Motivic Lefschetz Theorem for twisted Milnor Hypersurfaces arXiv:2404.07314

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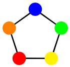
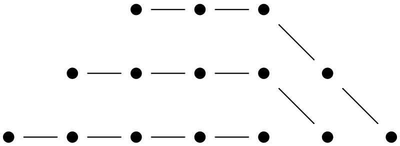




Simple algebras

Let F be a field. The study of simple algebras can be traced as follows.

- A simple F-algebra A, by Artin–Wedderburn theorem, is a matrix algebra over a *division algebra*:

$$A = M_n(D).$$

- A division algebra D is a central division algebra over its center:

$$Z(D) = E \supset F.$$

- central division algebras are classified by the Brauer group:

$$[D] \in \mathrm{Br}(E) = H^2(E, \mathbb{G}_m).$$

Brauer Groups

The classical definition of Brauer group is

$$\mathrm{Br}(F) = \frac{\{\text{central simple algebras over } F\}}{A \sim B \iff M_m(A) \cong M_n(B)}.$$

Let A be a central simple algebra of degree n over F. Then $\mathrm{Aut}(A)$ is a twisted form of PGL_n, thus defines a class in

$$[A] \in H^1(F, PGL_n) \subset H^2(F, \mathbb{G}_m).$$

Here the inclusion is induced by the long exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1.$$

Severi–Brauer variety

Let us denote the *Severi–Brauer* variety

$$\mathrm{SB}(A) = \{I \subset A\}, \qquad I \lhd_r A, \ \dim I = n.$$

This is a twisted form of the projective space $\mathbb{P}^{n-1}$. But the geometry is quit different from $\mathbb{P}^{n-1}$. For example, in general,

- There is no rational points. Actually,

$$\mathrm{SB}(A)(E) \neq \varnothing \iff A_E \simeq M_n(E) \iff \mathrm{SB}(A)_E \cong \mathbb{P}^{n-1}_E.$$

- There is no bundle behavior like $\mathcal{O}(1)$ over $\mathrm{SB}(A)$ in general. Otherwise, the intersection of hyperplane sections will produce a rational point.

Twisted Milnor Hypersurfaces

We can identify

$$\mathrm{SB}(A^{op}) = \{I \subset A\}, \qquad I \lhd_r A, \ \dim I = n(n-1)$$

which is also a twisted form of $\mathbb{P}^{n-1}$. Let us define the *twisted Milnor hypersurface*

$$X = \{I_1 \subset I_{n-1} \subset A\} \subset \mathrm{SB}(A) \times \mathrm{SB}(A^{op})$$

cut by the section of the line bundle

$$[\mathcal{I}_1 \subset A \to A/\mathcal{I}_{n-1}] \in \mathcal{H}om_A(\mathcal{I}_1, A/\mathcal{I}_{n-1})$$

Note that this is a twisted form of *incidence variety*

$$X_0 := Fl(1, n-1; n) = \{V_1 \subset V_{n-1} \subset F^n\}, \qquad \dim V_i = i.$$

Cyclic algebras

There is a huge source of central simple algebras known as *cyclic algebras*.

Assume F contains a primitive roots of unity ζ. We pick $a, b \in F$. Let A be a *cyclic algebra* of degree n

$$A = F\langle u, v\rangle / \langle u^n = a, v^n = b, uv = \zeta vu\rangle.$$

This algebra is known to be a central simple algebra.

Example

For $F = \mathbb{R}$, the quaternion

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

is a cyclic algebra of degree 2.

The Hyperplane section

We define a hyperplane section of the twisted Milnor hypersurfaces X to be

$$Y = \{(I_1 \subset I_{n-1}) \in X \colon uI_1 \subset I_{n-1}\} \subset X$$

cut by the section of the line bundle

$$[\mathcal{I}_1 \subset A \xrightarrow{u} A \to A/\mathcal{I}_{n-1}] \in \mathcal{H}om_A(\mathcal{I}_1, A/\mathcal{I}_{n-1}).$$

In other word, Y is a complete intersection of two sections from the same line bundle over $\mathrm{SB}(A) \times \mathrm{SB}(A^{op})$.

We will study the *motivic decomposition* of Y.

Motives

A *Chow motive* is a pair

$$(X, p) : \begin{array}{l} X \text{ is a smooth complete variety over } F, \\ p \in \mathrm{CH}(X \times X) \text{ is an idempotent.} \end{array}$$

A morphism $(X, p) \to (Y, q)$ is

$$q \circ \mathrm{CH}^*(X \times Y) \circ p.$$

The Chow motives form an additive category, so we want to study how to decompose

$$\mathcal{M}(X) = (X, \Delta_X) = (X, \mathrm{id}_X)$$

into smaller direct summands.

Motivic decomposition

It is known that $\mathcal{M}(X) =$

Theorem (Calmès, Petrov, Semenov, Zainoulline, 2006)

$\mathcal{M}(\mathrm{SB}(A)) \oplus \mathcal{M}(\mathrm{SB}(A))(1) \oplus \cdots \oplus \mathcal{M}(\mathrm{SB}(A))(n-3) \oplus \mathcal{M}(\mathrm{SB}(A))(n-2).$

Our result is $\mathcal{M}(Y) =$

Theorem (Xiong, Zainoulline, 2024)

$\mathcal{M}(\mathrm{SB}(A)) \oplus \mathcal{M}(\mathrm{SB}(A))(1) \oplus \cdots \oplus \mathcal{M}(\mathrm{SB}(A))(n-3) \oplus \mathcal{M}(\operatorname{Spec} L)(n-2).$

Here $L = F[\sqrt[n]{a}]$ is a field extension of F of degree n.

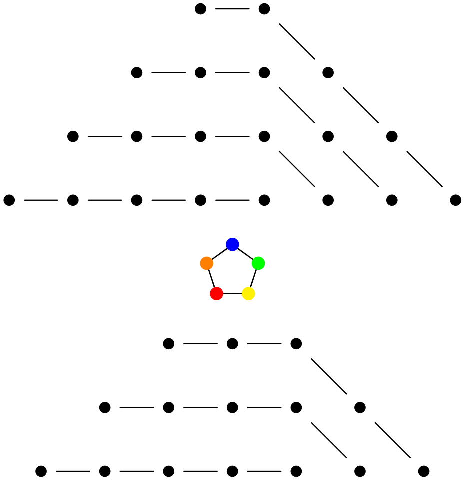
Since when A is a division algebra, $\mathcal{M}(\mathrm{SB}(A))$ is indecomposable, this is the best we can prove for general A.

Calmès, B.; Petrov, V.; Semenov, N.; Zainoulline, K. Chow motives of twisted flag varieties. *Compos. Math.* **142** (2006), no. 4, 1063–1080.

Example ($n = 5$)

$\mathcal{M}(X)$

$\mathcal{M}(Y)$



Hard Lefschetz

Recall the Hard Lefschetz theorem in complex algebraic geometry.

Theorem (Hard Lefschetz)

Let $\iota : Y \subset X$ be an ample smooth divisor. Then

the pullback

$$\iota^* : H^*(X) \longrightarrow H^*(Y)$$

is an isomorphism for $* < \dim Y$

the pushforward

$$\iota_* : H^*(Y) \longrightarrow H^{*+1}(X)$$

is an isomorphism for $* > \dim Y$.

The diagram is like this

$$\begin{tikzcd}
H^0(X) \arrow[dr, equal] & & H^2(X) \arrow[dr, equal] & & H^4(X) \arrow[dr, equal] & & H^6(X) \arrow[dr, dotted] & & H^8(X) \arrow[dr, dotted] & & H^{10}(X) \arrow[dr, dotted] & & H^{12}(X) \arrow[dr, dotted] & & H^{14}(X) \\
& H^0(X) \arrow[ur, dotted] & & H^2(X) \arrow[ur, dotted] & & H^4(X) \arrow[ur, dotted] & & H^6(X) \arrow[ur, dotted] & & H^8(X) \arrow[ur, equal] & & H^{10}(X) \arrow[ur, equal] & & H^{12}(X) \arrow[ur, equal] &
\end{tikzcd}$$

Severi–Brauer part

The Severi–Brauer part of the decompostion can be viewed as an analogy of this theorem.

- In the motivic decomposition of $\mathcal{M}(X)$, the idempotents is given by

$$g_i \circ f_i, \qquad 0 \leq i \leq n-2.$$

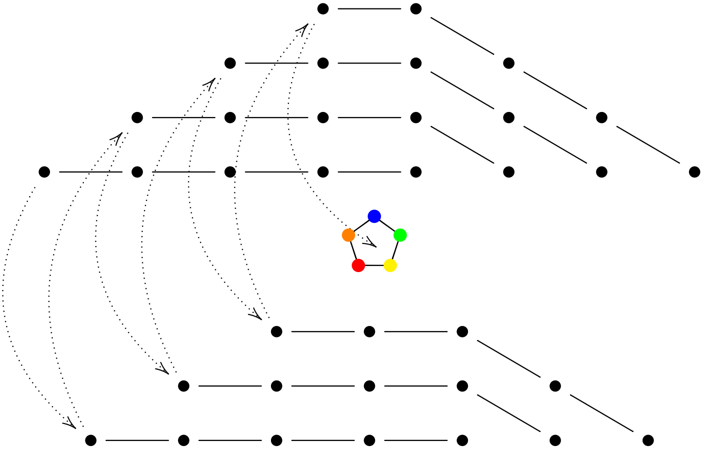
- In the motivic decomposition of $\mathcal{M}(Y)$, the idempotents is given by

$$\overline{g}_i \circ \overline{f}_{i+1}, \qquad 0 \leq i \leq n-3$$

  where $\overline{*}$ is the restriction from X to Y.

Note that the shift of index is a feature of Lefschetz type theorem.

Example ($n = 5$)



Monodromy Actions (I)

Hodge theory also gives us hint about how the middle dimension cohomology supposed to be.

Theorem (Deligne invariant cycle theorem)

$\operatorname{im}\left[H^*(X) \longrightarrow H^*(Y)\right] \; = \;$ *monodromy invariant component.*

In our case, the Galois group $\Gamma_L = \mathrm{Gal}(L/F)$ will be a part of monodromy.

$$\begin{tikzcd} Y_L \arrow[loop above, "\sigma"] \arrow[r] \arrow[d] & \operatorname{Spec} L \arrow[loop above, "\sigma"] \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} F \end{tikzcd}$$

Universal family

We can consider the *universal family of hyperplane sections*

$$\mathcal{Y} = \left\{(l_1 \subset l_{n-1}, a) \in X \times A : al_1 \subset l_{n-1}\right\} \xrightarrow{\mathrm{pr}_2} A.$$

Thus Y is the fibre at $u \in A$.

Let us assume at this moment $F = \mathbb{C}$.

Theorem (Ehresmann's fibration theorem)

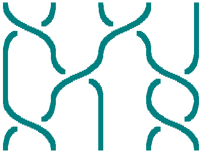
The space $\mathcal{Y}$ is a topological fibre bundle over the smooth locus A^{sm}.

In particular , the fundamental group $\pi_1(A^{sm})$ acts on the cohomlogy of the any smooth fibre (called the *monodromy*).

It is not hard to see, $a \in A^{sm}$ if and only if the reduced characteristic polynomial of u has no multiple roots (i.e. a is a regular semisimple element).

Over $\mathbb{C}$, it is well-known that

$$\pi_1(A^{sm}) = \mathfrak{B}_n = \left\{ \quad , \cdots \right\}$$



The monodromy action factors through the symmetric group S_n.

For general F, philosophically speaking, Galois group contributes plays the rôle of (at least, a part of) monodromy.

1-cocycle

Recall $L = F[\sqrt[n]{a}]$. We know there is an L-algebra isomorphism

$$\rho : A_L \simeq M_n(L).$$

But this does not commute with the obvious Galois group action. The obstruction is recorded in a 1-cocycle

$$\begin{aligned} \mathfrak{a} \colon \Gamma_L &\longrightarrow \mathrm{Aut}_L(M_n(L)) \simeq PGL_n(L), \\ \sigma &\longmapsto \mathfrak{a}_\sigma = \rho \circ \sigma \circ \rho^{-1} \circ \sigma^{-1}. \end{aligned}$$

In our case, we can compute this 1-cocycle explicitly:

- $\mathfrak{a}_\sigma \in N_T(PGL_n)(F) \subset PGL_n(F) \subset PGL_n(L)$
- $\mathfrak{a}$ is actually a group homomorphism.

Monodromy Actions (II)

As we mentioned, X is a twisted form of

$$X_0 = Fl(1, n-1; n) = \{V_1 \subset V_{n-1} \subset F^n\}, \qquad \dim V_i = i.$$

Thus ρ induces an isomorphism $X_L \simeq X_{0L}$. On the geometric side, we have

$$\begin{tikzcd} Y_L \arrow[r, "\subset"] \arrow[d, "\sigma"'] & X_L \arrow[r, "\simeq", "\rho"'] \arrow[d, "\sigma"'] & X_{0L} \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] \\ Y_L \arrow[r, "\subset"] & X_L \arrow[r, "\simeq", "\rho"'] & X_{0L} \end{tikzcd}$$

This will allow us computing the monodromy action.

Torus fixed points

Note that X_0 admits an action of the standard maximal torus $T \subset PGL_n$. The torus fixed point are given by

$$[ij] = \big(\operatorname{span}(e_i) \subseteq \operatorname{span}(e_1, \ldots, \cancel{e_j}, \ldots, e_n)\big), \qquad 1 \leq i \neq j \leq n.$$

Then by the isomorphism $X_L \simeq X_{0L}$, the torus T_L acts on X_L. Since L contains all eigenvalues of $u \in A$, we can assume the image of u is diagonal. So T_L also acts on Y_L and one can check directly that

$$Y_L^{T_L} = X_L^{T_L}$$

In particular, $\operatorname{rank} \mathrm{CH}(Y_L) = \operatorname{rank} \mathrm{CH}(X_L)$.

Monodromy Actions (III)

Now by taking torus fixed point, we get the following diagram

$$\begin{tikzcd}
Y_L^{T_L} \arrow[r, equal] \arrow[d, "\sigma"'] & X_L^{T_L} \arrow[r, "\rho"] \arrow[d, "\sigma"'] & X_0^T \times \operatorname{Spec} L \arrow[r, equal] \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] & \coprod_{1 \leq i \neq j \leq n} [ij]_L \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] \\
Y_L^{T_L} \arrow[r, equal] & X_L^{T_L} \arrow[r, "\rho"] & X_0^T \times \operatorname{Spec} L \arrow[r, equal] & \coprod_{1 \leq i \neq j \leq n} [ij]_L .
\end{tikzcd}$$

It is obvious that $\mathfrak{a}_\sigma$ permutes $[ij]$. Explicit computation shows that it is induced by the n-cycle

$$1 \stackrel{\eta}{\longmapsto} 2 \stackrel{\eta}{\longmapsto} \cdots n \stackrel{\eta}{\longmapsto} 1$$

where $\eta \in \Gamma_L$ such that $\eta(\sqrt[n]{a}) = \zeta \sqrt[n]{a}$.

Equivariant Chow ring

The T-invariance of the varieties allows us to consider T-equivariant Chow rings. Assume T splits. We have

Theorem (Brion, [Br97])

- $\mathrm{CH}_T(\mathrm{pt}) = \mathrm{Sym}_{\mathbb{Z}}\, T^*$;
- *the usual Chow ring* $\mathrm{CH}(X)$ *is a quotient of* $\mathrm{CH}_T(X)$.

The main benefit of considering equivariantly is the localization theorem.

Theorem (Brion, [Br97])

Let X be a projective, nonsingular variety with an action of T. Then the restriction $\mathrm{CH}_T(X) \longrightarrow \mathrm{CH}_T(X^T)$ *is injective.*

Brion, M. Equivariant Chow groups for torus actions. *Transform. Groups* **2** (1997), no. 3, 225–267.

Monodromy Actions (IV)

As a result, we can lift the Galois group action to equivariant Chow ring where we can play the trick of localization theorem.

$$\begin{tikzcd}
\mathrm{CH}_{T_L}(Y_L) \arrow[d, "\sigma"'] & \mathrm{CH}_{T_L}(X_L) \arrow[l, "\imath^*"'] \arrow[d, "\sigma"'] & \mathrm{CH}_{T_L}(X_{0L}) \arrow[l, "\rho^*"'] \arrow[r, hook] \arrow[d, "\mathfrak{a}_\sigma \times \sigma"] & \bigoplus_{1\le i\neq j\le n} \mathrm{Sym}_{\mathbb{Z}}\, T^* \arrow[d, "\widehat{\sigma}"] \\
\mathrm{CH}_{T_L}(Y_L) & \mathrm{CH}_{T_L}(X_L) \arrow[l, "\imath^*"'] & \mathrm{CH}_{T_L}(X_{0L}) \arrow[l, "\rho^*"'] \arrow[r, hook] & \bigoplus_{1\le i\neq j\le n} \mathrm{Sym}_{\mathbb{Z}}\, T^*.
\end{tikzcd}$$

where

$$\widehat{\sigma}(\varphi_{ij})_{ij} = (\sigma\,\varphi_{\sigma^{-1}(i)\sigma^{-1}(j)})_{ij}, \qquad \varphi_{ij} \in \mathrm{Sym}_{\mathbb{Z}}\, T^*.$$

This will allow us computing the monodromy action combinatorially.

T-stable curves

Let us consider the following curves over X_0

(i) a root-conic curve connecting $[ij]$ and $[ji]$:

$$\mathbb{P}^1 \ni [x:y] \mapsto \big(\mathrm{span}(xe_i+ye_j) \subset \mathrm{span}(e_1,\dots,\cancel{e_i},\dots,\cancel{e_j},\dots,e_n,xe_i+ye_j)\big),$$

(ii) a plane curve connecting $[ij]$ and $[ik]$ (for distinct i,j,k):

$$\mathbb{P}^1 \ni [x:y] \mapsto \big(\mathrm{span}(e_i) \subset \mathrm{span}(e_1,\dots,\cancel{e_j},\dots,\cancel{e_k},\dots,e_n,ye_j+xe_k)\big),$$

(iii) a plane curve connecting $[ij]$ and $[kj]$ (for distinct i,j,k):

$$\mathbb{P}^1 \ni [x:y] \mapsto \big(\mathrm{span}(xe_i+ye_k) \subset \mathrm{span}(e_1,\dots,\cancel{e_j},\dots,e_n)\big).$$

Theorem (Benedetti, Perrin, [BP22])

All T_L-stable curves over Y_L are plane curves.

Benedetti, V.; Perrin, N. Cohomology of hyperplane sections of (co)adjoint varieties. *arXiv:2207.02089*.

Equivariant cohomology

Let us define a graph with $n(n-1)$ vertices denoted $[ij]$, $1 \leq i \neq j \leq n$, which has two types of labelled edges

$$[ij] \xrightarrow{\alpha_{jk}} [ik] \quad \text{and} \quad [ij] \xrightarrow{\alpha_{ik}} [kj], \quad \text{where all } i, j, k \text{ are distinct,}$$

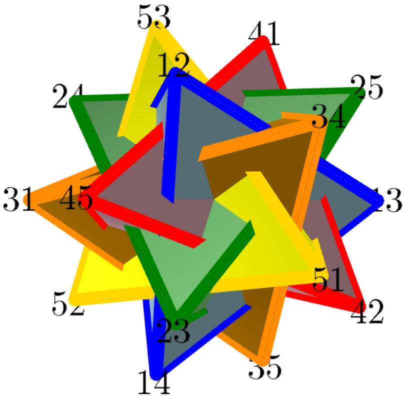
and $\alpha_{ij} = t_i - t_j \in T^*$.

Theorem

$$\mathrm{CH}_{T_L}(Y_L) \simeq \left\{ (\varphi_{ij})_{ij} : \alpha \mid \varphi_{ij} - \varphi_{kh} \textit{ for any edge } [ij] \xrightarrow{\alpha} [kh] \right\}.$$

This is a particular case [Br97, §3].

Example ($n = 5$)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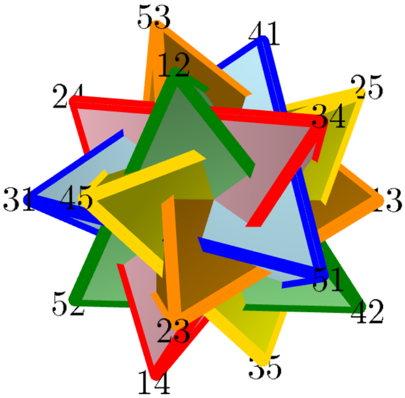
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Hodge–Riemann relation

The last piece of $\mathcal{M}(Y)$, the Artin motive $\mathcal{M}(\operatorname{Spec} L)$, is supported on the primitive space (in terms of Hodge theory).

The Hodge–Riemann relations predicts the index of the intersection form over the primitive space

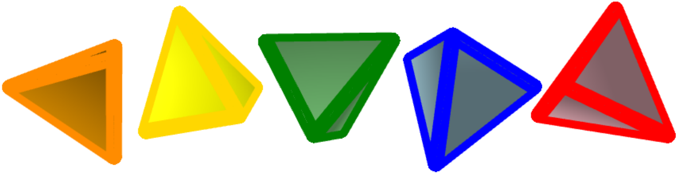
$$\begin{array}{ccccccc} & & - & - & - & & \\ & & & - \; - & & & \\ & & + & + & + & & \\ & - & - & - & - & - & \\ + & + & + & + & + & + & + \end{array}$$

So the intersection form should be of $(-1)^{n-2}$-definite.

Monodromy actions (V)

We then constructed cycles

$$\gamma_\ell \in \bigoplus_{1\le i\neq j\le n} \mathrm{Sym}_{\mathbb{Z}}\, T^* \quad \text{with properties} \begin{cases} \gamma_\ell \in \mathrm{CH}_T^{n-2}(Y_L), \\ \widehat{\sigma}\gamma_\ell = \gamma_{\sigma(\ell)},\ \sigma \in \Gamma_L, \\ \langle \gamma_k, \gamma_\ell \rangle_{Y_L} = (-1)^{n-2}\delta_{k,\ell}. \end{cases}$$



One can show further that they are orthogonal to the Severi–Brauer parts, and give the last motive $\mathcal{M}(\mathrm{Spec}\, L)$. Q.E.D.

Thanks