$$

$$\text{Proof:}$$
$$P_A(A) = A^n + \sum_{i=0}^{n-1} \alpha_i A^i = 0$$
$$\implies A^n = \sum_{i=0}^{n-1} -\alpha_i A^i \in sp\left\{I, A, A^2, \ldots, A^{n-1}\right\}$$
$$\text{By Induction starting from } n\text{:}$$
$$\text{Let } k \geq n$$
$$A^{k+1} = AA^k = A\left(\sum_{i=0}^{n-1} \beta_i A^i\right) =$$
$$= \sum_{i=0}^{n-1} \beta_i A^{i+1} \in sp\left\{I, A, \ldots, A^k\right\} = sp\left\{I, A, \ldots, A^{n-1}\right\}$$

Note: by using m_A instead of P_A we can bound the dimension even further by $\deg(m_A)$

Note: dimension is exactly $\deg(m_A)$ as otherwise m_A would not be minimal

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$$A^2 = A \implies A^2 - A = 0 \implies f(x) = x^2 - x, f(A) = 0$$
$$f(x) = x(x-1)$$
$$\implies m_A \in \{x, x-1, x(x-1)\}$$
$$\implies \text{Largest Jordan block in } A_J \text{ is of size } 1$$
$$\implies A_J \text{ is diagonal} \implies A \text{ is diagonalizable}$$

$$A \in \mathbb{R}^{7\times 7} \text{ is invertible}$$
$$(A^3 + A)(A - 2I) = 0$$
$$tr(A) = 2$$
$$\text{Find } P_A, m_A$$

Solution:

$$(A^3 + A)(A - 2I) = A(A^2 + I)(A - 2I) = 0$$
$$A \text{ is invertible} \implies (A^2 + I)(A - 2I) = 0$$
$$m_A(x) \mid (x^2 + 1)(x - 2)$$
$$\text{Let } m_A(x) = x^2 + 1$$
$$\implies P_A(x) = (x^2 + 1)^k$$
$$\implies 2k = 7 - \text{Contradiction!}$$
$$\text{Let } m_A(x) = x - 2$$
$$\implies P_A(x) = (x - 2)^7 \implies tr(A) = 14 - \text{Contradiction!}$$
$$\implies \boxed{m_A(x) = (x^2 + 1)(x - 2)}$$
$$\implies P_A(x) = (x^2 + 1)^k (x - 2)^{7-2k}$$
$$P_A(x) = \begin{cases} (x^2+1)(x-2)^5 \\ (x^2+1)^2(x-2)^3 \\ (x^2+1)^3(x-2) \end{cases}$$
$$\text{Let us examine this over } \mathbb{C}$$
$$m_A(x) = (x - i)(x + i)(x - 2)$$
$$\implies A \text{ is diagonalizable over } \mathbb{C}$$
$$P_A(x) = (x - i)^k (x + i)^k (x - 2)^{7-2k}$$
$$\implies tr(A) = ki - ki + 2(7 - 2k) = 2(7 - 2k)$$
$$\implies 7 - 2k = 1 \implies k = 3$$
$$\implies \boxed{P_A(x) = (x^2 + 1)^3 (x - 2)}$$

$$\text{Find Jordan form of } A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \text{ over } \mathbb{C}$$

Solution:

$$P_A(x) = \begin{vmatrix} x+2 & 0 & 0 & 0 \\ 1 & x-1 & 0 & 0 \\ -1 & 1 & x & 1 \\ -1 & -1 & -1 & x-2 \end{vmatrix} = (x+2)(x-1)\begin{vmatrix} x & 1 \\ -1 & x-2 \end{vmatrix} =$$
$$= (x+2)(x-1)(x^2 - 2x + 1) = (x+2)(x-1)^3$$
$$\lambda = 1 \implies \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \gamma_A(1) = 2$$
$$\implies A_J = J_1(-2) \oplus J_2(1) \oplus J_1(1)$$

$$\text{Let } A = \begin{pmatrix} n & n-1 & n-2 & \dots & 1 \\ 0 & n & n-1 & \ddots & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & n-1 \\ 0 & \dots & \dots & 0 & n \end{pmatrix} \in \mathbb{R}^{n\times n}$$

$$P_A(x) = (x-n)^n$$
$$rank(nI - A) = n-1 \implies \gamma_A(n) = 1$$
$$\implies A_J = J_n(n)$$

$$\text{Let } A, B \in \mathbb{F}^{n\times n}$$
$$\text{Let } P_A = P_B$$
$$\text{Let } m_A = m_B$$
$$\text{Let } \forall\lambda : \gamma_A(\lambda) = \gamma_B(\lambda)$$
$$\text{Prove or disprove: } A \sim B$$

Disproof:

$$A_J = J_3(\lambda) \oplus J_2(\lambda) \oplus J_2(\lambda)$$
$$B_J = J_3(\lambda) \oplus J_3(\lambda) \oplus J_1(\lambda)$$

$$\text{Let } A \in \mathbb{C}^{6\times 6}$$
$$\text{Let } P_A(x) = (x-1)^4(x-2)^2$$
$$\text{Let } m_A(x) = (x-1)^2(x-2)$$
$$\text{Let } \gamma_A(1) = 2$$
$$\text{Find } A_J$$

Solution:

$$A_J = J_2(1) \oplus J_2(1) \oplus J_1(2) \oplus J_1(2)$$

$$\text{Let } A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix} \in \mathbb{C}^{3\times 3}$$
$$\text{Find } m_A$$

Solution:

$$P_A(x) = \begin{vmatrix} x-3 & -1 & 0 \\ 4 & x+1 & 0 \\ -4 & 8 & x+2 \end{vmatrix} = (x+2)(x^2-2x+1) = (x+2)(x-1)^2$$
$$(A+2I)(A-I) = \begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ 4 & -8 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ -4 & 0 & 0 \\ 4 & -8 & -3 \end{pmatrix} = \begin{pmatrix} 6 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \neq 0$$
$$\implies \boxed{m_A(x) = (x+2)(x-1)^2}$$

Alternative approach:

$$\begin{pmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ -4 & 8 & 3 \end{pmatrix} \to \begin{pmatrix} -2 & -1 & 0 \\ 0 & 0 & 0 \\ -4 & 8 & 3 \end{pmatrix} \implies \gamma_A(1) = 1$$
$$\implies A_J = J_1(-2) \oplus J_2(1) \implies \boxed{m_A(x) = (x+2)(x-1)^2}$$
