2

$$egin{aligned} \operatorname{Let} V &= \mathbb{C}^{n imes n} \ \operatorname{Let} orall A, B \in V : \langle A, B 
angle &= tr(AB^*) \end{aligned}$$

**2**a

Prove:  $\langle A, B \rangle$  is an inner product

Proof:

$$\text{Let } A, B, C \in V, z \in \mathbb{C} \\ \langle A + zB, C \rangle = tr((A + zB)C^*) = tr(AC^*) + tr(zBC^*) = \langle A, C^* \rangle + z\langle B, C^* \rangle \\ \langle A, B \rangle = tr(AB^*) = tr((AB^*)^T) = tr(\overline{B}A^T) = \overline{tr(\overline{B}A^T)} = \overline{tr(BA^*)} = \overline{\langle B, A \rangle} \\ \langle A, A \rangle = tr(AA^*) = \sum_{i=1}^n R_i(A) \cdot C_i(A^*) = \sum_{i=1}^n R_i(A) \cdot \overline{R_i(A)} = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2 \geq 0 \\ \langle A, A \rangle = 0 \iff \forall i, j \in [1, n] : A_{ij} = 0 \iff A = 0$$

**2**b

Let  $W \subseteq V$  be a space of diagonal matrices

Find 
$$P_W \begin{pmatrix} 1 & 2i \\ 4i & 3 \end{pmatrix}$$

Solution:

This is a standard inner product  $\implies$  Standard basis is orthonormal

 $\implies$  Basis  $\{e_1, e_3\}$  is an orthonormal basis of W

$$\operatorname{Let} A = egin{pmatrix} 1 & 2i \ 4i & 3 \end{pmatrix} \ \implies P_W(A) = \langle A, e_1 
angle e_1 + \langle A, e_3 
angle e_3 = egin{pmatrix} tr(Ae_1^*) & 0 \ 0 & tr(Ae_3^*) \end{pmatrix} = egin{pmatrix} 1 & 0 \ 0 & 3 \end{pmatrix}$$

3

 $\mathrm{Let}\ A,B\in\mathbb{R}^{n\times n}$ 

Let A be invertible

Let 
$$\forall v \in \mathbb{R}^n : \|Av\| = \|Bv\|$$

3a

Prove or disprove: B is invertible

Proof:

Let B be non-invertible

$$\implies \exists v \neq 0 : Bv = 0$$

$$v \neq 0 \implies Av \neq 0 \implies \|Av\| > 0 \implies \|Bv\| = \|Av\| > 0 - ext{Contradiction!}$$
 $\implies B ext{ is invertible}$ 

3b

Prove or disprove: A, B are unitary

Disproof:

$$A=B=2I$$

Prove or disprove:  $AB^{-1}$  is unitary

$$egin{aligned} orall v \in \mathbb{R}^n : \|Av\| = \|Bv\| \ & ext{Let } v \in \mathbb{R}^n \ \Longrightarrow \ B^{-1}v \in \mathbb{R}^n \implies \|AB^{-1}v\| = \|BB^{-1}v\| = \|v\| \end{aligned}$$

$$\implies B^{-1}v \in \mathbb{R}^n \implies \|AB^{-1}v\| = \|BB^{-1}v\| = \|v\|$$
 $\forall v \in \mathbb{R}^n : \|AB^{-1}v\| = \|v\| \implies AB^{-1} \text{ is unitary}$ 

4

Let  $A \in \mathbb{R}^{n imes n}$  such that  $A^T A$  is a scalar matrix  $\exists lpha \in \mathbb{R} : A^T A = lpha I$ 

4a

Prove: 
$$\alpha = 0 \implies A = 0$$

Proof:

$$egin{aligned} lpha &= 0 \implies A^T A = 0 \implies orall i \in [1,n]: R_i(A^T) \cdot C_i(A) = 0 \ \implies orall i \in [1,n]: \sum_{j=1}^n (A_{ij})^2 = 0 \implies orall i, j \in [1,n]: A_{ij} = 0 \ \implies A = 0 \end{aligned}$$

4b

Prove: 
$$\alpha \neq 0 \implies \alpha > 0$$

Proof:

$$egin{aligned} lpha 
eq 0 &\Longrightarrow A^T A = lpha I &\Longrightarrow orall i \in [1,n]: R_i(A^T) \cdot C_i(A) = lpha \ &\Longrightarrow orall i \in [1,n]: \sum_{j=1}^n (A_{ij})^2 = lpha &\Longrightarrow lpha > 0 \end{aligned}$$

**4c** 

 $ext{Prove:} \, orall lpha \in \mathbb{R} : \exists b \in \mathbb{R}, Q ext{ orthogonal } : A = bQ$ 

$$\begin{array}{cccc} \mathrm{Let}\; \alpha = 0 \implies A = 0 \implies Q = I, b = 0 \\ \mathrm{Let}\; \alpha \neq 0 \implies \alpha > 0 \end{array}$$

Let 
$$\alpha \neq 0 \implies \alpha > 0$$

$$A^T A = lpha I \implies \left(rac{1}{\sqrt{lpha}}A
ight)^T \left(rac{1}{\sqrt{lpha}}A
ight) = I$$
 $\operatorname{Let} Q = rac{1}{\sqrt{lpha}}A$ 

$$Q^T = rac{\sqrt{lpha}}{\sqrt{lpha}} A^T$$

$$Q^TQ = rac{1}{lpha}A^TA = rac{lpha}{lpha}I = I \implies Q ext{ is orthogonal}$$

$$egin{array}{ccc} lpha & & lpha \ & b = \sqrt{lpha}, A = \sqrt{lpha}Q \end{array}$$

$$egin{aligned} \operatorname{Let} n &\geq 2 \ \operatorname{Let} A &\in \mathbb{R}^{n imes n} \end{aligned}$$
  $\operatorname{Let} tr(A) = 0, rank(A) = 1$ 

Find characteristic and minimal polynomials of A

Solution:

$$egin{aligned} rank(A) &= 1 \implies \dim N(A) = n-1 \implies k_0 \geq n-1 \ &\Longrightarrow P_A(x) = egin{bmatrix} x^{n-1}(x-\lambda) \ x^n \ &\Longrightarrow A ext{ is triangularizable} \ &\Longrightarrow tr(A) = tr(U) = \sum_{i=1}^{n-1} 0 + \lambda = 0 \ &\Longrightarrow \lambda = 0 \implies egin{bmatrix} P_A(x) = x^n \ \end{bmatrix} \end{aligned}$$

 $g_0 = n-1 \implies ext{Jordan form of } A ext{ has } n-1 ext{ blocks}$ 

 $\implies$  Exactly one of them is a block of size 2, all others are of size 1

$$\Longrightarrow \boxed{m_A(x)=x^2}$$

5b

Let V be a finite-dimensional inner product space over  $\mathbb F$ 

Let U be a subspace of V

$$egin{aligned} \operatorname{Let} U^0 &= \{T: V 
ightarrow \mathbb{F} | orall u \in U: T(u) = 0 \} \ \end{aligned} \ \operatorname{Prove:} \ orall T \in U^0: \exists w \in U^\perp: orall v \in V: T(v) = \langle v, w 
angle \end{aligned}$$

Proof:

Let 
$$T \in U^0$$

By Riesz theorem,  $\exists w \in V : \forall v \in V : T(v) = \langle v, w \rangle$ 

Let us prove that  $w \in U^{\perp}$ 

$$T \in U^0 \implies orall u \in U: T(u) = 0 \implies orall u \in U: \langle u, w 
angle = 0 \ \implies \boxed{w \in U^\perp}$$