2

$$\text{Let } V = \mathbb{C}^{n\times n}$$
$$\text{Let } \forall A,B \in V : \langle A,B\rangle = tr(AB^*)$$

2a

$$\text{Prove: } \langle A,B\rangle \text{ is an inner product}$$

$$\text{Proof:}$$
$$\text{Let } A,B,C \in V, z \in \mathbb{C}$$
$$\langle A+zB,C\rangle = tr((A+zB)C^*) = tr(AC^*) + tr(zBC^*) = \langle A,C^*\rangle + z\langle B,C^*\rangle$$
$$\langle A,B\rangle = tr(AB^*) = tr((AB^*)^T) = tr(\overline{B}A^T) = \overline{tr(\overline{\overline{B}A^T})} = \overline{tr(BA^*)} = \overline{\langle B,A\rangle}$$
$$\langle A,A\rangle = tr(AA^*) = \sum_{i=1}^n R_i(A)\cdot C_i(A^*) = \sum_{i=1}^n R_i(A)\cdot \overline{R_i(A)} = \sum_{i=1}^n\sum_{j=1}^n |A_{ij}|^2 \geq 0$$
$$\langle A,A\rangle = 0 \iff \forall i,j \in [1,n] : A_{ij} = 0 \iff A = 0$$

2b

$$\text{Let } W \subseteq V \text{ be a space of diagonal matrices}$$
$$\text{Find } P_W\begin{pmatrix}1 & 2i\\ 4i & 3\end{pmatrix}$$

$$\text{Solution:}$$
$$\text{This is a standard inner product} \implies \text{Standard basis is orthonormal}$$
$$\implies \text{Basis } \{e_1,e_3\} \text{ is an orthonormal basis of } W$$
$$\text{Let } A = \begin{pmatrix}1 & 2i\\ 4i & 3\end{pmatrix}$$
$$\implies P_W(A) = \langle A,e_1\rangle e_1 + \langle A,e_3\rangle e_3 = \begin{pmatrix}tr(Ae_1^*) & 0\\ 0 & tr(Ae_3^*)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$$

3

$$\text{Let } A,B \in \mathbb{R}^{n\times n}$$
$$\text{Let } A \text{ be invertible}$$
$$\text{Let } \forall v \in \mathbb{R}^n : \|Av\| = \|Bv\|$$

3a

$$\text{Prove or disprove: } B \text{ is invertible}$$

$$\text{Proof:}$$
$$\text{Let } B \text{ be non-invertible}$$
$$\implies \exists v \neq 0 : Bv = 0$$
$$v \neq 0 \implies Av \neq 0 \implies \|Av\| > 0 \implies \|Bv\| = \|Av\| > 0 - \text{Contradiction!}$$
$$\implies B \text{ is invertible}$$

3b

$$\text{Prove or disprove: } A,B \text{ are unitary}$$

$$\text{Disproof:}$$
$$A = B = 2I$$

3c

Prove or disprove: AB^{-1} is unitary

Proof:

$$\forall v \in \mathbb{R}^n : \|Av\| = \|Bv\|$$

Let $v \in \mathbb{R}^n$

$$\implies B^{-1}v \in \mathbb{R}^n \implies \|AB^{-1}v\| = \|BB^{-1}v\| = \|v\|$$

$$\forall v \in \mathbb{R}^n : \|AB^{-1}v\| = \|v\| \implies AB^{-1} \text{ is unitary}$$

4

Let $A \in \mathbb{R}^{n \times n}$ such that $A^T A$ is a scalar matrix

$$\exists \alpha \in \mathbb{R} : A^T A = \alpha I$$

4a

Prove: $\alpha = 0 \implies A = 0$

Proof:

$$\alpha = 0 \implies A^T A = 0 \implies \forall i \in [1, n] : R_i(A^T) \cdot C_i(A) = 0$$

$$\implies \forall i \in [1, n] : \sum_{j=1}^{n} (A_{ij})^2 = 0 \implies \forall i, j \in [1, n] : A_{ij} = 0$$

$$\implies A = 0$$

4b

Prove: $\alpha \neq 0 \implies \alpha > 0$

Proof:

$$\alpha \neq 0 \implies A^T A = \alpha I \implies \forall i \in [1, n] : R_i(A^T) \cdot C_i(A) = \alpha$$

$$\implies \forall i \in [1, n] : \sum_{j=1}^{n} (A_{ij})^2 = \alpha \implies \alpha > 0$$

4c

Prove: $\forall \alpha \in \mathbb{R} : \exists b \in \mathbb{R}, Q \text{ orthogonal} : A = bQ$

Proof:

Let $\alpha = 0 \implies A = 0 \implies Q = I, b = 0$

Let $\alpha \neq 0 \implies \alpha > 0$

$$A^T A = \alpha I \implies \left(\frac{1}{\sqrt{\alpha}} A\right)^T \left(\frac{1}{\sqrt{\alpha}} A\right) = I$$

$$\text{Let } Q = \frac{1}{\sqrt{\alpha}} A$$

$$Q^T = \frac{1}{\sqrt{\alpha}} A^T$$

$$Q^T Q = \frac{1}{\alpha} A^T A = \frac{\alpha}{\alpha} I = I \implies Q \text{ is orthogonal}$$

$$b = \sqrt{\alpha}, A = \sqrt{\alpha} Q$$

5a

$$\text{Let } n \geq 2$$
$$\text{Let } A \in \mathbb{R}^{n \times n}$$
$$\text{Let } tr(A) = 0, rank(A) = 1$$
Find characteristic and minimal polynomials of A

Solution:

$$rank(A) = 1 \implies \dim N(A) = n - 1 \implies k_0 \geq n - 1$$
$$\implies P_A(x) = \left[\begin{array}{c} x^{n-1}(x-\lambda) \\ x^n \end{array}\right.$$
$$\implies A \text{ is triangularizable}$$
$$\implies tr(A) = tr(U) = \sum_{i=1}^{n-1} 0 + \lambda = 0$$
$$\implies \lambda = 0 \implies \boxed{P_A(x) = x^n}$$
$$g_0 = n - 1 \implies \text{Jordan form of } A \text{ has } n-1 \text{ blocks}$$
$$\implies \text{Exactly one of them is a block of size 2, all others are of size 1}$$
$$\implies \boxed{m_A(x) = x^2}$$

5b

$$\text{Let } V \text{ be a finite-dimensional inner product space over } \mathbb{F}$$
$$\text{Let } U \text{ be a subspace of } V$$
$$\text{Let } U^0 = \{T : V \to \mathbb{F} | \forall u \in U : T(u) = 0\}$$
$$\text{Prove: } \forall T \in U^0 : \exists w \in U^\perp : \forall v \in V : T(v) = \langle v, w \rangle$$

Proof:

$$\text{Let } T \in U^0$$
$$\text{By Riesz theorem, } \exists w \in V : \forall v \in V : T(v) = \langle v, w \rangle$$
$$\text{Let us prove that } w \in U^\perp$$
$$T \in U^0 \implies \forall u \in U : T(u) = 0 \implies \forall u \in U : \langle u, w \rangle = 0$$
$$\implies \boxed{w \in U^\perp}$$