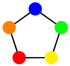
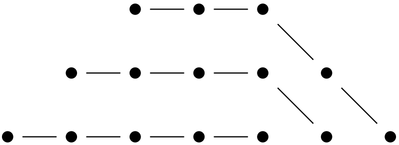
Motivic Lefschetz Theorem for twisted Milnor Hypersurfaces arXiv:2404.07314

with Kirill Zainoulline

Rui Xiong





Motivic Decomposition

The idea of the **category of Chow motives** is due to Grothendieck. By definition, it is the idempotent completion of the category of correspondence. Roughly speaking

$$\text{Chow motives} = \text{Linearization of varieties.}$$

As a result, a variety X, when putting into the category of motives, could be decomposed into a direct sum of smaller pieces (known as **motivic decomposition**). For example,

$$\mathcal{M}(\mathbb{P}^n) \cong \mathbb{Z} \oplus \mathbb{Z}(1) \oplus \cdots \oplus \mathbb{Z}(n) \quad \text{where} \quad \mathbb{Z} = \operatorname{Spec} k.$$

In general, a variety decomposes into direct sum $\mathbb{Z}(i)$ if it admitting an affine paving. However, it is very hard to decompose an arbitrary variety.

Homogeneous Varieties

Let us focus on **homogeneous varieties** G/P. Over algebraic closed fields, homogeneous varieties are classified by **Dynkin diagrams**. For example,

projective spaces

$$\circ - \bullet - \cdots - \bullet - \bullet$$

incidence varieties

$$\circ - \bullet - \cdots - \bullet - \circ$$

odd quadrics

$$\circ - \bullet - \cdots - \bullet \Rightarrow \bullet$$

even quadrics

$$\begin{array}{ccccccc} \circ - \bullet - \cdots - & \bullet & - \bullet \\ & | & \\ & \bullet & \end{array}$$

However, over non-algebraically closed fields, besides Dynkin diagrams, we need to classify the **twisted forms** of an algebraic group, e.g.

$$\begin{aligned} \text{twisted forms of } \mathrm{PGL}_n &\iff \text{central simple algebras} \\ &\qquad \text{(classified by Brauer groups)} \\ \text{twisted forms of } \mathrm{O}_n &\iff \text{quadratic forms} \\ &\qquad \text{(classified by Witt rings)} \end{aligned}$$

Milnor hypersurfaces

Let A be a **cyclic algebra** of degree n:

$$A = F\langle u, v\rangle / \langle u^n = a, v^n = b, uv = \zeta vu\rangle, \qquad \zeta = \sqrt[n]{1}, a, b \in F.$$

It is a **central simple algebra** of dimension n^2 over F. Let us denote the **Severi–Brauer varieties**

$$\begin{aligned} \mathrm{SB}(A) &= \{I \subset A\}, & I \lhd_{\mathrm{r}} A,\ \dim I = n, \\ \Rightarrow \mathrm{SB}(A^{op}) &= \{I \subset A\}, & I \lhd_{\mathrm{r}} A,\ \dim I = n(n-1). \end{aligned}$$

Note that Severi–Brauer variety is a homogeneous variety, a twisted form of $\mathbb{P}^{n-1}$.

Theorem (Karpenko, 1994)

When A is a central division algebra, $\mathcal{M}(\mathrm{SB}(A))$ is indecomposable.

Twisted Milnor hypersurfaces

Define the **twisted Milnor hypersurface** and a hypersurface of it

$$X = \{I_1 \subset I_{n-1} \subset A\} \subset \mathrm{SB}(A) \times \mathrm{SB}(A^{op}),$$
$$Y = \{(I_1 \subset I_{n-1}) \in X \colon uI_1 \subset I_{n-1}\} \subset X.$$

Note that X is a homogeneous variety, a twisted form of **incidence variety**. It is known that $\mathcal{M}(X) =$

Theorem (Calmès, Petrov, Semenov, Zainoulline, 2006)

$\mathcal{M}(\mathrm{SB}(A)) \oplus \mathcal{M}(\mathrm{SB}(A))(1) \oplus \cdots \oplus \mathcal{M}(\mathrm{SB}(A))(n-3) \oplus \mathcal{M}(\mathrm{SB}(A))(n-2).$

Our result is $\mathcal{M}(Y) =$ (assuming Rost nilpotence)

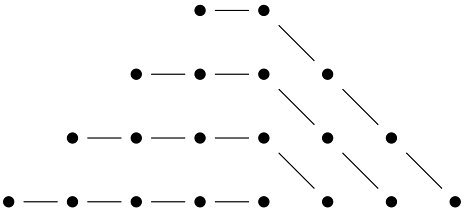
Theorem (Xiong, Zainoulline, 2024)

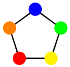
$\mathcal{M}(\mathrm{SB}(A)) \oplus \mathcal{M}(\mathrm{SB}(A))(1) \oplus \cdots \oplus \mathcal{M}(\mathrm{SB}(A))(n-3) \oplus \mathcal{M}(\mathrm{Spec}\, L)(n-2).$

Here $L = F[\sqrt[n]{a}]$ is a field extension of F of degree n.

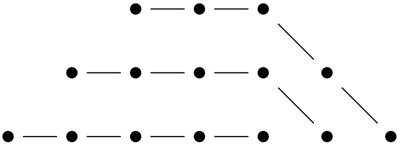
Example ($n = 5$)

$\mathcal{M}(X)$





$\mathcal{M}(Y)$



Hard Lefschetz

Recall the **Hard Lefschetz theorem** in complex algebraic geometry.

Theorem (Hard Lefschetz)

Let $\iota : Y \subset X$ be an ample smooth divisor. Then

the pullback $\iota^* : H^*(X) \longrightarrow H^*(Y)$ *is an isomorphism for* $* < \dim Y$;	*the pushforward* $\iota_* : H^*(Y) \longrightarrow H^{*+2}(X)$ *is an isomorphism for* $* > \dim Y$.

The Severi–Brauer part of the decompostion can be viewed as an analogy of this theorem:

$$\begin{tikzcd}
\mathcal{M}(SB(A)) \arrow[d] & \mathcal{M}(SB(A))(1) \arrow[d] & \cdots & \mathcal{M}(SB(A))(n-3) \arrow[d] & \mathcal{M}(SB(A))(n-2) \\
\mathcal{M}(SB(A)) \arrow[ur] & \mathcal{M}(SB(A))(1) \arrow[ur] & \cdots \arrow[ur] & \mathcal{M}(SB(A))(n-3) \arrow[ur] & {\color{blue}\mathcal{M}(\mathrm{Spec}\, L)(n-2)}
\end{tikzcd}$$

Monodromy actions

Hodge theory also gives us hint about how the middle dimension cohomology supposed to be.

Theorem (Deligne invariant cycle theorem)

$\mathrm{im}\left[H^*(X) \longrightarrow H^*(Y)\right] \;=\; \textit{monodromy invariant component}.$

In particular, the **primitive part** must have a non-trivial monodromy action. We can consider the **universal family of hyperplane sections**

$$\mathcal{Y} = \left\{(I_1 \subset I_{n-1}, x) \in X \times A : xI_1 \subset I_{n-1}\right\} \xrightarrow{\mathrm{pr}_2} A.$$

Thus Y is the fiber at $u \in A$. If all these are defined over $\mathbb{C}$, then $A \simeq M_n(\mathbb{C})$. Moreover, the monodromy comes from the fundamental group of the locus $M_n(\mathbb{C})^{\mathrm{rs}}$, the **Artin braid group**.

Galois group actions

In our case, the **Galois group** Γ_L plays the role of monodromy. Note that we have an isomorphism

$$\rho : A_L \simeq M_n(L) \qquad \text{(as } L\text{-algebras)}.$$

But this does not commute with the obvious Galois group actions. The obstruction is recorded in a 1-cocycle (thus representing a class in the Galois cohomology)

$$\begin{aligned} \mathfrak{a}\colon \Gamma_L &\longrightarrow \mathrm{Aut}_L(M_n(L)) \simeq PGL_n(L), \\ \sigma &\longmapsto \mathfrak{a}_\sigma = \rho \circ \sigma \circ \rho^{-1} \circ \sigma^{-1}. \end{aligned}$$

Now we have the following commutative diagram

$$\begin{tikzcd} Y_L \arrow[r, "\subset"] \arrow[d, "\sigma"'] & X_L \arrow[r, "\simeq", "\rho"'] \arrow[d, "\sigma"'] & X_{0L} \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] \\ Y_L \arrow[r, "\subset"] & X_L \arrow[r, "\simeq", "\rho"'] & X_{0L} \end{tikzcd}$$

$$X_0 = \{V_1 \subset V_{n-1} \subset F^n\}, \quad \dim V_i = i.$$

Torus actions

Let $T \subset PGL_n$ be the standard maximal torus. Since L contains all eigenvalues of $u \in A$, we can assume the image of $\rho(u) \in M_n(L)$ is diagonal. This implies T_L acts on Y_L and X_L such that the **torus fixed points**

$$Y_L^{T_L} = X_L^{T_L} \simeq X_{0L}^{T_L} = \{[ij] : 1 \leq i \neq j \leq n\}$$

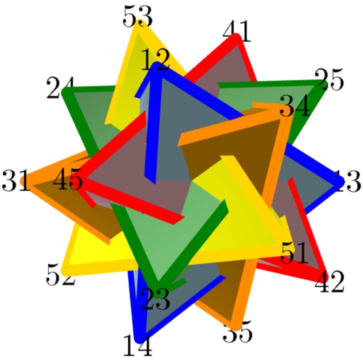
where $[ij] = \left(\mathrm{span}(e_i) \subseteq \mathrm{span}(e_1, \ldots, \cancel{e_j}, \ldots, e_n)\right)$. Moreover, the **$T_L$-stable curves** can be classified

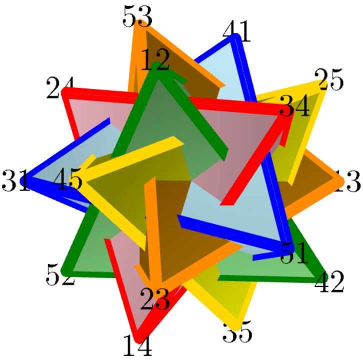
$$[ij] \xrightarrow{\alpha_{jk}} [ik] \quad \text{and} \quad [ij] \xrightarrow{\alpha_{ik}} [kj], \quad \text{where all } i, j, k \text{ are distinct,}$$

and $\alpha_{ij} = t_i - t_j \in T^*$. Using **localization theorems** of equivariant Chow ring, we are able to compute the (monodromy) Γ_L-actions and constructing algebraic cycles.

Example ($n = 5$)

The edges form two families of **compounds of five tetrahedra**





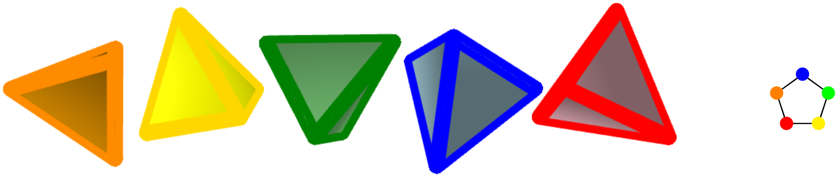
The Galois action corresponds to the rotations.

Artin motive part

We then constructed cycles $\{\gamma_\ell \in \mathrm{CH}^{n-2}_T(Y_L) : \ell \in \Gamma_L\}$ such that

$$\forall \sigma \in \Gamma_L,\ \sigma\gamma_\ell = \gamma_{\sigma(\ell)}, \qquad \langle \gamma_k, \gamma_\ell \rangle_{Y_L} = (-1)^{n-2}\delta_{k,\ell}.$$

The sign $(-1)^{n-2}$ reflects the **Hodge–Riemann relation** in Hodge theory. When $n = 5$, each class is supported over each tetrahedron



One can show further that they are orthogonal to the Severi–Brauer parts, and give the last motive $\mathcal{M}(\mathrm{Spec}\, L)(n-2)$. Q.E.D.

Thanks