1a

$$\int \tan^3(x)dx$$

Solution:

$$\tan^2(x) = \frac{\sin^2(x)}{\cos^2(x)} = \frac{1-\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2 x} - 1$$

$$\implies \int \tan^3(x)\,dx = \int \frac{\tan(x)}{\cos^2(x)}\,dx - \int \tan(x)\,dx$$

$$\int \tan(x)\,dx = \begin{Bmatrix} t = \cos(x) \\ dt = -\sin(x)dx \end{Bmatrix} = -\int \frac{1}{t}dt = -\ln|t| + C$$

$$\int \frac{\tan(x)}{\cos^2(x)}\,dx = \int \frac{\sin(x)}{\cos^3(x)}\,dx = \begin{Bmatrix} t = \cos(x) \\ dt = -\sin(x)dx \end{Bmatrix} = -\int \frac{1}{t^3}\,dt = \frac{1}{2t^2} + C$$

$$\implies \int \tan^3(x)\,dx = \frac{1}{2\cos^2(x)} + \ln|\cos(x)| + C$$

1b

Determine whether $\int_0^\infty x\sin(e^{2x})\,dx$ converges

Solution:

$$\int_0^\infty x\sin(e^{2x})\,dx = \begin{Bmatrix} t = e^{2x} \\ x = \frac{\ln(t)}{2} \\ dt = 2e^{2x}dx \\ dx = \frac{dt}{2t} \\ x \to \infty \implies t = e^{2x} \to \infty \\ x = 0 \implies t = e^0 = 1 \end{Bmatrix} = \frac{1}{4}\int_1^\infty \frac{\ln(t)}{t}\sin(t)\,dt$$

$$\int_0^x \sin(t)\,dt = -\cos(x) - 1 \text{ is bounded}$$

$$\frac{\ln(t)}{t} \text{ is monotonically decreasing to } 0$$

$$\implies \text{By Dirichlet's test } \int_0^\infty x\sin(e^{2x})\,dx \text{ converges}$$

2a

Let f be a function defined on I

Prove or disprove: f has a primitive $\implies$ f is integrable on I

Disproof:

$$\text{Let } F(x) = x^2\sin\left(\frac{1}{x^2}\right)$$

$$\implies F'(x) = 2x\sin\left(\frac{1}{x^2}\right) + x^2\left(\sin\left(\frac{1}{x^2}\right)\right)' = 2x\sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right)$$

$$\text{Let } f(x) = \begin{cases} F'(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Let $I = [-1, 1]$

f has a primitive, but is not bounded at 0 $\implies$ f is not integrable

2b

Let f be a function defined on I

Prove or disprove: f is integrable $\implies$ f has a primitive

Disproof:

$$\text{Let } f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \in (1,2] \end{cases}$$

f is bounded and has one discontinuity $\implies$ It is integrable

f has a jumo discontinuity $\implies$ f has no primitive by Darboux theorem

3a

Prove or disprove: $f_n(x) = \sqrt[n]{1+x^n}$ converges uniformly on $[0,\infty)$

Proof?:

$$\text{Let } x \le 1 \implies \underbrace{\sqrt[n]{1}}_{\to 1} \le f_n(x) \le \underbrace{\sqrt[n]{2}}_{\to 1} \implies f_n(x) \to 1 \text{ for } x \in [0,1]$$

$$\text{Let } x > 1 \implies x^n > 1$$

$$\implies \underbrace{\sqrt[n]{x^n}}_{\to x} \le \sqrt[n]{1+x^n} \le \underbrace{\sqrt[n]{2x^n}}_{\to x}$$

$$\implies f_n(x) \to f(x) = \begin{cases} 1 & x \in [0,1] \\ x & x \in (1,\infty) \end{cases}$$

$$d_n = \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \max\{\max_{x\in[0,1]} \sqrt[n]{1+x^n} - 1,\ \sup_{x\in(1,\infty)} \left|\sqrt[n]{1+x^n} - x\right|\}$$

$$\max_{x\in[0,1]} \sqrt[n]{1+x^n} - 1 = \sqrt[n]{2} - 1$$

$$\sup_{x\in(1,\infty)} \left|\sqrt[n]{1+x^n} - x\right|$$

$$\text{Let } t = \frac{1}{x^n}$$

$$x > 1 \implies 0 < t < 1$$

$$\sqrt[n]{1+t} = (1+t)^{1/n}$$

$$\implies 1 < (1+t)^{1/n} \le 1 + \frac{t}{n}$$

$$\implies 0 < (1+t)^{1/n} - 1 \le \frac{t}{n} \implies \left| x\left(1 + \frac{1}{x^n}\right)^{1/n} - x\right| \le \frac{1}{x^n n}$$

$$\implies |f_n(x) - x| \le \frac{1}{x^n n} \le \frac{1}{n}$$

$$\implies \sup_{x\in(1,\infty)} \left|\sqrt[n]{1+x^n} - x\right|\} \le \frac{1}{n}$$

$$\implies d_n \le \max\{\underbrace{\sqrt[n]{2}-1}_{\to 1-1=0},\ \underbrace{\frac{1}{n}}_{\to 0}\} \to 0$$

$$\implies \boxed{f_n(x) \rightrightarrows f(x)}$$

3b

Write down $\ln(13)$ as a series of rational numbers

Solution:

$$\text{Let } x \in (-1,1)$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \implies \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|$$

$$\text{Let } x = \frac{12}{13}$$

$$\implies \sum_{n=0}^{\infty} \frac{12^{n+1}}{13^{n+1}(n+1)} = -\ln\left|1 - \frac{12}{13}\right| = -\ln\left(\frac{1}{13}\right) = \ln(13)$$

4

$$\sum_{n=0}^{\infty} \frac{1}{16^n(4n+1)}$$

Solution:

$$x \in (-1,1) \implies \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\implies \frac{1}{1-x^4} = \sum_{n=0}^{\infty} x^{4n} \implies \sum_{n=0}^{\infty} \int_0^x t^{4n}\,dt = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1} = \int_0^x \frac{1}{1-t^4}\,dt =$$

$$= \int_0^x \frac{1}{(1-t)(1+t)(1+t^2)}\,dt$$

$$\frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1+x^2}$$

$$A(1+x)(1+x^2) + B(1-x)(1+x^2) + C(1-x)(1+x) = 1$$

$$\implies \begin{cases} A+B+C=1 \\ A-B=0 \\ A+B-C=0 \\ A-B=0 \end{cases} \implies \begin{cases} A=\frac{1}{4} \\ B=\frac{1}{4} \\ C=\frac{1}{2} \end{cases}$$

$$\implies \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1} = \frac{1}{4}\int_0^x \frac{1}{1-t} + \frac{1}{1+t} + \frac{2}{(1+t^2)}\,dt =$$

$$= \frac{1}{4}(-\ln|1-x| + \ln|1+x| + 2\arctan(x))$$

$$\text{Let } x = \frac{1}{2}$$

$$\implies \sum_{n=0}^{\infty} \frac{1}{2\cdot 16^n(4n+1)} = \frac{1}{4}\left(-\ln\frac{1}{2} + \ln\frac{3}{2} + 2\arctan\left(\frac{1}{2}\right)\right)$$

$$\implies \sum_{n=0}^{\infty} \frac{1}{16^n(4n+1)} = \frac{1}{2}\left(\ln 3 + 2\arctan\frac{1}{2}\right)$$

5a

Find critical points of function:

$$f(x,y) = xy^2 - 2x^2y - 4xy$$

Solution:

$$f_x = y^2 - 4xy - 4y = 0 \implies y(y - 4x - 4) = 0 \implies \left[\begin{array}{l} y = 0 \\ y = 4(x+1) \end{array}\right.$$

$$f_y = 2xy - 2x^2 - 4x \implies x(y - x - 2) = 0 \implies \left[\begin{array}{l} x = 0 \\ y = x+2 \end{array}\right.$$

$$y = 0 \implies \left[\begin{array}{l} x = 0 \\ x = -2 \end{array}\right.$$

$$y = 4(x+1) \implies \left[\begin{array}{l} x = 0 \implies y = 4 \\ y = x + 2 \implies 4x + 2 = x \implies x = -\frac{2}{3}, y = \frac{4}{3} \end{array}\right.$$

$$\implies \text{Critical points are:}$$

$$(0,0), (-2,0), (0,4), \left(-\frac{2}{3}, \frac{4}{3}\right)$$

$$f_{xx} = -4y$$
$$f_{xy} = 2y - 4x - 4$$
$$f_{yx} = 2y - 4x - 4$$
$$f_{yy} = 2x$$

$$\implies H_f = \begin{pmatrix} -4y & 2y - 4x - 4 \\ 2y - 4x - 4 & 2x \end{pmatrix}$$

$$M_1 = -4y$$
$$M_2 = -8xy - (2y - 4x - 4)^2$$

$$(0,0) \to M_1 = 0, M_2 = -16 \implies \text{Saddle}$$
$$(-2,0) \to M_1 = 0, M_2 = -16 \implies \text{Saddle}$$
$$(0,4) \to M_1 = -16, M_2 = -16 \implies \text{Saddle}$$
$$\left(-\frac{2}{3}, \frac{4}{3}\right) \to M_1 = -\frac{16}{3}, M_2 = \frac{64}{9} - \left(\frac{8}{3} + \frac{8}{3} - 4\right)^2 = \frac{16}{3} > 0 \implies \text{Local maximum}$$