Complexe analyse huistaak 7

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Let f be holomorphic and non-constant in a region Ω. Let C be a simple closed surve that encloses a region D with $c \cup D \subset \Omega$. Suppose f is injective on C. Then $f(C)$ is a simple closed curve that encloses a region $\widetilde{D}$.

a
(a) Show that the restriction of f to D is a bijection from D to $\widetilde{D}$

(b) Show that the inverse mapping $g : \widetilde{D} \to D$ (which exists since $f : D \to \widetilde{D}$ is a bijection by part (a)) is holomorphic and

$$g(w) = \frac{1}{2\pi i} \oint_C \frac{zf'(z)}{f(z) - w} dz$$

(a)

We apply the argument principle on the function $f(z) - w$, for a complex number $w \notin f(C)$. Because f is holomorphic and has no poles inside Ω, the number of zeros (N) of $f(z) - w$ in D is equal to:

$$N = \frac{1}{2\pi i} \int_C \frac{f'(z)dz}{f(z) - w}$$

by the argument principle. We can use the substitution $u = f(z)$, $du = f'(z)dz$. Observe that because f is injective and C is a simple closed curve, the curve $f(C)$ has no self-intersections and is also a simple closed curve. We get that:

$$N = \int_{f(C)} \frac{du}{u - w} \tag{1}$$

$$= \begin{cases} 0 \text{ if } u \notin \widetilde{D} \\ 1 \text{ if } u \in \widetilde{D} \end{cases} \tag{2}$$

We conclude that if z is enclosed by C, then $f(z)$ is enclosed by $f(C)$ and there are no other points $w \in D$ such that $f(z) = f(w)$. So $f : D \to \widetilde{D}$ is a bijection.

(b)

Because f is holomorphic and non-constant in a region Ω, we have by the open mapping theorem that f is open. Therefore $g = f^{-1}$ is continuous. To proof that g is holomorphic we try to calculate its derivative

$$\begin{aligned}\frac{g(w)-g(w_0)}{w-w_0} &= \frac{g(w)-g(w_0)}{f(g(w))-f(g(w_0))} \\ &= \frac{1}{\frac{f(g(w))-f(g(w_0))}{g(w)-g(w_0))}}\end{aligned}$$

Because g is continuous, we can calculate the limit for $w \to w_0$ as follows ($z = g(w), z_0 = g(w_0)$):

$$\begin{aligned}g'(w_0) &= \lim_{w\to w_0} \frac{1}{\frac{f(g(w))-f(g(w_0))}{g(w)-g(w_0)}} \\ &= \lim_{z\to z_0} \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}} \\ &= \frac{1}{f'(z_0)} \\ &= \frac{1}{f'(g(w_0))}\end{aligned}$$

Where we used that f is holomoprhic. This can only be done if the derivative of f vanishes nowhere on D. Suppose there is a $z_0 \in D$ such that $f'(z_z) = 0$, in this case the function $f(z) - f(z_0)$ has a zero with multiplicity at least 2 in z_0, but because of Equation 2 this is not possible. We conclude that g is holomorphic on $\widetilde{D}$.

Using the Cauchy integral formula we have that:

$$g(w) = \frac{1}{2\pi i}\int_{f(C)} \frac{g(\zeta)}{\zeta - w} d\zeta$$

Now substitute ζ by $f(z)$, $d\zeta = f'(z)dz$:

$$g(w) = \frac{1}{2\pi i}\int_{C} \frac{zf'(z)}{f(z) - w} dz \tag{3}$$

which is the desired result.