## Abstract

We derive formulae for the Zernike polynomials with cartesian coordinate arguments, without the use of trigenometric functions. In many computing applications, the input coordinates are cartesian to begin with. In these circumstances it is very undesirable to compute the polar representation and then evaluate the Zernike polynomial.

# Cartesian Zernike Polynomials

Adam Mendenhall

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## 1 Introduction

The Zernike polynomials are a set of scalar valued functions defined on the unit circle. The set is orthogonal with respect to the standard metric

$$\langle f, g \rangle = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} f(r,\theta) g(r,\theta) \, \mathrm{d}\theta \mathrm{d}r.$$

Most indexing schemes define a unique polynomial for each index tuple $(n, m)$ with $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $|n| \leq m$ and $n \equiv m \pmod 2$. For a given $(n, m)$, we define the Zernike polynomial

$$Z_n^m(r,\theta) = \begin{cases} R_n^m(r) \cos m\theta & \text{if } m > 0 \\ R_n^{-m}(r) \sin m\theta & \text{if } m < 0 \\ R_n^0(r) & \text{if } m = 0 \end{cases}$$

where the radial polynomials $R_n^m$ are defined for $n, m \geq 0$ as

$$R_n^m(r) = \sum_{k=0}^{\frac{n-m}{2}} (-1)^k \binom{n-k}{k} \binom{n-2k}{\frac{n-m}{2} - k} r^{n-2k}.$$

The radial polynomials are also defined recursively with $R_n^n(r) = r^n$ and $R_{n+2}^n(r) = \big((n+2)r^2 - (n+1)\big) r^n$, otherwise

$$R_n^m(r) = -\frac{n(n+m-2)(n-m-2)}{(n+m)(n-m)(n-2)} R_{n-4}^m(r) + \frac{2(n-1)(2n(n-2)r^2 - m^2 - n(n-2))}{(n+m)(n-m)(n-2)} R_{n-2}^m(r).$$

Finally, we have the recursive formulae

$$\cos((n+2)\theta) = 2\cos((n+1)\theta)\cos\theta - \cos(n\theta),$$

$$\sin((n+2)\theta) = 2\sin((n+1)\theta)\cos\theta - \sin(n\theta).$$

We are now ready to redefine the Zernike polynomials in cartesian coordinates.

## 2 Main Dish

Let $x = r\cos\theta$ and $y = r\sin\theta$, we have $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$ (with $x < 0 \iff \theta \in (\pi/2, 3\pi/2)$ and $\theta \in [0, 2\pi)$).

### 2.1 The Irrotational Functions

As in 1, for $n \geq 1$ in full generality,

$$Z_n^{\pm n}(r,\theta) = r^n \, {}^{\cos}_{\sin} n\theta,$$

so we have

$$Z_n^{\pm n}(r,\theta) = 2r\cos\theta \cdot r^{n-1} \, {}^{\cos}_{\sin} \big((n-1)\theta\big) - r^2 \cdot r^{n-2} \, {}^{\cos}_{\sin} \big((n-2)\theta\big).$$

Naturally,

$$Z_n^{\pm n}(x,y) = 2x Z_{n-1}^{\pm(n-1)} - (x^2+y^2) Z_{n-2}^{\pm(n-2)}.$$

Observe that the equations

$$Z_n^n(x,y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^{n-k} x^{n-2k} y^{2k}$$

$$Z_n^{-n}(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k x^{n-(2k+1)} y^{2k+1}$$

both satisfy the recursion formula and the initial conditions for $n = 2$ and $n = 1$. When $n = 0$, the first equation is correct: $Z_0^{+0} = 1$ (instead of $Z_0^{-0} = 0$).

### 2.2 The Solenoidal Functions

As in 1, the recursive definition of $R$ holds for $Z$ (when $n$ is held constant). Observe that the equation

$$Z_n^0(x,y) = (-1)^{\frac{n}{2}} \sum_{k=0}^{\frac{n}{2}} \sum_{l=0}^{\frac{n}{2}-k} x^{2k} y^{2l} \begin{cases} 1 \\ \frac{-n}{2(k+l)!^2} \binom{k+l}{l} \left(\frac{n}{2} + k + l\right) \prod_{\varsigma=1}^{k+l-1} \varsigma \end{cases}$$

satisfies both the recursion formula and the initial conditions for $n = 0$ and $n = 1$.