

### **Abstract**

We derive formulae for the Zernike polynomials with cartesian coordinate arguments, without the use of trigonometric functions. In many computing applications, the input coordinates are cartesian to begin with. In these circumstances it is very undesirable to compute the polar representation and then evaluate the Zernike polynomial.

# Cartesian Zernike Polynomials

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## 1 Introduction

The Zernike polynomials are a set of scalar valued functions defined on the unit circle. The set is orthogonal with respect to the standard metric

$$\langle f, g \rangle = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} f(r, \theta) g(r, \theta) \, d\theta dr.$$

Most indexing schemes define a unique polynomial for each index tuple  $(n, m)$  with  $n \in \mathbb{N}$  and  $m \in \mathbb{Z}$  with  $|m| \leq n$  and  $n \equiv m \pmod{2}$ . For a given  $(n, m)$ , we define the Zernike polynomial

$$Z_n^m(r, \theta) = \begin{cases} R_n^m(r) \cos m\theta & \text{if } m > 0 \\ R_n^{-m}(r) \sin m\theta & \text{if } m < 0 \\ R_n^0(r) & \text{if } m = 0 \end{cases}$$

where the radial polynomials  $R_n^m$  are defined for  $m \geq 0$  as

$$R_n^m(r) = \sum_{k=0}^{\frac{n-m}{2}} (-1)^k \binom{n-k}{k} \binom{n-2k}{\frac{n-m}{2}-k} r^{n-2k}.$$

The radial polynomials are also defined recursively with  $R_n^n(r) = r^n$  and  $R_{n+2}^n(r) = ((n+2)r^2 - (n+1))r^n$ , otherwise

$$R_n^m(r) = -\frac{n(n+m-2)(n-m-2)}{(n+m)(n-m)(n-2)} R_{n-4}^m(r) + \frac{2(n-1)(2n(n-2)r^2 - m^2 - n(n-2))}{(n+m)(n-m)(n-2)} R_{n-2}^m(r).$$

We will use expressions for  $\frac{\cos}{\sin}(n\theta)$  as polynomials in  $\cos \theta$  to derive explicit and recursive formulae for  $Z_n^m$  as polynomials in  $x$  and  $y$ .

## 2 Main Dish

Let  $x = r \cos \theta$  and  $y = r \sin \theta$ , we have  $r = \sqrt{x^2 + y^2}$ .

## 2.1 Positive $m$

Take the formula due to Chebyshev

$$\cos(m\theta) = \frac{m}{2} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k}{m-k} \binom{m-k}{k} (2 \cos \theta)^{m-2k}$$

and the radial Zernike polynomial formula to define for  $m > 0$

$$Z_n^m(x, y) = \frac{m}{2} \sum_{\zeta=0}^{\frac{n-m}{2}} \sum_{\xi=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^{\zeta+\xi}}{m-\xi} \binom{m-\xi}{\xi} \binom{n-\zeta}{\zeta} \binom{n-2\zeta}{\frac{n-m}{2}-\zeta} (2x)^{m-2\xi} (x^2 + y^2)^{\xi-\zeta+\frac{n-m}{2}}$$

By binomial expansion, we have  $Z_n^m(x, y) =$

$$\frac{m}{2} \sum_{\zeta=0}^{\frac{n-m}{2}} \sum_{\xi=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\varsigma=0}^{\xi-\zeta+\frac{n-m}{2}} 2^{m-2\xi} \frac{(-1)^{\zeta+\xi}}{m-\xi} \binom{m-\xi}{\xi} \binom{n-\zeta}{\zeta} \binom{n-2\zeta}{\frac{n-m}{2}-\zeta} \binom{\xi-\zeta+\frac{n-m}{2}}{\varsigma} x^{n-2\zeta-2\varsigma} y^{2\varsigma}$$

We perform the substitution

$$\begin{cases} \zeta' = \zeta + \varsigma \\ \xi' = \xi \\ \varsigma' = \varsigma \end{cases} \quad \left| \begin{array}{l} 0 \leq \zeta' \leq \lfloor \frac{n}{2} \rfloor \\ \max(0, \zeta' - \frac{n-m}{2}) \leq \xi' \leq \lfloor \frac{m}{2} \rfloor \\ \max(0, \zeta' - \frac{n-m}{2}) \leq \varsigma' \leq \zeta' \end{array} \right.$$

to achieve (variable domains not repeated for brevity)  $Z_n^m(x, y) =$

$$\frac{m}{2} \sum_{\zeta'} \sum_{\xi'} \sum_{\varsigma'} 2^{m-2\xi'} \frac{(-1)^{\zeta'-\varsigma'+\xi'}}{m-\xi'} \binom{m-\xi'}{\xi'} \binom{n-\zeta'+\varsigma'}{\zeta'-\varsigma'} \binom{n-2\zeta'+2\varsigma'}{\frac{n-m}{2}-\zeta'+\varsigma'} \binom{\xi'-\zeta'+\varsigma'+\frac{n-m}{2}}{\varsigma'} x^{n-2\zeta'} y^{2\varsigma'}$$

The diligent reader will identify a bijection between the sets  $\{(\zeta, \xi, \varsigma)\}$  and  $\{(\zeta' - \varsigma', \xi', \varsigma')\}$ , keeping in mind  $\lfloor \frac{n}{2} \rfloor = \frac{n-m}{2} + \lfloor \frac{m}{2} \rfloor$ .

It is clear that for  $Z_n^{>0}(x, y)$ , the powers of  $x$  are  $X = [0, n] \cap \{x \equiv n \pmod{2}\}$  while the powers of  $y$  are  $Y = [0, n] \cap \{x \equiv 0 \pmod{2}\}$  and that the exponents of  $Z$  as a polynomial in  $x$  and  $y$  are  $X \times Y \cap \{(x, y) | x+y \leq n\}$ .

## 2.2 Negative $m$

We repeat the process from section 2.1 for the Zernike polynomials with negative  $m$ . Take the formula due to Chebyshev

$$\sin(m\theta) = \left( \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^k \binom{m-1-k}{k} (2 \cos \theta)^{m-1-2k} \right) \sin \theta$$

and the radial Zernike polynomial formula to define for  $m > 0$

$$Z_n^{-m}(x, y) = y \sum_{\zeta=0}^{\frac{n-m}{2}} \sum_{\xi=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{\zeta+\xi} \binom{n-\zeta}{\zeta} \binom{n-2\zeta}{\frac{n-m}{2}-\zeta} \binom{m-1-\xi}{\xi} (2x)^{m-1-2\xi} (x^2 + y^2)^{\xi-\zeta+\frac{n-m}{2}}$$

By binomial expansion, we have  $Z_n^{-m}(x, y) =$

$$y \sum_{\zeta=0}^{\frac{n-m}{2}} \sum_{\xi=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{\varsigma=0}^{\xi-\zeta+\frac{n-m}{2}} 2^{m-1-2\xi} (-1)^{\zeta+\xi} \binom{n-\zeta}{\zeta} \binom{n-2\zeta}{\frac{n-m}{2}-\zeta} \binom{m-1-\xi}{\xi} \binom{\frac{n-m}{2}-\zeta+\xi}{\varsigma} x^{n-2\zeta-2\varsigma-1} y^{2\varsigma}$$

We perform the substitution

$$\begin{cases} \zeta' = \zeta + \varsigma \\ \xi' = \xi \\ \varsigma' = \varsigma \end{cases} \left| \begin{array}{l} 0 \leq \zeta' \leq \lfloor \frac{n}{2} \rfloor \\ \max(0, \zeta' - \frac{n-m}{2}) \leq \xi' \leq \lfloor \frac{m-1}{2} \rfloor \\ \max(0, \zeta' - \frac{n-m}{2}) \leq \varsigma' \leq \xi' \end{array} \right.$$

to achieve (variable domains not repeated for brevity)  $Z_n^{-m}(x, y) =$

$$y \sum_{\zeta'} \sum_{\varsigma'} \sum_{\xi'} 2^{m-1-2\xi'} (-1)^{\zeta'-\varsigma'+\xi'} \binom{n-\zeta'+\varsigma'}{\zeta'-\varsigma'} \binom{n-2\zeta'+2\varsigma'}{\frac{n-m}{2}-\zeta'+\varsigma'} \binom{m-1-\xi'}{\xi'} \binom{\frac{n-m}{2}-\zeta'+\varsigma'+\xi'}{\varsigma'} x^{n-2\zeta'-1} y^{2\varsigma'}$$