From “Two Points Determine a Line” To “Higher Hook Formulas”

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(Joint work with Neil J.Y. Fan and Peter L.Guo)

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Abstract

We know from the classical Euclidean geometry that two points determine a line. A less well-known result due to Schubert is that there are 2 lines passing through 4 general lines in the 3D space. In this talk, we will answer this question for higher dimensions. Surprisingly, Grassmannian plays an important role in this question, and lead us to the famous Hook formulas. Later, we will discuss how we can generalize the geometric method to higher degrees. This a joint work with Neil J.Y. Fan and Peter L.Guo.

An old problem

Question

On the plane ($\dim = 2$)*, how many lines intersect* 2 *given points?*

The answer to this question is

$$1,$$

since there is a unique line passing through two given points.

An old problem

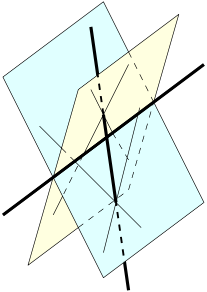
Question

In the space ($\dim = 3$)*, how many lines intersect* 4 *given lines?*

The answer to this question is

$$2,$$

as presented on the right.



Generalization

To ask a proper question for higher dimension, we should ...

plane	space	$\cdots$	$\Rightarrow$	vector space
point	line	$\cdots$	$\Rightarrow$	**codim2 plane**
codim2 plane = translation of subspace of dimension two less than the underlying vector space				
lines	lines	$\cdots$	$\Rightarrow$	lines
line = translation of subspace of dimension one				

We should ask

Question (To be filled)

In a vector space V, how many lines intersect __ given codim2 planes?

How to ask a correct question

We hope to fill the blank such that the answer is not 0 or ∞.

For example,

On the plane, how many lines intersect __ given points?						
__	0	1	**2**	3	4	$\cdots$
answer	∞	∞	**1**	0	0	$\cdots$

In fact, the correct number should be put in __ is the first number such that the answer is **NOT** infinity.

Grassmannians

To study how many lines, we need a space of lines, i.e. a space whose points are "lines".

Let V be a n-dimensional (projective) space. Introduce

$$\mathbb{G}\mathrm{r}(1, n) = \{\text{lines in the } V\}$$

the **Grassmanian manifold**.

Let F be a codim2 plane of V. Denote

$$D_F = \{\text{line } \ell : \ell \cap F \neq \varnothing\}$$

the divisor of F.

Our problem is to ask the number

$$\#\left(D_{F_1} \cap D_{F_2} \cap \cdots \cap D_{F_r}\right)$$

for $F_1, \ldots, F_r$. We want (1) the smallest number r such that this is finite; (2) the finite number.

A general principle

The standard way to compute the intersection is to use **cohomology**.

When $F_1, \ldots, F_r$ are at the general position, we have

$$\text{(counting)} \qquad \#\big(D_{F_1} \cap \cdots \cap D_{F_r}\big) = d \iff$$
$$\iff [D_{F_1}] \smile \cdots \smile [D_{F_r}] = d \cdot [\mathrm{pt}] \qquad \text{(cohomology)}$$

where $[\cdots]$ is the **fundamental class**.

Cohomology of $\mathbb{G}\mathrm{r}(1, n)$

The cohomology of $\mathbb{G}\mathrm{r}(1, n)$ is very funny. It admits a basis

$$H^\bullet(\mathbb{G}\mathrm{r}(1, n)) = \mathbb{Z}\varnothing \oplus \mathbb{Z}\ydiagram{1} \oplus \mathbb{Z}\ydiagram{1,1} \oplus \mathbb{Z}\ydiagram{2} \oplus \mathbb{Z}\ydiagram{2,1} \oplus \mathbb{Z}\ydiagram{2,2} \oplus \cdots .$$

It is a little bit technical to describe all of them. But some special cases are easier,

$$[X] = 1 = \varnothing$$

$$[D_F] = \square \qquad \text{(a single box)}$$

$$[\mathrm{pt}] = \begin{ytableau} \ & \cdots & \ \\ \ & \cdots & \ \end{ytableau} \qquad (n-1 \text{ many boxed in each row})$$

For example,

$$\text{when } n = 3 \qquad \ydiagram{2,2} = [\mathrm{pt}].$$

Chevalley Formula

The multiplication structure can be partially described by a famous **Chevalley formula** that

$$\lambda \cdot \square = \sum_{\mu=\lambda+\square} \mu \qquad \text{(obtained by adding a box).}$$

For example, when $n = 3$,

λ	$1 = \varnothing$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{2}$	$\ydiagram{2,1}$	$\ydiagram{2,2}$
$\lambda \cdot \begin{ytableau} *(yellow) \end{ytableau}$	$\begin{ytableau} *(yellow) \end{ytableau}$	$\begin{ytableau} \, & *(yellow) \end{ytableau} + \begin{ytableau} \, \\ *(yellow) \end{ytableau}$	$\begin{ytableau} \, & *(yellow) \\ \, \end{ytableau}$	$\begin{ytableau} \, & \, \\ *(yellow) \end{ytableau}$	$\begin{ytableau} \, & \, \\ \, & *(yellow) \end{ytableau}$	0

Computation

Now we are at the position to compute

$$[D_{F_1}] \smile \cdots \smile [D_{F_r}] = ??[\mathrm{pt}].$$

For example, when $n = 3$,

$$\begin{aligned}
[D_{F_1}] \smile \cdots \smile [D_{F_4}] &= \boxed{1} \smile \boxed{2} \smile \boxed{3} \smile \boxed{4} \\
&= \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right) \smile \boxed{3} \smile \boxed{4} \\
&= \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) \smile \boxed{4} \\
&= \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right) = 2 \cdot [\mathrm{pt}]
\end{aligned}$$

As a result,

$$\text{the answer } 2 = \# \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\}.$$

The answer to this question

In general, for dim $= n$, we should ask

Question

In a vector space V, how many lines intersect $2(n-1)$ given codim2 planes?

The answer is the number of **standard Young tableaux**

> fulfilling $2 \times (n-1)$ table by $\{1, 2, 3, \ldots\}$ such that the numbers increase in row and column respectively.

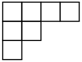
For example, when $n = 4$,

$$\#\left\{ \begin{array}{c} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}, \\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \end{array} \right\} = 5.$$

These numbers are actually **Catalan numbers**.

General question

We want to ask a general question, for a partition, for example


.

what is the number of **standard Young tableaux**?

Actually, this question is not only asked for fun, since

- **A**lgebraically, it corresponds to the dimension of some representation of symmetric group;
- **C**ombinatorially, it corresponds to the number of reduced word for some shuffle permutation.
- **G**eometrically, it corresponds to the degree of some Schubert variety under the Plücker embedding;

Classic Hook Formulas

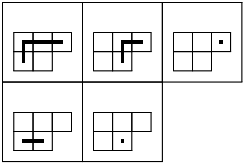
The number of **standard Young tableaux** is given by hook length formula

$$\#\left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\;\; \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\;\; \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\;\; \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&5\\\cline{1-2}\end{array}\;\; \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} \right\} = \frac{5!}{4\times 3\times 1\times 2\times 1},$$

where the denominator is the product of all **hook length**.

For example, we label the hook length at each box of the partition

$$\begin{array}{|c|c|c|}\hline 4&3&1\\\hline 2&1\\\cline{1-2}\end{array} =$$



Review of the proof

We know standard Young tableaux is nothing but a record of adding box each time

$$\varnothing \xrightarrow{\text{add } \square} \boxed{1} \xrightarrow{\text{add } \square} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{\text{add } \square} \cdots \xrightarrow{\text{add } \square} \begin{array}{|c|c|c|c|}\hline 1&3&4&7\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array}.$$

We also know Chevalley formula is

$$\text{product with } \square = \text{add box}$$

So the number we want is the coefficients of

$$\square^k = \sum_{|\lambda|=k} \#\,\mathrm{SYT}(\lambda) \cdot \lambda.$$

But in usual cohomology, it is hard to **extract** the coefficients.

Naruse's trick

The idea is to lift cohomology to **equivariant cohomology**

I guess we do not have enough time for discussing equivariant cohomology. But let me mention, this trick is very successful. The philosophy is

a (generalized) equivariant Chevalley formula	$\approx$	a (generalized) hook formula

It applies to cohomology, K-theory, CSM classes, motive Chern classes, etc (whatever it means).

Our work

Our work is to generalize this philosophy to **higher degrees**, by viewing Murnaghan–Nakayama rule as the higher degree analogue. We will take $\deg = 2$ as an example.

Let $p_2 = \begin{array}{|c|c|}\hline & \\ \hline \end{array} - \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$. We have

$$\lambda \cdot p_2 = \sum_{\mu = \lambda + \begin{array}{|c|c|}\hline & \\ \hline \end{array}} \mu - \sum_{\nu = \lambda + \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}} \nu.$$

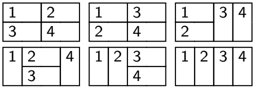
Briefly,

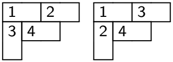
$$\text{product with } p_2 = \pm \text{add donimo}.$$

We should get a hook formula for **Donimo tableaux**.

Domino Tableaux

The number of **domino tableaux** is given by

$$\# \Bigg\{$$

$$\Bigg\} = \frac{8!!}{4 \times 2 \times 4 \times 2}.$$

$$\# \Bigg\{$$

$$\Bigg\} = \frac{8!!}{6 \times 4 \times 4 \times 2}.$$

where the denominator is the product of all **even** hook lengths.

$$\begin{array}{|c|c|c|c|}\hline 5 & \boxed{4} & 3 & \boxed{2} \\ \hline \boxed{4} & 3 & \boxed{2} & 1 \\ \hline \end{array}, \qquad \begin{array}{|c|c|c|c|}\hline \boxed{6} & \boxed{4} & 3 & 1 \\ \hline \boxed{4} & \boxed{2} & 1 & \\ \hline 1 & & & \\ \hline \end{array}.$$

Our formulation

Actually, we deal with the general cases of **r-rim hook tableaux** for skew shapes (donimo case is when $r = 2$).

Theorem (Fan, Guo and Xiong)

For a skew shape Λ/λ of size dr, we have the following Laurant expansion

$$\left.\frac{[Y(\lambda)]_T|_\Lambda}{[Y(\Lambda)]_T|_\Lambda}\right|_{t_i=z^i} = \frac{1}{(z^r-1)^d}\left(\pm\frac{\#\mathrm{RHT}^r(\Lambda/\lambda)}{r^d d!}+o(1)\right),$$

near a primitive r-th root of unity.

We use our equivariant MN rule to show this identity.

Thanks