Puzzles in Schubert Calculus

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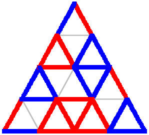


Outline

Grassmannians $\mathrm{Gr}(k, n)$

Puzzles for $\mathrm{Gr}(k, n)$

Flag varieties

Puzzles for Seperated Descents

Grassmannians

The Grassmannian

$$\mathrm{Gr}(k, n) = \{V \subseteq \mathbb{C}^n \mid \dim V = k\}.$$

Denote the *standard opposite flag*

$$\mathbb{C}^n = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{n-1} \supseteq F^n = 0,$$

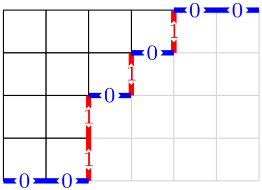
where $F^i = \langle \boldsymbol{e}_n, \ldots, \boldsymbol{e}_{n-i} \rangle$ for $i = 1, \ldots, n-1$.

For each $V \subseteq \mathbb{C}^n$ of dimension k, we have a decreasing flag

$$V = F^0 \cap V \supseteq \cdots \supseteq F^{n-2} \cap V \supseteq F^{n-1} \cap V \supseteq F^n \cap V = 0.$$

Assign the set of "jumping indices" $\lambda = (\lambda_1, \ldots, \lambda_n)$, where

$$\lambda_i = \begin{cases} 0, & \text{if } \dim(F^{i-1} \cap V) > \dim(F^i \cap V); \\ 1, & \text{if } \dim(F^{i-1} \cap V) = \dim(F^i \cap V). \end{cases}$$



$$\lambda = (0,0,1,1,0,1,0,1,0,0) \leftrightarrow (4,3,2,2).$$

$$\text{Let } P(n,k) = \left\{ \text{partitions in the } k \times (n-k) \text{ square} \right\}.$$

The *Schubert cell* for $\lambda \in P(n,k)$

$$\Sigma_\lambda^\circ = \left\{ V \in \mathrm{Gr}(k,n) \,\middle|\, \text{jumping indices of } V = \lambda \right\}.$$

$$\Sigma_\lambda = \text{closure of } \Sigma_\lambda^\circ, \qquad \sigma_\lambda = [\Sigma_\lambda^\circ] \in H^\bullet(\mathrm{Gr}(k,n)).$$

It is well known that

$$\mathrm{Gr}(k,n) = \bigcup_{\lambda \in P(n,k)} \Sigma_\lambda^\circ$$

and

$$H^\bullet(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda \in P(n,k)} \mathbb{Q} \cdot \sigma_\lambda.$$

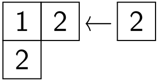
Littlewood–Richardson coefficients

Assume

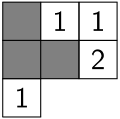
$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\mu \in \binom{[n]}{k}} c_{\lambda\mu}^{\nu} \cdot \sigma_\nu .$$

The coefficients $c_{\lambda\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ are known as *Littlewood–Richardson (LR) coefficients*.

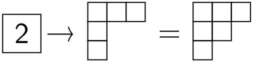
It also appears in the study of representation theory and symmetric functions. These coefficients admit a lot of combinatorial models like



Robinson–Schensted correspondence



jeu de taquin



crystal

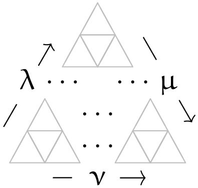
Schur operators

Puzzles for $H^\bullet(\mathrm{Gr}(k,n))$

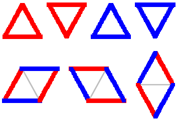
Let red $= 1$, blue $= 0$.

Theorem (Knutson–Tao–Woodward, 2003)

The number $c^{\nu}_{\lambda\mu}$ is the number of puzzles

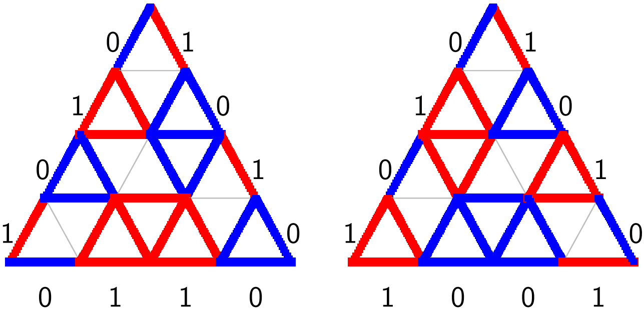


using



[1] A. Knutson, T. Tao, C. Woodward, The honeycomb model of $GL(n)$ tensor products II: Puzzles determine facets of the Littlewood–Richardson cone, J. Amer. Math. Soc. 17 (1) (2004) 19–48.

Examples



$$\sigma_{1010} \cdot \sigma_{1010} = \sigma_{0110} + \sigma_{1001}.$$

Puzzles for $K(\mathrm{Gr}(k,n))$

Let us denote

$$\mathcal{O}_\lambda = [\mathcal{O}_{\Sigma_\lambda}] = \text{structure sheaf for } \Sigma_\lambda.$$

$$\mathcal{I}_\lambda = [\mathcal{O}_{\Sigma_\lambda}(-\partial\Sigma_\lambda)] = \text{ideal sheaf for } \partial\Sigma_\lambda = \Sigma_\lambda \setminus \Sigma_\lambda^\circ.$$

It is known that they are dual basis under the Poincaré pairing.

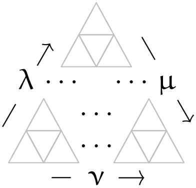
Similarly, we have

$$K(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q} \cdot \mathcal{O}_\lambda = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q} \cdot \mathcal{I}_\lambda.$$

We call the coefficients of their expansion the *structure constants*.

Theorem (Vakil, 2006)

The structure constant for $\mathcal{O}_\lambda$ is the number of puzzles



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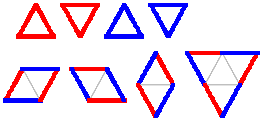
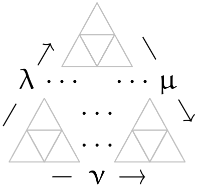


[1] R. Vakil, A geometric Littlewood–Richardson rule, Ann. of Math. 164 (2006), 371–421.

Theorem (Wheeler–Zinn-Justin, 2019)

The structure constant for $\mathcal{I}_\lambda$ is the number of puzzles



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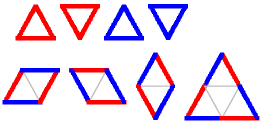
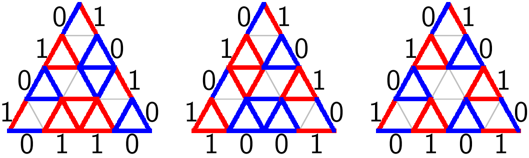
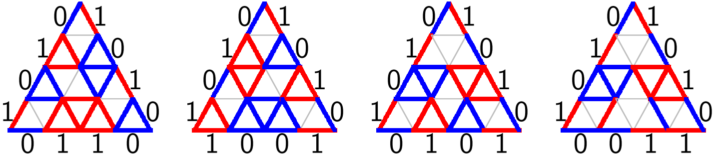


[1] M.Wheeler and P. Zinn-Justin, Littlewood–Richardson coefficients for Grothendieck polynomials from integrability, J. Reine Angew. Math. 757 (2019), 159-195.

Examples



$$\mathcal{O}_{1010} \cdot \mathcal{O}_{1010} = \mathcal{O}_{0110} + \mathcal{O}_{1001} + \mathcal{O}_{0101}$$



$$\mathcal{I}_{1010} \cdot \mathcal{I}_{1010} = \mathcal{I}_{0110} + \mathcal{I}_{1001} + \mathcal{I}_{0101} + \mathcal{I}_{0011}.$$

Puzzles for $H_T^\bullet(\mathrm{Gr}(k,n))$

Consider the *toric equivariant cohomology*, we have

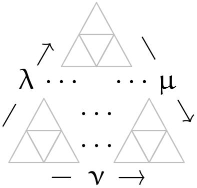
$$H_T^\bullet(\mathrm{Gr}(k,n))) = \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q}[t_1, \dots, t_n] \cdot \sigma_\lambda$$

Similarly, we have the *toric equivariant K-theory*

$$\begin{aligned} K_T(\mathrm{Gr}(k,n))) &= \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q}[\tau_1^{\pm 1}, \dots, \tau_n^{\pm 1}] \cdot \mathcal{O}_\lambda, \\ &= \bigoplus_{\lambda \in \binom{[n]}{k}} \mathbb{Q}[\tau_1^{\pm 1}, \dots, \tau_n^{\pm 1}] \cdot \mathcal{I}_\lambda. \end{aligned}$$

Theorem (Knutson–Tao, 2003)

The structure constant for $H_T^\bullet(\mathrm{Gr}(k,n))$ *can be computed by*



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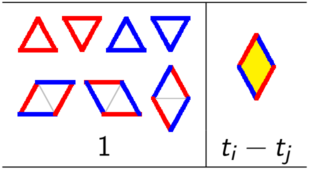
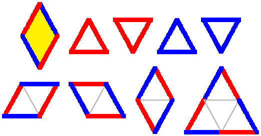


[1] A. Knutson and T. Tao, Puzzles and (equivariant) cohomology of Grassmannians, Duke Math. J. 119 (2003), 221-260.

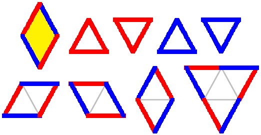
Puzzles for $K_T(\mathrm{Gr}(k,n))$

Theorem (Pechenik–Yong, Wheeler–Zinn-Justin)

The structure constant for $K_T(\mathrm{Gr}(k,n))$ can be computed by



for $\mathcal{O}_\lambda$

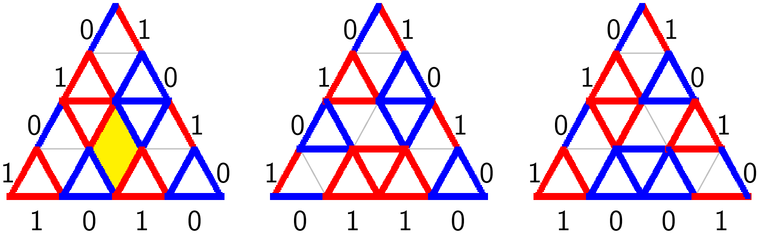


for $\mathcal{I}_\lambda$

[1] M.Wheeler and P. Zinn-Justin, Littlewood–Richardson coefficients for Grothendieck polynomials from integrability, J. Reine Angew. Math. 757 (2019), 159–195.

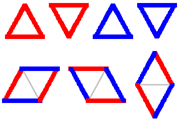
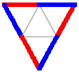
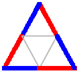
[2] O. Pechenik and A. Yong, Equivariant K-theory of Grassmannians II: The Knutson–Vakil conjecture, Compos. Math. 153 (2017), 667-–677.

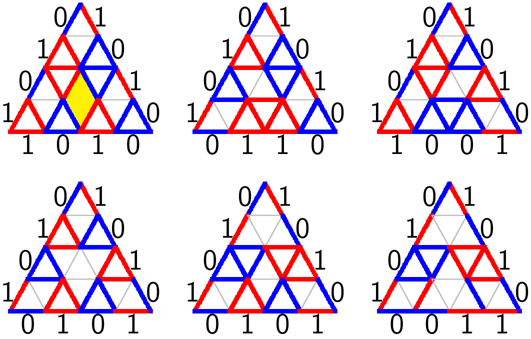
Examples



$$\mathcal{O}_{1010} \cdot \mathcal{O}_{1010} = (t_3 - t_2) \cdot \mathcal{O}_{1010} + \mathcal{O}_{0110} + \mathcal{O}_{1001}.$$

Summary

tiles	K-tiles	equivariant tiles
	 or 	



Flag varieties

Now we turn to *flag varieties*

$$\mathrm{Fl}(n) = \{0{=}V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n{=}\mathbb{C}^n\}.$$

For each flag $V_\bullet \in \mathrm{Fl}(n)$, we can similarly assign a permutation w such that

$$w(i) = j \iff \dim \frac{F^{i-1} \cap V_j + F^i}{F^{i-1} \cap V_{j-1} + F^i} = 1.$$

We can similarly define

$$\Sigma_w^\circ = \{V_\bullet \in \mathrm{Fl}(k,n) \,|\, \text{permutations of } V = w\}.$$

$$\Sigma_w = \text{closure of } \Sigma_w^\circ, \qquad \sigma_w = [\Sigma_w^\circ] \in H^\bullet(\mathrm{Fl}(n)).$$

Littlewood–Richardson coefficients

It is known that

$$H^\bullet(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q} \cdot \sigma_w \qquad \text{(as a vector space)}$$

The central problem in Schubert calculus is to compute the coefficients c_{uv}^w in the expression

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{uv}^w \cdot \sigma_w.$$

There is no general combinatorial model for c_{uv}^w up to now.

Schubert polynomials

Lascoux and Schutzenberger introduced the *Schubert polynomials*. For $w \in S_\infty$,

$$\mathfrak{S}_{n\cdots 21} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

$$\mathfrak{S}_{w(i,i+1)} = \frac{\mathfrak{S}_w - \mathfrak{S}_w|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}, \qquad w_i < w_{i+1}.$$

It turns out the structure constant can be computed by

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_\infty} c_{uv}^w \cdot \mathfrak{S}_w.$$

Thus we translate a geometric problem to an algebraic problem.

Examples

$$
\begin{tikzcd}[column sep=small]
 & \mathfrak{S}_{321} = x_1^2 x_2 \arrow[dl, dash] \arrow[dr, dash] & \\
x_1 x_2 = \frac{x_1^2 x_2 - x_2^2 x_1}{x_1 - x_2} = \mathfrak{S}_{231} \arrow[d, dash] \arrow[drr, dash] & & \mathfrak{S}_{312} = \frac{x_1^2 x_2 - x_1^2 x_3}{x_2 - x_3} = x_1^2 \arrow[d, dash] \arrow[dll, dash] \\
x_1 = \frac{x_1 x_2 - x_1 x_3}{x_2 - x_3} = \mathfrak{S}_{213} \arrow[dr, dash] & & \mathfrak{S}_{132} = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2 \arrow[dl, dash] \\
 & \mathfrak{S}_{123} = 1 &
\end{tikzcd}
$$

We have

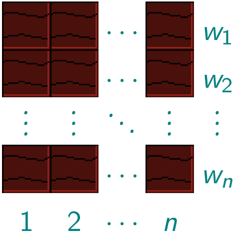
$$\mathfrak{S}_{213} \cdot \mathfrak{S}_{132} = \mathfrak{S}_{231} + \mathfrak{S}_{312}.$$

Bumpless pipe dream

There is an amazing combinatorial model for Schubert polynomials called *bumpless pipe dream*.

Theorem (Lam, Lee, and Shimozono)

Schubert polynomial $\mathfrak{S}_w$ is the weighted sum of



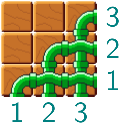
using

	
1	x_i

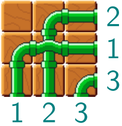
such that each pair of pipes crosses at most once

Examples

$\mathfrak{S}_{321} = x_1^2 x_2$

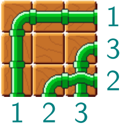


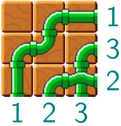
$\mathfrak{S}_{213} = x_1$



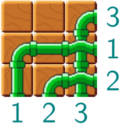
$\mathfrak{S}_{231} = x_1 x_2$

$\mathfrak{S}_{132} = x_1 + x_2$

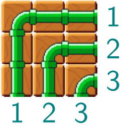




$\mathfrak{S}_{312} = x_1^2$



$\mathfrak{S}_{123} = 1$



Generalization A $\quad H^\bullet(\mathrm{Fl}(n)) \rightsquigarrow K(\mathrm{Fl}(n))$

Similarly,

$$K(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q} \cdot \mathcal{O}_w \qquad \text{(as a vector space).}$$

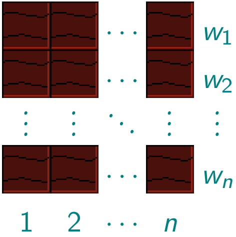
The structure constant of $\mathcal{O}_w$ is the same as the the structure constant of *Grothendieck polynomials*:

$$\mathfrak{G}_{n\cdots 21} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

$$\mathfrak{G}_{w(i,i+1)} = \frac{(1+\beta x_{i+1})\mathfrak{G}_w - (1+\beta x_i)\mathfrak{G}_w|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}}, \qquad w_i < w_{i+1}.$$

Theorem (Weigandt)

Grothendieck polynomial $\mathfrak{G}_w$ is the weighted sum of



using

			
1	β	$1+\beta x_i$	x_i

such that:

(i) two pipes cross at most once;

(ii) in each *, the labels*  $>$ .

Generalization B $\quad H^\bullet(\mathrm{Fl}(k,n)) \rightsquigarrow H_T^\bullet(\mathrm{Fl}(n))$

We have

$$H_T^\bullet(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q}[t_1, \ldots, t_n] \cdot \sigma_w$$

$$K_T(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} \mathbb{Q}[\tau_1^{\pm 1}, \ldots, t_n^{\pm 1}] \cdot \mathcal{O}_w$$

The corresponding polynomial is known as *double Schubert/Grothendieck polynomial*.

Theorem (Lam, Lee, and Shimozono)

Double Schubert polynomial $\mathfrak{S}_w$ is the weighted sum of bumpless pipe dreams but with double weight:

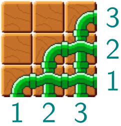
	
1	$x_i - t_j$

Theorem (Weigandt)

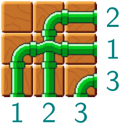
Double Grothendieck polynomial $\mathfrak{G}_w$ is the weighted sum of bumpless pipe dreams but with double weight:

			
1	β	$1 + \beta(x_i - t_j)$	$x_i - t_j$

Examples

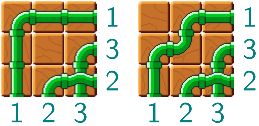


$(x_1 - t_1)(x_1 - t_2)(x_2 - t_1)$

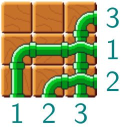


$(x_1 - t_1)$

$(x_1 - t_1)(x_2 - t_1)$



$(x_2 - t_2) + (x_1 - t_1)$



$(x_1 - t_1)(x_1 - t_2)$

1

Seperated descents

Assume $u, v \in S_n$ have seperated descents

$$\max(\mathrm{des}(u)) \leq k \leq \min(\mathrm{des}(v)).$$

There is a very recent combinatorial rule by Knutson and Zinn-Justin for the expansion of

$$\mathcal{O}_u \cdot \mathcal{O}_v = \sum_w c_{uv}^w(t) \cdot \mathcal{O}_w,$$

We generalize it to the *triple version*.

Our main result

single Schubert calculus	double Schubert calculus	triple Schubert calculus
non-equivariant $\mathfrak{S}_u(x)\mathfrak{S}_v(x)$	equivariant $\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,t)$	★ $\mathfrak{S}_u(x,t)\mathfrak{S}_v(x,y)$

We can view triple Schubert calculus as the *universal rule* for

$$\mathfrak{S}_u(x,t)\cdot\mathfrak{S}_v(x,wt)$$

which geometrically corresponds to the intersection of Schubert varieties of different transversality.

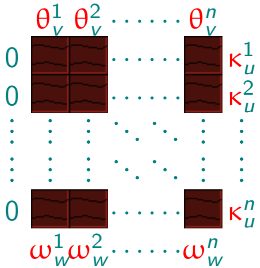
Theorem (FGX)

There is a combinatorial rule for $c_{uv}^w(y,t)$ in the expansion

$$\mathfrak{S}_u(x,y)\cdot\mathfrak{S}_v(x,t)=\sum_{w\in S_\infty} c_{uv}^w(y,t)\cdot\mathfrak{S}_w(x,t).$$

Pipe Puzzles

Let us first state the rule for cohomology, i.e. $\beta = 0$.



using

in each , the labels  $<$ .

For K-theory, it can be computed by using one more piece .

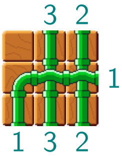
Example

Recall

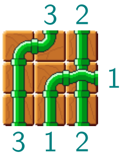
$$\mathfrak{S}_{213}(x, y) = x_1 - y_1 \qquad \mathfrak{S}_{132}(x, t) = x_1 + x_2 - t_1 - t_2$$

$$\mathfrak{S}_{231}(x, t) = (x_1 - t_1)(x_2 - t_1) \quad \mathfrak{S}_{312}(x, t) = (x_1 - t_1)(x_1 - t_2)$$

$$k = 1, \qquad u = 2 \mid 13, \qquad v = 1 \mid 32.$$



$t_1 - y_1$



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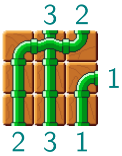

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$$\mathfrak{S}_{213}(x, y) \cdot \mathfrak{S}_{132}(x, t) = (t_1 - y_1)\mathfrak{S}_{132}(x, t) + \mathfrak{S}_{231}(x, t) + \mathfrak{S}_{312}(x, t).$$

On the proof

Our proof is based on the classical *6-vertex model*, and is significantly simple! What we need is to prove

I. induction on y II. induction on t III. initial cases.

Historically, people realized that equivariant cohomology is usually easier than usual cohomology.

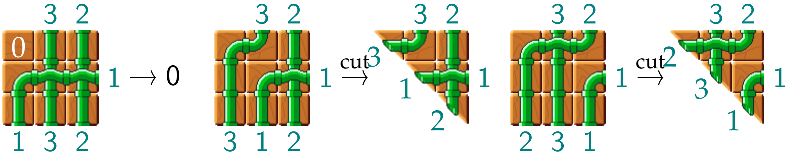
$$\boxed{\text{single}} \Longrightarrow \boxed{\text{double}}$$

It turns out the same happens for

$$\boxed{\text{double}} \Longrightarrow \boxed{\text{triple}}$$

Specialization A — seperated descents puzzles

If we set $y_i = t_i$, then on the diagonal  has weight 0. So it suffices to count those with  or  on the diagonal; so all pipes must go straight down under the diagonal. So we only need the upper triangle. This specializes to Knutson and Zinn-Justin's puzzle.



$$\mathfrak{S}_{213}(x,t)\cdot\mathfrak{S}_{132}(x,t) = \mathfrak{S}_{231}(x,t) + \mathfrak{S}_{312}(x,t).$$

Specialization B — bumpless pipe dream

If we set $k = n$, then $v = \mathrm{id}$. Taking $x = t$ on both sides of

$$\mathfrak{S}_u(x, y) \cdot \mathfrak{S}_v(x, t) = \sum_{w \in S_\infty} c_{uv}^w(y, t) \cdot \mathfrak{S}_w(x, t),$$

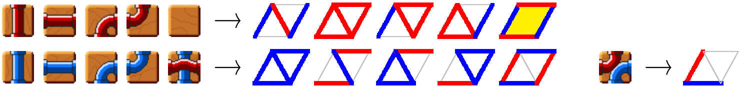
we will get

$$\mathfrak{S}_u(t, y) = c_{u\,\mathrm{id}}^{\mathrm{id}}(y, t).$$

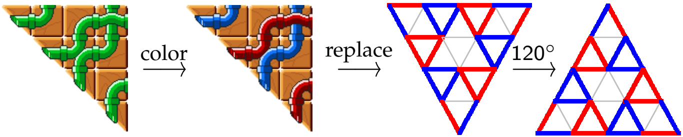
By reflecting against the diagonal and changing the labels, we recover the Weigandt's model of bumpless pipe dream.

Specialization C — classical puzzles

When u and v are both k-Grassmannian (i.e. at most one descent at k), we can recover the Grassmannian puzzles introduced in the first part. First, let us color pipes $\leq k$ by red and $\geq k$ by blue. Then we replace



Then rotate 120° anticlockwise.



Thanks