Take home 2

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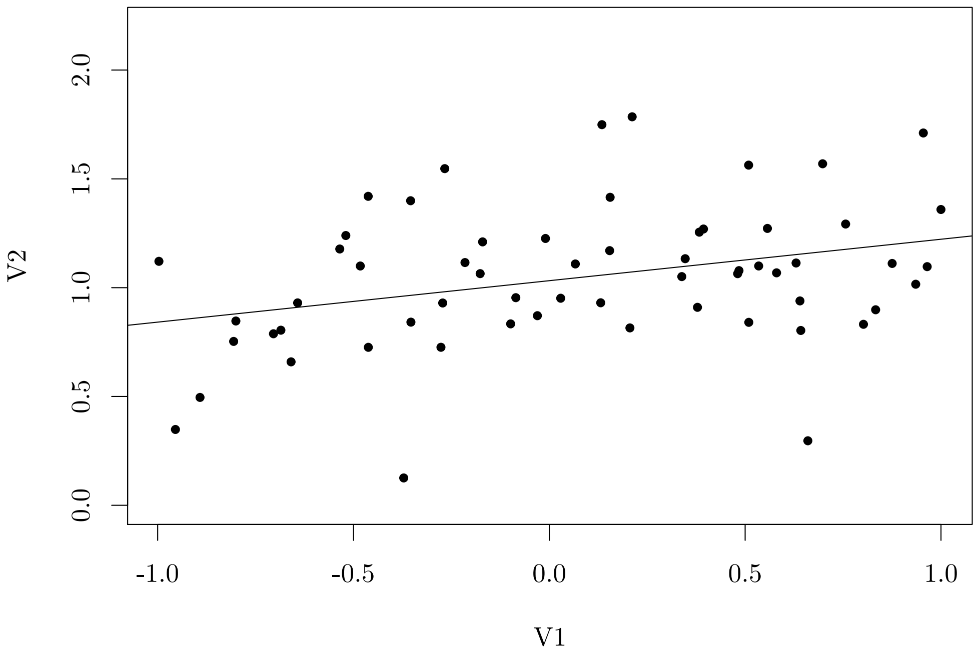
Question 1

(a)

The value for the intercept of the fit is 1.0321764 and for the rico of the fit is 0.1904323



Figuur 1: line fit throught the data in Ex1.txt

(b)

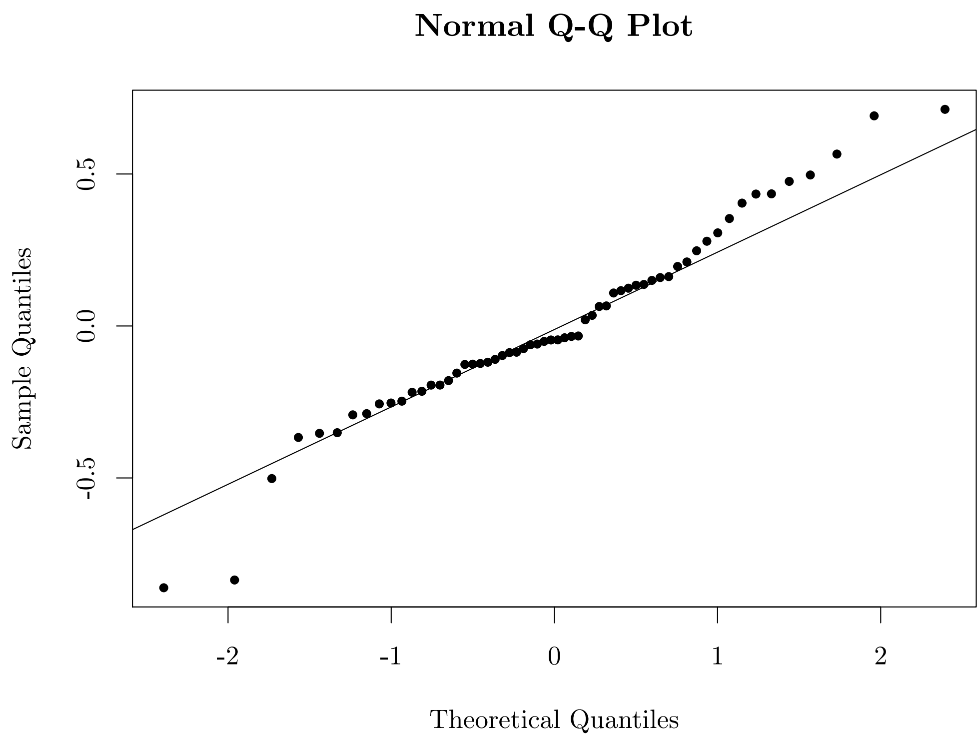
Using the result from section 7.4.1 in <REFERENCE cursus> we know that the random variable

$$T = \frac{\beta_1}{\sqrt{\frac{S^2}{\sum_{i=1}^{n}(x_i - \bar{x}^2)}}} \tag{1}$$

has a Student distribution with $n-2$ degrees of freedom. The test value is 2.603. Using a student-t distribution with $60-2=58$ degrees of freedom we find a p-value of 0.0117. At the confidence level $\alpha = 0.01$, the null hypothesis that $\beta_1 = 0$ holds. The 99% confidence region for the test value is $[-2.663, 2.663]$.

(c)

A q-q plot is a plot where the quantiles of the assumed distribution are plotted against the quantiles from the sample. So in a sample of n points where we label the observation $x_i, i \in \{1, 2, \ldots, n\}$ from lowest to highest, the ith point will be plotted at the $(Q(i/n), x_i)$. Here $Q(x)$ is the quantile function of the assumed distribution. This is a function such that $P(X < Q(p)) = p$. If the assumed distribution is a good model for the observed sample, the points will well fitted by a linear function.



Figuur 2: qq plot for the errors on the fit

(d)

We use the formula from section 7.5.4 in the book to find a confidence region for β. THis region is of the form:

$$\left\{ \beta \middle| \frac{(\hat{\beta}-\beta)^T (X^T X)(\hat{\beta}-\beta)}{pS^2} \leq F_{p,n-p}(1-\alpha) \right\} \tag{2}$$

Because the matrix $X^T X$ is positive definite, this is an ellips with center β and axes along the eigenvectors of $X^T X$. The upper bound for this expression is in this case: $F_{2,58(0.99)=4.9910}$ Filling in the values for $X, p = 2, n = 60$ and S^2 we get the following result (obtained using the cAS system sympy):

$$\{\beta | 5.2596\alpha^2 + 0.8222\alpha\beta - 0.4950\alpha + 1.6711\beta^2 - 0.6629\beta + 0.0711 \leq 4.9910\}$$

Question 2

(a)

The value for $\hat{\beta}_{OLS}$ obtained using ordinary least squares with the correlated errors is:

$$\hat{\beta}_{OLS} = (3.923161, 1.214355)^T \tag{3}$$

(b)

we know that the variance is the same for all ϵ_i and that they obey the recursive relation:

$$\epsilon_i = \rho\epsilon_{i-1} + \eta_i$$

with η_i standard normaly distributed. Using $Var(\epsilon_i) = Var(\epsilon_j)$ for all $i, j \in \{1, 2, \ldots 80\}$ we find:

$$\begin{aligned}
& Var(\epsilon_i) = \rho^2 Var(\epsilon_{i-1}) + Var(\eta_i) \\
\Leftrightarrow\ & Var(\epsilon_i) = \rho^2 Var(\epsilon_i) + 1 \\
\Leftrightarrow\ & Var(\epsilon_i) = \frac{1}{1-\rho^2} \\
& \qquad\quad\ = \frac{25}{9} \\
& \qquad\quad\ = 2.777\ldots
\end{aligned}$$

We conclude that the variance of ϵ is $\frac{25}{9}$.

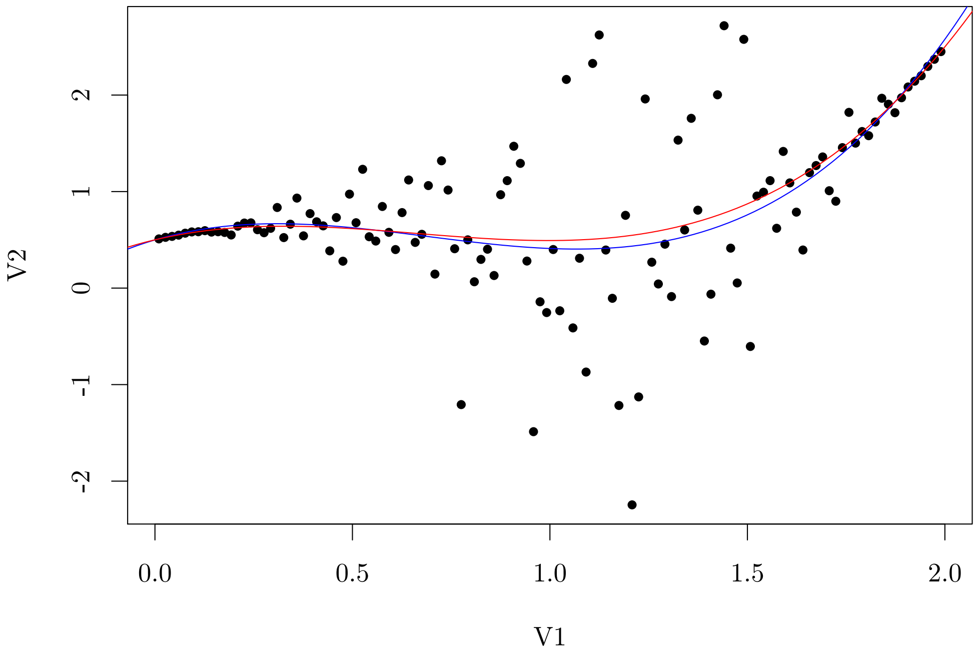
(c)

To transform the errors to a basis where they are no longer correlated we will use the variance-covariance matrix V of the errors. The off-diagonal elements, so the covariances, are (suppose $i < j$:

$$\begin{aligned}
Cov(\epsilon_i, \epsilon_j) &= E[(\epsilon_i - E(\epsilon_i))(\epsilon_j - E(epsilon_j))] \\
&= E[\epsilon_i\epsilon_j] \\
&= E[\epsilon_i(\rho\epsilon_{j-1}\eta_j)] \\
&\vdots \\
&= E[\epsilon_i(\rho^{j-i}\epsilon_i + \rho^{j-i-1}\eta_i + \rho^{j-i-2}\eta_{i-2} + \ldots + \eta_j)] \\
&= E[\epsilon_i\rho^{i-j}\epsilon_i] \\
&= \rho^{i-j}Var(\epsilon) \\
&= \rho^{i-j}\frac{25}{9}
\end{aligned}$$

So the matrix V has $\frac{25}{9}$ on the diagonal, $\frac{25}{9}\rho$ on the two sub-diagonals next to the diagonal, $\frac{25}{9}\rho^k$ on the kth sub-diagonal. the matrix V can be diagonalized with an orthogonal (or unitary) transformation U: $\Lambda = MVM^T$. When we use the definition of the variance-covariance matrix (where ϵ denotes the column vector with the errors) we find:

$$\begin{aligned}
& V = E((\epsilon - E(\epsilon))(\epsilon - E(\epsilon))^T) \\
\Leftrightarrow\ & \Lambda = UE(\epsilon\epsilon^T)U^T \\
\Leftrightarrow\ & \Lambda = E((U\epsilon)(U\epsilon)^T)
\end{aligned}$$



Figuur 3: fits of a cubic function using ordinary least square (blue) and weighted least squares (red)

Where we used that $E(\epsilon = \mathbf{0})$. The vector $U\epsilon$ has variance-covariance matrix Λ so we conclude that it is a vector of uncorrelated stochastic variables.

The value for $\hat{\beta}_{OLS}$ that we find using these uncorrelated errors is exactly the same as with the uncorrelated errors:

$$\hat{\beta}_{OLS} = (3.923161, 1.214355)^T$$

This is expected because we used a unitary transformation to get uncorrelated errors and by filling this in the expression for the Sum of squares of the remainders we find:

$$\begin{aligned} RSS(\hat{\beta}_{Uncorrelated}) &= (UY - UX\beta)^T(UY - UX\beta) \\ &= (Y - X\beta)^T U^T U (Y - X\beta) \\ &= (Y - X\beta)^T I (Y - X\beta) \\ &= RSS(\hat{\beta}_{Correlated}) \end{aligned}$$

which means the RSS for the correlated and the uncorrelated errors will have the same optimal $\hat{\beta}$.

Question 3

(a)

The values for the fitted coefficients with ordinary least squares are: $[0.4994, 1.1985, -2.4959, 1.2082]$. In figure 3 the fit is plotted along with the data.

(b)

The coeffitients fitted with the weighted least square algorithm are $[0.5003, 0.9669, -1.9655, 0.9910]$.

(c)

The coefficients that are calculated using the least squares method are closer to the real values then the ones calculated using the ordinary least squares. To really say something about the difference and how well they match the given real values of the function we need more information about the distribution of the parameters.

(d)

Using the result from class that the distribution is given by:

$$\hat{\beta}_{WLS} = (X^T W^{-1} X)^{-1} X^T W^{-1} Y \sim N_p(\beta, (X^T W^{-1} X)^{-1})$$

This result can be derived by noting that the vector Y has a multinormal distribution with mean $\bar{Y}$ and as variance-covariance matrix W. Therefore $\hat{\beta}_{WLS}$ is a linear combination of multinormal distributions and is itself again a multinormal distribution. <REF NAAR FORMULE?>.

Question 4

(a)

To generate a sample from a multi-normal distribution we work analogues to the derivation in section 6.2 in the course notes, but backwards. We are given the variance-covariance matrix Σ_X of a multivariate normal model. This matrix must be positive definite, so the square root of this matrix exists. Let A be a matrix such that $AA^T = \Sigma_X$. This matrix can be found by diagonalizing Σ_X: $\Sigma_X = U\Lambda U^T$ and than taking $A = U\sqrt{(\Lambda)}$. We can do this because all eigenvalues of Σ_X are positive. The result from section 6.2 now tells us that $X = AY + \mu$ where Y is a vector of standard normal distributions. This observation can be used to draw a sample from X by drawing n independent samples from a standard normal distribution, filling them in in a vector y and the calculating the sample from X as $x = Ay + \mu$. This was implemented in R

Acknowledgements

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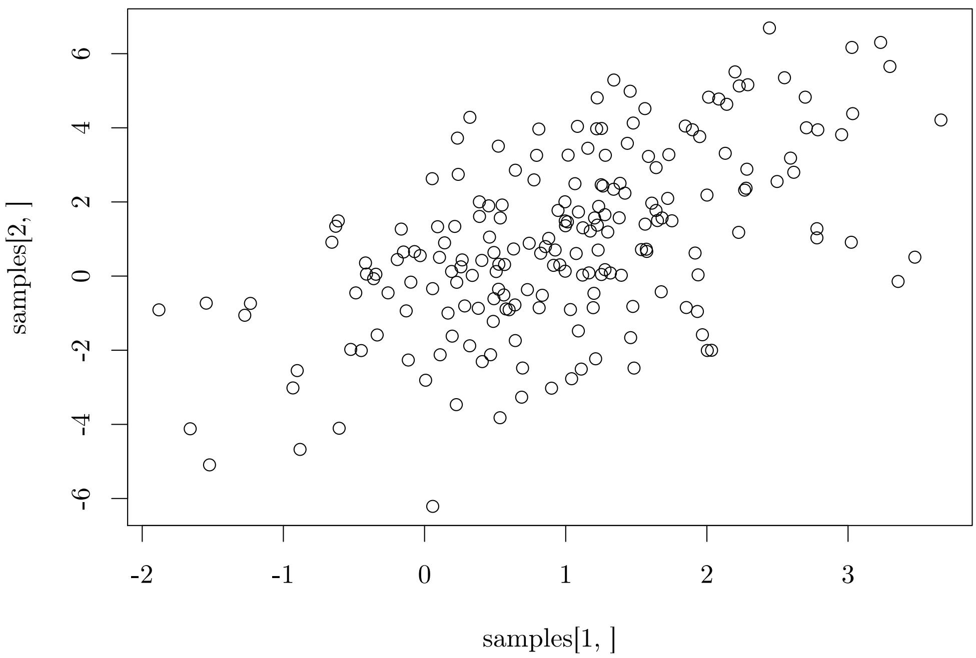

Figuur 4: Projection on the xy-plane of the sample from the multivariate normal distribution