1a

Linear-2 5

1b

$$A=\begin{pmatrix}4 & x & 2\\ 2 & y & 3\\ -5 & 6 & 4\end{pmatrix}$$
$$\text{Let } \det(A)=d$$
$$B=\begin{pmatrix}4 & x+1 & 0 & 2\\ -5 & 2 & 1 & 4\\ 2 & y-2 & 0 & 3\\ -5 & 2 & 2 & 4\end{pmatrix}$$
$$\text{Find } \det(B) \text{ as a function of } d$$

$$\text{Solution:}$$
$$\begin{vmatrix}4 & x+1 & 0 & 2\\ -5 & 2 & 1 & 4\\ 2 & y-2 & 0 & 3\\ -5 & 2 & 2 & 4\end{vmatrix}=\begin{vmatrix}4 & x+1 & 0 & 2\\ 2 & y-2 & 0 & 3\\ -5 & 2 & 2 & 4\\ -5 & 2 & 1 & 4\end{vmatrix}=\begin{vmatrix}4 & x+1 & 0 & 2\\ 2 & y-2 & 0 & 3\\ 0 & 0 & 1 & 0\\ -5 & 2 & 1 & 4\end{vmatrix}=$$
$$=\begin{vmatrix}4 & x+1 & 0 & 2\\ 2 & y-2 & 0 & 3\\ 0 & 0 & 1 & 0\\ -5 & 2 & 1 & 4\end{vmatrix}=\begin{vmatrix}4 & x+1 & 2\\ 2 & y-2 & 3\\ -5 & 2 & 4\end{vmatrix}=\begin{vmatrix}4 & x+3 & 2\\ 2 & y+1 & 3\\ -5 & 6 & 4\end{vmatrix}=$$
$$=\begin{vmatrix}4 & x & 2\\ 2 & y+1 & 3\\ -5 & 6 & 4\end{vmatrix}+\begin{vmatrix}0 & 3 & 0\\ 2 & y+1 & 3\\ -5 & 6 & 4\end{vmatrix}=\begin{vmatrix}4 & x & 2\\ 2 & y & 3\\ -5 & 6 & 4\end{vmatrix}+\begin{vmatrix}4 & x & 2\\ 0 & 1 & 0\\ -5 & 6 & 4\end{vmatrix}+9=d+26-69=$$
$$=\boxed{d-43}$$

2a

$$\text{Let } A\in\mathbb{C}^{5\times 5}:\forall i,j:A_{ij}\in\mathbb{R}$$
$$\text{Let } A^4=-A^2$$
$$\text{Prove: } det(A)=tr(A)$$

$$\text{Proof:}$$
$$A^4=-A^2 \implies (A^2+I)A^2=0$$
$$\implies (A-iI)(A+iI)A^2=0$$
$$\implies m_A(\lambda)\mid(\lambda-i)(\lambda+i)\lambda^2$$
$$\implies \text{Eigenvalues of } A \text{ can only be } \{0,i,-i\}$$
$$\forall i,jA_{ij}\in\mathbb{R}\implies k_i=k_{-i}$$
$$\implies P_A(\lambda)=\lambda^{k_0}(\lambda^2+1)^{k_i}=(\lambda-i)^{k_i}(\lambda+i)^{k_i}\lambda^{k_0}$$
$$0 \text{ is an eigenvalue of } A \implies \det(A)=0$$
$$A\sim U\implies tr(A)=tr(U)=k_i\cdot i+k_i\cdot(-i)+k_0\cdot 0=0$$
$$\implies \boxed{\det(A)=tr(A)}$$

2b

In addition to 2a, $rank(A) = 2$

Find all possible Jordan forms of A

Solution:

$$rank(A) = 2 \implies g_0 = \dim N(A) = 5 - rank(A) = 3 \implies k_0 \geq 3$$

$$\implies P_A(\lambda) = \lambda^{k_0}(\lambda^2+1)^{k_i} = \lambda^{k_0}(\lambda - i)^{k_i}(\lambda + i)^{k_i}$$

$$k_i > 1 \implies \sum k_{\lambda_i} > 5 \implies k_i \leq 1$$

$$k_i = 1 \implies \forall\lambda : g_\lambda = k_\lambda, \sum g_\lambda = 5 = n \implies A \text{ is diagonalizable over } \mathbb{C}$$

$$\implies A_J = D = \begin{pmatrix} i & & & & \\ & -i & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} = J_1(i) \oplus J_1(-i) \oplus J_1(0) \oplus J_1(0) \oplus J_1(0)$$

$$k_i = 0 \implies P_A(\lambda) = \lambda^5$$

$$m_A(x) \mid x^2(x^2+1) \implies m_A(x) = x \text{ or } m_A(x) = x^2$$

$$m_A(x) = x \implies \text{There are three Jordan blocks of size 1 and that's not enough}$$

$$\implies m_A(x) = x^2 \implies A_J = \begin{pmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} = J_2(0) \oplus J_2(0) \oplus J_1(0)$$

3a

Let $A \in \mathbb{F}^{n\times n}$ be triangularizable

Let $B \in \mathbb{F}^{n\times n} : P_A(B) = 0$

Prove or disprove: B is triangularizable

Proof:

$$P_A(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$$

$$P_A(B) = 0$$

$$\implies m_B(x) \mid P_A(x)$$

$$\implies P_B(x) \mid m_B^n(x) \mid P_A^n(x)$$

$$\implies P_B(x) \text{ is factorizable into linear factors } \implies \boxed{B \text{ is triangularizable}}$$

3b

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