Modeling of MHD waves in the solar corona

Pieter Luyten, Tomas Reunbrouck

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1 Introduction

The lay of the land

This report discusses simulation results of the plasma in the solar corona in accordance with the theory of magnetohydrodynamics (MHD). We aim give an qualitative interpretation of our simulation data based on the MHD-equations for an ideal plasma.

Studying the behaviour of a plasma based on analytically acquired solutions of the MHD equations is very complicated. Thus, it is useful to attain some solutions numerically and interpret the results by means of visualizations.

In this manner we have completed an analysis of the behaviours of an idealised plasma for different boundary conditions. For each case we have investigated the influence of changes in the initial conditions by running the simulation in parallel for different values of the initial variables and comparing the outputs.

Physics of the solar corona

A brief justification of some basic assumptions is in place. As mentioned, we presuppose that the solar corona consists of an ideal plasma. That is, a highly ionised gas with smooth background conditions. A gas of this type should demonstrate a 'collective behaviour' which is necessary for the the ideal MHD-equations to be sufficiently accurate. By 'collective behaviour', we mean the following.

A plasma consists of positively charged ions and negatively charges electrons. To be able to assume the ideal MHD theory we must have that the kinetic energy of these particles sufficiently outweighs the potential energy produced by the pairwise Coulomb interactions. In other words, the ratio $\frac{KE}{PE}$ must be very big. If this is indeed the case we may presume that we are working with a collection of particles that interact with smooth a background.

A smooth background is achieved by a phenomenon called 'electric screening'. This is the effect by which positively charges ions are electrically screened from each other - in that their Coulomb interaction becomes negligible - by a cloud of electrons.

Consider such an ion. We can write the electric potential in a system of mobile charged particles as

$$\phi \sim \frac{1}{r} e^{-rk_D}$$

where $k_D = \frac{1}{\lambda_D}$ and λ_D is the Debye length. The reason why the Debye length is important for our purposes is that it traces the boundary of where the motion of the particles begins to outweigh the electric potential. We can see this because for a given charge Q

$$\frac{KE}{PE} \sim \frac{T}{\frac{Q}{\lambda_D}} \sim \frac{\lambda_{MFP}}{\lambda_D}$$

where λ_{MHD} is the mean length of a free particle path.

Since the temperature in the solar corona is of the order of $10^6 K$ we may assume that these ratios are sufficiently large to consider the ion gas as a collection of charges particles which behave collectively, which is what constitutes a plasma.

Goal of this report

As mentioned the main purpose of the project was to gain a basic understanding of the nature of the plasma in the solar corona. Specifically MHD-waves are of great interest as they are directly observable in satellite observations of the sun. Therefore, we shall focus our report on them. Concretely, we discuss two compelling examples of MHD-waves: the MHD-blastwave and the interaction of an MHD-wave with a coronal hole.

For the first problem we have simulated an MHD-blastwave under the influence of a very powerful magnetic field. Visualizations of the output data have shown that the results are quite distinctive for different values of the magnetic field's strength. We have also simulated a blastwave under normal hydrodynamic conditions. This HD-blastwave provides the case where the field strength is 0.

Secondly, we discuss a simulation where we had an MHD-wave run into a coronal hole. This hole is part of the boundary conditions for this problem. It consists of a sharp drop in pressure and density. As one would expect, its effect on the wave is quite striking and we shall discuss it in detail.

The software

The software that was used for the numerical solutions is called PLUTO. This is an open source code written specifically for this purpose. It is a piece of code designed to solve the HD and MHD conservation laws for arbitrary initial conditions in a finite volume.

PLUTO was developed by the Department of Physics at Torino University.

[**notes-fluid-dynamics**]

2 Theoretical background

While the main focus of this bachelor project is the numerical modeling of waves in the solar corona, some theoretical background is important to frame the results of our simulations. Furthermore, this knowledge gives some insight in the assumptions that are made in deriving the magnetohydrodynamic (MHD) equations and when they are valid.

Hydrodynamics

2.1 Hydrodynamic fluid equations

The theory in this section is adapted from [**notes-fluid-dynamics**]. For the first task a non-viscous Newtonian fluid is considered. Heat conduction and dissipation is neglected as well. This type of fluid obeys the Euler equations for conservation of mass, momentum and internal energy:

$$\boxed{\begin{aligned} \frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} &= 0 \\ \rho\frac{d\mathbf{v}}{dt} &= -\nabla p + \mathbf{F} \\ \frac{dp}{dt} - \frac{\gamma p}{\rho}\frac{d\rho}{dt} &= 0 \end{aligned} \qquad (1)}$$

These are the Euler equations in Lagrangian form, with time derivatives following the fluid, hence the total derivatives with respect to time. PLUTO does the fluid simulation using a static grid, so we need the equations in Eulerian form with partial derivatives with respect to time. This change of derivatives can be carried out using the following relation, found in [**notes-fluid-dynamics**] in section 2.4:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f \tag{2}$$

where $f(x, y, z, t)$ is a function that describes a property of the fluid. The equations can also be rederived using an Eulerian view. In any case the result is the same:

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \frac{\partial}{\partial t}(\rho \mathbf{v}) &= \nabla \cdot (-p - \rho \mathbf{v}\mathbf{v}) + \mathbf{F} \\ \frac{\partial}{\partial t}\left(\rho\left(\frac{v^2}{2} + \mathcal{U}\right)\right) &= \mathbf{F} \cdot \mathbf{v} - \nabla \cdot \left(\rho\left(\frac{v^2}{2} + \mathcal{U}\right)\mathbf{v} + p\mathbf{v}\right)\end{aligned}}$$

Next introduce the variable $\mathbf{m} = \rho \mathbf{v}$, the momentum density. The energy density $\mathcal{U}$ can be split in the thermal energy ρe and gravitational potential energy $\rho \Phi$. Let $E_t = e\rho + \frac{v^2}{2}$ The only external force is $\mathbf{F} = \rho \mathbf{g}$ carrying out these substitutions leads to the equations in section 6 in the PLUTO manual [**pluto-manual**]:

is the expression for $\mathbf{F}$ correct? Looks to be different in equation 2 and 3

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{m} &= 0 \\ \frac{\partial \mathbf{m}}{\partial t} + \nabla \cdot (\mathbf{m}\mathbf{v} + p) &= -\rho \nabla \Phi + \rho \mathbf{g} \\ \frac{\partial}{\partial t}(E_t + \rho \Phi) + \nabla \cdot ((E_t + p + \rho \Phi)\mathbf{v}) &= \mathbf{m} \cdot \mathbf{g}\end{aligned} \quad (3)}$$

Together with an equation of state $\rho e = \rho e(p, \rho)$, which gives the thermal energy as a function of p and ρ. In the remainder of this paper a calorically ideal gas is assumed. This is a gas for which the adiabatic constant γ obeys:

$$\gamma = \frac{f + 2}{f} \tag{4}$$

where f is the number of degrees of freedom. the previous relation can be rewritten as

$$f = \frac{2}{\gamma - 1}$$

And by substituting this equation in the equation that expresses the energy as a function of degrees of freedom the closure relation $\rho e = \rho e(\rho, p)$ is found:

Add a reference for this energy equation

$$E_t = \rho e = \frac{f}{2} n k_B T = \frac{p}{\gamma - 1}$$

short discussion of assumptions made (no viscosity and heat conduction, callorically ideal gas)

afleiding golven, groepssnelheid

2.2 Hydrodynamic linear waves

We start again from the ideal fluid equations as given in Equation 1 and linearize them. For this we rewrite the quantities ρ and p as a background density ρ_0 and pressure p_0 with slight deviations ρ_1,

p_1. Furthermore it is assumed that there are no external forces acting on the fluid. The Linearized equations are:

$$\boxed{\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} &= 0 \\ \rho_0 \frac{\partial \mathbf{v}}{\partial t} &= -\nabla p_1 \\ \frac{\partial p_1}{\partial t} &= \frac{\gamma p_0}{\rho_0} \frac{\partial \rho_1}{\partial t} \end{aligned} \qquad (5)}$$

By acting with ∇ on the second equation and using the first to substitute $\rho_0 \nabla \cdot \mathbf{v}$ we find the following relation:

$$\frac{\partial^2 \rho_1}{\partial t^2} = -\nabla^2 p_1$$

Acting with $\frac{\partial}{\partial t}$ on the last equation and substituting the previous expression yields

$$\frac{\partial^2 p_1}{\partial t^2} + \frac{\gamma p_0}{\rho_0} \nabla^2 p_1 = 0$$

which is the wave equation with $v_s = \sqrt{\frac{\gamma p_0}{\rho_0}}$ the phase speed of the wave. Similar expressions are found for the other variables. this wave speed can be found by substituting a plan wave of the form $p_1 = A \exp\left(i(\omega t - \mathbf{k} \cdot \mathbf{x})\right)$. Substituting this expression in the wave equation for p_1 leads to the dispersion relation:

Referentie?

$$\omega^2 = k^2 v_s^2. \qquad (6)$$

The phase velocity is given by

$$v_{ph} = \frac{\partial \omega}{\partial k} = v_s \qquad (7)$$

from which we conclude that these waves are non-dispersive.

reference for this relation for the phase speed?

2.3 Hydrodynamic shocks

Now we shall reconsider one of the least convincing assumptions made for the derivations of the fluid equations: that of perfectly continuous background variables. In reality, we might encounter very sudden changes in the scalar variable density ρ and vectorial variable velocity $\mathbf{v}$. To have the theory of ideal fluids take this into account, we can introduce these jumps in the variables as mathematical discontinuities. This discontinuity is appropriately called a 'shock'. We are interested in how this shock moves through the fluid. The derivation of its motion is quite straight forward.

Start from the continuity equation in its Eulerian form in 1D

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0 \qquad (8)$$

Of course, this equation assumes that ρ and ρv are continuous variables with continuous partial derivatives. Rewrite the equation so that over a distance Δx and a duration Δt the variables ρ and ρv experience a change $\Delta \rho$ and $\Delta \rho v$. This gives the much less elegant version

$$\frac{\Delta \rho}{\Delta t} + \frac{\Delta (\rho v)}{\Delta x} = 0 \ .$$

If this were the perfectly continuous case we would now let $\Delta x, \Delta t \to 0$, resulting in Equation 8. However, we might also say that the transition is not smooth and that for $\Delta x, \Delta t \to 0$ the jump remains: $\Delta \rho, \Delta \rho v \to \Delta \rho, \Delta \rho v$. Rewrite the equations to see what this means:

$$\frac{\Delta x}{\Delta t}\Delta\rho + \Delta(\rho v).$$

Then for $\Delta x, \Delta t \to 0$ we get

$$\frac{\partial x}{\partial t}\Delta\rho + \Delta(\rho v) = -V_S\Delta\rho + \Delta(\rho v) = 0 \tag{9}$$

where V_S is the shock speed. This relation is the hydrodynamic shock condition. To generalize it beyond 1D, it suffices to take $\mathbf{v}\cdot\mathbf{n}$ instead of v where $\mathbf{n}$ is the unit normal vector on the shock wave front pointing towards the region with lower pressure. It looks as follows

$$-V_S\Delta(\rho v) + \Delta(\rho\ \mathbf{v}\cdot\mathbf{n}) = 0\ . \tag{10}$$

The minus sign in front of V_S is merely a matter of orientation. In Equation 9 the orientation is along the positive x-axis. In Equation 10 it is along the unit vector $\mathbf{n}$. This is the first of the three *Rankine–Hugoniot* relations. The other two can analogously be derived from the Eulerian form of momentum and energy equations in Equation 1. The three Rankine–Hugoniot conditions are

$$\begin{aligned} V_S\Delta\rho &= \mathbf{n}\cdot\Delta(\rho\ \mathbf{v}) \\ V_S\Delta(\rho\ \mathbf{v}) &= \mathbf{n}\cdot\Delta(\rho\ \mathbf{v}\ \mathbf{v} + p\ \mathbb{I}) \\ V_S\Delta E_t &= \mathbf{n}\cdot\Delta\big(\rho\ (e + \frac{v^2}{2} + \frac{p}{\rho})\ \mathbf{v}\big) \end{aligned} \tag{11}$$

Derivation of MHD equation, discussion of the integration scheme used

Test informatievaardigheden!!!

2.4 Magnetohydrodynamic fluid equations

There are two approaches commonly taken in the literature to derive the MHD equations. They are either derived from kinetic gass theory, or postulated with added justification of why they can accuratly describe plasmas.

A plasma is an ionised gas consisting of postive and negative ions. In the case of the corona of the sun this is mainly ionised hydrogen. Therefore the negative ions are free electrons and the positive ions protons, which are a lot heavier than electrons. When the characteristic timescales τ_e and τ_i between two collisions of electrons, respectively ions, is much shorter than characteristic timescales τ_f at which macroscopic variables change we can use a fluid description. At these timescales the individual interactions of individual particles are not relevant anymore.

The plasma can then be described as two different fluids, commonly refered to as the *two-fluid theory*. The electron gas is one fluid and the proton gas the other. The next assumption that is made, is that the relaxation time τ_T until the electron fluid and ion fluid are in thermal equilibrium after a slight disturbance is also a lot smaller than τ_f. Finally, we assume that the fluid has no net charge. Not globally, but also not locally. This means that in every large enough volume, for every ion with charge Z, there are also about Z electrons in this volume. When all this applies, the variables describing the different fluids can be averaged or added together, to describe the plasma as one fluid.

add reference to cursus Poedts and course notes arxiv

The MHD equations can then be found by adding the maxwell equations to the HD equations. Because the HD equations are invariant under Galilean transformations. However the Maxwell equations are invariant under Lorentz transformations, so we cannot simply add them to the HD equations and expect a consistent picture. Understanding the averaging process is important for understanding what the plasma variables actually represent. Denote with n_α the number density of a certain type of particle, m_α the mass, $\mathbf{u}_\alpha$ the velocity of a fluid element and with p_α the pressure of the gas of these particles. Let the subscript e denote variables concerning the electrongas and i variables describing the iongas. The variables describing the plasma are the following linear combinations of variables describing the electron and ion gas:

$$\boxed{\begin{aligned} \rho &= n_e m_e + n_i m_i \\ \mathbf{v} &= (n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i)/\rho \\ \mathbf{J} &= -e(n_e \mathbf{u}_e - Z n_i \mathbf{u}_i) \\ p &= p_e + p_i \end{aligned} \qquad (12)}$$

Where e is the charge of an electron and Z the charge of an ion as a multiple of the electron charge. The first equation is the *total mass density*, the second the *center of mass velocity*, the third the *current density* and the last one describes the *total pressure*.

For a consistent Newtonian theory of MHD, the displacement current $\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ is neglected.

reference to lecture notes arXiv

Finally, the viscosity and heat flow are neglected like in the HD case. Furthermore, for the ideal MHD case the resistivity of the fluid is neglected. The extra equations we need are then:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \end{aligned}$$

We do not need an equation relating the charge distribution to the electric field in the first equation since we assumed the fluid is locally neutral. Furthermore the displacement term in the third equation was neglected.

Adding everything together such as in [REFERENCE TO ONE OF THE COURSES] yields the ideal MHD equations:

reference to course notes poedts

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p - \mathbf{J} \times \mathbf{B} &= 0 \\ \frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} &= 0 \\ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) &= 0 \end{aligned} \qquad (13)}$$

Where

$$\mathbf{J} = \frac{\nabla \times \mathbf{B}}{\mu_0}.$$

We need one additional equation which the inital condition has to satisfy:

$$\nabla \cdot \mathbf{B} = 0$$

which expresses that there are no magnetic monopoles. By acting with $\nabla \cdot$ on the fourth equation in Equation 13 we see that if the initial equation satisfies $\nabla \cdot \mathbf{B} = 0$, it is automatically satisfied for all later times:

$$\frac{\partial (\nabla \cdot \mathbf{B})}{\partial t} = 0$$

The equations used by PLUTO in the ideal case have a slightly different form:

reference to user guide

$$\boxed{\begin{aligned}\frac{\partial\rho}{\partial t}+\nabla\cdot(\mathbf{m})&=0\\ \frac{\partial\mathbf{m}}{\partial t}+\nabla\cdot\left[\mathbf{mv}-\mathbf{BB}+I\left(p+\frac{B^2}{2}\right)\right]^T&=-\rho\nabla\Phi+\rho\mathbf{g}\\ \frac{\partial\mathbf{B}}{\partial t}+\nabla\times(c\mathbf{E})&=0\\ \frac{\partial(E_t+\rho\Phi)}{\partial t}+\nabla\cdot\left[(E_T+p_t+\rho\Phi)\,\mathbf{v}-\mathbf{B}(\mathbf{v}\cdot\mathbf{B})\right]&=\mathbf{m}\cdot\mathbf{g}\end{aligned}} \tag{14}$$

where, as with the HD equations, $\mathbf{m}=\rho\mathbf{v}$ and E_t is again the total energy density, this time with an extra term for the magnetic field:

$$E_t=\rho e+\frac{\rho v^2+B^2}{2}$$

$c\mathbf{E}$ is given by:

$$c\mathbf{E}=-\mathbf{v}\times\mathbf{B}$$

note that the equations do not formally depend on the speed of light, but it is kept in the equations for consistency with the relativistic case.

2.5 Magnetohydrodynamic waves

For our discussion of magnetohydrodynamic we rewrite the basic MHD equations into a form which is easier to linearalize. We also assume that the plasma is completely homogeneous and in equilibrium. In practice this means that we shall remove the cross products in the original equations as follows

$$\begin{aligned}-\mathbf{J}\times\mathbf{B}&=-(\nabla\times\mathbf{B})\times\mathbf{B}=(\nabla\mathbf{B})\cdot\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{B}\\ \nabla\times\mathbf{E}&=-\nabla\times(\mathbf{v}\times\mathbf{B})=\mathbf{B}\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{v}\end{aligned} \tag{15}$$

The MHD equations now become

$$\boxed{\begin{aligned}\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})&=0\\ \rho\frac{\partial\mathbf{v}}{\partial t}+\rho\mathbf{v}\cdot\nabla\mathbf{v}+(\gamma-1)\nabla(\rho e)+(\nabla\mathbf{B})\cdot\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{B}&=0\\ \frac{\partial e}{\partial t}+\mathbf{v}\cdot\nabla e+(\gamma-1)e\nabla\cdot\mathbf{v}&=0\\ \frac{\partial\mathbf{B}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{B}+\mathbf{B}\nabla\cdot\mathbf{v}-\mathbf{B}\cdot\nabla\mathbf{v}&=0\end{aligned}} \tag{16}$$

Of course the condition that $\nabla\cdot\mathbf{B}=0$ remains. In this form the equations are much easier to linearalize:

$$\boxed{\begin{aligned} \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) &= 0 \\ \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + (\gamma - 1)\nabla(\rho_1 e_0) + \nabla(\rho_0 e_1) + (\nabla \mathbf{B}_1) \cdot \mathbf{B}_0 - \mathbf{B}_0 \cdot \nabla \mathbf{B}_1 &= 0 \\ \frac{\partial e_1}{\partial t} + (\gamma - 1) e_0 \nabla \cdot \mathbf{v}_1 &= 0 \\ \frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0 \nabla \cdot \mathbf{v}_1 - \mathbf{B}_0 \cdot \nabla \mathbf{v}_1 &= 0 \end{aligned} \qquad (17)}$$

The second one of these equations Equation 17 is the linearalized momentum equation. We shall work from this one as it lends itself the most for our discussion of ideal MHD waves. This is because it directly describes flow velocity. Plugging the other three into this equation gives us the essential equation for ideal MHD waves:

$$\frac{\partial^2 \mathbf{v}_1}{\partial^2 t} = \Big((\mathbf{b}\nabla)^2 \mathbb{I} + (b^2 + c^2)\nabla\nabla - \mathbf{b} \cdot \nabla(\nabla \mathbf{b} + \mathbf{b}\nabla) \Big) \cdot \mathbf{v}_1 \tag{18}$$

where

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}} \tag{19}$$

and

$$\mathbf{b} = \frac{\mathbf{b}_0}{\sqrt{\rho_0}}. \tag{20}$$

We introduce this constants c and $\mathbf{b}$ as they will be the wave velocities of the solutions of the wave equation Equation 18. The constant c is the acoustic speed known from regular hydrodynamics. The constant $\mathbf{b}$ is know as the *Alfvén* velocity and it is a vector in the same direction as the background magnetic field $\mathbf{B}_0$.

Notice that if we set $\mathbf{B} = 0$ Equation 18 becomes

$$\frac{\partial^2 \mathbf{v}_1}{\partial^2 t} = c^2 \nabla^2 \mathbf{v}_1$$

which is exactly what we would expect as this is wave equation in the normal hydrodynamic case. This is an important sanity check for our method.

We shall be looking for sinusoidal wave solutions. For now we shall also limit the discussion the waves in the velocity vector field as the waves in the scalar pressure and density fields and the magnetic vector field can easily be expressed in terms of the velocity field using the equations Equation 17. The solutions we look for are of the form

$$\mathbf{v}_1 = \bar{v} \exp(i(\omega t - \mathbf{k} \cdot \mathbf{x})) \ .$$

Under the constrain of having to provide sinusoidal wave solutions Equation 18 becomes

$$\Big((\omega^2 - (\mathbf{k} \cdot \mathbf{b})^2)\mathbb{I} - (b^2 + c^2)\mathbf{k}\mathbf{k} + \mathbf{k} \cdot \mathbf{b}(\mathbf{k}\mathbf{b} + \mathbf{b}\mathbf{k}) \Big) \cdot \bar{v} = 0 \ . \tag{21}$$

Without any loss of generality we may assume that $\mathbf{b} = (b, 0, 0)$ and $\mathbf{k} = (k_x, k_y, 0) = (k\cos\theta, k\sin\theta, 0)$ where θ is the angle between $\mathbf{b}$ and $\mathbf{k}$. Filling in these into Equation 21 results in

$$\begin{pmatrix} \omega^2 - k_x^2 c^2 & -k_y k_x c^2 & 0 \\ -k_y k_x c^2 & \omega^2 - k_y^2(b^2 + c^2) - k_x^2 b^2 & 0 \\ 0 & 0 & \omega^2 - k_x^2 b^2 \end{pmatrix} \begin{pmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{v}_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

In order to solve this we need the determinant of the matrix in Equation 22 to be 0. This results in the dispersion relation

$$\left(\omega^2 - k^2 b^2 \cos^2\theta\right)\left(\omega^4 - \omega^2 k^2 (b^2 + c^2) + b^2 c^2 k^4 \cos^2\theta\right) = 0\ . \tag{23}$$

We shall first discuss the factor $\left(\omega^4 - \omega^2 k^2 (b^2 + c^2) + b^2 c^2 k^4 \cos^2\theta\right)$ of which the roots are

$$\omega_{F,S}^2 = \frac{k^2}{2}\left[b^2 + c^2 \pm \sqrt{(b^2 + c^2)^2 - 4b^2 c^2 \cos^2\theta}\right] \tag{24}$$

These solutions correspond to the so-calles fast (+) and slow (-) magnetosonic waves. Notice that because $(v_a^2 + v_s^2)^2 - 4v_a^2 v_s^2 \cos^2\theta \geq (v_a^2 - v_s^2)^2) \geq 0$ the square root can always be taken. for the negative sign however

One can readily see that they are the result of a quite complicated interplay between the hydrodynamic and magnetic sides of the story. To help us see this better, the other factor will help.

The only root of the firs factor in Equation 23 is $\omega^2 = k_x^2 b^2$. This solution is of great interest as it does not contain the same complicated magnetosonic interaction and solely depends on the nature of the magnetic field. The density irregularities only provide the wave's momentum. The restoring force is entirely generated by the tension in the magnetic field.

The wave corresponding to $\omega_A^2 = k_x^2 b^2$ is called the *Alfvén* wave. Notice that its direction corresponds to that of the magnetic field, where $\omega_A = k_x b$ lies in the same direction and $-\omega_A$ in the opposite direction. It should be noted that this solution is non-relativistic. As the magnetic field becomes stronger in comparison to the density the Alfén wave becomes a regular electromagnetic wave.

Now, the first roots we had - the magnetosonic waves - are combinations of Alvén waves and ordinary sound waves. There are two types because the Alfvén and sound waves can either be in fase or in antifase to one another. In the first case ω_F the region of high pressure will correspond to a high magnetic field density, which causes the resulting wave to be driven forward by both ordinary hydrodynamic pressure and the tension of the concentrated magnetic field lines. In the other case ω_S these same two forces work against each other, slowing the wave.

Magnetohydrodynamic shocks

To derive the Rankine–Hugoniot conditions for MHD shocks, one simply uses Equation 13 as it is already in its Eulerian form and preforms the same calculations as for the Rankine–Hugoniot conditions for HD shocks. This yields

$$\begin{aligned} V_S \Delta\rho &= \mathbf{n} \cdot \Delta(\rho\ \mathbf{v}) \\ V_S \Delta(\rho\ \mathbf{v}) &= \mathbf{n} \cdot \Delta(\rho\ \mathbf{v}\ \mathbf{v} + (p + \frac{B^2}{2})\ \mathbb{I} - \mathbf{BB}) \\ V_S \Delta E_t &= \mathbf{n} \cdot \Delta\big((\rho\frac{v^2}{2} + \frac{\gamma}{\gamma - 1}p + B^2)\ \mathbf{v} - \mathbf{v} \cdot \mathbf{BB}\big) \\ V_S \Delta\mathbf{B} &= \mathbf{n} \cdot \Delta\big(\mathbf{vB} - \mathbf{Bv}\big) \end{aligned} \tag{25}$$

Now, for the purposes of this report we would like to write a shock condition in terms of pressure as this is the variable on which we shall be focusing in the simulation data results. Consider a pressure discontinuity $\Delta p = p_1 - p_0$ for some constant background density ρ. Notice that this implies a shock in the velocity field as well since we get from the ideal gas law that $v^2 = \frac{\gamma p}{\rho}$.

Use the momentum equations to obtain in the 1D case that

$$\begin{aligned} V_S = \frac{\Delta \rho v^2 + p}{\Delta \rho v} &= \frac{\Delta(\gamma+1)p}{\Delta\sqrt{\gamma\rho p}} \\ &= \frac{(\gamma-1)(p_1-p_0)}{\sqrt{\gamma\rho}(\sqrt{p_1}-\sqrt{p_0})} \\ &= \frac{\gamma-1}{\sqrt{\gamma\rho}}(\sqrt{p_1}+\sqrt{p_0}) \end{aligned} \tag{26}$$

Lifting this to 3D contains a small subtlety. Without loss of generality we may say that the unit normal on the shock front is $\mathbf{n} = (1, 0, 0)$. From the 3D momentum equation we then get that

$$V_S \begin{pmatrix} \Delta\rho v_1 \\ \Delta\rho v_2 \\ \Delta\rho v_3 \end{pmatrix} = \begin{pmatrix} \Delta\rho v_1^2 + p \\ \Delta\rho v_1 v_2 \\ \Delta\rho v_1 v_3 \end{pmatrix}$$

Only the first components in this equation are of concern to us since the second and third will be 0 due to our choice of $\mathbf{n}$. Now, one think this will yield the same equation as in the 1D case but in should be taken into account that $v_1 \neq v$. In general, we cannot say what the precise relationship between v_1 and $\mathbf{v}$ is but look at the average case $v_1 = \frac{1}{\sqrt{3}}v$. This then gives us the pressure shock condition for 3D

$$V_S = \frac{1}{\sqrt{3}}\frac{\gamma-1}{\sqrt{\gamma\rho}}(\sqrt{p_1}+\sqrt{p_0}) \tag{27}$$

2.6 Units

The PLUTO code works, in general, with dimensionless code-units. This is done by defining a unit density ρ_0, unit velocity v_0 and unit length L_0. From these, unit time can be defined as $t_0 = L_0/v_0$. Inspired by Equation 19 and Equation 20 we defin $p_0 = \rho_0 v_0^2$ and $B_0 = v_0\sqrt{4\pi\rho_0}$. Next, we use the substitutions $v_0\mathbf{v}_u = \mathbf{v}$, $\rho_0\rho_u = \rho$, $L_0\mathbf{x}_u = \mathbf{x}$, $t_0 t_u = t$, $p_0 p_u = p$ and $B_0\mathbf{b}_u = \mathbf{B}$ in Equation 13. Here the subscript u denotes that these numbers or vectors are dimensionless, the units are contained in the factors with subscript 0.

$$\boxed{\begin{aligned} \frac{\rho_0}{t_0}\frac{\partial\rho_u}{\partial t_u} + \frac{\rho_0 v_0}{L_0}\nabla\cdot(\rho_u\mathbf{v}_u) &= 0 \\ \frac{\rho_0 v_0^2}{L_0}\rho_u\left(\frac{\partial\mathbf{v_u}}{\partial t_u} + \mathbf{v}_u\cdot\nabla_u\mathbf{v}_u\right) + \frac{\rho_o v_0^2}{L_0}\nabla_u p_u - \frac{\rho_0 v_0^2}{L_0}\frac{\nabla_u\times\mathbf{B}_u}{\mu_0}\times\mathbf{B}_u &= 0 \\ \frac{p_0 v_0}{L_0}\frac{\partial p_u}{\partial t_u} + \frac{p_0 v_0}{L_0}\mathbf{v}_u\cdot\nabla_u p_u + \frac{p_0 v_0}{L_0}\gamma p_u\nabla_u\cdot\mathbf{v}_u &= 0 \\ \frac{B_0 v_0}{L_0}\frac{\partial\mathbf{B}_u}{\partial t_u} - \frac{B_0 v_0}{L_0}\nabla_u\times(\mathbf{v}_u\times\mathbf{B}_u) &= 0 \end{aligned}} \tag{28}$$

We see that all the units drop out of the equations and we are left with a set of dimensionless equations. This highlights an important fact for ideal MHD: the equations are scale-invariant. For the behaviour of waves the absolute scales are not important, but only the relative magnitude of the characteristic scales for a wave. When we have a wave with wavelength λ_1 and frequency f_1, it will show the same behaviour as a wave with wavelength λ_2 and frequency f_1/λ_1 in the same medium (that is, same Alfvén speed).

Because the ideal MHD equations are in fact dimensionless (without sources), we can carry out the simulations in dimensionless units and later scale our results to match conditions as found in e.g. the solar corona, under the condition that the relative magnitudes of ρ_0, v_0 and L_0 are correct.

3 Shock waves

As a testcase to learn how to use the PLUTO software we simulated hydrodynamic and magneto-hydrodynamic blastwaves in a 2D domain. The initial condition for the simulations is a circle with radius 0.3 (in code units) with higher pressure. The pressure outside the circle is 1, in code units. For the pressure inside the circle two scenarios were simulated. One with a large pressure difference, where the pressure inside the circle is 5. In the other scenario a lower pressure difference was used: the pressure inside the circle was 1.5 The reason for using different pressure differences is to see the non-linear effects in the shock-wave with high pressure difference.

3.1 Hydrodynamic shockwave

First some technical details of the simulation (all physical quantities are in code units). The simulation was done on a 1024×1024 grid for a period of $t = 1.5$ with an initial time step of $1e - 4$. A snapshot of the variables describing the system was saved every 0.03 time units, for a total of 50 snapshots (excluding the initial condition). The Pluto simulation used 996 steps.

In Figure 3.1 the pressureprofile of both scenarios is plot for the initial condition and two frames right after the start of the simulation. In Figure 3.1 the profile is plotted for later times. Its imedeatly clear that the shock wave with the higher pressure difference travels faster then the other one, but the general shape of the wave is the same.

In Figure 3.1 the speed of the wave front along the positive x-axis is plotted as a function of time, and this confirms that the wave speed is lower with a lower pressure difference. The group speed was calculated in two ways. The first was to start at a point at the edge of the domain and find the first point on the line from this point to the center with a pressure higher than 1.01 times the background pressure. This gives a relation $x(t)$ for the position of the wave-front along this line. The speed was calculated using numerical differentiation with the following central difference:

$$f'(t_0) = \frac{f(t+dt) - f(t-dt)}{2dt} + O(dt^2)$$

found in [REF]. This is the simulated speed, represented by the dots.

reference naar cursus numerieke wiskunde

Using [FORMULA SHOCK SPEED RIEMANN PROBLEM] the theoretical speed of a shock in a Riemann problem was calculated using the highest pressure in the domain as max pressure and 1 as the lowe pressure. This is plotted as the line.

calculate shock speed in a Riemann problem as function of the two pressures

Firstly, we notice that the speeds calculated with the first metod tend to lie on some lines. This is a numerical from the discretization of time and space. It has nothing to do with the physics of the problem. The slight deviations from this lines are because PLUTO does not use fixed time steps for integration, therefore the time between two snapshots is not always exactly the same.

We see that the curve closely follows the dots with the lower pressure difference, whereas this is not the case for the higher pressure difference. There are two main reasons for this discrepancy. Firstly, this is not exactly the same as a Riemann problem, certainly at a later stage when the sharp boundary has smoothend out. Secondly, due to non-linear effects we cannot assume that the wave is non-dispersive. This means that the first edge of the wave is probably traveling faster than the group speed of the center of this shockwave, which is the value calculated in the Riemann problem. These two effects together can explain the difference in speed between the value for a Riemann problem and the value calculated from the simulation data.

We remark that approximating the shock as a Riemann problem is a crude first-order approximation. The PLUTO code calculates flow velocities like this in every integration step. This small example highlights the importance of robust integrators and Riemann solvers, together with small enought time steps, to acuratly model flow variables in the vicinity of large gradients.

Finally Figure 3.1 shows that the wave is isotropic. The speed of the wavefront along lines making an angle θ with the x-axis through the origin was calculated. The slight variations in speed are again artifacts of the space discretization.

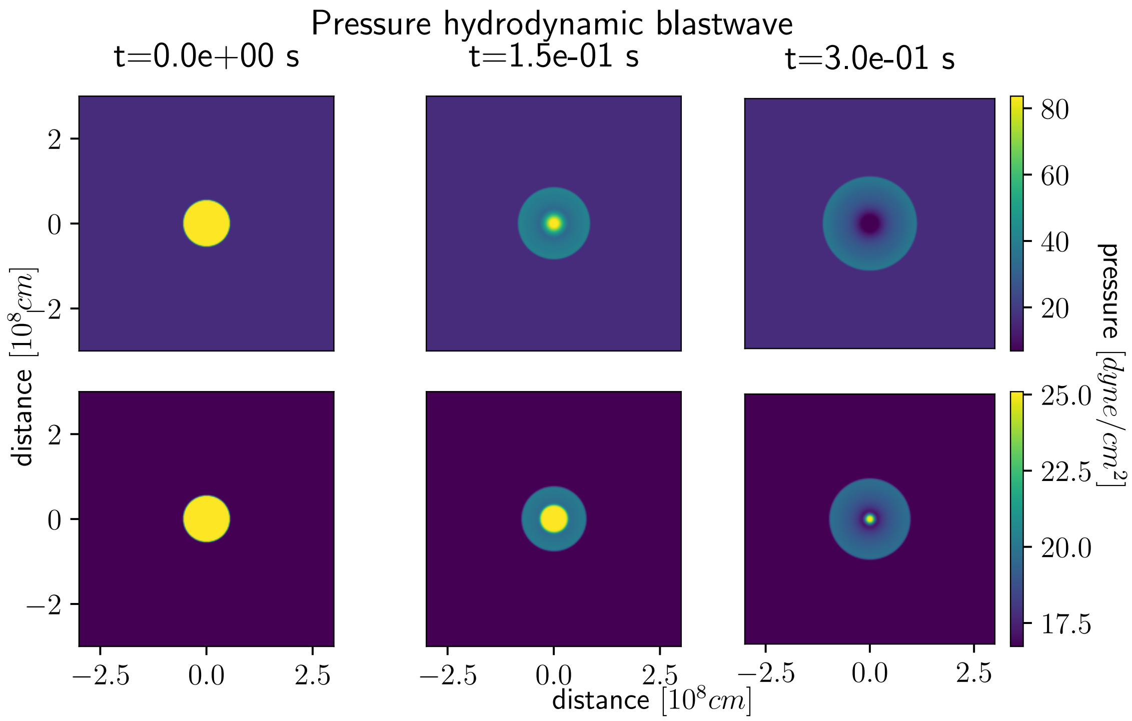

Figure 1: Pressure profile for a blastwave in an ideal fluid at different times. The top row start with the larger pressure difference of 5/1, the bottom row is the blastwave with smaller pressure difference of 1.5/1.

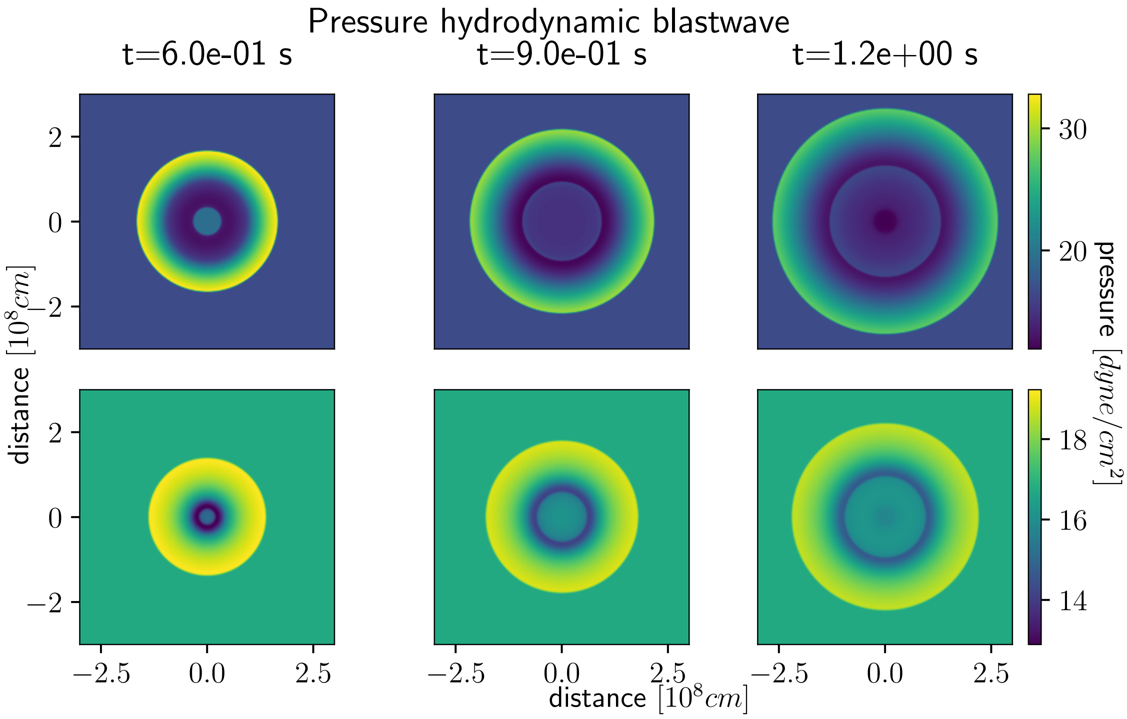

Figure 2: Pressure profile for a blastwave in an ideal fluid at larger timescales. Initial conditions for each row are the same as in Figure 3.1.

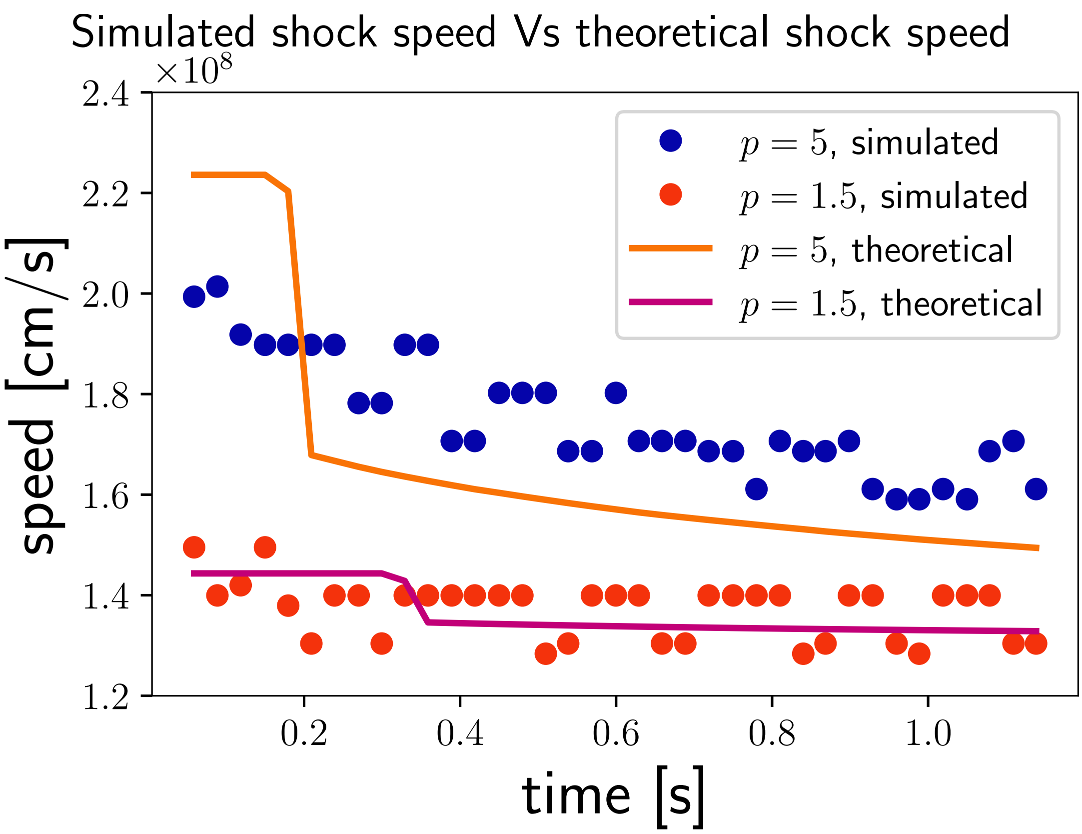

Figure 4: the speed of the wave as a function of time, along the x-axis. The dots represent the speed calculated from the simulation data by deriving the function $x(t)$ that represents the position of the wave front. The lines represent the theoretical shock speed in a Riemann-problem with as high pressure the highest pressure in the domain, and low pressure 1.

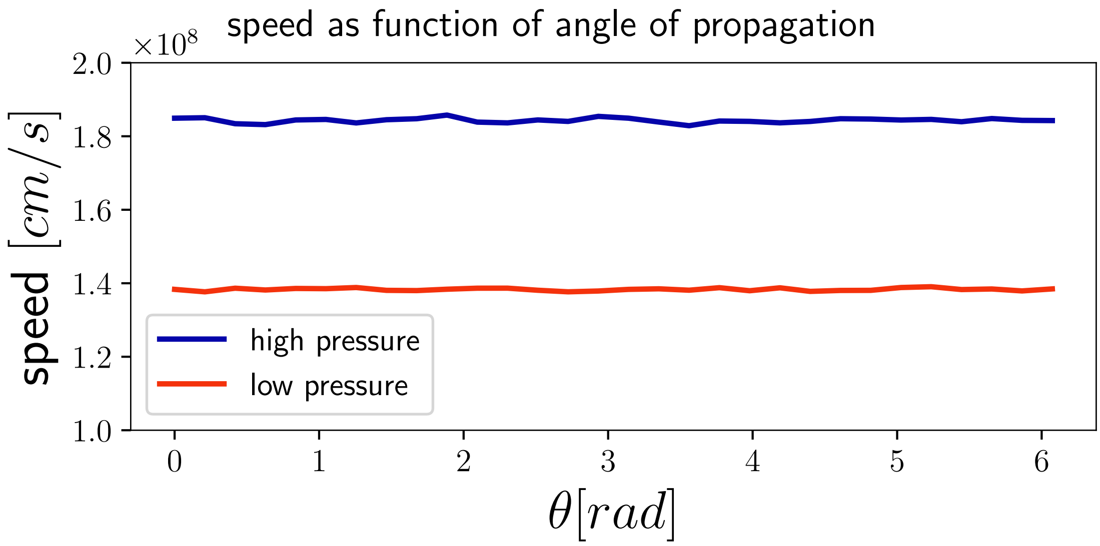

Figure 3: The wave speed as a function of the angle between the x-axis and the line connecting a point on the wave-front with the center of the domain. The wavespeed is calculated from the simulation data.

test

3.2 Magnetohydrodynamic shock wave

In the magnetohydrodynamic case the same initial conditions where used, but a uniform magnetic field in the y-direction was added. The simulation was done for varying strengths of the magnetic field.