Abstract

We derive formulae for the Zernike polynomials with cartesian coordinate arguments, without the use of trigenometric functions. In many computing applications, the input coordinates are cartesian to begin with. In these circumstances it is very undesirable to compute the polar representation and then evaluate the Zernike polynomial.

Cartesian Zernike Polynomials

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1 Introduction

The Zernike polynomials are a set of scalar valued functions defined on the unit circle. The set is orthogonal with respect to the standard metric

$$\langle f, g\rangle = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} f(r,\theta)g(r,\theta)\ \mathrm{d}\theta\mathrm{d}r.$$

Most indexing schemes define a unique polynomial for each index tuple (n, m) with $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $|m| \leq n$ and $n \equiv m \pmod 2$. For a given (n, m), we define the Zernike polynomial

$$Z_n^m(r,\theta) = \begin{cases} R_n^m(r)\cos m\theta & \text{if } m > 0 \\ R_n^{-m}(r)\sin m\theta & \text{if } m < 0 \\ R_n^0(r) & \text{if } m = 0 \end{cases}$$

where the radial polynomials R_n^m are defined for $m \geq 0$ as

$$R_n^m(r) = \sum_{k=0}^{\frac{n-m}{2}} (-1)^k \binom{n-k}{k}\binom{n-2k}{\frac{n-m}{2}-k} r^{n-2k}.$$

The radial polynomials are also defined recursively with $R_n^n(r) = r^n$ and $R_{n+2}^n(r) = \left((n+2)r^2 - (n+1)\right) r^n$, otherwise

$$R_n^m(r) = -\frac{n(n+m-2)(n-m-2)}{(n+m)(n-m)(n-2)} R_{n-4}^m(r) + \frac{2(n-1)(2n(n-2)r^2 - m^2 - n(n-2))}{(n+m)(n-m)(n-2)} R_{n-2}^m(r).$$

We will use expressions for $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(n\theta)$ as polynomials in $\cos\theta$ to derive explicit and recursive formulae for Z_n^m as polynomials in x and y.

2 Main Dish

Let $x = r\cos\theta$ and $y = r\sin\theta$, we have $r = \sqrt{x^2 + y^2}$.

2.1 Positive $\boldsymbol{m}$

Take the formula due to Chebyshev

$$\cos(m\theta) = \frac{m}{2}\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k}{m-k}\binom{m-k}{k}(2\cos\theta)^{m-2k}$$

and the radial Zernike polynomial formula to define for $m > 0$

$$Z_n^m(x,y) = \frac{m}{2}\sum_{\zeta=0}^{\frac{n-m}{2}}\sum_{\xi=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^{\zeta+\xi}}{m-\xi}\binom{m-\xi}{\xi}\binom{n-\zeta}{\zeta}\binom{n-2\zeta}{\frac{n-m}{2}-\zeta}(2x)^{m-2\xi}\left(x^2+y^2\right)^{\xi-\zeta+\frac{n-m}{2}}$$

By binomial expansion, we have $Z_n^m(x,y) =$

$$\frac{m}{2}\sum_{\zeta=0}^{\frac{n-m}{2}}\sum_{\xi=0}^{\lfloor \frac{m}{2} \rfloor}\sum_{\varsigma=0}^{\xi-\zeta+\frac{n-m}{2}} 2^{m-2\xi}\frac{(-1)^{\zeta+\xi}}{m-\xi}\binom{m-\xi}{\xi}\binom{n-\zeta}{\zeta}\binom{n-2\zeta}{\frac{n-m}{2}-\zeta}\binom{\xi-\zeta+\frac{n-m}{2}}{\varsigma}x^{n-2\zeta-2\varsigma}y^{2\varsigma}$$

We perform the substitution

$$\left.\begin{array}{c} \zeta' = \zeta+\varsigma \\ \xi' = \xi \\ \varsigma' = \varsigma \end{array}\right| \begin{array}{c} 0 \leq \zeta' \leq \lfloor \frac{n}{2} \rfloor \\ \max\left(0, \zeta' - \frac{n-m}{2}\right) \leq \xi' \leq \lfloor \frac{m}{2} \rfloor \\ \max\left(0, \zeta' - \frac{n-m}{2}\right) \leq \varsigma' \leq \zeta' \end{array}$$

to achieve (variable domains not repeated for brevity) $Z_n^m(x,y) =$

$$\frac{m}{2}\sum_{\zeta'}\sum_{\varsigma'}\sum_{\xi'} 2^{m-2\xi'}\frac{(-1)^{\zeta'-\varsigma'+\xi'}}{m-\xi'}\binom{m-\xi'}{\xi'}\binom{n-\zeta'+\varsigma'}{\zeta'-\varsigma'}\binom{n-2\zeta'+2\varsigma'}{\frac{n-m}{2}-\zeta'+\varsigma'}\binom{\xi'-\zeta'+\varsigma'+\frac{n-m}{2}}{\varsigma'}x^{n-2\zeta'}y^{2\varsigma'}$$

The diligent reader will identify a bijection between the sets $\{(\zeta,\xi,\varsigma)\}$ and $\{(\zeta'-\varsigma',\xi',\varsigma')\}$, keeping in mind $\lfloor \frac{n}{2} \rfloor = \frac{n-m}{2} + \lfloor \frac{m}{2} \rfloor$.

It is clear that for $Z_n^{m>0}(x,y)$, the powers of x are $X = [0,n] \cap \{x \equiv n \pmod 2\}$ while the powers of y are $Y = [0,n] \cap \{x \equiv 0 \pmod 2\}$ and that the exponents of Z as a polynomial in x and y are $X \times Y \cap \{(x,y) | x+y \leq n\}$.

2.2 Negative $\boldsymbol{m}$

We repeat the process from section 2.1 for the Zernike polynomials with negative m. Take the formula due to Chebyshev

$$\sin(m\theta) = \left(\sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^k\binom{m-1-k}{k}(2\cos\theta)^{m-1-2k}\right)\sin\theta$$

and the radial Zernike polynomial formula to define for $m > 0$

$$Z_n^{-m}(x,y) = y\sum_{\zeta=0}^{\frac{n-m}{2}}\sum_{\xi=0}^{\lfloor \frac{m-1}{2} \rfloor}(-1)^{\zeta+\xi}\binom{n-\zeta}{\zeta}\binom{n-2\zeta}{\frac{n-m}{2}-\zeta}\binom{m-1-\xi}{\xi}(2x)^{m-1-2\xi}\left(x^2+y^2\right)^{\xi-\zeta+\frac{n-m}{2}}$$

By binomial expansion, we have $Z_n^{-m}(x,y) =$

$$y\sum_{\zeta=0}^{\frac{n-m}{2}}\sum_{\xi=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\sum_{\varsigma=0}^{\xi-\zeta+\frac{n-m}{2}} 2^{m-1-2\xi}(-1)^{\zeta+\xi}\binom{n-\zeta}{\zeta}\binom{n-2\zeta}{\frac{n-m}{2}-\zeta}\binom{m-1-\xi}{\xi}\binom{\frac{n-m}{2}-\zeta+\xi}{\varsigma}x^{n-2\zeta-2\varsigma-1}y^{2\varsigma}$$

We perform the substitution

$$\begin{array}{c|c} \zeta'=\zeta+\varsigma & 0\le\zeta'\le\left\lfloor\frac{n}{2}\right\rfloor \\ \xi'=\xi & \max\left(0,\zeta'-\frac{n-m}{2}\right)\le\xi'\le\left\lfloor\frac{m-1}{2}\right\rfloor \\ \varsigma'=\varsigma & \max\left(0,\zeta'-\frac{n-m}{2}\right)\le\varsigma'\le\zeta' \end{array}$$

to achieve (variable domains not repeated for brevity) $Z_n^{-m}(x,y) =$

$$y\sum_{\zeta'}\sum_{\varsigma'}\sum_{\xi'}2^{m-1-2\xi'}(-1)^{\zeta'-\varsigma'+\xi'}\binom{n-\zeta'+\varsigma'}{\zeta'-\varsigma'}\binom{n-2\zeta'+2\varsigma'}{\frac{n-m}{2}-\zeta'+\varsigma'}\binom{m-1-\xi'}{\xi'}\binom{\frac{n-m}{2}-\zeta'+\varsigma'+\xi'}{\varsigma'}x^{n-2\zeta'-1}y^{2\varsigma'}$$

The following two subsections are crappy.

2.3 The Irrotational Functions

As in 1, for $n\ge 1$ in full generality,

$$Z_n^{\pm n}(r,\theta) = r^n \,{}^{\cos}_{\sin}\, n\theta,$$

so we have

$$Z_n^{\pm n}(r,\theta) = 2r\cos\theta\cdot r^{n-1}\,{}^{\cos}_{\sin}\left((n-1)\theta\right) - r^2\cdot r^{n-2}\,{}^{\cos}_{\sin}\left((n-2)\theta\right).$$

Naturally,

$$Z_n^{\pm n}(x,y) = 2xZ_{n-1}^{\pm(n-1)}(x,y) - \left(x^2+y^2\right)Z_{n-2}^{\pm(n-2)}(x,y).$$

Observe that the equations

$$Z_n^{n}(x,y) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}(-1)^{n-k}x^{n-2k}y^{2k}$$

$$Z_n^{-n}(x,y) = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}(-1)^{k}x^{n-(2k+1)}y^{2k+1}$$

both satisfy the recursion formula and the initial conditions for $n=2$ and $n=1$. When $n=0$, the first equation is correct: $Z_0^{+0}(x,y)=1$.

2.4 The Solenoidal Functions

As in 1, the recursion relation for R holds for Z. Observe that the equation

$$Z_n^0(x,y) = (-1)^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}\sum_{l=0}^{\frac{n}{2}-k}x^{2k}y^{2l}\frac{\left(\frac{n}{2}+k+l\right)!(-1)^{\frac{n}{2}+k+l}}{\left(\frac{n}{2}-(k+l)\right)!(k+l)!k!l!}$$

satisfies both the recursion formula and the initial conditions $Z_0^0(x,y)=1$ and $Z_2^0(x,y)=2\left(x^2+y^2\right)-1$.

$$Z_n^m(x,y) = m\mathrm{sign}(x) \sum_{\vartheta=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k=0}^{\frac{n-m}{2}} \sum_{\varkappa=0}^{\frac{n-m}{2}+\vartheta-k} \left(\binom{\frac{n-m}{2}+\vartheta-k}{\varkappa} \left(x^2\right)^{\frac{m}{2}-\vartheta+\varkappa} \left(y^2\right)^{\frac{n-m}{2}+\vartheta-k-\varkappa} \right)$$
$$\left(\frac{(-1)^k}{k!} \frac{2^m\,(n-k)!}{\left(\frac{n+m}{2}-k\right)!\left(\frac{n-m}{2}-k\right)!} \sum_{\varsigma=0}^{2\vartheta} (-1)^{\varsigma}\, 2^{\varsigma-2\vartheta} \frac{(2m-2\vartheta+\varsigma-1)!\,(\varsigma+m-2\vartheta)!}{(2\vartheta-\varsigma)!\,(2\varsigma+2m-4\vartheta)!\varsigma!\,(m-2\vartheta)!} \right)$$