Pieri Rules over Grassmannian and Applications
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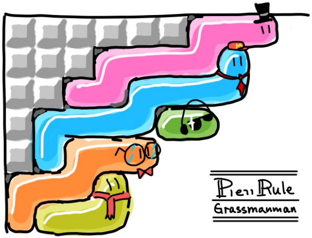


Grassmannian

Recall that **Grassmannian manifold**

$$\mathrm{Gr}(k,n) = \{V \subseteq \mathbb{C}^n : \dim V = k\}.$$

We have the following **Bruhat decomposition**

$$\mathrm{Gr}(k,n) = \bigcup_{\lambda \subseteq (n-k)^k} Y(\lambda)^\circ \quad \text{(disjoint)},$$

where $Y(\lambda)^\circ$ is the **opposite Schubert cell**.

Description

For example, $V \in Y\left(\ydiagram{4,4,3}\right)^{\circ} \iff$

$$
\begin{array}{lccccc}
\text{codim}=4 & & & & & \phi_1 \cap V = \phi_0 \cap V \\
 & & & & & \cap \\
\text{codim}=3 & & & & & \phi_2 \cap V \\
 & & & & & \cap \\
\text{codim}=2 & & & & \phi_4 \cap V = & \phi_3 \cap V \\
 & & & & \cap & \\
\text{codim}=1 & \phi_8 \cap V = & \phi_7 \cap V = & \phi_6 \cap V = & \phi_5 \cap V & \\
 & \cap & & & & \\
\text{codim}=0 & \phi_9 \cap V & & & &
\end{array}
$$

where $\phi_i = \text{span}(\mathbf{e}_n, \ldots, \mathbf{e}_{n-i+1})$.

Bruhat Decomposition

Denote **Schubert variety**

$$Y(\lambda) = \overline{Y(\lambda)^\circ} = \bigcup_{\mu \supseteq \lambda} Y(\mu)^\circ.$$

The **cohomology group** and **K-group** can be computed to be

$$H^*\big(\mathrm{Gr}(k,n)\big) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q} \cdot [Y(\lambda)].$$

$$K\big(\mathrm{Gr}(k,n)\big) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q} \cdot [\mathcal{O}_{Y(\lambda)}]$$

Chern Classes

Let $\mathcal{V}$ be the **tautological bundle** over $\mathrm{Gr}(k,n)$. We denote

$$c_r = c_r(\mathcal{V}^\vee) = \text{the } r\text{-th equivariant } \textbf{Chern classes} \text{ of } \mathcal{V}^\vee.$$

It is known that

$$H^*\big(\,\mathrm{Gr}(k,n)\big) = \mathbb{Q}[c_1,\dots,c_k]\big/\text{some ideal}.$$

$$K\big(\,\mathrm{Gr}(k,n)\big) = \mathbb{Q}[c_1,\dots,c_k]\big/\text{some ideal}.$$

Geometry of cohomology

Roughly speaking **cohomology**

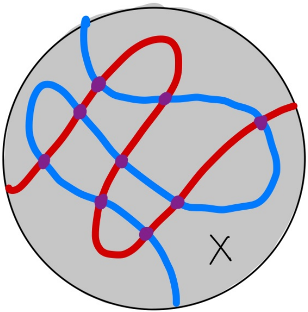
$$H^\bullet(X) = \bigoplus_{Y \text{ closed } \subseteq X} \mathbb{Q}\cdot[Y] \Big/ \text{HOMOTOPY EQUIVALENCE}$$

with product **the transversal intersection**

$$[Y_1]\cdot[Y_2] = [Y_1 \pitchfork Y_2].$$

Over $\mathrm{Gr}(k,n)$, we have **Schubert class**

$$[Y(\lambda)] \in H^{2\ell(\lambda)}(\mathrm{Gr}(k,n)).$$



Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

We have

$$Y(\square) = \text{the point } \infty,$$
$$Y(\varnothing) = \text{the entire } \mathbb{P}^1.$$

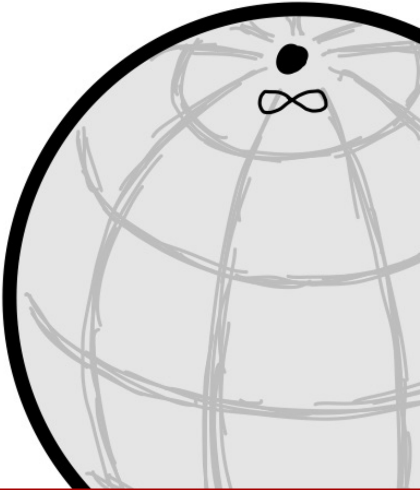
The intersection

$\pitchfork$	pt	$\mathbb{P}^1$
pt	$\varnothing$	pt
$\mathbb{P}^1$	pt	$\mathbb{P}^1$

The cohomology

$$H^*(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2)$$

where $x = [Y(\square)]$.

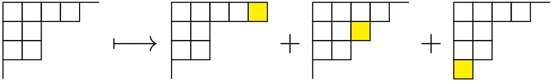


Classical Chevalley Formula

Theorem (Chevalley Formula)

$$c_1(\mathcal{V}^\vee) \cdot [Y(\lambda)] = \sum_{\mu = \lambda + \Box} [Y(\mu)].$$

Example:



Classical Pieri Rule

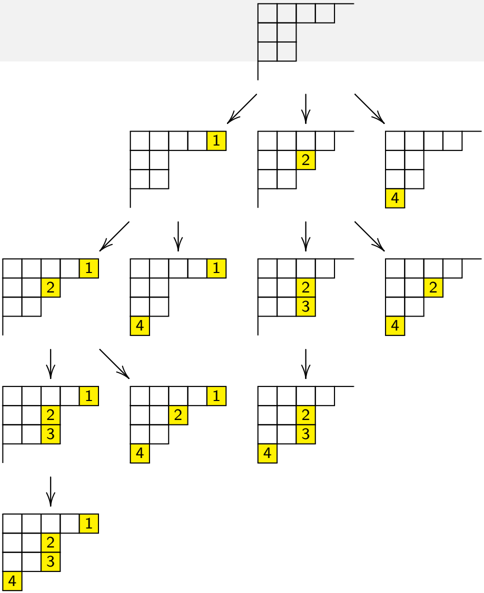
Let us denote **Schur operators**

$$[i] \to [Y(\lambda)] = \begin{cases} [Y(\mu)], & \mu = \lambda + \square \text{ in the } i\text{-th row}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (Pieri Rule)

$$c_r(\mathcal{V}^\vee) \cdot [Y(\lambda)] = \sum_{1 \le i_1 < \cdots < i_r \le k} [i_r] \to \cdots [i_1] \to [Y(\lambda)].$$

Example



Geometry of K-theory

Roughly speaking, the **K-theory**

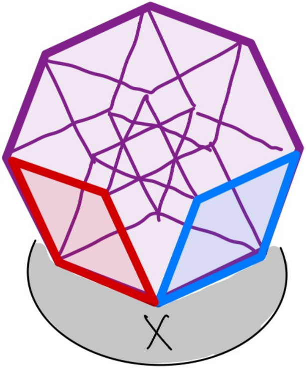
$$K(X) = \bigoplus_{\mathcal{F} \in \mathcal{C}\mathrm{oh}\, X} \mathbb{Q} \cdot [\mathcal{F}] \Big/ \begin{matrix}\text{EXACT}\\ \text{SEQUENCES}\end{matrix}$$

with product **the tensor product**

$$[\mathcal{F}_1] \cdot [\mathcal{F}_2] = [\mathcal{F}_1 \otimes \mathcal{F}_2].$$

Over $\mathrm{Gr}(k,n)$, we have **structure sheaves**

$$[\mathcal{O}_{Y(\lambda)}] \in K(\mathrm{Gr}(k,n)).$$



Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

We have

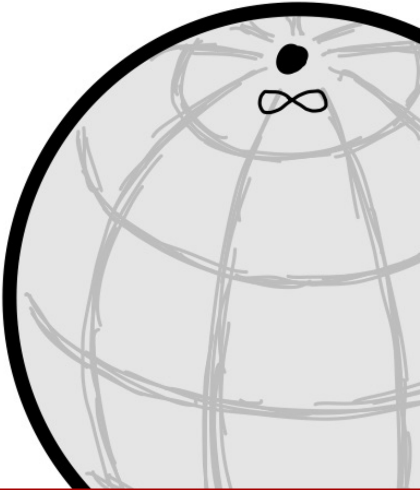
$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_\infty \to 0.$$

Thus $[\mathcal{O}_\infty] = 1 - \mathcal{O}(-1)$.

The K-theory

$$K(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2)$$

where $x = 1 - \mathcal{O}(-1) = [\mathcal{O}_{Y(\square)}]$.

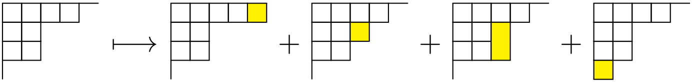


K-theory Chevalley Formula

Theorem (Lenart [1])

$$c_1(\mathcal{V}^\vee) \cdot [\mathcal{O}_{Y(\lambda)}] = \sum_{\mu=\lambda+\square} [\mathcal{O}_{Y(\mu)}].$$

Example:



K-theory Pieri Rule

Let us denote **Schur operators**

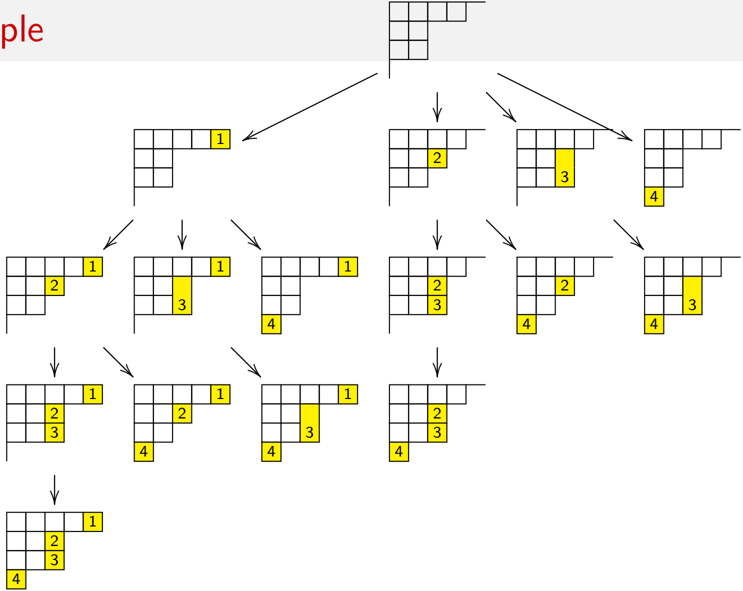
$$[\,i\,] \to [\mathcal{O}_{Y(\lambda)}] = [\mathcal{O}_{Y(\mu)}]$$

where $\mu = \lambda +$ a vertical strip with its tail at the i-th row.

Theorem (Lenart [1])

$$c_r(\mathcal{V}^\vee) \cdot [\mathcal{O}_{Y(\lambda)}] = \sum_{1 \le i_1 < \cdots < i_r \le k} [\,i_r\,] \to \cdots [\,i_1\,] \to [\mathcal{O}_{Y(\lambda)}].$$

Example



Constructible Functions

Consider

$$\begin{aligned}\mathrm{Fun}(X) &= \{\text{constructible functions over } X\} \\ &= \mathrm{span}(\mathbf{1}_A : A \subseteq X \text{ closed}).\end{aligned}$$

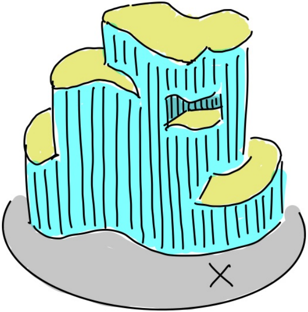
For any proper map $f : X \to Y$, we have a **push-forward**

$$f_* : \mathrm{Fun}(X) \to \mathrm{Fun}(Y)$$

defined such that

$$(f_*(\mathbf{1}_A))(y) = \chi_c(A_y)$$

the Euler characteristic of fibre A_y.



CSM classes

By MacPherson [2], there is a natural transform (wrt push-forward) called **Chern–Schwartz–MacPherson classes**

$$c_{\mathrm{SM}} : \mathrm{Fun}(-) \to H_\bullet(-),$$

such that when X is smooth

$$c_{\mathrm{SM}}(X) = \text{total Chern class of the tangent bundle of } X.$$

Over $\mathrm{Gr}(k,n)$, we have **CSM classes**

$$c_{\mathrm{SM}}(Y(\lambda)^\circ) := c_{\mathrm{SM}}(\mathbf{1}_{Y(\lambda)^\circ}) \in H^*(\mathrm{Gr}(k,n)).$$

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

Recall

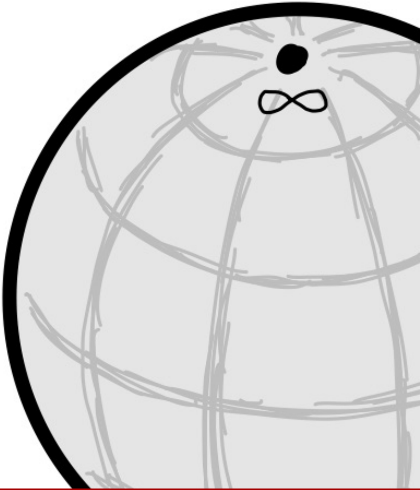
$$Y(\square)^\circ = \text{the point } \infty,$$
$$Y(\varnothing)^\circ = \mathbb{P}^1 \setminus \{\infty\}.$$

So by definition,

$$c_{\mathrm{SM}}(Y(\square)^\circ) = [Y(\square)] = x.$$

Since $\mathscr{T}_{\mathbb{P}^1} = \mathcal{O}(2)$, we have

$$\begin{array}{r l} \text{total Chern class} = & 1 + 2x \\ \hline c_{\mathrm{SM}}(Y(\square)^\circ) = & \phantom{1 + {}} x \\ c_{\mathrm{SM}}(Y(\varnothing)^\circ) = & 1 + x \end{array}$$

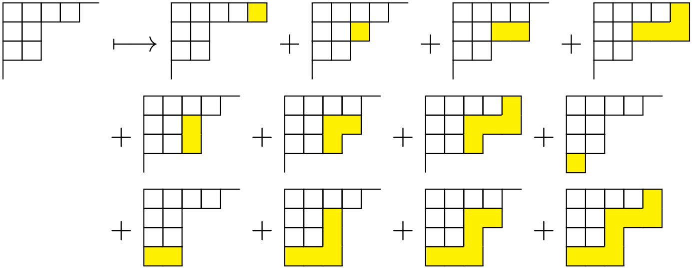


CSM Chevalley formula

Theorem (Aluffi, Mihalcea, Schürmann and Su [3])

$$c_1(\mathcal{V}^\vee) \cdot c_{\mathrm{SM}}(Y(\lambda)^\circ) = \sum_{\mu=\lambda+} c_{\mathrm{SM}}(Y(\mu)^\circ).$$

Example:



CSM Pieri Rule

Let us denote **ribbon Schubert operators**

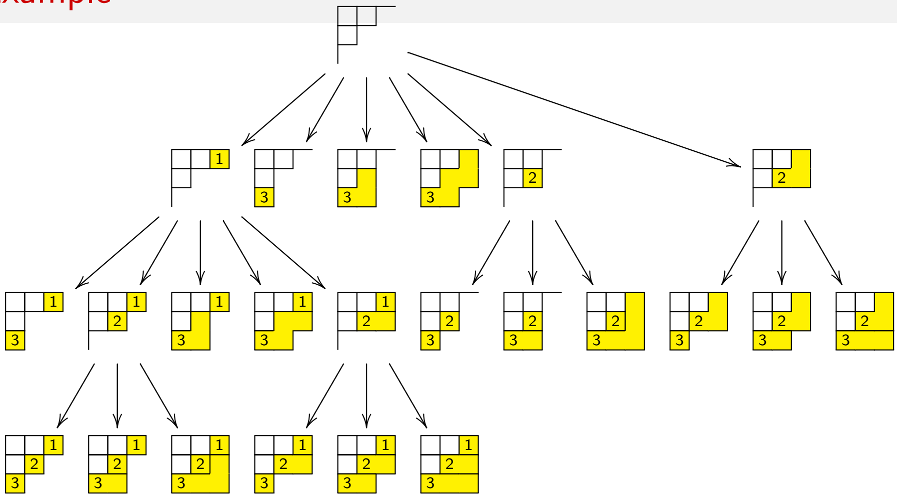
$$[\, i\, | \to c_{\mathrm{SM}}(Y(\lambda)^\circ) = \sum_{\mu} c_{\mathrm{SM}}(Y(\mu)^\circ)$$

where the sum over $\mu = \lambda +$ a ribbon strip with its tail at the i-th row.

Theorem (Fan, Guo and Xiong [4])

$$c_r(\mathcal{V}^\vee) \cdot c_{\mathrm{SM}}(Y(\lambda)^\circ) = \sum_{1 \le i_1 < \cdots < i_r \le k} [\, i_r\, | \to \cdots [\, i_1\, | \to c_{\mathrm{SM}}(Y(\lambda)^\circ).$$

Example



Grothendieck group

Consider the **Grothendieck group of varieties**

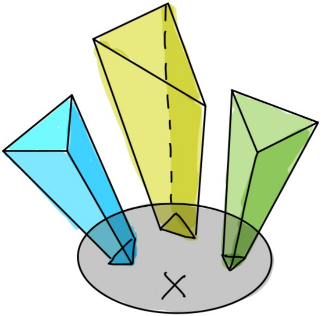
$$\mathrm{G}(X) = \bigoplus_{\text{variety } Z \to X} \mathbb{Z} \cdot [Z \to X] \Big/ [U \to X] + [Z \setminus U \to X] = [Z \to X].$$

For any proper map $f : X \to Y$, we have a **push-forward**

$$f_* : \mathrm{G}(X) \to \mathrm{G}(Y)$$

with

$$f_*[Z \to X] = [Z \to X \to Y].$$



Motivic Chern classes

By Brasselet, Schürmann and Yokura [5], there is a natural transform (wrt push-forward) called **motivic Chern classes**

$$\mathrm{MC}_y : \mathrm{G}(-) \to K(-)[y],$$

such that when X is smooth,

$$\mathrm{MC}_y(X) = \lambda\text{-class} = \sum_{k=1}^{\dim X} y^k [\Lambda^k \mathscr{T}_X^\vee].$$

Over $\mathrm{Gr}(k,n)$, we have **motivic Chern classes**

$$\mathrm{MC}_y(Y(\lambda)^\circ) := \mathrm{MC}_y([Y(\lambda)^\circ \to \mathrm{Gr}(k,n)]) \in K(\mathrm{Gr}(k,n)).$$

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

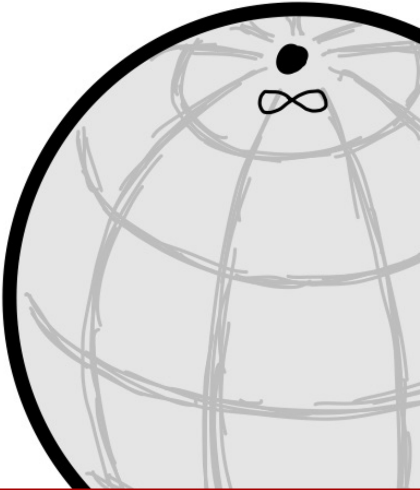
Similarly,

$$\mathrm{MC}_y(Y(\square)^\circ) = [\mathcal{O}_{Y(\square)}] = x.$$

Recall that $x = 1 - \mathcal{O}(-1)$.

Since $\mathscr{T}_{\mathbb{P}^1} = \mathcal{O}(2)$, we have

$$\begin{array}{rl} \lambda\text{-class} & = 1 + y\mathcal{O}(-2) \\ & = (1+y) - 2yx \\ \hline \mathrm{MC}_y(Y(\varnothing)^\circ) & = (1+y) - (2y+1)x \\ \mathrm{MC}_y(Y(\square)^\circ) & = \qquad\qquad\qquad x \end{array}$$

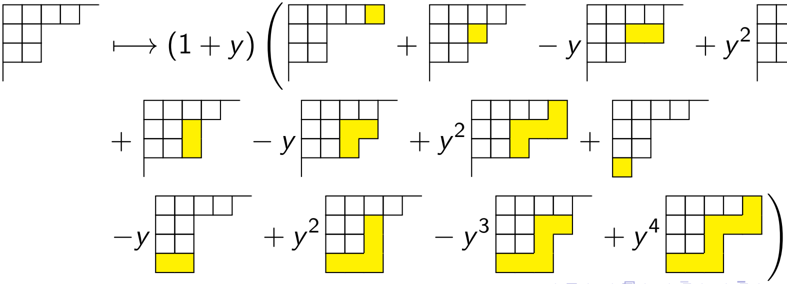


MC Chevalley formula

Theorem (Fan, Guo, Su and Xiong)

$$c_1(\mathcal{V}^\vee)\cdot \mathrm{MC}_y(Y(\lambda)^\circ) = (1+y) \sum_{\mu=\lambda+} (-y)^{\mathrm{wd}(\mu/\lambda)-1}\,\mathrm{MC}_y(Y(\mu)^\circ).$$

Example:



MC Pieri Rule

Let us denote **ribbon Schubert operators**

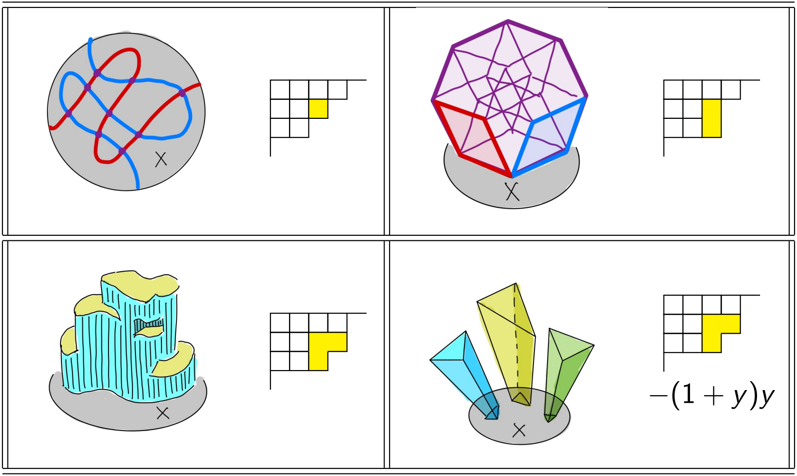
$$[i\,| \to MC_y(Y(\lambda)^\circ) = (1+y)\sum_{\mu}(-y)^{\mathrm{wd}(\mu/\lambda)-1}\,MC_y(Y(\mu)^\circ)$$

where the sum over $\mu = \lambda +$ a ribbon strip with its tail at the i-th row.

Theorem (Fan, Guo, Su and Xiong)

$$c_r(\mathcal{V}^\vee)\cdot MC_y(Y(\lambda)^\circ) = \sum_{1\le i_1<\cdots<i_r\le k} [i_r\,| \to \cdots [i_1\,| \to MC_y(Y(\lambda)^\circ).$$

Summary



Affine Hecke algebra

Our approach is by introducing a version of affine Hecke algebra of three parameters

$$
\begin{aligned}
T_i^2 &= -(p-q)T_i + pq \\
T_i T_j &= T_j T_i, \quad |i-j| > 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\
x_i x_j &= x_j x_i, \\
T_i x_j &= x_j T_i, \quad j \neq i, i+1, \\
T_i x_i &= x_{i+1} T_i + (\hbar - (p-q)x_i), \\
T_i x_{i+1} &= x_i T_i - (\hbar - (p-q)x_i).
\end{aligned}
$$

Rôles of $p, q, \hbar$

It turns out that $p, q, \hbar$ control the following ribbon statistics

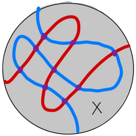
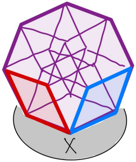
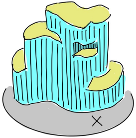
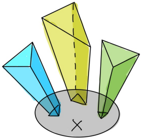
$$p : \text{height} - 1, \qquad q : \text{width} - 1, \qquad \hbar : \text{number of ribbons}.$$

We have the following table

classes	$(p, q, \hbar)$	Pieri rule
$[Y(\lambda)]$	$(0, 0, 1)$	adding boxes $\square$
$[\mathcal{O}_{Y(\lambda)}]$	$(1, 0, 1)$	adding vertical strips
$c_{\mathrm{SM}}(Y(\lambda)^\circ)$	$(1, 1, 1)$	adding ribbons
$\mathrm{MC}_y(Y(\lambda)^\circ)$	$(1, -y, 1+y)$	adding ribbons and counting width

Dual Basis

In all four cases, we have another choice of basis

Theory				
basis	$[Y(\lambda)]$	$[\mathcal{I}_{\partial Y(\lambda)}]$	$c_{\mathrm{SM}}(Y(\lambda)^\circ)$	$\mathrm{MC}_y(Y(\lambda)^\circ)$
opposite dual basis	$[Y(\lambda)]$	$[\mathcal{O}_{Y(\lambda)}]$	$s_{\mathrm{SM}}(Y(\lambda)^\circ)$	$\mathrm{SMC}_y(Y(\lambda)^\circ)$

Theorem (Fan, Guo, Su and Xiong)

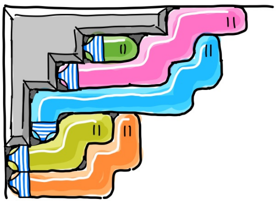
The ***opposite dual basis*** *has the same Pieri rule as* ***basis****.*

Discussion of the proof

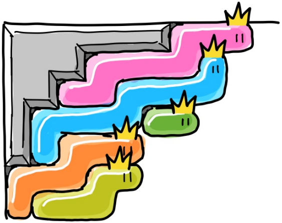
A priori, the Pieri rule for the opposite dual basis is given by $|\,i]$, the adjoint operator on the 180° rotated complement.

$[i\,|$ ...with its **tail** at the i-th row ... $\longleftrightarrow$ $|\,i]$ ...with its **head** at the i-th row ...

But they are equivalent:



v.s.



Equivariant version

All the basis are defined in equivariant cohomology/K-theory.

Theorem (Fan, Guo, Su and Xiong)

The equivariant classes

$$[Y(\lambda)], \qquad [\mathcal{I}_{\partial Y(\lambda)}], \qquad c_{\mathrm{SM}}(Y(\lambda)^{\circ}), \qquad \mathrm{MC}_y(Y(\lambda)^{\circ})$$

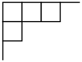
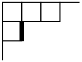
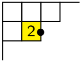
satisfy the **head-valued Pieri rule**, *i.e.* $|\, i\,]\!\bullet$ *or equivalently* $[\, i\, |\!\bullet$.

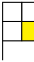
Theorem (Fan, Guo, Su and Xiong)

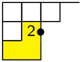
The equivariant classes

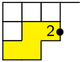
$$[Y(\lambda)], \qquad [\mathcal{O}_{Y(\lambda)}], \qquad s_{\mathrm{SM}}(Y(\lambda)^{\circ}), \qquad \mathrm{SMC}_y(Y(\lambda)^{\circ})$$

satisfy the **tail-valued Pieri rule**, *i.e.* $\bullet\!|\, i\,]$ *or equivalently* $\bullet\![\, i\, |$.

$|2]$ $\rightarrow$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_4) \cdot$ 

$+ (\hbar - (p-q)t_5) \cdot$ 

$+ (\hbar - (p-q)t_4)pq \cdot$ 

$+ (\hbar - (p-q)t_5)pq^2 \cdot$ 

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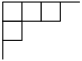
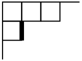
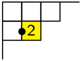
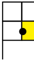
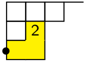
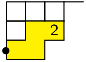
$|2]$ $\rightarrow$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_3) \cdot$ 

$+ (\hbar - (p-q)t_3) \cdot$ 

$+ (\hbar - (p-q)t_1)pq \cdot$ 

$+ (\hbar - (p-q)t_1)pq^2 \cdot$ 

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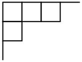
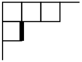
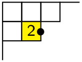
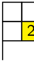
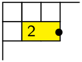
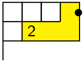
$[2|$ $\rightarrow$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_4) \cdot$ 

$+ (\hbar - (p-q)t_5) \cdot$ 

$+ (\hbar - (p-q)t_5)q \cdot$ 

$+ (\hbar - (p-q)t_7)pq^2 \cdot$ 

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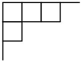
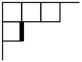
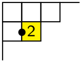
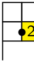
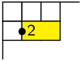
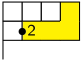
$[2|$ $\rightarrow$  $= t_3 \cdot$  $+ (\hbar - (p-q)t_3) \cdot$ 

$+ (\hbar - (p-q)t_3) \cdot$ 

$+ (\hbar - (p-q)t_3)q \cdot$ 

$+ (\hbar - (p-q)t_3)pq^2 \cdot$ 

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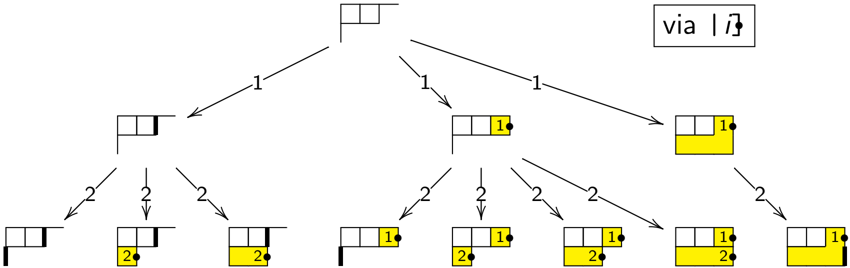
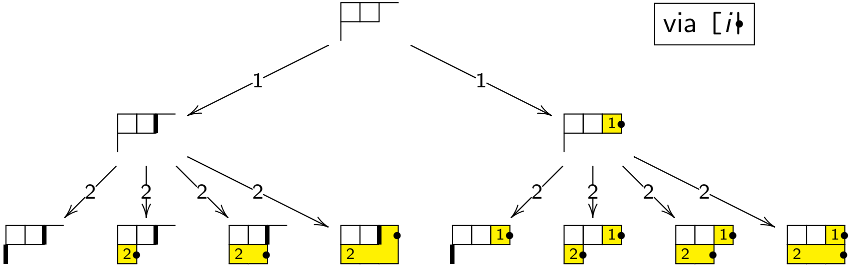




Application A

There is a classic relation between ideal sheaves and structure sheaves over Grassmannian.

Theorem (Buch [6], see also [7, Prop. 4.2])

$$(1-[\mathcal{O}_{Y(\square)}])\cdot[\mathcal{O}_{Y(\lambda)}]=[\mathcal{I}_{\partial Y(\lambda)}].$$

This can be generalized to equivariant K-theory.

$$\frac{(1-[\mathcal{O}_{Y(\square)}])\cdot[\mathcal{O}_{Y(\lambda)}]}{1-[\mathcal{O}_{Y(\square)}]|_\lambda}=[\mathcal{I}_{\partial Y(\lambda)}]\in K_T(\mathrm{Gr}(k,n)).$$

Relation between MC and SMC

Using our Pieri rule, we can prove the following analogy for MC and SMC classes.

Theorem (Fan, Guo, Su and Xiong)

$$\lambda_y(\mathscr{T}^\vee_{\mathrm{Gr}(k,n)}) \cdot (1 - [\mathcal{O}_{Y(\square)}]) \cdot \mathrm{SMC}_y(Y(\lambda)^\circ) = \mathrm{MC}_y(Y(\lambda)^\circ).$$

This can be generalized to equivariant K-theory.

$$\lambda_y(\mathscr{T}^\vee_{\mathrm{Gr}(k,n)}) \cdot \frac{(1 - [\mathcal{O}_{Y(\square)}]) \cdot \mathrm{SMC}_y(Y(\lambda)^\circ)}{1 - [\mathcal{O}_{Y(\square)}]|_\lambda} = \mathrm{MC}_y(Y(\lambda)^\circ).$$

If we set $y = 0$, we will recover the result in the previous page.

Discussion of the proof

The proof is by one sentence:

> both sides have the same Pieri rule and
> they agree after certain specialization.

Precisely:

- the factor
$$1 - [\mathcal{O}_{Y(\square)}]|_\lambda$$
intertwines $\bullet\!\lfloor i \,|$ and $\lfloor i \,|\!\bullet$;
- the factor rest gives normalization by looking at localization.

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

Recall

$$K(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2), \qquad x = 1 - [\mathcal{O}(-1)].$$

We have

$$\begin{tikzcd}
1+\dfrac{y}{1+y}x \arrow[r, equal] & \mathrm{SMC}_y(Y(\varnothing)^\circ) \arrow[rr, bend left, "(1-x)\big((1+y)-2yx\big)"] & & \mathrm{MC}_y(Y(\varnothing)^\circ) \arrow[r, equal] & (1+y)-(2y+1)x \\
& & \text{dual basis} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & & \\
\dfrac{1}{1+y}x \arrow[r, equal] & \mathrm{SMC}_y(Y(\square)^\circ) \arrow[rr, bend right, "(1-x)\big((1+y)-2yx\big)"'] & & \mathrm{MC}_y(Y(\square)^\circ) \arrow[r, equal] & x
\end{tikzcd}$$

Application B

Recall the **stable grothendieck polynomial** is defined using set-valued tableaux:

$$\tilde{G}_\lambda = \sum_{T \in \mathrm{SVT}(\lambda)} x^T, \quad \text{e.g.} \quad \begin{array}{|c|c|c|c|} \hline 1 & 123 & 35 & 6 \\ \hline 234 & 46 & \multicolumn{2}{c}{} \\ \cline{1-2} 5 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \left\{ \begin{array}{l} \text{filled by nonempty sets} \\ \text{strictly increasing in column} \\ \text{weakly increasing in row} \end{array} \right.$$

Theorem (Buch [6])

$$(-1)^{|\lambda|} \tilde{G}_\lambda(-x_1, \cdots, -x_k, 0, \ldots) = [\mathcal{O}_{Y(\lambda)}] \in K(\mathrm{Gr}(k, n)).$$

Dualizing Sheaves

In Lam and Pylyavskyy [8], the omega involution of $\tilde{G}_\lambda$ was studied. It is given by a sum over weak set-valued tableaux:

$$J_\lambda = \sum_{T \in \mathrm{WSVT}(\lambda)} x^T, \quad \text{e.g.} \quad \begin{array}{|c|c|c|c|} \hline 11 & 334 & 55 & 6 \\ \hline 12 & 4 \\ \cline{1-2} 223 \\ \cline{1-1} \end{array} \quad \left\{ \begin{array}{l} \text{filled by nonempty multi-sets} \\ \text{strictly increasing in row} \\ \text{weakly increasing in column} \end{array} \right.$$

Theorem (Fan, Guo, Su and Xiong)

$$\big((1 - G_\square)^n J_{\lambda'}\big)(x_1, \ldots, x_k, 0, \ldots) = [\omega_{Y(\lambda)}] \in K(\mathrm{Gr}(k,n))$$

where $\omega_{Y(\lambda)}$ is the dualizing sheaf of $Y(\lambda)$.

Discussion of the proof

By [9],

$$\mathrm{MC}_y(Y(\lambda)^\circ) = y^{\mathrm{dim}}[\omega_{Y(\lambda)}] + (\text{lower } y\text{-degree}).$$

In the Pieri rule of motivic Chern classes, only the horizontal strip contributes the highest y-degree. Thus

$$\text{Pieri rule of } [\omega_{Y(\lambda)}] = \text{adding horizontal strips}.$$

Compare:

$$\text{Pieri rule of } [\mathcal{O}_{Y(\lambda)}] = \text{adding vertical strips}.$$

The omega involution switches two kind of strips.

Example: $\mathrm{Gr}(1,2) = \mathbb{P}^1$

From the (weak) set-tableaux model, $\tilde{G}_\square = J_\square = 1$, and

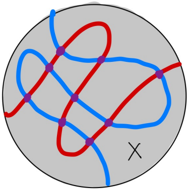
$$\tilde{G}_{\square} = \sum_A x^A = -1 + \prod_{i=1}^{\infty}(1+x_i), \qquad \boxed{A} \quad \text{nonempty sets } A \text{ of positive integers,}$$

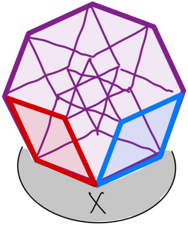
$$J_{\square} = \sum_B x^B = -1 + \prod_{i=1}^{\infty}\frac{1}{1-x_i}, \qquad \boxed{B} \quad \text{nonempty multisets } B \text{ of positive integers.}$$

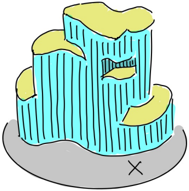
Thus for $\mathbb{P}^1$

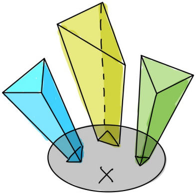
$$(-1)^0 \tilde{G}_{\varnothing}(-x,0,\cdots) = 1 = [\mathcal{O}_{Y(\varnothing)}],$$
$$(-1)^1 \tilde{G}_{\square}(-x,0,\cdots) = x = [\mathcal{O}_{Y(\square)}],$$
$$\left((1-G_\square)^2 \cdot J_\varnothing\right)(x,0,\ldots) = 1-2x = [\omega_{Y(\varnothing)}],$$
$$\left((1-G_\square)^2 \cdot J_\square\right)(x,0,\ldots) = x = [\omega_{Y(\square)}].$$

Thank You!









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