Complex Analysis: Week 7

Exercise 7.1. Suppose $f : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous, analytic in the unit disk $\mathbb{D}$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions to the equation $f(z) = z^n$, where n is a positive integer.

Exercise 7.2. For $\lambda > 1$, show that the equation

$$\lambda - z - e^{-z} = 0$$

has exactly one solution in the half plane $\operatorname{Re} z > 0$. Show that this solution is real. What happens to the solution as $\lambda \to 1$?

Exercise 7.3. Determine the number of zeros of the polynomial $p(z) = z^7 + 5z^3 - z - 2$ in the unit disk.

Exercise 7.4. Show that $f(z) = z^6 - 5z^4 + 10$

(a) has no zeros in $|z| < 1$,

(b) has four zeros in $1 < |z| < 2$,

(c) has two zeros in $2 < |z| < 3$.

Exercise 7.5. Let $P_t(z)$ be a polynomial in z of degree $\leq N$ for each value of $t \in [0, 1]$. Suppose that $P_t(z)$ depends continuously on t in the sense that

$$P_t(z) = \sum_{j=0}^{N} a_j(t) z^j$$

and each $a_j(t)$ is continuous for $t \in [0, 1]$.

(a) Suppose z_0 is a simple zero of P_0. Show that there exist $r > 0$ and $t_0 > 0$ such that for every $t < t_0$ the polynomial P_t has exactly one zero in $D_r(z_0)$.

(b) What can you say if z_0 is a zero of order $k \geq 2$ of P_0?

Exercise 7.6. Use Rouché's theorem to prove the fundamental theorem of algebra.

Hint: for the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ (with $n \geq 1$ and $a_n \neq 0$) we take $g(z) = a_n z$. Then for $|z| = R > 1$ we have

$$|f(z) - g(z)| \leq (|a_0| + |a_1| + \cdots + |a_{n-1}|) R^{n-1},$$

and this is smaller than $|g(z)|$ if R is large enough.

Exercise 7.7. Prove that, for any given $\varepsilon > 0$, the function

$$z \mapsto \frac{1}{z+i} + \sin z$$

has an infinite number of zeros in the strip $|\operatorname{Im} z| < \varepsilon$.

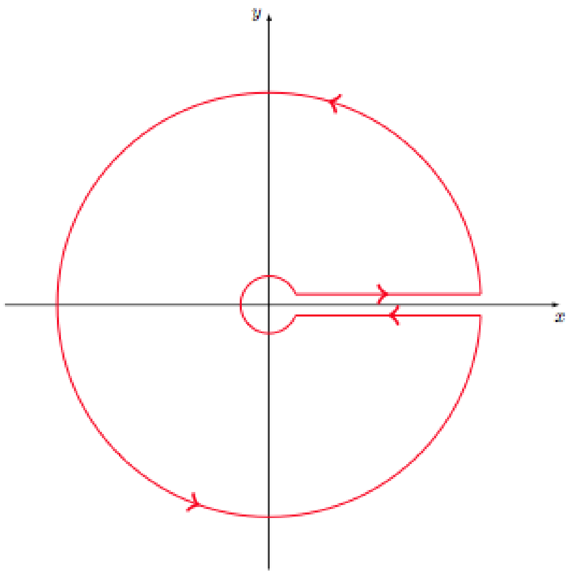
Exercise 7.8. We are going to evaluate

$$I_p := \int_0^\infty \frac{x^{p-1}}{1+x} dx$$

by means of the meromorphic function on $\mathbb{C} \setminus [0, \infty)$ defined by

$$f(z) = \frac{z^{p-1}}{1+z},$$

with $z^p = r^{p-1}e^{i(p-1)\theta}$ if $z = re^{i\theta}$ with $0 < \theta < 2\pi$. We use the following keyhole contour $\gamma_{\delta,\varepsilon,R}$ (see figure) that depends on three parameters $0 < \delta < \varepsilon < R$.



The contour has two segments that are at a distance δ from the positive real axis. The intervals connect a small circle around zero with radius ε with a big circle of radius R. The orientation of the contour is as shown in the figure.

(a) Show that the improper integral I_p converges if and only if $0 < p < 1$.

(b) Assume $\varepsilon < 1 < R$. Compute $\displaystyle\oint_{\gamma_{\delta,\varepsilon,R}} f(z)dz$ by means of a residue calculation.

(c) We first let $\delta \to 0+$. Show that the integral of f over the horizontal segment in the upper half plane tends to $\displaystyle\int_\varepsilon^R f(x)dx$, while the integral over the horizontal segment in the lower half plane tends to $-e^{2\pi ip}\displaystyle\int_\varepsilon^R f(x)dx$

(d) Show that $\displaystyle\lim_{\varepsilon\to 0+} \int_{C_\varepsilon} f(z)dz = 0$ and $\displaystyle\lim_{R\to+\infty} \int_{C_R} f(z)dz = 0$, where we assume that $0 < p < 1$.

(e) Combine parts (b), (c), and (d) to compute I_p for $0 < p < 1$. Write the answer in such a way that it is clear that I_p is real and positive.

Exercise 7.9. For which values of $p \in \mathbb{R}$ is the integral

$$I = \int_0^\infty \frac{x^p}{x^2 + a^2} dx$$

convergent? Evaluate the integral by residue calculations for those values for which it is convergent.

Exercise 7.10. Evaluate the integrals

$$I_1 = \int_0^\infty \frac{\log x}{x^2 + 1} dx \quad \text{and} \quad I_2 = \int_0^\infty \frac{\log^2 x}{x^2 + 1} dx.$$

Hint: use residue integration for the integral

$$\oint_{\gamma_{\varepsilon,R}} \frac{\log^2 z}{z^2 + 1} dz$$

where $\gamma_{\varepsilon,R}$ is the closed path consisting of the intervals $[-R, -\varepsilon]$ and $[\varepsilon, R]$, and the two semi-circles of radius ε and R in the upper half-plane.

Exercise 7.11. For $p, q > 0$, compute

(a) $\displaystyle\int_{-\infty}^\infty \frac{\cos(px) - \cos(qx)}{x^2 + \varepsilon^2} dx$ where $\varepsilon > 0$, and

(b) $\displaystyle\int_{-\infty}^\infty \frac{\cos(px) - \cos(qx)}{x^2} dx.$

(c) Is it true that

$$\lim_{\varepsilon \to 0+} \int_{-\infty}^\infty \frac{\cos(px) - \cos(qx)}{x^2 + \varepsilon^2} dx = \int_{-\infty}^\infty \frac{\cos(px) - \cos(qx)}{x^2} dx\,?$$

Exercise 7.12. Show that

$$\int_0^\infty \frac{x^{-p}}{x^2 + 2x\cos\theta + 1} dx = \frac{\pi}{\sin(p\pi)} \frac{\sin(p\theta)}{\sin\theta}$$

for $0 < p < 1$ and $0 < \theta < \pi$.

Exercise 7.13. (Difficult) Show that

$$\int_0^\infty \frac{x}{\sinh kx} dx = \frac{\pi^2}{4k^2}, \qquad k > 0$$

Hint: first reduce to the case $k = 1$. For $k = 1$, use an indented rectangular contour with corners at $\pm R$ and $\pm R + i\pi$.

Exercise 7.14. An annulus is a region of the form

$$A(z_0; R_1, R_2) = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$$

with $z_0 \in \mathbb{C}$ and $0 \leq R_1 < R_2 \leq +\infty$. The aim of this exercise is to prove the following.

Theorem 2. *Suppose f is holomorphic on $\Omega = A(z_0; R_1, R_2)$. Then there exist a_k for $k \in \mathbb{Z}$ such that*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \qquad z \in \Omega. \tag{2}$$

The series in (2) is called a **Laurent series**.

(a) Let $C_r = C_r(z_0)$ be the circle of radius r centered at z_0. Pick r_1, r_2 with $R_1 < r_1 < r_2 < R_2$ and show that

$$\oint_{C_{r_1}} g(\zeta) d\zeta = \oint_{C_{r_2}} g(\zeta) d\zeta$$

for every holomorphic $g : \Omega \to \mathbb{C}$.

(b) Apply (a) to $g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$ (why is it holomorphic at $\zeta = z$?) and deduce that

$$f(z) = \frac{1}{2\pi i} \oint_{C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{C_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

whenever $z \in A(z_0; r_1, r_2)$.

(c) Show that

$$\frac{1}{2\pi i} \oint_{C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad z \in A(z_0; 0, r_2),$$

and

$$-\frac{1}{2\pi i} \oint_{C_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k, \qquad z \in A(z_0; r_1, \infty),$$

with coefficients a_k that do not depend on the radii r_1 and r_2 that we picked in part (a).

(d) Deduce that (2) holds.

Note: The theorem applies in particular to the case $R_1 = 0$. Then z_0 is an isolated singularity of f. If f has a pole at z_0 of order n, then we know that for some $R_2 > 0$,

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k, \qquad z \in \Omega = D(z, R_2) \setminus \{0\},$$

which is of the form (2) with $a_k = 0$ whenever $k < -n$. If f has an essential singularity at z_0, then the Laurent series has an infinite number of non-zero a_k with $k < 0$.

Exercise 7.15. Let f be holomorphic and non-constant in a region Ω. Let C be a simple closed curve that enclosed a region D with $C \cup D \subset \Omega$. Suppose f is injective on C. Then $f(C)$ is a simple closed curve that encloses a region $\widetilde{D}$.

(a) Show that the restriction of f to D is a bijection from D to $\widetilde{D}$.

(b) Show that thee inverse mapping $g : \widetilde{D} \to D$ (which exists since $f : D \to \widetilde{D}$ is a bijection by part (a)) is holomorphic and

$$g(w) = \frac{1}{2\pi i} \oint_C \frac{z f'(z)}{f(z) - w} dz, \qquad w \in \widetilde{D}.$$