NOTES ON QUANTUM COHOMOLOGY

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Contents

1. Quantum product

The moduli space of stable maps.

1.1. Stable maps. A quasi-stable curve with n-marked point is

$$(C, p_1, \dots, p_n)$$

where C is a projective, connected, reduced, (at worst) nodal curve of arithmetic genus 0, $p_1, \dots, p_n \in C$ are distinct regular points on C. We call

$$\{\text{special points}\} = \{\text{marked points}\} \cup \{\text{nodal points}\}.$$

For a variety X, $\beta \in \mathrm{Eff}(X)$, we define the moduli space of stable maps

$$\overline{\mathcal{M}}_n(X, \beta) = \left\{ (f, C, p_1, \dots, p_n) : \begin{array}{c} (C, p_1, \dots, p_n) \text{ is quasi-stable} \\ f : C \to X \text{ with } f_*[C] = \beta, \\ \text{and the stability condition} \end{array} \right\} /\text{re-parametrization}.$$

Here the stability condition is

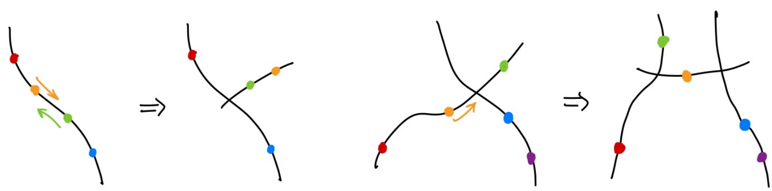
$$\boxed{\begin{array}{c} \text{If } f \text{ is constant over an irreducible component of } C, \\ \text{then there must be at least 3 special points on it.} \end{array}}$$

Equivalently, the automorphism group $\mathrm{Aut}(f, C, p_1, \dots, p_n)$ is finite. We denote

$$\overline{\mathcal{M}}_n(X) = \bigcup_{\beta} \overline{\mathcal{M}}_n(X, \beta), \qquad \overline{\mathcal{M}}_n = \overline{\mathcal{M}}_n(\mathrm{pt}).$$

1.2. Compactification. It turns out $\overline{\mathcal{M}}_n(X, \beta)$ is a compactification of

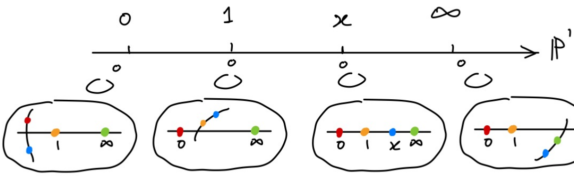
$$\left\{ (f, \mathbb{P}^1, p_1, \dots, p_n) : \begin{array}{c} f : \mathbb{P}^1 \to X \text{ with } f_*[\mathbb{P}^1] = \beta \\ p_1, \dots, p_n \in \mathbb{P}^1 \text{ distinct} \end{array} \right\} /\text{re-parametrization}.$$



When $n = 3$, as any three points can be moved to $(0, 1, \infty)$ by a re-parametrization $\mathrm{Aut}(\mathbb{P}^1)$, the moduli space $\overline{\mathcal{M}}_3(X)$ is a compactification of $\mathrm{Mor}(\mathbb{P}^1, X)$.

1.3. Example. We have

$$\overline{\mathcal{M}}_3 = \mathrm{pt}, \qquad \overline{\mathcal{M}}_4 = \mathbb{P}^1.$$



1.4. Example. We have

$$\overline{\mathcal{M}}_3(\mathbb{P}^1, 1) = \mathrm{pt}, \qquad \overline{\mathcal{M}}_3(X, 0) = X.$$

1.5. Expected dimension. At the point $(f, C, p_1, \ldots, p_n)$, the tangenet space is the difference of the following

$$\begin{aligned}
(\text{deforming } f) &= \text{tangent fields of } X \text{ along } C \\
&= H^0(C, f^*\mathscr{T}_X). \\
(\text{infinitesimal automorphisms}) &= (\text{infinitesimal reparametrization}) \\
&= \text{tangent fields of } C \text{ vanishing at } p_1, \ldots, p_n \\
&= H^0(C, \mathscr{T}_C(-p_1 - \cdots - p_n)) \\
&= \mathrm{Ext}^0(\omega_C(p_1 + \cdots + p_n), \mathcal{O}_C).
\end{aligned}$$

By Riemann–Roch

$$\chi(C, f^*\mathscr{T}_X) = \dim X + \langle \beta, c_1(\mathscr{T}_X) \rangle$$
$$\chi(C, \mathscr{T}_C(-p_1 - \cdots - p_n)) = -n + 3.$$

So the expected dimension of $\overline{\mathcal{M}}_n(X, \beta)$ is

$$\dim X + \langle \beta, c_1(\mathscr{T}_X) \rangle + n - 3.$$

Gromov–Witten invariants.

1.6. Morphisms. We have a morphism called evaluation

$$\mathrm{ev} : \overline{\mathcal{M}}_n(X, \beta) \longrightarrow X \times \cdots \times X : \qquad (f, C, p_1, \ldots, p_n) \longmapsto (f(p_1), \ldots, f(p_n)).$$

We denote ev_i the i-th component. We have a forgetful morphism ft

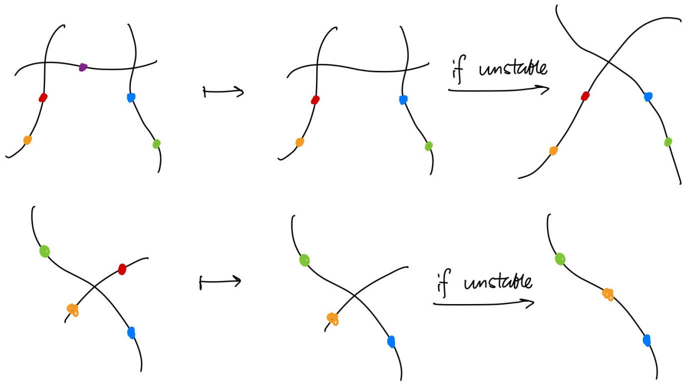
$$\mathrm{ft}_i : \overline{\mathcal{M}}_{n+1}(X, \beta) \longrightarrow \overline{\mathcal{M}}_n(X, \beta)$$

by forgetting the i-th marked point and collapsing branches if necessary to get a stable map. Note that this map is not defined for $\beta = 0$ and $n = 2$, as $\overline{\mathcal{M}}_2(X, 0) = \varnothing$. Similarly for $f : X \to Y$, we have

$$f_* : \overline{\mathcal{M}}_n(X, \beta) \longrightarrow \overline{\mathcal{M}}_n(Y, f_*\beta).$$

In particular, we have

$$\mathrm{ft}_X : \overline{\mathcal{M}}_n(X, \beta) \longrightarrow \overline{\mathcal{M}}_n.$$



1.7. Gromov–Witten invariants. For $\gamma_1, \gamma_2, \gamma_3 \in H^*(X)$, we define

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta := \int_{\overline{\mathcal{M}}_n(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3).$$

Note that $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta = 0$ unless

$$(\deg \gamma_1 + \deg \gamma_2 + \deg \gamma_3) = \dim X + \langle \beta, c_1(\mathscr{T}_X) \rangle.$$

Here $\deg \gamma = k$ if $\gamma \in H^{2k}(X)$.

1.8. Meaning. Assume $\gamma_i = [Z_i]$ for subvariety $Z_i \subset X$. Then the meaning of Gromov–Witten invariant can be understood as

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta = \# \left\{ \mathbb{P}^1 \xrightarrow{f} X : \begin{array}{c} f_*[\mathbb{P}^1] = \beta,\ f(0) \in Z_1, \\ f(1) \in Z_2,\ f(\infty) \in Z_3 \end{array} \right\}.$$

Note that now

$$\text{reparametrization} = \mathrm{Aut}(\mathbb{P}^1, 0, 1, \infty) = \text{trivial group}.$$

1.9. Novikov Ring. Denote Novikov ring

$$\mathbb{Q}[\![\mathrm{Eff}(X)]\!] = \mathbb{Q}[\![q^\beta]\!]_{\beta \in \mathrm{Eff}(X)} / \langle q^0 = 1,\ q^{\beta_1} q^{\beta_2} = q^{\beta_1 + \beta_2} \rangle.$$

We will equip the degree

$$\deg q^\beta = \langle \beta, c_1(\mathscr{T}) \rangle.$$

Quantum cohomology.

1.10. Quantum cohomology. We define

$$QH^*(X) = H^*(X, \mathbb{Q})[\![\mathrm{Eff}(X)]\!]$$

with the quantum product $*$ uniquely determined by

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle = \sum_{\beta \in \mathrm{Eff}(X)} q^\beta \langle \gamma_1, \gamma_2, \gamma_3 \rangle_\beta,$$

where $\langle\, ,\, \rangle$ is the Poincaré pairing. As $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_0 = \langle \gamma_1\gamma_2, \gamma_3 \rangle$, quantum product is a q-deformation of classical product

$$\gamma_1 * \gamma_2 = \gamma_1\gamma_2 + (\text{quantum correction})$$

with

$$(\text{quantum correction}) \in \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta H^*(X)$$

which tends to 0 under the limit $\lim_{q \to 0} : H^*(X, \mathbb{Q})[\![\mathrm{Eff}(X)]\!] \to H^*(X, \mathbb{Q})$.

1.11. Commutativity. Note that this expression is symmetric under any permutation of $\gamma_1, \gamma_2, \gamma_3$, so quantum product is commutative

$$\gamma_1 * \gamma_2 = \gamma_2 * \gamma_1$$

and satisfies the **Frobenius property**

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle = \langle \gamma, \gamma_2 * \gamma_3 \rangle.$$

1.12. Associativity. Let us consider

$$\mathrm{ft}_X : \overline{\mathcal{M}}_4(X) \longrightarrow \overline{\mathcal{M}}_4 = \mathbb{P}^1.$$

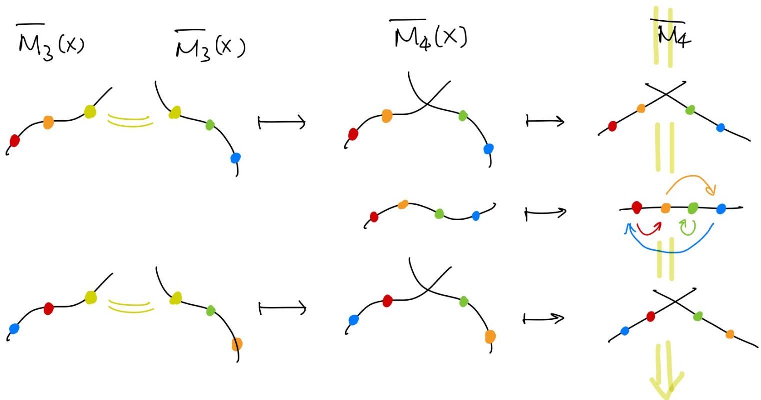
For the nodal curve C on $\overline{\mathcal{M}}_4$, we have

$$\mathrm{ft}_X^{-1}(\{C\}) = \bigcup_{\beta_1 + \beta_2 = \beta} \overline{\mathcal{M}}_3(X, \beta_1) \times_X \overline{\mathcal{M}}_3(X, \beta_2).$$

Here

$$\begin{tikzcd}
\overline{\mathcal{M}}_3(X, \beta_1) \times_X \overline{\mathcal{M}}_3(X, \beta_2) \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\text{fibre product}"] & \overline{\mathcal{M}}_3(X, \beta_1) \arrow[d, "ev_3"] & \overline{\mathcal{M}}_4(X, \beta) \\
\overline{\mathcal{M}}_3(X, \beta_2) \arrow[r, "ev_3"] & X. &
\end{tikzcd}$$

The map is given by gluing the last marked points.



Let us compute

$$
\begin{aligned}
&\int_{\overline{\mathcal{M}}_4(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3 \boxtimes \gamma_4)\, \mathrm{ft}^*([\mathrm{pt}])\\
&= \sum_{\beta_1+\beta_2=\beta} \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times_X \overline{\mathcal{M}}_3(X,\beta_2)} (\mathrm{ev}\boxtimes\mathrm{ev})^*(\gamma_1 \boxtimes \gamma_2 \boxtimes 1 \boxtimes \gamma_3 \boxtimes \gamma_4 \boxtimes 1)\\
&= \sum_{\beta_1+\beta_2=\beta} \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times \overline{\mathcal{M}}_3(X,\beta_2)} (\mathrm{ev}\boxtimes\mathrm{ev})^*(\gamma_1 \boxtimes \gamma_2 \boxtimes 1 \boxtimes \gamma_3 \boxtimes \gamma_4 \boxtimes 1)(\mathrm{ev}_3\boxtimes\mathrm{ev}_3)^*(\Delta_X)\\
&= \sum_{\beta_1+\beta_2=\beta} \sum_w \int_{\overline{\mathcal{M}}_3(X,\beta_1)\times \overline{\mathcal{M}}_3(X,\beta_2)} (\mathrm{ev}\boxtimes\mathrm{ev})^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \sigma_w \boxtimes \gamma_3 \boxtimes \gamma_4 \boxtimes \sigma^w)\\
&= \sum_{\beta_1+\beta_2=\beta} \sum_w \langle \gamma_1, \gamma_2, \sigma_w\rangle_{\beta_1} \langle \gamma_3, \gamma_4, \sigma^w\rangle_{\beta_2},
\end{aligned}
$$

where $\{\sigma_w\} \subset H^*(X)$ is a basis and $\{\sigma^w\}$ is its dual basis under Poincaré duality. Note that

$$
\Delta_X = \sum_w \sigma_w \otimes \sigma^w \in= H^*(X) \otimes H^*(X) = H^*(X \times X).
$$

As a result, we have

$$\begin{aligned}
&\sum_{\beta\in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_4(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3 \boxtimes \gamma_4)\, \mathrm{ft}^*([\mathrm{pt}]) \\
&= \sum_{\beta_1+\beta_2} \sum_{w} q^{\beta_1} \langle \gamma_1, \gamma_2, \sigma_w\rangle_{\beta_1} q^{\beta_2} \langle \gamma_3, \gamma_4, \sigma^w\rangle_{\beta_2} \\
&= \sum_{w} \langle \gamma_1 * \gamma_2, \sigma_w\rangle \langle \gamma_3 * \gamma_4, \sigma^w\rangle \\
&= \langle \gamma_1 * \gamma_2, \gamma_3 * \gamma_4\rangle = \langle (\gamma_1 * \gamma_2) * \gamma_3, \gamma_4\rangle.
\end{aligned}$$

Note that this is invariant under any permutation of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. In particular, we have associativity

$$(\gamma_1 * \gamma_2) * \gamma_3 = (\gamma_2 * \gamma_3) * \gamma_1 = \gamma_1 * (\gamma_2 * \gamma_3).$$

1.13. Remark. When $\gamma_i = [Z_i]$ for subvariety $Z_i \subset X$. This also tells

$$\langle \gamma_1 * \gamma_2, \gamma_3 * \gamma_4\rangle = \sum_{\beta} q^\beta \# \left\{ \mathbb{P}^1 \xrightarrow{f} X : f_*[\mathbb{P}^1] = \beta, f(c_i) \in Z_i \right\}$$

for any given four points $c_1, \dots, c_4 \in \mathbb{P}^1$.

1.14. Identity. Let $\beta > 0$. Let us consider

$$\mathrm{ft}_3 : \overline{\mathcal{M}}_3(X, \beta) \longrightarrow \overline{\mathcal{M}}_2(X, \beta).$$

Then

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes 1) \\
&= \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ft}_3^*(\mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2)) \\
&= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2)\, \mathrm{ft}_{3*}(1) = 0
\end{aligned}$$

Here $\mathrm{ft}_{3*}(1) = 0$ by degree reason. When $\beta = 0$,

$$\int_{\overline{\mathcal{M}}_3(X,0)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes 1) = \int_X \gamma_1\gamma_2 = \langle \gamma_1, \gamma_2\rangle.$$

This proves

$$\langle \gamma_1 * 1, \gamma_2\rangle = \langle \gamma_1, \gamma_2\rangle.$$

So $1 \in H^*(X) \subset QH^*(X)$ is the identity

$$\gamma_1 * 1 = \gamma_1.$$

2. Properties and Examples

Divisor equation.

2.1. Divisor. Let λ be a divisor. When $\beta > 0$,

$$\begin{aligned}\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \lambda) &= \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ft}_3^*(\mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2))\, \mathrm{ev}_3^*(\lambda) \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2)\, \mathrm{ft}_{3*}(\mathrm{ev}_3^*(\lambda)).\end{aligned}$$

By degree reason, $\mathrm{ft}_{3*}(\mathrm{ev}_3^*(\lambda))$ is a number. So it equals to the intersecting number of the generic fibre and $\mathrm{ev}_3 *(\lambda)$. For a generic stable map $(f, \mathbb{P}^1, p_1, p_2)$, the fibre along ft_3 is $\mathbb{P}^1$ itself, and ev_3 is identified with f. So the intersecting number is $\langle \beta, \lambda \rangle$. We conclude that

$$\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \lambda) = \langle \lambda, \beta \rangle \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2).$$

In other word,

$$\langle \gamma_1 * \lambda, \gamma_2 \rangle = \langle \gamma_1 \lambda, \gamma_2 \rangle + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} \langle \lambda, \beta \rangle q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2).$$

Remark. This can be understood as follows. Assume $\lambda = [D]$ for a codimension 1 subvariety $D \subset X$.

$$\langle \gamma_1, \gamma_2, \lambda \rangle_\beta = \# \left\{ \mathbb{P}^1 \xrightarrow{f} X : \begin{matrix} f_*[\mathbb{P}^1] = \beta, & f(0) \in Z_1, \\ f(1) \in D, & f(\infty) \in Z_2 \end{matrix} \right\}.$$

Note D intersects any $\mathbb{P}^1 \to X$ by $\langle \beta, \lambda \rangle$ points. Thus

$$\langle \gamma_1, \gamma_2, \lambda \rangle_\beta = \langle \beta, \lambda \rangle \# \left\{ \mathbb{P}^1 \xrightarrow{f} X : \begin{matrix} f_*[\mathbb{P}^1] = \beta, \\ f(0) \in Z_1, \quad f(\infty) \in Z_2 \end{matrix} \right\} / \mathbb{C}^\times.$$

Note that now

$$\text{reparametrization} = \mathrm{Aut}(\mathbb{P}^1, 0, \infty) = \mathbb{C}^\times.$$

Product.

2.2. Product. Let X and Y be two varieties. We have a birational

$$\overline{\mathcal{M}}_3(X \times Y, (\beta, \beta')) \longrightarrow \overline{\mathcal{M}}_3(X, \beta) \times \overline{\mathcal{M}}_3(Y, \beta')$$

induced by two projections. Note that, this is birational only for $n = 3$ in which case $\overline{\mathcal{M}}_3(X)$ is a compactification of $\mathrm{Mor}(\mathbb{P}^1, X)$. We can conclude

$$QH^*(X \times Y) \longrightarrow QH^*(X) \otimes QH^*(Y)$$

is an algebra isomorphism.

2.3. Corollary. When $\beta_1, \beta_2 > 0$

$$\int_{\overline{\mathcal{M}}_2(X\times Y,(\beta_1,\beta_2))} \mathrm{ev}^*\left((\gamma_1\otimes\gamma_1')\boxtimes(\gamma_2\otimes\gamma_2')\right) = 0.$$

This can be proved using divisor equation. For any ample divisor $\lambda \in H^2(X)$,

$$\begin{aligned}
&\langle(\gamma_1\otimes\gamma_1') * (\lambda\otimes 1), \gamma_2\otimes\gamma_2'\rangle \\
&= \langle\gamma_1\lambda, \gamma_1'\rangle + \sum_{\beta_1,\beta_2}\langle\lambda,\beta_1\rangle q^{\beta_1}q^{\beta_2}\int_{\overline{\mathcal{M}}_2}\mathrm{ev}^*((\gamma_1\otimes\gamma_1')\boxtimes(\gamma_2\otimes\gamma_2')).
\end{aligned}$$

Note that $\langle\lambda, \beta_1\rangle > 0$. On the other hand,

$$\langle(\gamma_1\otimes\gamma_1') * (D\otimes 1), \gamma_2\otimes\gamma_2'\rangle = \langle\gamma_1 * \lambda, \gamma_2\rangle\langle\gamma_2, \gamma_2'\rangle$$

having no q^{β_2}-term.

2.4. Remark. Let us give a direct proof of this fact. When $\beta_1, \beta_2 > 0$, we have the following diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_2(X\times Y,(\beta,\beta')) \arrow[r, "(*)"] \arrow[d, "\mathrm{ev}"] & \overline{\mathcal{M}}_2(X,\beta)\times\overline{\mathcal{M}}_2(Y,\beta) \arrow[d, "\mathrm{ev}\boxtimes\mathrm{ev}"] \\
X\times Y\times X\times Y \arrow[r] & X\times X\times Y\times Y
\end{tikzcd}$$

Note that

$$\dim \text{left-hand side of } (*) - \dim \text{right-hand side of } (*) = 1.$$

By degree reason, the Gromov–Witten invariant vanishes.

Projective spaces.

2.5. Example. We have

$$\mathbb{P}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^\times.$$

We know

$$\begin{aligned}
H^2(\mathbb{P}^n) &= \mathbb{Z}\cdot H, \qquad H = [\text{a hyperplane}] = c_1(\mathcal{O}(1)) \\
H^2(\mathbb{P}^n) &= \mathbb{Z}\cdot\ell, \qquad \ell = [\text{a straight line}].
\end{aligned}$$

Recall that

$$H^*(\mathbb{P}^n) = \mathbb{Z}[H]/\langle H^n\rangle, \qquad \langle H^a, H^b\rangle = \delta_{a+b=n}.$$

Since the tangent bundle $\mathscr{T}_X$ can be put into the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}(1)^{N+1} \longrightarrow \mathscr{T}_X \longrightarrow 0,$$

we have $c_1(\mathscr{T}_X) = (n+1)H$. As a result, $q := q^\ell$ has degree $n+1$.

2.6. Approach A. Let us compute when $a+b=n+1$

$$H^a * H^b = (??)q.$$

That is,

$$\langle H^a * H^b, H^n \rangle = (??).$$

Note that H^k is represented by a codimension k-plane, and in particular, H^n is represented by a point. By the geometric meaning,

$$(??) = \# \left\{ \begin{array}{c} \text{straight lines going through a point } P \\ \text{a } (n-a)\text{-plane } A \text{ and a } (n-b)\text{-plane } B \end{array} \right\}$$

Note that the affine span of P and A intersects a unique point Q with B. Then PQ is the straight line going through P, A and B. So $(??) = 1$. Thus when $a+b=n+1$, we have

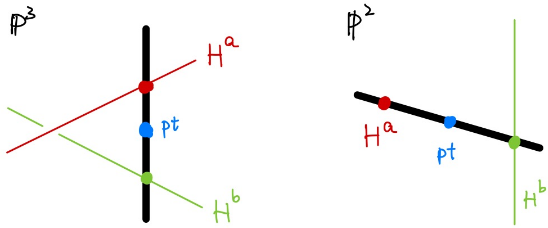
$$H^a * H^b = q.$$

By degree reason, we can conclude that, for $0 \le a, b \le n$,

$$H^a * H^b = \begin{cases} H^{a+b}, & a+b \le n \\ qH^{a+b-n-1}, & a+b > n. \end{cases}$$

So we have the following presentation of quantum cohomology

$$QH^*(\mathbb{P}^n) = \mathbb{Q}[H, q]/\langle H^{n+1} = q \rangle.$$



2.7. Approach B. There is anther approach of doing this. Let us compute

$$\underbrace{H * \cdots * H}_{n+1} = (??)q.$$

Recall that

$$\begin{aligned} \mathrm{Mor}_{\deg=1}(\mathbb{P}^1, \mathbb{P}^N) &= \{\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^N : f_*[\mathbb{P}^1] = \ell\} \\ &= \left\{ (s_0, \ldots, s_n) : \begin{array}{l} s_i \in H^0(\mathbb{P}^1, \mathcal{O}(1)) \\ s_0 \cdots s_n \text{ vanishes nowhere} \end{array} \right\} / \mathbb{C}^\times. \end{aligned}$$

Actually, for any $f : \mathbb{P}^1 \to \mathbb{P}^n$ of degree 1, the corresponding $(s_0, \ldots, s_n)$ is given by

$$s_i = f^*(x_i), \qquad \text{the } i\text{-th coordinate } x_i \in H^0(\mathbb{P}^1, \mathcal{O}(1)) = \mathbb{C}^{n+1}.$$

Conversely, f is defined by

$$f(x) = [s_0(x) : \cdots : s_n(x)] \in \mathbb{P}^n, \qquad x \in \mathbb{P}^1.$$

Let $H_i = \{x_i = 0\} \subset \mathbb{P}^n$ be the coordinate hyperplane. Let $c_0, \ldots, c_n \in \mathbb{P}^1$ be given points. Then

$$\left\{ \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^N : \begin{array}{l} f_*[\mathbb{P}^1] = \ell \\ f(c_i) \in H_i \end{array} \right\} = \left\{ (s_0, \ldots, s_n) : \begin{array}{l} s_i \in H^0(\mathbb{P}^1, \mathcal{O}(1)) \\ s_0 \cdots s_n \text{ vanishes nowhere} \\ s_i(c_i) = 0 \end{array} \right\} / \mathbb{C}^\times.$$

Note that

$$s_i(c_i) = 0 \iff s_i \in \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(c_i), \mathcal{O}(1)) \cong \mathbb{C}.$$

For a given generic $x \in \mathbb{P}^1$, we see that

$$\left\{ \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^n : \begin{array}{l} f_*[\mathbb{P}^1] = \ell \\ f(c_i) \in H_i \end{array} \right\} \xrightarrow{\mathrm{ev}_x} \mathbb{P}^N$$

is an isomorphism. Thus

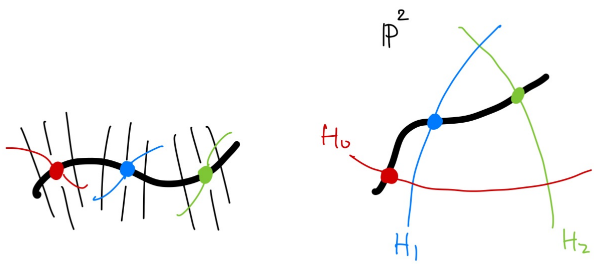
$$\# \left\{ \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^n : \begin{array}{l} f_*[\mathbb{P}^1] = \ell, \ f(c_i) \in H_i \\ f(x) = \text{a given point} \end{array} \right\} = 1.$$

This proves

$$\langle H * \cdots * H, [\mathrm{pt}] \rangle = q.$$

That is,

$$\underbrace{H * \cdots * H}_{n+1} = q.$$



Full flag variety in $\mathbb{C}^3$.

2.8. Example. Let us consider the full flag variety

$$X = \mathcal{F}\ell_2 = \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3\}.$$

We have a tautological flag bundle

$$0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{O}_X^3.$$

Let us denote

$$x_1 = -c_1(\mathcal{V}_1), \qquad x_2 = -c_1(\mathcal{V}_2/\mathcal{V}_1), \qquad x_3 = -c_1(\mathcal{O}_X^3/\mathcal{V}_2).$$

The usual cohomology is given by

$$H^*(\mathcal{F}\ell_2) = \mathbb{Z}[x_1, x_2, x_3] \Big/ \left\langle \begin{array}{l} x_1 + x_2 + x_3 = 0 \\ x_1x_2 + x_1x_3 + x_1x_2 = 0 \\ x_1x_2x_3 = 0 \end{array} \right\rangle.$$

We have the following dual basis

$$1 \leftrightarrow x_1^2x_2, \qquad x_1 \leftrightarrow x_1x_2, \qquad x_1 + x_2 \leftrightarrow x_1^2.$$

Let us consider

$$X_1 = \mathbb{P}^2 = \{0 \subset V_1 \subset \mathbb{C}^3\}, \qquad X_2 = (\mathbb{P}^2)^\vee = \{0 \subset V_2 \subset \mathbb{C}^3\}.$$

We have forgetful map $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$. Denote

$$\beta_1 = \text{fibre of } \pi_1, \quad q_1 = q^{\beta_1}, \qquad \beta_2 = \text{fibre of } \pi_2, \quad q_2 = q^{\beta_2}.$$

The intersection form is

$\langle\,,\rangle$	x_1	x_2	x_3
β_1	1	-1	0
β_2	0	1	-1

Since

$$c_1(\mathscr{T}_X) = (x_1 - x_2) + (x_2 - x_3) + (x_1 - x_3) = 2x_1 - 2x_3.$$

We have

$$\deg q_1 = \deg q_2 = 2.$$

By degree reason,

$$\lambda_1 * \lambda_2 = \lambda_1\lambda_2 + (\mathsf{a\ number})q_1 + (\mathsf{a\ number})q_2.$$
$$\lambda_1 * \lambda_2 * \lambda_3 = \lambda_1\lambda_2\lambda_3 + (\mathsf{a\ divisor})q_1 + (\mathsf{a\ divisor})q_2.$$

2.9. Relation A. We can get the quadratic relation as follows. For two divisors λ_1, λ_2, by using the divisor equation twice, we have

$$\begin{aligned}\langle \lambda_1 * \lambda_2, \gamma\rangle &= \langle \lambda_1\lambda_2, \gamma\rangle + \sum_{\beta\in \mathrm{Eff}(X)\setminus\{0\}} q^{\beta}\langle\lambda_1, \beta\rangle \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\lambda_1 \boxtimes \gamma) \\ &= \langle \lambda_1\lambda_2, \gamma\rangle + \sum_{\beta\in \mathrm{Eff}(X)\setminus\{0\}} q^{\beta}\langle\lambda_1, \beta\rangle\langle\lambda_2, \beta\rangle \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(\gamma).\end{aligned}$$

The key observation is, we can identify

$$\begin{tikzcd} \overline{\mathcal{M}}_1(X,\beta_1) \arrow[r, equal, "\mathrm{ev}"] \arrow[d, "\mathrm{ft}_1"'] & X \arrow[d, "\pi_1"] \\ \overline{\mathcal{M}}_0(X,\beta_1) \arrow[r, equal] & X_1 \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathcal{M}}_1(X,\beta_2) \arrow[r, equal, "\mathrm{ev}"] \arrow[d, "\mathrm{ft}_2"'] & X \arrow[d, "\pi_2"] \\ \overline{\mathcal{M}}_0(X,\beta_2) \arrow[r, equal] & X_2 \end{tikzcd}$$

By taking $\gamma = [\mathrm{pt}]$, we get

$$\lambda_1 * \lambda_2 = \lambda_1\lambda + \langle\lambda_1, \beta_1\rangle\langle\lambda_2, \beta_1\rangle q_1 + \langle\lambda_1, \beta_2\rangle\langle\lambda_2, \beta_2\rangle q_2.$$

We can now compute

$*$	x_1	x_2	x_3
x_1	$x_1^2 + q_1$	$x_1x_2 - q_1$	x_1x_3
x_2	$x_1x_2 - q_1$	$x_2^2 + q_1 + q_2$	$x_2x_3 - q_2$
x_3	x_1x_3	$x_2x_3 - q_2$	$x_3^2 + q_2$

So we can conclude that

$$x_1x_2 + x_2x_3 + x_3x_1 + q_1 + q_2 = 0.$$

2.10. Relation B. We further have

$$\overline{\mathcal{M}}_2(X,\beta_1) = X \times_{X_1} X, \qquad \overline{\mathcal{M}}_2(X,\beta_2) = X \times_{X_2} X.$$

We have

$$\begin{tikzcd}
\overline{\mathcal{M}}_2(X,\beta_1) \arrow[rrr, "\mathrm{ft}_2"] \arrow[dddd, "\mathrm{ft}_1"'] \arrow[dr, equal] \arrow[drr, bend left, "\mathrm{ev}_1"] \arrow[ddr, bend right, "\mathrm{ev}_2"'] & & & \overline{\mathcal{M}}_1(X,\beta_1) \arrow[dddd, "\mathrm{ft}"] \arrow[dl, equal, "\mathrm{ev}"'] \\
& X \times_{X_1} X \arrow[r] \arrow[d] & X \arrow[d, "\pi_1"] & \\
& X \arrow[r, "\pi_1"] \arrow[ddl, equal, "\mathrm{ev}"'] & X_1 \arrow[ddr, equal] & \\
& & & \\
\overline{\mathcal{M}}_1(X,\beta_1) \arrow[rrr, "\mathrm{ft}"] & & & \overline{\mathcal{M}}_0(X,\beta_1)
\end{tikzcd}$$

It is well-known that the composition

$$\begin{aligned}
&[H^*(X) \xrightarrow{\text{pull}} H^*(X \times_{X_i} X) \xrightarrow{\text{push}} H^{*-2}(X)] \\
&= [H^*(X) \xrightarrow{\text{push}} H^{*-2}(X_i) \xrightarrow{\text{pull}} H^*(X)] \\
&= \partial_i \text{ the BGG Demazure operator.}
\end{aligned}$$

The BGG Demazure operator acts as

$$\partial_1 f = \frac{f - f|_{x_1 \leftrightarrow x_2}}{x_1 - x_2}, \qquad \partial_2 f = \frac{f - f|_{x_2 \leftrightarrow x_3}}{x_2 - x_3}.$$

For a divisor λ, by divisor relation,

$$\lambda * \gamma = \lambda\gamma + q_1\langle \lambda, \beta_1 \rangle \partial_1(\gamma) + q_2 \langle \lambda, \beta_2 \rangle \partial_2(\gamma) + (\text{other quantum terms}).$$

But by degree reason, there will be no other quantum terms. As a result,

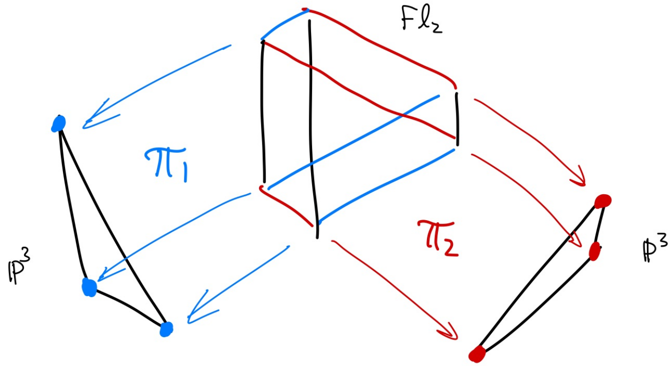
$$\begin{aligned}
x_1 * (x_2 * x_3) &= x_2 * (x_1 * x_3) = x_2 * (x_1 x_3) \\
&= x_1 x_2 x_3 + q_1 \langle x_2, \beta_1 \rangle \partial_1(x_1 x_3) + q_2 \langle x_2, \beta_2 \rangle \partial_2(x_1 x_3) \\
&= 0 - q_1 x_3 - q_2 x_1.
\end{aligned}$$

This proves

$$x_1 * x_2 * x_3 + q_1 x_3 + q_2 x_1 = 0.$$

In summary, the relations are given by the coefficients of characteristic polynomial of

$$\begin{bmatrix} x_1 & q_1 & \\ -1 & x_2 & q_2 \\ & -1 & x_3 \end{bmatrix}.$$



Grassmannian in $\mathbb{C}^4$.

2.11. Example. Let us consider

$$X = \mathrm{Gr}(2,4) = \{V \subset \mathbb{C}^4 : \dim V = 2\}.$$

We have a tautological exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_X^4 \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Let us denote

$$D = e_1 = h_1 = -c_1(\mathcal{V}) = c_1(\mathcal{Q}), \qquad e_2 = c_2(\mathcal{V}), \qquad h_2 = c_2(\mathcal{Q}).$$

The relation is

$$(1 - e_1 y + e_2 y^2)(1 + h_1 y + h_2 y^2) = 1 \quad \text{(as a polynomial in } y\text{)}.$$

We have $\mathscr{T}_X = \mathcal{H}om(\mathcal{V}, \mathcal{Q})$, so $c_1(\mathscr{T}_X) = nD$. Let ℓ be the primitive generator of $\mathrm{Eff}(X)$, we denote $q = q^\ell$. We have $\deg q = n$. Now let us consider

$$\begin{tikzcd}
\mathcal{F}\ell_4 = \{0 \subset V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4\} \arrow[r, equal] & \overline{\mathcal{M}}_1(X,\ell) \arrow[dl, "\mathrm{ev}"'] \arrow[dr, "\mathrm{ft}"] & \\
\mathrm{Gr}(2,4) & & \overline{\mathcal{M}}_0(X,\ell) = \{0 \subset V_1 \subset V_3 \subset \mathbb{C}^4\}
\end{tikzcd}$$

We can identify

$$Y = \overline{\mathcal{M}}_0(X,\ell), \qquad \mathcal{F}\ell_4 = \overline{\mathcal{M}}_1(X,\ell).$$

2.12. Relation. By degree reason, we have

$$e_2 * h_2 = e_2h_2 + (\text{a number})q.$$

Note that

$$\text{the number} = \int_{\overline{\mathcal{M}}_3(X,\ell)} \mathrm{ev}^*(e_2 \boxtimes h_2 \boxtimes [\mathrm{pt}]).$$

We can identify

$$\overline{\mathcal{M}}_3(X,\ell) = \mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4\,.$$

We have

$$\begin{aligned} H^*(\overline{\mathcal{M}}_3(X,\ell)) &= H^*(\mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4 \times_Y \mathcal{F}\ell_4) \\ &= H^*(\mathcal{F}\ell_4) \otimes_{H^*(Y)} H^*(\mathcal{F}\ell_4) \otimes_{H^*(Y)} H^*(\mathcal{F}\ell_4) \\ H^*(Y) &= \text{invariant algebra of } H^*(\mathcal{F}\ell_4) \text{ under } x_2 \leftrightarrow x_3. \end{aligned}$$

Let us denote

$$x_i = x_i \otimes 1 \otimes 1, \qquad y_i = 1 \otimes x_i \otimes 1, \qquad z_i = 1 \otimes 1 \otimes x_i.$$

Note that

$$x_1 = y_1 = z_1, \qquad x_4 = y_4 = z_4.$$

We can represent

$$e_2 = x_1x_2, \qquad h_2 = x_1^2 + x_1x_2 + x_2^2, \qquad [\mathrm{pt}] = x_1^2x_2^2.$$

As a result,

$$\mathrm{ev}^*(\cdots) = (x_1x_2)(x_1^2 + x_1y_2 + y_2^2)(x_1^2z_2^2).$$

The pushforward is given by

$$\partial_2^x\partial_2^y\partial_2^z, \qquad \partial_2^x = \frac{f - f|_{x_2\leftrightarrow x_3}}{x_2 - x_3}, \text{ etc.}$$

So

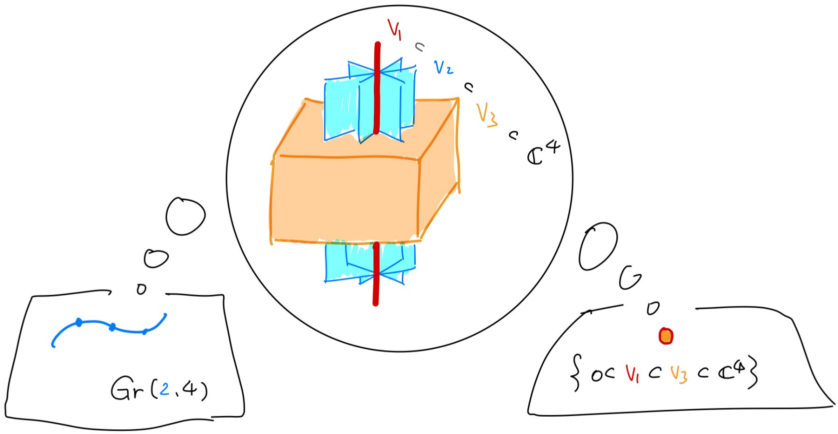
$$\begin{aligned} \mathrm{ft}_*(\mathrm{ev}^*(\cdots)) &= (x_1)(x_1 + y_2 + y_3)(x_1^2(z_2 + z_3)) \\ &= (x_1)(x_1 + x_2 + x_3)(x_1^2(x_2 + x_3)) = [\mathrm{pt}]. \end{aligned}$$

As a result,

$$e_2 * h_2 = q.$$

So the relation is

$$(1 - e_1y + e_2y^2)(1 + h_1y + h_2y^2) = 1 + q.$$


V1 ⊂ V2 ⊂ V3 ⊂ ℂ4
Gr(2,4)
{0 ⊂ V1 ⊂ V3 ⊂ ℂ4}

3. Fundamental solutions

The purpose of this section is to establish the theory of fundamental solution of quantum differential equations.

Psi class.

3.1. Universal curve. We could view the forgetful morphism

$$\mathrm{ft}_{n+1} : \overline{\mathcal{M}}_{n+1}(X,\beta) \longrightarrow \overline{\mathcal{M}}_n(X,\beta)$$

the universal curve. That is, the fibre of a stable map $(f, C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_n(X,\beta)$ is C itself. We also have universal sections σ_i $(1 \leq i \leq n)$

$$\sigma_i : \overline{\mathcal{M}}_n(X,\beta) \longrightarrow \overline{\mathcal{M}}_{n+1}(X,\beta)$$

by attaching a

$$\mathbb{P}^1 \ni p_{n+1}, \ (\text{new } p_i), \ (\text{attaching point})$$

on the i-th marked point.

3.2. Universal cotangent line. We define the **universal cotangent line** to be

$$\mathbb{L}_i = \sigma_i^*(\text{relative dualizing sheaf of } \mathrm{ft}_{n+1})$$

a line bundle over $\overline{\mathcal{M}}_n(X,\beta)$. In particular, at each point $(f, C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_n(X,\beta)$, the fibre of $\mathbb{L}_i$ is the cotangent line at $p_i \in C$. The **psi-class** is defined to be

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_n(X,\beta), \mathbb{Q}).$$

3.3. Local computation. The following computation is very important in the computation of psi-classes. Consider the family of curves with 1 marked point

$$(1,h) \in C_h = \{(x,y) : xy = h\} \subset \mathbb{C}^2, \qquad h \in \mathbb{C}.$$

Then we have

$$\begin{aligned} \upsilon &: \mathbb{C}^2 \longrightarrow \mathbb{C}, & (x,y) &\longmapsto xy; & &(\text{universal family}) \\ \sigma &: \mathbb{C} \longrightarrow \mathbb{C}^2, & h &\longmapsto (1,h). & &(\text{universal section}) \end{aligned}$$

We denote $\mathbb{L}$ the universal cotangent line. Note that the 2-nd projection defines a morphism $\mathbb{L}^* \longrightarrow \mathscr{T}_{\mathbb{C}}$, i.e.

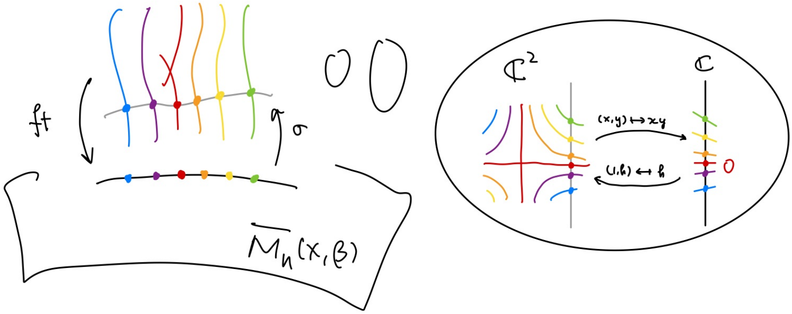
$$\text{tangent line of } C_h \text{ at } (1,h) \xrightarrow{\mathrm{pr}_2} \text{tangent line of } \mathbb{C} \text{ at } h.$$

Note that this morphism has a zero at $h = 0$. So we have

$$\mathbb{L} \otimes \mathscr{T}_{\mathbb{C}} \simeq \mathcal{O}(\{0\}), \qquad \text{i.e.} \qquad \mathbb{L} \simeq \Omega_{\mathbb{C}}(\{0\}).$$

The principle is

$$\boxed{\psi_i - \mathrm{ft}^* \psi_i = \begin{bmatrix} \text{locus of curves collapsing on} \\ \text{the branch of the } i\text{-th marked point} \end{bmatrix}}$$



3.4. Example. Let $C = (f, \mathbb{P}^1, p_1, \ldots, p_n)$ be a generic stable map on $\overline{\mathcal{M}}_n(X, \beta)$. We know $\mathbb{P}^1 \simeq \mathrm{ft}_{n+1}^{-1}(C)$. Let us compute the restriction of $\mathbb{L}_i$ to $\mathbb{P}^1$. The first guess is

$$\mathbb{L}_i|_{\mathbb{P}^1} \quad \text{``="} \quad \Omega_{\mathbb{P}^1} = \mathcal{O}(-2).$$

But this is not true. At the point $p_i \in \mathbb{P}^1$, the corresponding curve is $\sigma_i(C) \in \mathrm{ft}_{n+1}^{-1}(C)$, whose i-th marked point is not p_i. From the local computation above, we actually have

$$\mathbb{L}_i|_{\mathbb{P}^1} = \Omega_{\mathbb{P}^1}(p_1 + \cdots + p_n) = \mathcal{O}(n-2).$$

3.5. Example. Recall the forgetful map

$$\mathrm{ft}_{n+1} : \overline{\mathcal{M}}_{n+1}(X, \beta) \longrightarrow \overline{\mathcal{M}}_n(X, \beta).$$

We shall compare psi classes for different number of marked points. The first guess is

$$\mathrm{ft}_{n+1}^* \psi_i \quad \text{``="} \quad \psi_i.$$

But this is not true. When forgetting the $(n+1)$-th marked point, we might need collapsion to get a stable map. The local computation shows

$$\psi_i - \mathrm{ft}_{n+1}^* \psi_i = \left[\text{image of } \sigma_i : \overline{\mathcal{M}}_n(X, \beta) \to \overline{\mathcal{M}}_{n+1}(X, \beta)\right]$$

3.6. Example. Consider the forgetful map

$$\mathrm{ft}_X : \overline{\mathcal{M}}_3(X, \beta) \longrightarrow \overline{\mathcal{M}}_3.$$

We shall compare psi classes between them. The first guess is

$$\mathrm{ft}_X^* \psi_i \quad \text{``="} \quad \psi_i = 0.$$

But this is not true. When forgetting the underlying space X, we might need collapsion to get a stable map. The local computation shows

$$\psi_3 = \psi_3 - \mathrm{ft}_X^* \psi_3 = \sum_{\beta = \beta_1 + \beta_2} \left[\overline{\mathcal{M}}_3(X, \beta_1) \underset{\Delta_X}{\times} \overline{\mathcal{M}}_2(X, \beta_2)\right].$$

Here

$$\begin{tikzcd}
\overline{\mathcal{M}}_3(X,\beta_1) \underset{\Delta_X}{\times} \overline{\mathcal{M}}_2(X,\beta_2) \arrow[r] \arrow[d] \arrow[rr, hook, bend left, "\text{glue}"] & \overline{\mathcal{M}}_3(X,\beta_1) \arrow[d, "\mathrm{ev}_3"] & \overline{\mathcal{M}}_3(X,\beta_1+\beta_2) \\
\overline{\mathcal{M}}_2(X,\beta_2) \arrow[r, "\mathrm{ev}_2"'] \arrow[ur, phantom, "\text{fibre product}"] & X &
\end{tikzcd}$$

Fundamental solution.

3.7. GW invariant twisted by psi class. For $\gamma_1,\gamma_2,\gamma_3 \in H^*(X)$, let us consider a gravitational correlator

$$\langle \gamma_1,\gamma_2,\tau_a\gamma_3\rangle_\beta := \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\gamma_3)\psi_3^a.$$

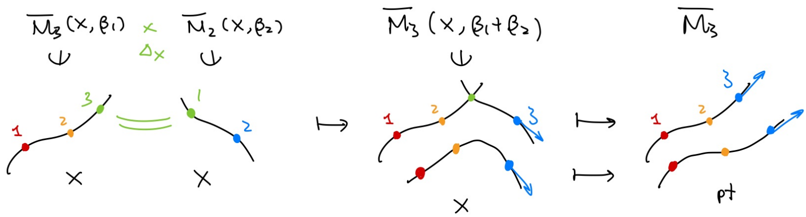
Let us pick a basis $\{\sigma_w\}\subset H^*(X)$ with dual basis $\{\sigma^w\}$.

3.8. Appraoch A. Let us apply Example 3.6. When $a\geq 1$,

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\gamma_3)\psi_3^a \\
&= \sum_{\beta=\beta_1+\beta_2}\int_{\overline{\mathcal{M}}_3(X,\beta)} [\overline{\mathcal{M}}_3(X,\beta_1)\times_{\Delta_X}\overline{\mathcal{M}}_2(X,\beta_2)]\cdot \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\gamma_3)\psi_3^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2}\int_{\overline{\mathcal{M}}_3(X,\beta_1)\times_{\Delta_X}\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\gamma_3)(1\boxtimes\psi_2)^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2}\int_{\overline{\mathcal{M}}_3(X,\beta_1)\times\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\Delta_X\boxtimes\gamma_3)(1\boxtimes\psi_2)^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2}\sum_w\int_{\overline{\mathcal{M}}_3(X)\times\overline{\mathcal{M}}_2(X)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\sigma_w\boxtimes\sigma^w\boxtimes\gamma_3)\psi_3^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2}\sum_w\int_{\overline{\mathcal{M}}_3(X,\beta_1)} \mathrm{ev}^*(\gamma_1\boxtimes\gamma_2\boxtimes\sigma_w)\int_{\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\sigma^w\boxtimes\gamma_3)\psi_2^{a-1} \\
&= \sum_{\beta=\beta_1+\beta_2}\sum_w\langle\gamma_1,\gamma_2,\sigma_w\rangle_{\beta_1}\int_{\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\sigma^w\boxtimes\gamma_3)\psi_2^{a-1}
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\beta\in \mathrm{Eff}(X)} q^{\beta} \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3)\psi_3^a \\
&= \sum_{w} q^{\beta_1}\langle \gamma_1,\gamma_2,\sigma_w\rangle_{\beta_1} \sum_{\beta_2} q^{\beta_2} \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \mathrm{ev}^*(\sigma^w \boxtimes \gamma_3)\psi_2^{a-1} \\
&= \sum_{\beta} q^{\beta} \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 * \gamma_2 \boxtimes \gamma_3)\psi_2^{a-1}
\end{aligned}$$



3.9. Approach B.

Let us apply Example 3.5. Let us denote

$$D = \left[\text{image of } \sigma_2 : \overline{\mathcal{M}}_2(X,\beta) \to \overline{\mathcal{M}}_3(X,\beta)\right].$$

Note that $\sigma_2^*\mathbb{L}_2$ is trivial, i.e. $D\cdot\psi_2 = 0$. When $a \geq 1$,

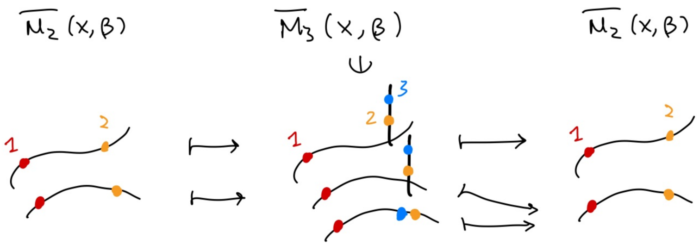
$$\psi_2^a = (\mathrm{ft}_3^*\,\psi_2 + D)\psi_2^{a-1} = \mathrm{ft}_3^*\,\psi_2\cdot\psi_2^{a-1} = \cdots = \mathrm{ft}_3^*\,\psi_2^a + D\cdot \mathrm{ft}_3^*\,\psi_2^{a-1}.$$

Let us assume $\gamma_2 = \lambda$ is a divisor. When $\beta > 0$,

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2 \boxtimes \gamma_3)\psi_3^a \\
&= \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \boxtimes \lambda)\psi_2^a \\
&= \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \boxtimes \lambda)\big(\mathrm{ft}_3^*\,\psi_2^a + D\cdot \mathrm{ft}_3^*\,\psi_2^{a-1}\big) \\
&= \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \boxtimes \lambda)\,\mathrm{ft}_3^*\,\psi_2^a + \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \boxtimes \lambda)D\cdot \mathrm{ft}_3^*\,\psi_2^{a-1} \\
&= \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ft}_3^*\left(\mathrm{ev}(\gamma_1 \boxtimes \gamma_3)\psi_2^a\right)\mathrm{ev}_3^*(\lambda) + \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3)\sigma_2^*(\mathrm{ev}_3^*\,\lambda)\psi_2^{a-1} \\
&= \langle\lambda,\beta\rangle \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3)\psi_2^a + \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3\cdot\lambda)\psi_2^{a-1} \\
&= \int_{\overline{\mathcal{M}}_2(X,\beta)} \langle\lambda,\beta\rangle\,\mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3)\psi_2^a + \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3\cdot\lambda)\psi_2^{a-1}
\end{aligned}$$

Here we use the facts

$$\mathrm{ft}_{3*}\,\mathrm{ev}_3(\lambda) = \langle \lambda, \beta \rangle, \qquad \mathrm{ev}_3 \circ \sigma_3 = \mathrm{ev}_2, \qquad \mathrm{ft}_3\, \sigma_2 = \mathrm{id}\,.$$



3.10. Summary. By equalizing the results by two approaches, we get ($a \geq 1$)

$$\begin{aligned}
&\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \lambda \boxtimes \gamma_3)\psi_3^a \\
&= \sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 * \lambda \boxtimes \gamma_3)\psi_2^{a-1} \\
&= \sum_{\beta \in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \langle \lambda, \beta \rangle\, \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3)\psi_2^a + \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \cdot \lambda)\psi_2^{a-1}.
\end{aligned}$$

When $\beta = 0$, $\overline{\mathcal{M}}_2(X, \beta) = \varnothing$, so the integral is understood as 0. Recall

$$\begin{aligned}
&\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \lambda \boxtimes \gamma_3) \\
&= \langle \gamma_1, \lambda \cdot \gamma_3 \rangle + \sum_{\beta \in \mathrm{Eff}(X)\setminus 0} q^\beta \langle \lambda, \beta \rangle \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3) \\
&= \langle \gamma_1 * \lambda, \gamma_3 \rangle
\end{aligned}$$

For any polynomial (or a power series) $T(\psi)$, we denote $T^{\downarrow}(\psi) = \frac{T(\psi)-T(0)}{\psi}$. We have

$$\begin{aligned}
&\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_3(X,\beta)} \mathrm{ev}^*(\gamma_1 \boxtimes \lambda \boxtimes \gamma_3) T(\psi_3) \\
&= \langle \gamma_1 * \lambda, \gamma_3 \rangle T(0) + \sum_{\beta \in \mathrm{Eff}(X)} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma_1 * \lambda \boxtimes \gamma_3) T^{\downarrow}(\psi_2) \\
&= \langle \gamma_1, \lambda \cdot \gamma_3 \rangle T(0) + \sum_{\beta \in \mathrm{Eff}(X)\setminus 0} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \langle \lambda, \beta \rangle\, \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3) T(\psi_2) + \mathrm{ev}^*(\gamma_1 \boxtimes \gamma_3 \cdot \lambda) T
\end{aligned}$$

3.11. Notations. Let us introduce more notations

- Let us take a formal variable z. Now let us consider
$$T(\psi) = \frac{1}{z-\psi} = \frac{1/z}{1-\psi/z} = \frac{1}{z} + \frac{\psi}{z^2} + \frac{\psi^2}{z^3} + \cdots .$$
Then
$$T^{\downarrow}(\psi) = \frac{1}{\psi}\left(\frac{1}{z-\psi} - \frac{1}{z}\right) = \frac{1}{z(z-\psi)} = \frac{1}{z}T(\psi).$$
- For any divisor λ denote ∂_λ the differential operator on $QH^*(X)$ with
$$\partial_\lambda q^\beta = \langle \lambda, \beta \rangle q^\beta.$$
Here, a differential operator is an $H^*(X)$-linear operators with Leibniz rule.
- Let us denote $p \ln q$ the unique function with
$$\partial_\lambda (p \ln q) = \lambda.$$
It can be constructed by $p \ln q = \sum p_i \ln q^{\beta_i}$ for $\{\beta_i\} \subset \mathrm{Eff}(X) \subset H_2(X)$ a basis with $\{p_i\} \subset H^2(X)$ its dual basis. In particular,
$$\partial_\lambda (e^{p \ln q/z}) = \frac{1}{z} e^{p \ln q/z}\lambda.$$

3.12. Fundamental solution. Let us denote a functional S as follows. For $\gamma, \gamma' \in H^*(X)$,
$$S(\gamma, \gamma') = \langle \gamma, e^{p \ln q/z}\gamma' \rangle + \sum_{\beta \in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes e^{p \ln q/z}\gamma') \frac{1}{z-\psi_2}.$$
Then we can write down the equation
$$\frac{1}{z} S(\gamma * \lambda, \gamma') = \partial_\lambda S(\gamma, \gamma').$$
In particular, let us denote an operator S such that
$$\langle \gamma, S(\gamma') \rangle = S(\gamma, \gamma') \qquad \text{i.e.} \quad S(\gamma') = \sum_w \sigma_w \cdot S(\sigma^w, \gamma').$$
In particular,
$$S(\gamma * \lambda, \gamma') = \langle \gamma * \lambda, S(\gamma') \rangle = \langle \gamma, \lambda * S(\gamma') \rangle$$
$$\partial_\lambda S(\gamma, \gamma') = \partial_\lambda \langle \gamma, S(\gamma') \rangle = \langle \gamma, \partial_\lambda S(\gamma') \rangle.$$
Thus for any $\gamma' \in H^*(X)$, we have
$$\partial_\lambda S(\gamma') - \frac{1}{z}\lambda * S(\gamma') = 0.$$
In particular, $S(\gamma')$ solves the quantum differential equation (discussed later). We call the operator S the **fundamental solution**.

3.13. Remark. Since

$$\lim_{z\to\infty} S(\gamma') = \gamma'$$

the operator S is nondegenerate.

J-function.

3.14. J-function. Let us define J to be the unique class such that

$$\langle J, \gamma' \rangle = S(1, \gamma') = \langle 1, S(\gamma') \rangle, \qquad \text{i.e.} \qquad J = \sum_w \sigma_w \cdot S(1, \sigma^w).$$

If we think S as a matrix, then each column of S is a solution of quantum differential equation. The J-function is by definition the row of S corresponding to $1 \in H^*(X)$.

3.15. Simplification. By definition

$$\begin{aligned} J &= \sum_w \sigma_w \cdot S(1, \sigma^w) \\ &= \sum_w \sigma_w \left(\langle 1, e^{p \ln q/z} \sigma^w \rangle + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(1 \boxtimes e^{p \ln q/z} \sigma^w) \frac{1}{z - \psi_2} \right). \end{aligned}$$

More general, for $\beta > 0$, let us denote

$$D = [\text{image of } s_1 : \overline{\mathcal{M}}_1(X, \beta) \to \overline{\mathcal{M}}_2(X, \beta)].$$

Similar as what we did in Approach B 3.9, we have

$$\begin{aligned} &\int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(1 \boxtimes \gamma) \psi_2^a \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes 1) \psi_1^a \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes 1)(\mathrm{ft}_2^* \psi_1^a + D \cdot \mathrm{ft}_2^* \psi_1^{a-1}) \\ &= \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ft}_2^*(\mathrm{ev}^*(\gamma) \psi_1^a) + \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(\gamma) \psi_1^{a-1} \\ &= 0 + \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(\gamma) \psi_1^{a-1}. \end{aligned}$$

Let us denote $\psi = \psi_1 \in H^2(\overline{\mathcal{M}}_1(X,\beta))$. So

$$\begin{aligned}
J &= \sum_w \sigma_w \langle 1, e^{p \ln q/z} \sigma^w \rangle + \sum_{\beta \in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \sum_w \sigma_w \int_{\overline{\mathcal{M}}_1(X,\beta)} \mathrm{ev}^*(e^{p \ln q} \sigma^w) \frac{1}{z(z-\psi)} \\
&= e^{p \ln q/z} + e^{p \ln q/z} \sum_{\beta \in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \, \mathrm{ev}_* \frac{1}{z(z-\psi)} \\
&= e^{p \ln q/z} \left(1 + \sum_{\beta \in \mathrm{Eff}(X)\setminus\{0\}} q^\beta \, \mathrm{ev}_* \frac{1}{z(z-\psi)} \right).
\end{aligned}$$

Relations. Let $D = f(z\partial_\lambda, q)$ be a differential operator with f a noncommutative polynomial. If

$$DJ = 0$$

then $\lim_{z\to 0} f(\lambda, q) = 0$ in $QH^*(X)$.

Proof. Note that

$$z\partial_\lambda S(\gamma') = \lambda * S(\gamma').$$

When f takes form of

$$\sum (\text{a function in } q) \cdot (\text{differential operators}),$$

we have

$$DS(\gamma') = f(\lambda *, q) S(\gamma').$$

Thus

$$\begin{aligned}
0 = \langle DJ, \gamma' \rangle &= D\langle J, S(\gamma') \rangle = D\langle 1, S(\gamma') \rangle \\
&= \langle 1, DS(\gamma') \rangle = \langle 1, f(\lambda *, q) S(\gamma') \rangle = \langle f(\lambda *, q), S(\gamma') \rangle.
\end{aligned}$$

Since $S(\gamma')$ is non-degenerate, $f(\lambda, q) = 0$ in $QH^*(X)$.

The general case follows from the fact that

$$[z\partial_\lambda, \text{multiplcation by } q^\beta] = z \cdot \text{multiplication by } \partial_\lambda q^\beta,$$

which is killed by $\lim_{z\to 0}$. □

4. Quasi maps

Normal bundle in terms of Psi class.

4.1. Local computation. Recall the family of curves

$$C_h = \{(x, y) : xy = h\} \subset \mathbb{C}^2, \qquad h \in \mathbb{C}.$$

The ideal for $C_0 = (x\text{-axis}) \cup (y\text{-axis})$ is

$$\mathfrak{m} = \langle xy \rangle \subset R := \mathbb{C}[x, y].$$

So the normal bundle of C_0 is

$$\mathfrak{m}/\mathfrak{m}^2 = xyR/\mathfrak{m} = \mathcal{O}_{C_0}(x) \otimes \mathcal{O}_{C_0}(y).$$

Thus we can naturally identify the normal bundle of the singleton $C_0 \in \{C_h\}$ with

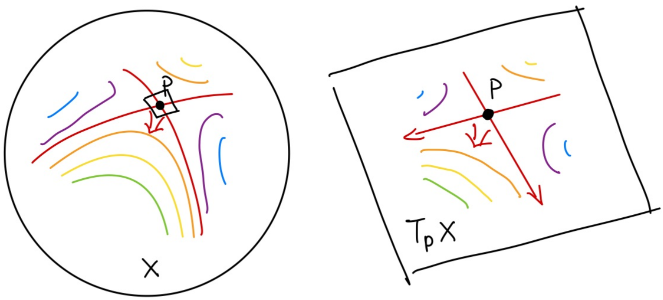
$$(\text{tangent line of } 0 \text{ along } x\text{-axis}) \otimes (\text{tangent line of } 0 \text{ along } y\text{-axis}).$$

Say, by the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[d] & \times & \mathbb{C} \arrow[d] \arrow[r] & \mathbb{C}^2 \arrow[d] & & \\
\{x\text{-axis}\} & \times & \{y\text{-axis}\} \arrow[r] & \{C_h\} & \simeq & \mathbb{C}.
\end{tikzcd}$$

The principle is

$$\boxed{\text{smoothing of the nodal point} = \text{tensor product of two tangent directions}}$$



4.2. Example. Let us consider the morphism

$$\overline{\mathcal{M}}_{n+1}(X, \beta_1) \times_X \overline{\mathcal{M}}_{m+1}(X, \beta_2) \longrightarrow \overline{\mathcal{M}}_{m+n}(X, \beta_1 + \beta_2)$$

by gluing the first marked points. Then the normal bundle of this morphism is the restriction of $(\mathbb{L}_1 \boxtimes \mathbb{L}_1)^*$.

4.3. Example. Let us consider the morphism

$$\overline{\mathcal{M}}_{n+1}(X,\beta_1)\times\overline{\mathcal{M}}_{m+1}(Y,\beta_2)\longrightarrow\overline{\mathcal{M}}_{m+n}(X\times Y,(\beta_1,\beta_2))$$

by gluing the first marked points. Then the normal bundle is the restriction of $(\mathbb{L}_1\boxtimes\mathbb{L}_1)^*$.

4.4. Example. Let us consider

$$\overline{\mathcal{M}}_n(X,\beta)\times\mathbb{P}^1\longrightarrow\overline{\mathcal{M}}_{n-1}(X\times\mathbb{P}^1,(\beta,1))$$

by sending (C,x) to the curve obtained by first putting C vertically at the point $x\in\mathbb{P}^1$ and then gluing a $\mathbb{P}^1$ horizontally at the first marked point. Then the normal bundle is $\mathbb{L}_1^*\boxtimes\mathscr{T}_{\mathbb{P}^1}$.

Quasi-maps.

4.5. Remark. Let $\mathcal{L}$ and $\mathcal{V}$ be two vector bundles. For a sheaf morphism $s:\mathcal{L}\to\mathcal{V}$, we have (by Nakayama lemma)

$$s\text{ is surjective}\iff s\text{ is fibrewise surjective.}$$

While we only have

$$s\text{ is injective}\Leftarrow s\text{ is fibrewise injective.}$$

Actually, when $\mathcal{L}$ is a line bundle,

$$s\text{ is injective}\iff s\text{ is nonzero (on each connected component).}$$

4.6. Quasi maps for projective space. Recall that

$$\mathrm{Mor}(C,\mathbb{P}^N)=\bigcup_{\mathcal{L}\in\mathrm{Pic}(C)}\mathrm{Surj}(\mathcal{O}_C^{N+1}\to\mathcal{L})/\mathbb{C}^*.$$

By taking dual,

$$\mathrm{Surj}(\mathcal{O}_C^{N+1}\to\mathcal{L})/\mathbb{C}^*\hookrightarrow\mathrm{Inj}(\mathcal{L}^\vee\to\mathcal{O}_C^{N+1})/\mathbb{C}^*=\mathbb{P}(H^0(C,\mathcal{L})^{N+1}).$$

We define the quasi-map space by

$$\mathrm{QM}(C,\mathbb{P}^N)=\bigcup_{\mathcal{L}}\mathbb{P}(H^0(C,\mathcal{L})^{N+1}).$$

When $C=\mathbb{P}^1$, we define

$$\mathrm{QM}(\mathbb{P}^N)=\bigcup_{d\geq0}\mathrm{QM}(\mathbb{P}^N,d)=\bigcup_{d\geq0}\mathbb{P}(\mathbb{C}[x]^{N+1}_{\deg\leq d}).$$

It is a compactification of the space of $\mathbb{P}^1\to\mathbb{P}^N$ of degree d.

4.7. Quasi maps for general X. Assume we can embed

$$X \longrightarrow \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$$

using primitive nef divisors $D_1, \ldots, D_m$. For $\beta \in \mathrm{Eff}(X)$, denote

$$\beta_1 = \langle D_1, \beta\rangle, \ldots, \beta_m = \langle D_m, \beta\rangle.$$

We can view

$$\begin{aligned} \mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, X) &\subset \mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, \mathbb{P}^{N_1} \times \cdots \mathbb{P}^{N_m}) \\ &= \mathrm{Mor}_{\deg=\beta_1}(\mathbb{P}^1, \mathbb{P}^{N_1}) \times \cdots \times \mathrm{Mor}_{\deg=\beta_m}(\mathbb{P}^1, \mathbb{P}^{N_m}) \\ &\subset \mathrm{QM}(\mathbb{P}^{N_1}, \beta_1) \times \cdots \times \mathrm{QM}(\mathbb{P}^{N_m}, \beta_m). \end{aligned}$$

We define

$$\mathrm{QM}(X, \beta) = \text{closure of } \mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, X) \text{ in } \mathrm{QM}(\mathbb{P}^{N_1}, \beta_1) \times \cdots \times \mathrm{QM}(\mathbb{P}^{N_m}, \beta_m)$$

and $\mathrm{QM}(X) = \bigcup_{\beta \in \mathrm{Eff}(X)} \mathrm{QM}(X, \beta)$.

4.8. Remark. We can think as follows. For sections $s_0, \ldots, s_N \in H^0(C, \mathcal{L})$, we define a rational map

$$C \longrightarrow \mathbb{P}^N, \qquad x \mapsto [s_0(x) : \cdots : s_N(x)].$$

This defines a morphism when $s_0, \ldots, s_N$ has no common zeros. In general, the closure of C defines a morphism $C \to \mathbb{P}^N$ but with class

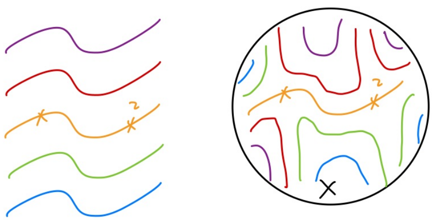
$$\mathcal{L}(-\text{common zeros}).$$

We call those common zeros by **marked points** (with multiplicity). So we have

$$\mathrm{QM}(\mathbb{P}^n, d) = \bigsqcup_{0 \le d' \le d} \mathrm{Mor}_{\deg=d'}(\mathbb{P}^1, \mathbb{P}^n) \times \mathrm{Sym}^{d-d'} C.$$

A quasi map can be uniquely recorded as a morphism $C \to \mathbb{P}^N$ and marked zeros. Generally, a quasi map over X can be uniquely recorded as a morphism $\mathbb{P}^1 \to X$ with colored marked point. That is,

$$\mathrm{QM}(X, \beta) = \bigsqcup_{0 \le \beta' \le \beta} \mathrm{Mor}_{\deg=\beta}(C, \mathbb{P}^n) \times \prod_{i=1}^{m} \mathrm{Sym}^{\langle \beta - \beta', D_i\rangle} \mathbb{P}^1.$$



4.9. Fixed locus. There is $\mathbb{C}^\times$-action on $QM(X)$ induced from $\mathbb{P}^1$. Firstly, let us look at

$$QM(\mathbb{P}^N, d) = \mathbb{P}(\mathbb{C}[x]^{N+1}_{\deg \le d}).$$

We have

$$QM(\mathbb{P}^N, d)^{\mathbb{C}^\times} = \bigcup_{0 \le d' \le d} x^{d'}\mathbb{P}(\mathbb{C}^{N+1}) = \bigcup_{0 \le d' \le d} \mathbb{P}^N.$$

That is, it is set of constant quasi-map with d marked point at 0 and $d - d'$ marked point at ∞. More generally, we have

$$QM(X, \beta)^{\mathbb{C}^\times} = \bigcup_{0 \le \beta' \le \beta} x^{\beta'} \cdot X.$$

4.10. Pseudo evaluation. Recall we have a morphism

$$\epsilon\upsilon^* : \mathrm{Pic}(X) \to \mathrm{Pic}(QM(X, \beta))$$

such that the restricting to any fixed component

$$\mathrm{Pic}(QM(X, \beta)) \longrightarrow \mathrm{Pic}(x^{\beta'}X) \simeq \mathrm{Pic}(X)$$

is identity. For any polynomial $f(x_1, \dots, x_m)$, we want to compute

$$\int_{QM(X,\beta)} f(\epsilon\upsilon^* D_1, \dots, \epsilon\upsilon^* D_m).$$

Graph Space.

4.11. Graph Space. Let us consider the graph space

$$G_0(X, \beta) = \overline{\mathcal{M}}_0(\mathbb{P}^1 \times X, (1, \beta)).$$

Note that $G_0(X)$ admits a $\mathbb{C}^\times$ action, so we can compute pushforward via localization. We view the projection

$$\mathbb{P}^1 \times X \to \mathbb{P}^1$$

as a fibre bundle. Every stable map in $G_0(X, \beta)$ is a union of a section and vertical curves.

4.12. Fixed component. For any $x \in X$, we denote $[x]$ the graph of constant map

$$[x] = \left[\mathbb{P}^1 \to \mathbb{P}^1 \times \{x\} \subset \mathbb{P}^1 \times X\right].$$

Assume $\beta > 0$. Let $\beta_1, \beta_2 > 0$. We have a morphism

$$i_{\beta_1,\beta_2} : \overline{\mathcal{M}}_1(X, \beta_1) \times_X \overline{\mathcal{M}}_1(X, \beta_2) \longrightarrow G_0(X, \beta_1 + \beta_2)$$

by putting two stable maps with same marked point on X horizontally at 0 and ∞ respectively, and gluing them by $[x]$. We also have

$$i_{\beta,0} : \overline{\mathcal{M}}_1(X, \beta) \longrightarrow G_0(X, \beta)$$

by putting a stable map at 0. We similarly define $i_{0,\beta}$. Then

$$G_0(X,\beta)^{\mathbb{C}^\times} = \bigcup_{\beta_1+\beta_2=\beta} (\text{image of } i_{\beta_1,\beta_2}).$$

4.13. Dimension estimation. Let us estimate the dimension. We have

$$\begin{aligned}\dim G_0(X,\beta) &= \dim X + 1 + \langle c_1(\mathscr{T}_X),\beta\rangle + \langle c_1(\mathscr{T}_{\mathbb{P}^1}),1\rangle + 0 - 3\\ &= \dim X + \langle c_1(\mathscr{T}_X),\beta\rangle.\end{aligned}$$

For $\beta_1,\beta_2>0$ with $\beta_1+\beta_2=\beta$,

$$\begin{aligned}\dim \overline{\mathcal{M}}_1(X,\beta_1)\times_X \overline{\mathcal{M}}_1(X,\beta_2) &= \dim X + \langle c_1(\mathscr{T}_X),\beta\rangle + 1 - 3 + 1 - 3\\ &= \dim X + \langle c_1(\mathscr{T}_X),\beta\rangle - 4.\end{aligned}$$

On the other hand,

$$\begin{aligned}\dim \overline{\mathcal{M}}_1(X,\beta) &= \dim X + \langle c_1(\mathscr{T}_X),\beta\rangle + 1 - 3\\ &= \dim X + \langle c_1(\mathscr{T}_X),\beta\rangle - 2.\end{aligned}$$

4.14. Normal bundle. Denote ξ the natural representation of $\mathbb{C}^\times$. For $\beta_1,\beta_2>0$, the normal bundle along i_{β_1,β_2}.

$$\begin{aligned}(\text{smoothing the gluing point at } 0) &= (\mathbb{L}^{-1}\otimes\xi)\boxtimes\mathcal{O}.\\ (\text{moving the vertical curve at } 0) &= \xi\boxtimes\mathcal{O} = \xi.\end{aligned}$$

Similarly for the gluing point at ∞

$$\begin{aligned}(\text{smoothing the gluing point at } \infty) &= \mathcal{O}\boxtimes(\mathbb{L}^{-1}\otimes\xi^{-1}).\\ (\text{moving the vertical curve at } \infty) &= \mathcal{O}\boxtimes\xi^{-1} = \xi^{-1}.\end{aligned}$$

Thus the Euler class

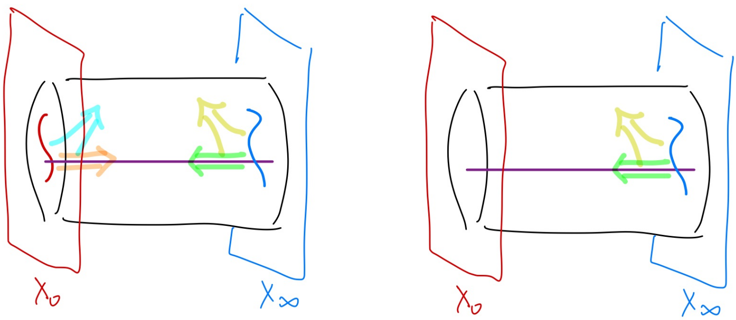
$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2})) = \text{restriction of } z(z-\psi)\otimes(-z(-z-\psi)).$$

When $\beta_2=0$, we do not need to smooth and move ∞, so

$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta,0})) = \text{restriction of } z(z-\psi)\otimes 1.$$

Similarly, when $\beta_1=0$,

$$\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta})) = \text{restriction of } 1\otimes(-z(-z-\psi)).$$

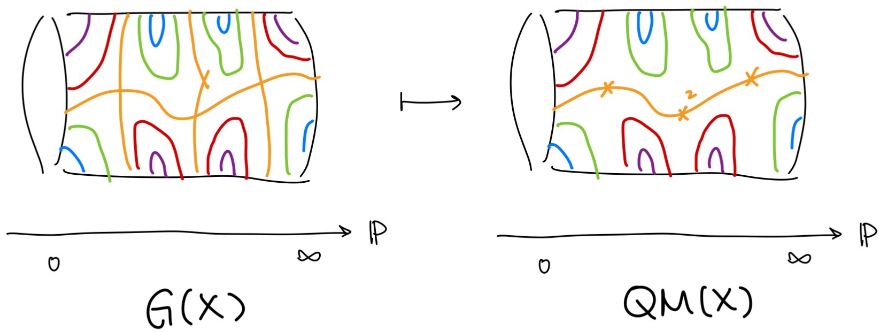


Comparison.

4.15. Comparison. Note that both $G(X,\beta)$ and $QM(X,\beta)$ are compatification of $\mathrm{Mor}_{\deg=\beta}(\mathbb{P}^1, X)$. We actually have a birational morphism

$$\mu : G(X,\beta) \longrightarrow QM(X,\beta)$$

by changing the vertical curves by marked points.



4.16. Localization. Let

$$\phi = f(D_1, \ldots, D_m).$$

As μ is birational,

$$
\begin{aligned}
&\int_{QM(X,\beta)} f(\epsilon\upsilon^* D_1,\dots,\epsilon\upsilon^* D_m)\\
&= \int_{G(X,\beta)} \mu^* f(\epsilon\upsilon^* D_1,\dots,\epsilon\upsilon^* D_m)\\
&= \sum_{\beta_1+\beta_2=\beta} \int \frac{i^*_{\beta_1,\beta_2}\mu^* f(\epsilon\upsilon^* D_1,\dots,\epsilon\upsilon^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))}.
\end{aligned}
$$

When $\beta_1,\beta_2>0$, we have the following diagram

$$
\begin{tikzcd}
G(X,\beta) \arrow[r, "\mu"] & QM(X,\beta) \\
\text{image of } i_{\beta_1,\beta_2} \arrow[u] \arrow[r] & \chi^{\beta_1}X \arrow[u] \\
\overline{\mathcal{M}}_1(X,\beta_!)\times_X \overline{\mathcal{M}}_1(X,\beta_2) \arrow[u, equal] \arrow[r, "\mathrm{ev}\boxtimes 1=1\boxtimes \mathrm{ev}"] & X \arrow[u, equal]
\end{tikzcd}
$$

Thus

$$
\begin{aligned}
&\int_{\overline{\mathcal{M}}_1(X,\beta_1)\times_X\overline{\mathcal{M}}_1(X,\beta_2)} \frac{i^*_{\beta_1,\beta_2}\mu^* f(\epsilon\upsilon^* D_1,\dots,\epsilon\upsilon^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))}\\
&= \int_{\overline{\mathcal{M}}_1(X,\beta_1)\times_X\overline{\mathcal{M}}_1(X,\beta_2)} \frac{(\mathrm{ev}\boxtimes 1)^*\big(f(D_1,\dots,D_m)\big)}{z(z-\psi)\otimes(-z)(-z-\psi)}\\
&= \int_{\overline{\mathcal{M}}_1(X,\beta_1)\times\overline{\mathcal{M}}_1(X,\beta_2)} \frac{(\mathrm{ev}\boxtimes 1)^*(\phi)}{z(z-\psi)\otimes(-z)(-z-\psi)}(\mathrm{ev}\boxtimes\mathrm{ev})^*(\Delta_X)\\
&= \int_{\overline{\mathcal{M}}_1(X,\beta_1)\times\overline{\mathcal{M}}_1(X,\beta_2)} \frac{(\mathrm{ev}\boxtimes 1)^*(\phi)}{z(z-\psi)\otimes(-z)(-z-\psi)}\sum_w(\mathrm{ev}\boxtimes\mathrm{ev})^*(\sigma_w\boxtimes\sigma^w)\\
&= \sum_w \int_{\overline{\mathcal{M}}_1(X,\beta_1)} \frac{\mathrm{ev}^*(\phi\cdot\sigma_w)}{z(z-\psi)}\int_{\overline{\mathcal{M}}_1(X,\beta_2)}\frac{\mathrm{ev}^*(\sigma^w)}{z(z-\psi)}\\
&= \sum_w \left\langle \mathrm{ev}_*\left(\frac{1}{z(z-\psi)}\right),\phi\cdot\sigma_w\right\rangle\left\langle \mathrm{ev}_*\left(\frac{1}{-z(-z-\psi)}\right),\sigma^w\right\rangle
\end{aligned}
$$

Similarly, when $\beta'=\beta$,

$$
\begin{tikzcd}
\text{image of } i_{\beta,0} \arrow[r] & \chi^{\beta}X \\
\overline{\mathcal{M}}_1(X,\beta) \arrow[u, equal] \arrow[r, "\mathrm{ev}_2"] & X. \arrow[u, equal]
\end{tikzcd}
$$

We have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i^*_{\beta,0}\mu^* f(\epsilon\upsilon^* D_1, \dots, \epsilon\upsilon^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))} \\
&= \int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{\mathrm{ev}^*(\phi)}{z(z-\psi)} = \left\langle \mathrm{ev}_* \left(\frac{1}{z(z-\psi)} \right), \phi \right\rangle \\
&= \sum_w \left\langle \mathrm{ev}_* \left(\frac{1}{z(z-\psi)} \right), \phi\sigma_w \right\rangle \langle 1, \sigma^w \rangle
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i^*_{0,\beta}\mu^* f(\epsilon\upsilon^* D_1, \dots, \epsilon\upsilon^* D_m)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))} \\
&= \int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{\mathrm{ev}^*(\phi)}{-z(-z-\psi)} = \left\langle \mathrm{ev}_* \left(\frac{1}{-z(-z-\psi)} \right), \phi \right\rangle \\
&= \sum_w \langle 1, \phi \cdot \sigma_w \rangle \left\langle \mathrm{ev}_* \left(\frac{1}{-z(-z-\psi)} \right), \sigma^w \right\rangle.
\end{aligned}$$

4.17. J-function again. Let us denote

$$\tilde{J}(z) = 1 + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta \, \mathrm{ev}_* \left(\frac{1}{z(z-\psi)} \right).$$

Recall that

$$J = e^{p \ln q / z} \tilde{J}.$$

Then above computation shows

$$\begin{aligned}
&\sum_{q \in \mathrm{Eff}(X)} q^\beta \int_{QM(X,\beta)} f(\epsilon\upsilon^* D_1, \dots, \epsilon\upsilon^* D_m) \\
&= \sum_w \langle \tilde{J}(z), \phi \cdot \sigma_w \rangle \langle \tilde{J}(-z), \sigma^w \rangle = \langle \tilde{J}(z)\tilde{J}(-z), \phi \rangle.
\end{aligned}$$

5. Properties and Applications

Quantum connection.

5.1. Remark. Recall that a connection of a vector bundle $\mathcal{V}$ over a real manifold M is an $\mathbb{R}$-bilinear morphism

$$\nabla : \mathcal{V} \longrightarrow \Omega_M \otimes_{\mathcal{O}_M} \mathcal{V}.$$

with the Leibniz rule

$$\nabla(fs) = df \otimes s + f \cdot \nabla s.$$

For a local vector field $X \in \mathscr{T}_M$, we deonte $\nabla_X s = \langle X, \nabla s\rangle$, with the pairing induced by the natural pairing $\langle\, , \,\rangle : \mathscr{T}_M \otimes \Omega_M \otimes \mathcal{V} \longrightarrow \mathcal{V}$. Then $\nabla_X s$ satisfies

- $\nabla_{fX+Y} s = f\nabla_X s + \nabla_Y s;$ (linearity)
- $\nabla_X(fs + t) = (Xf)s + f\nabla_X s + \nabla_X t;$ (Leibinize rule)

To define a connection locally, it suffices to define ∇_X for those X forming a basis of $\mathscr{T}_M$ over $\mathcal{O}_M$ (called a frame) and check the second condition.

5.2. Quantum connection. Let us consider

the trivial vector bundle $\mathcal{V}$ over $M = H^2(X)$ with fibre $H^*(X)$.

Note that we can view q^β as a function over $H^2(X)$ for $\beta \in \mathrm{Eff}(X) \subset H_2(X, \mathbb{Z})$. Thus

$$H^0(M, \mathcal{V}) = H^*(X) \otimes_{\mathbb{C}} \mathcal{O}(M) = QH^*(X) \otimes_{\mathbb{C}(q)} \mathcal{O}(M).$$

The **quantum connection** is defined to be (z is a formal variable)

$$\nabla_\lambda = \partial_\lambda - \frac{1}{z}\lambda *,$$

where

- ∂_λ is the differential operator over M such that $\partial_\lambda q^\beta = \langle\lambda, \beta\rangle q^\beta$;
- $\lambda*$ is the $\mathcal{O}$-linear map of quantum product with divisor $\lambda \in H^2(X)$ fibre-wise.

This is a connection:

$$\begin{aligned}
\nabla_\lambda(fs + t) &= \partial_\lambda(fs + t) - \frac{1}{z}\lambda * (fs + t) \\
&= (\partial_\lambda f) + f(\partial_\lambda s) + \partial_\lambda t - \frac{1}{z} f\lambda * s - \frac{1}{z}\lambda * t \\
&= (\partial_\lambda f) + f\nabla_\lambda s + \nabla_\lambda t.
\end{aligned}$$

Here we use the fact that the quantum product is $\mathbb{C}(q)$-linear.

5.3. Remark. For a connection ∇ of a vector bundle $\mathcal{V}$ over M, we can extend

$$0 \longrightarrow \mathcal{V} \xrightarrow{\nabla} \Omega_M \otimes_{\mathcal{O}_M} \mathcal{V} \xrightarrow{\nabla} \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{V} \xrightarrow{\nabla} \cdots$$

by

$$\nabla(\omega \wedge s) = d\omega \otimes s + (-1)^{\deg \alpha} \omega \wedge \nabla s.$$

The map $\nabla^2 : \mathcal{V} \to \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{V}$ is $\mathcal{O}_M$-linear, called the curvature. A connection is flat if $\nabla^2 = 0$, equivalently, the above chain is a complex. In terms of $\nabla_X s$, it is equivalent to say

$$\langle X \wedge Y, \nabla^2 s\rangle = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = 0.$$

If we define ∇_X for a frame forming a basis of $\mathscr{T}_M$, then it suffice to check for all pairing of vector fields from the frame. For a flat connection ∇, the following differential equation has a local solution

$$\nabla(f) = 0, \qquad f \in H^0(M, \mathcal{V})$$

for any given initial value of f at a point $x \in M$.

5.4. Flatness. The quantum connection is flat.

$$\begin{aligned}
&\nabla_\lambda \nabla_\mu s - \nabla_\mu \nabla_\lambda s - \nabla_{[\lambda,\mu]} s \\
&= \nabla_\lambda \nabla_\mu s - \nabla_\mu \nabla_\lambda s \\
&= (\partial_\lambda - \tfrac{1}{z}\lambda *)(\partial_\mu - \tfrac{1}{z}\mu *)s - (\partial_\mu - \tfrac{1}{z}\mu *)(\partial_\lambda - \tfrac{1}{z}\lambda *)s \\
&= \big(\partial_\lambda \partial_\mu s - \tfrac{1}{z}\mu * \partial_\lambda s - \tfrac{1}{z}\lambda * \partial_\mu s + \tfrac{1}{z^2}\lambda * \mu * s\big) \\
&- \big(\partial_\mu \partial_\lambda s - \tfrac{1}{z}\lambda * \partial_\mu s - \tfrac{1}{z}\mu * \partial_\lambda s + \tfrac{1}{z^2}\mu * \lambda * s\big) \\
&= \frac{1}{z^2}(\lambda * \mu * s - \mu * \lambda * s) = 0.
\end{aligned}$$

Here we use the associativity and commutativity of the quantum product.

5.5. Remark. As we mentioned, $S(\gamma')$ solves the quantum differential equation,

$$\nabla_\lambda(f) = 0, \quad \text{i.e.} \quad \partial_\lambda f = \frac{1}{z}\lambda * \gamma.$$

It is actually the fundamental solution.

5.6. Remark. Note that if we replace quantum product by usual product, then the fundamental solution is easy seen to be

$$S(\gamma') = e^{p \ln q/z}\gamma'.$$

Applications.

5.7. Remark. Let F be a component of X^T. Then the push forward

$$i_* : H_T^*(F) \longrightarrow H_T^*(X)$$

is an isomorphism after localization. The inverse is given by

$$H_T^*(X) \longrightarrow H_T^*(F), \qquad \gamma \longmapsto \frac{\gamma|_F}{\mathrm{Eu}(\mathrm{Nm}_F X)}.$$

5.8. Embedding. We have an embedding

$$i_{\beta,0} : \overline{\mathcal{M}}_1(X,\beta) \longrightarrow G_0(X,\beta).$$

For two varieties X and Y, we have

$$\begin{tikzcd}
G_0(X\times Y,(\beta_X,\beta_Y)) \arrow[r, dashed, "\text{birational}"] & G_0(X,\beta_X)\times G_0(Y,\beta_Y) \\
\overline{\mathcal{M}}_1(X\times Y,(\beta_X,\beta_Y)) \arrow[u, "i_{X\times Y}"] \arrow[r, dashed, "\Pi"] \arrow[d, "\mathrm{ev}"'] & \overline{\mathcal{M}}_1(X,\beta_X)\times\overline{\mathcal{M}}_1(Y,\beta_Y) \arrow[u, "i_X\times i_Y"'] \arrow[d, "\mathrm{ev}"] \\
X\times Y \arrow[r, dashed] & X\times Y.
\end{tikzcd}$$

This implies

$$\Pi_*\left(\frac{1}{z(z-\psi)}\boxtimes\frac{1}{z(z-\psi)}\right)=\frac{1}{z(z-\psi)}.$$

This shows the J-function of the product is the product of J-functions.

5.9. J-function of projective space. Recall we have

$$\begin{tikzcd}
G(\mathbb{P}^N,d) \arrow[r, dashed, "\text{birational}"] & QM(\mathbb{P}^N,d) \\
\overline{\mathcal{M}}_1(X,d) \arrow[u] \arrow[r, dashed, "\mathrm{ev}"] & \mathbb{P}^N \arrow[u, "i"']
\end{tikzcd}$$

As a result,

$$\mathrm{ev}_*\left(\frac{1}{z(z-\psi)}\right)=\frac{1}{\mathrm{Eu}(i)}.$$

Recall

$$QM(\mathbb{P}^N,d)=\mathbb{P}(H^0(\mathbb{C}[x]_{\deg\le d})^{N+1}).$$

Note that $\mathbb{P}^N\subset QM(\mathbb{P}^N,d)$ is induced by

$$\mathbb{C}^{N+1}\simeq(\mathbb{C}x^d)^{N+1}\subset(\mathbb{C}[x]_{\deg\le d})^{N+1}.$$

So it is defined by

$$\text{coefficients of } x^0,\dots,x^{d-1} \text{ of every } N+1 \text{ component} = 0.$$

So

$$\mathrm{Eu}(i) = \prod_{k=1}^{d} (H + kz).$$

As a result, we have

$$\tilde{J} = 1 + \sum_{d>1} \frac{q^d}{\prod_{k=1}^{d} (H + kz)}.$$

That is,

$$J = q^{H/z} \left(1 + \sum_{d>1} \frac{q^d}{\prod_{k=1}^{d} (H + kz)^{N+1}} \right).$$

5.10. Remark. Let us compute

$$\partial_H J = \frac{H}{z} q^{H/z} + \sum_{d>1} \frac{\left(d + \frac{H}{z}\right) q^{d+H/z}}{\prod_{k=1}^{d} (H + kz)^{N+1}}.$$

Similarly,

$$\begin{aligned} (z\partial_H)^{N+1} J &= H^{N+1} q^{H/z} + \sum_{d>1} \frac{(H + dz)^{N+1} q^{d+H/z}}{\prod_{k=1}^{d} (H + kz)^{N+1}} \\ &= \sum_{d>1} \frac{q^{d+H/z}}{\prod_{k=1}^{d-1} (H + kz)^{N+1}} = qJ. \end{aligned}$$

So we have

$$H^{N+1} = q \qquad \text{(quantum product)}.$$

Unitary property.

5.11. A twisted fundamental solution. Let us denote

$$\mathcal{M}(\gamma, \gamma') = \langle \gamma, \gamma' \rangle + \sum_{\beta \in \mathrm{Eff}(X) \setminus \{0\}} q^\beta \int_{\overline{\mathcal{M}}_2(X,\beta)} \mathrm{ev}^*(\gamma \boxtimes \gamma') \frac{1}{z - \psi_2}.$$

Let us denote the operator M by

$$\langle M(\gamma), \gamma' \rangle = \mathcal{M}(\gamma, \gamma').$$

5.12. Equation for M. Then

$$\partial_\lambda \langle M(\gamma), \gamma' \rangle = \frac{1}{z}\langle M(\lambda * \gamma), M(\gamma')\rangle - \frac{1}{z}\langle \lambda M(\gamma), \gamma' \rangle.$$

Thus

$$\partial_\lambda M(\gamma) = \frac{1}{z} M(\lambda * \gamma) - \frac{1}{z}\lambda M(\gamma).$$

For general f, i.e. possibly involving quantum parameters,

$$\partial_\lambda M(f) = \frac{1}{z} M(\lambda * f) - \frac{1}{z}\lambda M(f) + M(\partial_\lambda f).$$

5.13. Summary. We have the following commutative diagram

$$\begin{tikzcd}
H_{\mathbb{T}}(X) \arrow[r, "M(-{,}z)"] \arrow[d, "\partial_\lambda + \frac{1}{z}\lambda *"'] & H_{\mathbb{T}}(X)(q) \arrow[d, "\partial_\lambda + \frac{1}{z}\lambda"] \\
H_{\mathbb{T}}(X) \arrow[r, "M(-{,}z)"] & H_{\mathbb{T}}(X)(q)
\end{tikzcd}$$

5.14. Equation for inverse. By substituting f by $M^{-1}(f)$, we get

$$\partial_\lambda f = \frac{1}{z} M(\lambda * M^{-1}(f)) - \frac{1}{z}\lambda f + M(\partial_\lambda M^{-1}(f)).$$

Applying M^{-1}, we get

$$M^{-1}(\partial_\lambda f) = \frac{1}{z}\lambda * M^{-1}(f) - \frac{1}{z} M^{-1}(\lambda f) + \partial_\lambda M^{-1}(f).$$

That is,

$$\partial_\lambda M^{-1}(f) = -\frac{1}{z}\lambda * M^{-1}(f) + \frac{1}{z} M^{-1}(\lambda f) + M^{-1}(\partial_\lambda f).$$

5.15. Equation for adjoint. On the other hand, denote the operator M' by

$$\langle \gamma, M'(\gamma')\rangle = M(\gamma, \gamma').$$

Then

$$\partial_\lambda \langle \gamma, M'(\gamma')\rangle = \frac{1}{z}\langle \gamma, \lambda * M'(\gamma')\rangle - \frac{1}{z}\langle \gamma, M'(\lambda\gamma')\rangle.$$

Thus

$$\partial_\lambda M'(\gamma') = \frac{1}{z}\lambda * M'(\gamma') - \frac{1}{z} M'(\lambda\gamma').$$

For general f, i.e. possibly involving quantum parameters,

$$\partial_\lambda M'(f) = \frac{1}{z}\lambda * M'(f) - \frac{1}{z} M'(\lambda f) + M'(\partial_\lambda f).$$

5.16. Conclusion. Let us denote $M(\gamma) = M(\gamma, z)$ to empathise the dependence of z. By comparing the differential equation, we have

$$M'(\gamma, z) = M^{-1}(\gamma, -z).$$

As a result, we have

$$\langle M(\gamma, z), M(\gamma', -z)\rangle = \langle \gamma, \gamma'\rangle.$$

In the rest of this section, we are going to give a geometric proof of this identity.

5.17. A pairing. Let us denote similarly

$$G_2(X, \beta) = \overline{\mathcal{M}}_2(\mathbb{P}^1 \times X, (1, \beta)).$$

We define for $\gamma_1, \gamma_2 \in H^*(X)$

$$G(\gamma_1, \gamma_2) = \langle \gamma_1, \gamma_2\rangle + \sum_{\beta>0} q^\beta \int_{G_2(X,\beta)} \mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)$$

where $i_0 : X \to \mathbb{P}^1 \times X$ and $i_\infty : X \to \mathbb{P}^1 \times X$ the inclusion of the fibre at 0 and ∞ respectively. Note that by 2.3, we just have $G(\gamma_1, \gamma_2) = \langle \gamma_1, \gamma_2\rangle$.

5.18. Components. Let us use localization to compute this pairing. Let us denote for $\beta_1, \beta_2 > 0$

$$i_{\beta_1,\beta_2} : \overline{\mathcal{M}}_2(X, \beta_1) \times_X \overline{\mathcal{M}}_2(X, \beta_2) \longrightarrow G_0(X, \beta_1 + \beta_2)$$

by gluing the second marked points. Similarly we define $i_{\beta,0}$ and $i_{0,\beta}$. Then

$$G_2(X, \beta)^{\mathbb{C}^\times} = (\cdots) \cup \bigcup_{\beta_1+\beta_2=\beta} (\text{image of } i_{\beta_1,\beta_2}).$$

Here $(\cdots)$ is the component does not contribute the pushforward.

5.19. Dimension estimation. Let us estimate the dimension. We have

$$\begin{aligned} \dim G_2(X, \beta) &= \dim X + 1 + \langle c_1(\mathscr{T}_X), \beta\rangle + \langle c_1(\mathscr{T}_{\mathbb{P}^1}), 1\rangle + 2 - 3 \\ &= \dim X + \langle c_1(\mathscr{T}_X), \beta\rangle + 2. \end{aligned}$$

For $\beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = \beta$,

$$\begin{aligned} \dim \overline{\mathcal{M}}_2(X, \beta_1) \times_X \overline{\mathcal{M}}_2(X, \beta_2) &= \dim X + \langle c_1(\mathscr{T}_X), \beta\rangle + 2 - 3 + 2 - 3 \\ &= \dim X + \langle c_1(\mathscr{T}_X), \beta\rangle - 2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \dim \overline{\mathcal{M}}_2(X, \beta) &= \dim X + \langle c_1(\mathscr{T}_X), \beta\rangle + 2 - 3 \\ &= \dim X + \langle c_1(\mathscr{T}_X), \beta\rangle - 1. \end{aligned}$$

5.20. Normal bundle. Similarly, when $\beta_1, \beta_2 > 0$,

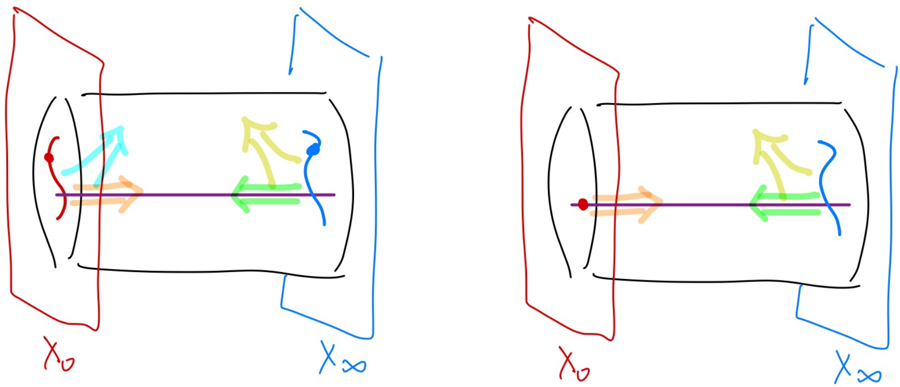
$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2})) = \text{restriction of } z(z-\psi) \otimes (-z(-z-\psi)).$$

When $\beta_2 = 0$, we do not need to smooth the marked point on ∞, so

$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta,0})) = \text{restriction of } z(z-\psi) \otimes (-z).$$

Similarly, when $\beta_1 = 0$,

$$\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta})) = \text{restriction of } z \otimes (-z(-z-\psi)).$$



5.21. Localization. When $\beta > 0$, using localization, we have

$$\int_{G_2(X,\beta)} \mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2) = \sum_{\beta_1+\beta_2=\beta} \int \frac{i^*_{\beta_1,\beta_2}(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))}.$$

When $\beta_1, \beta_2 > 0$, we have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_2(X,\beta_1)\times_X \overline{\mathcal{M}}_2(X,\beta_2)} \frac{i^*_{\beta_1,\beta_2}(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)}{\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2}))} \\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_1)\times_X \overline{\mathcal{M}}_2(X,\beta_2)} \frac{(\mathrm{ev}_1 \boxtimes \mathrm{ev}_1)^*(i_0^* i_{0*}\gamma_1 \boxtimes i_\infty^* i_{\infty*}\gamma_2)}{z(z-\psi_2)\otimes(-z(-z-\psi_2))} \\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_1)\times \overline{\mathcal{M}}_2(X,\beta_2)} \frac{(\mathrm{ev}_1 \boxtimes \mathrm{ev}_1)^*(i_0^* i_{0*}\gamma_1 \boxtimes i_\infty^* i_{\infty*}\gamma_2)}{z(z-\psi_2)\otimes(-z(-z-\psi_2))} (\mathrm{ev}_2 \boxtimes \mathrm{ev}_2)^*(\Delta_X) \\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_1)\times \overline{\mathcal{M}}_2(X,\beta_2)} \frac{(\mathrm{ev}_1 \boxtimes \mathrm{ev}_1)^*(z\gamma_1 \boxtimes (-z)\gamma_2)}{z(z-\psi_2)\otimes(-z(-z-\psi_2))} \sum_w (\mathrm{ev}_2 \boxtimes \mathrm{ev}_2)^*(\sigma_w \boxtimes \sigma^w) \\
&= \sum_w \int_{\overline{\mathcal{M}}_2(X,\beta_1)} \frac{\mathrm{ev}^*(\gamma_1 \boxtimes \sigma_w)}{z-\psi_2} \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \frac{\mathrm{ev}^*(\gamma_2 \boxtimes \sigma^w)}{-z-\psi_2}.
\end{aligned}$$

Here $\{\sigma_w\} \subset H^*(X)$ is a basis and $\{\sigma^w\} \subset H^*(X)$ is its dual basis. Similarly, when β_2, we have

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i_{\beta,0}^*(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)}{\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta}))} \\
&= \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \frac{\mathrm{ev}^*(\gamma_1 \boxtimes \gamma_2)}{-z-\psi_2} = \sum_w \int_{\overline{\mathcal{M}}_2(X,\beta_1)} \frac{\mathrm{ev}^*(\gamma_1 \boxtimes \sigma_w)}{z-\psi_2} \langle \gamma_2, \sigma^w \rangle, \\
&\int_{\overline{\mathcal{M}}_1(X,\beta)} \frac{i_{0,\beta}^*(\mathrm{ev}^*(i_{0*}\gamma_1 \boxtimes i_{\infty*}\gamma_2)}{\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta}))} = \sum_w \langle \gamma_1, \sigma_w \rangle \int_{\overline{\mathcal{M}}_2(X,\beta_2)} \frac{\mathrm{ev}^*(\gamma_2 \boxtimes \sigma^w)}{-z-\psi_2}.
\end{aligned}$$

5.22. Conclusion. As a result,

$$\begin{aligned}
\langle \gamma_1, \gamma_2 \rangle = G(\gamma_1, \gamma_2) &= \sum_w \mathcal{M}(\gamma_1, \sigma_w)\mathcal{M}(\gamma_2, \sigma^w)|_{z \mapsto -z} \\
&= \sum_w \langle M(\gamma_1, z), \sigma_w \rangle \langle M(\gamma_2, -z), \sigma^w \rangle \\
&= \langle M(\gamma_1, z), M(\gamma_2, -z) \rangle.
\end{aligned}$$

6. Shift operators

Shift operator.

6.1. Setup. Assume T acts on X. We are going to define a family of operators for any $k \in 1\mathrm{PS}(T)$. Let $\mathbb{T} = T \times \mathbb{C}^\times$. We denote z the canonical generator in $H^2_{\mathbb{C}^\times}(\mathrm{pt})$.

6.2. Twisted action. For any $k \in 1\mathrm{PS}(T)$, we have a twisted $\mathbb{T}$-action by

$$\rho_k(t,u)\cdot x = t \cdot k(u) \cdot x.$$

We have

$$\begin{tikzcd} H^*_{\mathbb{T}}(X,\rho_0) \arrow[r, "\sim"] & H^*_{\mathbb{T}}(X,\rho_k) \\ H^*_{\mathbb{T}}(\mathrm{pt}) \arrow[u] \arrow[r, "\lambda\mapsto\lambda+\langle k,\lambda\rangle z", "\sim"'] & H^*_{\mathbb{T}}(\mathrm{pt}) \arrow[u] \end{tikzcd}$$

Let us denote the isomorphism by $\gamma \mapsto \gamma[k]$.

6.3. Bundle. Let us denote

$$E_k = \left(\mathbb{C}^2 \setminus \{0\}\right) \underset{\mathbb{C}^\times}{\times} X,$$

with the action induced by k. Then $\mathbb{T}$ acts on E_k. We have a projection

$$\pi : E_k \to \left(\mathbb{C}^2 \setminus \{0\}\right)/\mathbb{C}^\times = \mathbb{P}^1$$

with

$$\pi^{-1}(0) \simeq (X,\rho_0) =: X_0, \qquad \pi^{-1}(\infty) \simeq (X,\rho_k) =: X_\infty.$$

6.4. Section class. Let us denote

$$\mathrm{Eff}(E_k)_{\mathrm{sec}} = \text{preimage of } [\mathbb{P}^1 \xrightarrow{\mathrm{id}} \mathbb{P}^1] \in \mathrm{Eff}(\mathbb{P}^1) \text{ under } \pi_* : \mathrm{Eff}(E_k) \to \mathrm{Eff}(\mathbb{P}^1).$$

6.5. Shift operator. Let us define

$$\iota_0 : X_0 \to E_k, \qquad \iota_\infty : X_\infty \to E_k.$$

Let us define the **shifted operator**

$$\tilde{\mathbb{S}}_k : H^*_{\mathbb{T}}(X,\rho_0) \longrightarrow H^*_{\mathbb{T}}(X,\rho_k)$$

by

$$\langle \tilde{\mathbb{S}}_k(\gamma), \gamma'[k]\rangle = \sum_{\tilde{\beta}\in\mathrm{Eff}(E_k)_{\mathrm{sec}}} q^{\tilde{\beta}} \int_{\overline{\mathcal{M}}_2(E_k,\tilde{\beta})} \mathrm{ev}^*(\iota_{0*}\gamma, \boxtimes\iota_{\infty*}\gamma'[k]).$$

Let us use localization to compute $\tilde{\mathbb{S}}_k$.

6.6. Example. When $k=0$, then

$$E_k = \mathbb{P}^1 \times X.$$

Applying the same trick to $\mathbb{C}^\times$ fixed locus as in the previous section, we get

$$\langle \tilde{\mathbb{S}}_k(\gamma), \gamma' \rangle = \langle M(\gamma, z), M(\gamma', -z) \rangle = \langle \gamma, \gamma' \rangle.$$

Thus $\tilde{\mathbb{S}}_0 = \mathrm{id}$. In general, we have to consider the T-fixed locus.

6.7. Fixed locus. Let $F \in \pi_0(X^T)$ be a connected component of X^T. We denote $\sigma_F \in \mathrm{Eff}(E_k)$ to be the class of σ_x for any $x \in F$. For $\beta_1, \beta_2 > 0$, let us denote

$$\overline{\mathcal{M}}_2(X_0, \beta_1) \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2) = (\mathrm{ev}_2 \boxtimes \mathrm{ev}_2)^{-1}(\Delta_F)$$

the space of stable maps with the second marked points the same in F. For (C_1, C_2) in this space with $\mathrm{ev}_2(C_1) = \mathrm{ev}_2(C_2) = x \in F$, by gluing $\sigma_x \subset E_k$, we have a $\mathbb{T}$-invariant stable maps over E_k. This defines

$$i_{\beta_1,\beta_2} : \overline{\mathcal{M}}_2(X_0, \beta_1) \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2) \longrightarrow \overline{\mathcal{M}}_2(E_k, i_{0*}\beta_1 + i_{\infty*}\beta_2 + \sigma_F).$$

It induces

$$\overline{\mathcal{M}}_2(X_0, \beta_1)^T \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2)^T \longrightarrow \overline{\mathcal{M}}_2(E_k, i_{0*}\beta_1 + i_{\infty*}\beta_2 + \sigma_F)^{\mathbb{T}}.$$

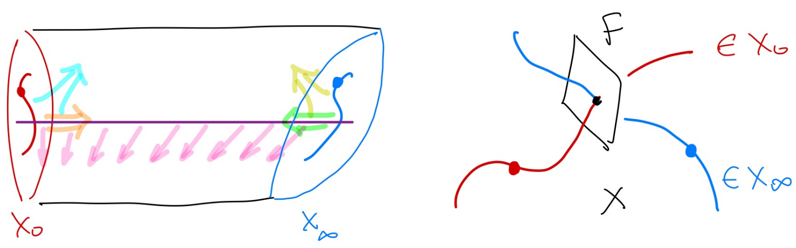
We similarly denote

$$i_{\beta_1,0}, i_{0,\beta_2} : \overline{\mathcal{M}}_2(X_0, \beta) \cap \mathrm{ev}_2^{-1}(F) \longrightarrow \overline{\mathcal{M}}_2(E_k, \beta_1 + \sigma_F).$$

We have the following decomposition

$$\overline{\mathcal{M}}_2(E_k, \tilde{\beta})^{\mathbb{T}} = (\cdots) \cup \bigcup_{i_{0*}\beta_1 + i_{\infty*}\beta_2 + \sigma_F = \tilde{\beta}} \text{image of } i_{\beta_1,\beta_2}.$$

Here $(\cdots)$ are those components not in $\mathrm{ev}^{-1}(X_0 \times X_\infty)$, which does not contribute the integral.



6.8. Computation. Let us compute the normal bundle of

$$\overline{\mathcal{M}}_2(X_0, \beta_1) \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2).$$

It contains the fixed component. Denote ξ the natural representation of $\mathbb{C}^\times$.

$$(\text{smoothing the gluing point at } 0) = (\mathbb{L}_2^{-1} \otimes \xi) \boxtimes \mathcal{O}.$$
$$(\text{moving the gluing point at } 0) = \xi \boxtimes \mathcal{O} = \xi.$$

Similarly for the gluing point at ∞

$$(\text{smoothing the gluing point at } \infty) = \mathcal{O} \boxtimes (\mathbb{L}_2^{-1} \otimes \xi^{-1}).$$
$$(\text{moving the gluing point at } \infty) = \mathcal{O} \boxtimes \xi^{-1} = \xi^{-1}.$$

Thus the Euler class

$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,\beta_2})) = z(z-\psi_2) \otimes (-z(-z-\psi_2)).$$

When $\beta_1 = 0$, the computation will be different. Now 0 is a marked point, so we do not need to smooth it. The Euler class

$$\mathrm{Eu}(\mathrm{Nm}(i_{0,\beta_2})) = z \otimes (-z(-z-\psi_2)).$$

Similarly for $\beta_2 = 0$,

$$\mathrm{Eu}(\mathrm{Nm}(i_{\beta_1,0})) = z(z-\psi) \otimes (-z).$$

6.9. Lemma. The normal bundle of $F \times \mathbb{P}^1$ is

$$\mathrm{Nm}_{F\times\mathbb{P}^1} E_k = \bigoplus_{\lambda \in \mathsf{char}(T)} (\mathrm{Nm}_F X)_\lambda \boxtimes \mathcal{O}_{\mathbb{P}^1}(-\langle \lambda, k\rangle),$$

where $(\mathrm{Nm}_F X)_\lambda = \mathrm{Hom}_T(\mathbb{C}_\lambda, \mathrm{Nm}_F X)$. Actually, it is characterized by (as $\mathbb{C}^\times$-equivariant bundles)

$$\mathrm{Nm}_{F\times\mathbb{P}^1} E_k|_{F\times 0} = \mathrm{Nm}_F X_0 = \mathrm{Nm}_F X = \bigoplus_{\lambda \in \mathsf{char}(T)} (\mathrm{Nm}_F X)_\lambda$$
$$\mathrm{Nm}_{F\times\mathbb{P}^1} E_k|_{F\times \infty} = \mathrm{Nm}_F X_\infty = (\mathrm{Nm}_F X)[k] = \bigoplus_{\lambda \in \mathsf{char}(T)} (\mathrm{Nm}_F X)_\lambda(\langle \lambda, k\rangle z).$$

6.10. Moving the horizontal cruve. Now let us compute the part of moving the horizontal curve. We have

$$\begin{aligned}&(\text{moving the horizontal curve})\\ &= (\text{moving to be non-constant inside } F) \oplus (\text{moving out of } F)\end{aligned}$$

Note that

$$(\text{moving to be non-constant inside } F) = \mathrm{Mor}(\mathbb{P}^1, H^0(F, \mathscr{T}_F))/\text{constant} = 0.$$

Note that

$$(\text{moving out of } F) = \bigoplus_{\lambda \in \mathrm{char}(T)} \mathrm{ev}^*(\mathrm{Nm}_F X)_\lambda \cdot \chi\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-\langle \lambda, k\rangle)\big)$$

where $\mathrm{ev} = \mathrm{ev}_2 \boxtimes 1 = 1 \boxtimes \mathrm{ev}_2$. Here $(\mathrm{Nm}_F X)_\lambda$ has trivial $\mathbb{C}^\times$-action, so ev^* induced by two maps do not differ. By localization theorem, we have

$$\chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) = \frac{1-\xi^{-i-1}}{1-\xi^{-1}} = \sum_{c \le 0} \xi^c - \sum_{c < -i} \xi^c.$$

So

$$(\text{moving the horizontal curve}) = \bigoplus_{\lambda \in \mathrm{char}(T)} \mathrm{ev}^*(\mathrm{Nm}_F X)_\lambda \cdot \left(\sum_{c \le 0} \xi^c - \sum_{c < \langle \lambda, k\rangle} \xi^c \right).$$

Note that its Euler class is

$$\prod_{\lambda \in \mathrm{char}(T)} \prod_{x \in \sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c \le 0} (\mathrm{ev}^* x + \lambda + cz)}{\prod_{c < \langle \lambda, k\rangle} (\mathrm{ev}^* x + \lambda + cz)} = (\mathrm{ev}_2 \boxtimes 1)^*(\cdots),$$

$$= (\mathrm{ev}_2 \boxtimes 1)^* \left(\prod_{\lambda \in \mathrm{char}(T)} \prod_{x \in \sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c \le 0} (x + \lambda + cz)}{\prod_{c < \langle \lambda, k\rangle} (x + \lambda + cz)} \right) =: (\mathrm{ev}_2 \boxtimes 1)^*(\cdots)$$

where $\sqrt{(\mathrm{Nm}_F X)_\lambda}$ means the Chern roots of the bundle.

6.11. Computation. Now, let us evaluate

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_2(E_k, \tilde{\beta})} \mathrm{ev}^*(\iota_{0*}\gamma, \boxtimes \iota_{\infty*}\gamma'[k]) \\
&= \sum_{\beta_1, \beta_2, F} \int_{\overline{\mathcal{M}}_2(X_0, \beta_1) \times_F \overline{\mathcal{M}}_2(X_\infty, \beta_2)} \frac{(\mathrm{ev}_1 \boxtimes \mathrm{ev}_1)^*(\iota_0^* \iota_{0*}\gamma \boxtimes \iota_\infty^* \iota_{\infty*}\gamma'[k])}{\mathrm{Nm}(\cdots)} \\
&= \sum_{\beta_1, \beta_2, F} \int_{\overline{\mathcal{M}}_2(X_0, \beta_1) \times \overline{\mathcal{M}}_2(X_\infty, \beta_2)} \frac{(\iota_0^* \iota_{0*}\gamma \boxtimes \iota_\infty^* \iota_{\infty*}\gamma'[k])}{\mathrm{Nm}(\cdots)} (\mathrm{ev}_2 \boxtimes \mathrm{ev}_2)^*(\Delta_F) \\
&= \sum_{\beta_1, \beta_2, F} \sum_{w} \int_{\overline{\mathcal{M}}_2(X_0, \beta_1)} \frac{z\,\mathrm{ev}^*(\gamma \boxtimes i_{F*}\sigma_w^F)}{z(z-\psi_1)} \cdot \mathrm{ev}_2^* \frac{1}{(\cdots)} \int_{\overline{\mathcal{M}}_2(X_\infty, \beta_2)} \frac{-z\,\mathrm{ev}^*(\gamma'[k] \boxtimes i_{F*}\sigma_F^w)}{-z(-z-\psi_1)} \\
&= \sum_{\beta_1, \beta_2, F} \sum_{u, w} \int_{\overline{\mathcal{M}}_2(X_0, \beta_1)} \frac{z\,\mathrm{ev}^*(\gamma \boxtimes i_{F*}\sigma_w^F)}{z(z-\psi_2)} \int_{\overline{\mathcal{M}}_2(X_\infty, \beta_2)} \frac{-z\,\mathrm{ev}^*(\gamma'[k] \boxtimes i_{F*}\sigma_F^u)}{-z(-z-\psi_2)} \int_F \frac{\sigma_F^w \sigma_u^F}{(\cdots)}
\end{aligned}$$

Here we omit the summand of $\beta_1, \beta_2 = 0$. Here we assume

$$[\Delta_F] = \sum_w \sigma_w^F \boxtimes \sigma_F^w \in H^*(F) \subset H_T^*(X).$$

We find

$$\begin{aligned}
\langle \tilde{\mathbb{S}}_k(\gamma), \gamma'[k]\rangle &= \sum_F q^{\sigma_F} \sum_{w,u} \langle M(\gamma,z), i_{F*}\sigma_w^F\rangle \langle M(\gamma',-z), i_{F*}\sigma_F^u\rangle [k] \int_F \frac{\sigma_F^w \sigma_u^F}{(\cdots)} \\
&= \sum_u \left\langle M(\gamma,z), \sum_F q^{\sigma_F} \sum_w i_{F*}\sigma_w^F \int_F \frac{\sigma_F^w \sigma_u^F}{(\cdots)} \right\rangle \langle M(\gamma',-z), i_{F*}\sigma_F^u\rangle [k] \\
&= \sum_u \left\langle M(\gamma,z), \sum_F q^{\sigma_F} \frac{i_{F*}\sigma_u^F}{(\cdots)} \right\rangle \langle M(\gamma',-z), \sigma_F^u\rangle [k].
\end{aligned}$$

We have

$$\begin{aligned}
\langle (\tilde{\mathbb{S}}_k(\gamma))[-k], \gamma'\rangle &= \sum_u \left\langle M(\gamma,z), \sum_F q^{\sigma_F} \frac{i_{F*}\sigma_u^F}{(\cdots)} \right\rangle [-k] \langle M(\gamma',-z), \sigma_F^u\rangle \\
&= \sum_u \left\langle M(\gamma,z)[-k], \sum_F q^{\sigma_F} \frac{i_{F*}\sigma_u^F}{(\cdots)}[-k] \right\rangle \langle M(\gamma',-z), \sigma_F^u\rangle
\end{aligned}$$

Let us compute

$$\frac{i_{F*}\sigma_u^F}{(\cdots)}[-k] = \frac{i_{F*}\sigma_u^F}{\mathrm{Eu}(\mathrm{Nm}_F X)} \prod_{\lambda \in \mathsf{char}(T)} \prod_{x \in \sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c \le 0}(x+\lambda+cz)}{\prod_{c \le -\langle \lambda, k\rangle}(x+\lambda+cz)}.$$

Let us denote

$$\Delta_F = \prod_{\lambda \in \mathsf{char}(T)} \prod_{x \in \sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\prod_{c \le 0}(x+\lambda+cz)}{\prod_{c \le -\langle \lambda, k\rangle}(x+\lambda+cz)}.$$

Note that $\{i_{F*}\sigma_F^u\}$ is dual to $\left\{\frac{i_{F*}\sigma_u^F}{\mathrm{Eu}(\mathrm{Nm}_F X)}\right\}$, so

$$\langle (\tilde{\mathbb{S}}_k(\gamma))[-k], \gamma'\rangle = \left\langle \sum_F q^{\sigma_F} \Delta_F M(\gamma,z)[-k], M(\gamma',-z) \right\rangle.$$

By 5.16,

$$(\tilde{\mathbb{S}}_k(\gamma))[-k] = M^{-1}\left(\sum_F q^{\sigma_F} \Delta_F \cdot M(\gamma,z)[-k], z\right).$$

6.12. Summary. Let us denote $\mathbb{S}_k$ by

$$\mathbb{S}_k(\gamma) = (\tilde{\mathbb{S}}_k \gamma)[-k].$$

We have the following commutative diagram

$$\begin{tikzcd} H_{\mathbb{T}}(X) \arrow[r, "M(-,z)"] \arrow[d, "\mathbb{S}_k"'] & H_{\mathbb{T}}(X)(q) \arrow[d, "\gamma\mapsto\bigoplus_F q^{\sigma_F}\Delta_F(\gamma[-k])"] \\ H_{\mathbb{T}}(X)(q) \arrow[r, "M(-,z)"] & H_{\mathbb{T}}(X)(q) \end{tikzcd}$$

6.13. Corollary. We have

$$\mathbb{S}_k \circ \mathbb{S}_\ell = q^{(\cdots)}\mathbb{S}_{k+\ell}.$$

Since M is non-degenerate, this reduces to the following easy identity

$$\Delta_F^\ell \cdot \Delta_F^k[-\ell] = \Delta_F^{k+\ell}.$$

6.14. Seidel element. Define

$$S_k = \lim_{z\to 0} \mathbb{S}_k(1) \in QH_T^*(X).$$

Note that

$$\left[z\partial_\lambda + \lambda, \sum_F q^{\sigma_F}\Delta_F\right] = z\sum_F (\partial_\lambda q^{\sigma_F})\Delta_F = o(z).$$

So

$$\left[\mathbb{S}_k, z\nabla_\lambda + \lambda *\right] = o(z).$$

Then by taking $z \to 0$, we see $\lim_{z\to 0}\mathbb{S}_k$ commutes with the quantum product with a divisor. When $H_T^*(X)$ is generated by divisor (after localization), it is given by the quantum product with S_k.

6.15. Remark. When $z = 1$, we can write Δ_F in terms of Gauss Gamma function

$$\Gamma(s) = \int_0^\infty t^s e^{-t}\frac{dt}{t}.$$

Recall that

$$\Gamma(s+1) = s\Gamma(s).$$

So when $a, b \in \mathbb{Z}$

$$\begin{aligned}\frac{\Gamma(s+a+1)}{\Gamma(s+b+1)} &= \frac{(s+a)\Gamma(s+a)}{(s+b)\Gamma(s+b)} = \cdots \\ &= \frac{(s+a)\cdots(s+c)\Gamma(s+c)}{(s+b)\cdots(s+c)\Gamma(s+c)} = \frac{\prod_{c\le a}(s+c)}{\prod_{c\le b}(s+c)}.\end{aligned}$$

As a result,

$$\begin{aligned}
\Delta_F|_{z=1} &= \prod_{\lambda\in\mathrm{char}(T)} \prod_{x\in\sqrt{(\mathrm{Nm}_F X)_\lambda}} \frac{\Gamma(x+\lambda+1)}{\Gamma(x+\lambda-\langle\lambda,k\rangle+1)} \\
&= \frac{\prod_{x\in\sqrt{\mathrm{Nm}_F(X,\rho_k)}}\Gamma(x+1)}{\prod_{x\in\sqrt{\mathrm{Nm}_F(X,\rho_0)}}\Gamma(x+1)}[-k] \\
&=: \frac{\Gamma\big(1+\mathrm{Nm}_F(X,\rho_k)\big)}{\Gamma\big(1+\mathrm{Nm}_F(X,\rho_0)\big)}[-k].
\end{aligned}$$

7. Quotient schemes

Hyperquot scheme.

7.1. Definition. Let us consider the case $X = \mathrm{Gr}(k,n)$ the Grassmannian variety. Then

$$\mathrm{Mor}_{\deg=d}(\mathbb{P}^1, \mathrm{Gr}(k,n)) = \{\text{locally split } \mathcal{V} \subset \mathcal{O}^n_{\mathbb{P}^1} \text{ of rank } k \text{ of degree } -d\}.$$

Let us denote

$$\mathrm{HQ}_d = \{\mathcal{V} \subset \mathcal{O}^n_{\mathbb{P}^1} \text{ of rank } k \text{ of degree } -d\}.$$

This is anther compactification of $\mathrm{Mor}_{\deg=d}(\mathbb{P}^1, \mathrm{Gr}(k,n))$. Denote $\mathrm{HQ} = \bigcup_{d\geq 0} \mathrm{HQ}_d$.

7.2. Universal bundle and tangent bundle. We have the universal exact sequence over $\mathbb{P}^1 \times \mathrm{HQ}$

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}^n_{\mathbb{P}^1\times \mathrm{HQ}} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

It is well-known that the tangent bundle

$$\mathscr{T}_{\mathrm{HQ}} = \mathrm{pr}_{2*}\, \mathcal{H}om_{\mathbb{P}^1\times \mathrm{HQ}}(\mathcal{V}, \mathcal{Q}).$$

We have

$$0 \longrightarrow \mathcal{V}\otimes \mathcal{V}^\vee \longrightarrow (\mathcal{V}^\vee)^{\oplus n} \longrightarrow \mathcal{H}om_{\mathbb{P}^1\times \mathrm{HQ}}(\mathcal{V}, \mathcal{Q}) \longrightarrow 0.$$

As $R^1\pi_{2*}\mathcal{V}^\vee = 0$, we have

$$R^1\,\mathrm{pr}_{2*}\, \mathcal{H}om_{\mathbb{P}^1\times \mathrm{HQ}}(\mathcal{V}, \mathcal{Q}) = 0.$$

This proves

$$[\mathcal{T}_{\mathrm{HQ}}] = \mathrm{pr}_{2*}(\mathcal{Q}\otimes \mathcal{V}^\vee) \in K_{\mathbb{C}^\times}(\mathrm{HQ}).$$

7.3. Fixed points. Over $\mathrm{Gr}(k,n)$, we have a $T = (\mathbb{C}^\times)^n$-action. Note that

$$\begin{aligned} \mathrm{Gr}(k,n)^T &= \left\{\text{coordinate } k\text{-subspaces of } \mathbb{C}^n\right\} \\ &= \{\mathbb{C}e_{a_1}\oplus\cdots\oplus\mathbb{C}e_{a_k} : a_1<\cdots<a_k\} \overset{1:1}{\longleftrightarrow} \binom{[n]}{k}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{HQ}^{T\times\mathbb{C}^\times} &= \left\{\text{cooridnate graded subsheaf of } \mathcal{O}^{\oplus n}_{\mathbb{P}^1}\right\} \\ &= \left\{\mathcal{I}_{m_1}e_{a_1}\oplus\cdots\oplus\mathcal{I}_{m_k}e_{a_k} : \begin{array}{l} a_1<\cdots<a_k, \\ m_i = (m_i^+, m_i^-)\in\mathbb{Z}^2_{\geq 0}. \end{array}\right\} \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_m &= \text{ideal sheaf of the cycle } m^+\cdot\{0\} + m^-\cdot\{\infty\} \\ &= \mathfrak{m}_0^{m^+}\cdot \mathfrak{m}_\infty^{m^-} \subset \mathcal{O}_{\mathbb{P}^1}. \end{aligned}$$

The sheaves over HQ_d are those with $\sum_{i=1}^k (m_i^+ + m_i^-) = d$.

7.4. Translating J-function. Recall the following diagram

$$\begin{tikzcd} G(X,d) \arrow[r, "\text{birational}"] & QM(X,d) \\ \overline{\mathcal{M}}_1(X,d) \arrow[u, "p"] \arrow[r, "ev"] & X. \arrow[u, "i"'] \end{tikzcd}$$

It implies

$$ev_*\left(\frac{1}{z-\psi}\right) = ev_*\left(\frac{1}{\mathrm{Eu}(p)}\right) = \frac{1}{\mathrm{Eu}(i)}.$$

We have a similar diagram for $X = \mathrm{Gr}(k,n)$.

$$\begin{tikzcd} QM(X,d) & HQ_d \arrow[l, "\text{birational}"'] \\ X \arrow[u, "i"] & \begin{array}{c}\text{one } \mathbb{C}^\times\text{-fixed} \\ \text{component}\end{array} \arrow[l, "j"'] \arrow[u, "q"'] \end{tikzcd}.$$

We have

$$\frac{1}{\mathrm{Eu}(i)} = j_*\left(\frac{1}{\mathrm{Eu}(q)}\right).$$

We can take the T-fixed points.

$$\left(\begin{array}{c}\text{one } \mathbb{C}^\times\text{-fixed} \\ \text{component}\end{array}\right)^T = \left\{\mathcal{I}_{m_1} e_{a_1} \oplus \cdots \oplus \mathcal{I}_{m_k} e_{a_k} : \text{as above, but all } m_i^- = 0\right\}.$$

By localization theorem, the localization at the fixed point $\phi \in X$ is

$$j_*\left(\frac{1}{\mathrm{Eu}(q)}\right)\Big|_\phi = \sum_{j(\Phi)=\phi} \frac{1}{\mathrm{Eu}(q)} \frac{\mathrm{Eu}\left(\{\phi\} \subset X\right)}{\mathrm{Eu}\left(\{\Phi\} \subset \mathrm{comp}\right)} = \sum_{j(\Phi)=\phi} \frac{\mathrm{Eu}(\mathscr{T}_X)|_\phi}{\mathrm{Eu}(\mathscr{T}_{HQ})|_\Phi}.$$

7.5. Computation of localization. Let us compute the localization at the fixed point

$$\phi = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k \in \mathrm{Gr}(k,n)$$

as an example. Let us consider

$$\Phi = \mathfrak{m}^{d_1} e_1 \oplus \cdots \oplus \mathfrak{m}^{d_k} e_k,$$

where $\mathfrak{m}$ is the ideal of $\{0\}$. Firstly,

$$\mathcal{H}om(\mathcal{V},\mathcal{Q})|_{\mathcal{P}^1\times\{\Phi\}} = \left(\sum_{i=1}^{n}\mathcal{O}(t_i)-\sum_{i=1}^{k}\mathfrak{m}^{d_i}(t_i)\right)\left(\sum_{j=1}^{k}\overline{\mathfrak{m}^{d_j}(t_j)}\right)$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{k}\mathfrak{m}^{-d_j}T_i/T_j-\sum_{i=1}^{k}\sum_{j=1}^{k}\mathfrak{m}^{d_i-d_j}T_i/T_j.$$

7.6. Lemma. We have

$$\chi(\mathbb{P}^1,\mathcal{F})=\frac{\mathcal{F}|_0}{1-\xi^{-1}}+\frac{\mathcal{F}|_\infty}{1-\xi}.$$

In our case,

$$\mathfrak{m}|_0=\xi^{-1},\qquad \mathfrak{m}|_\infty=1.$$

For example,

$$\chi(\mathbb{P}^1,\mathfrak{m})=\frac{\xi^{-1}}{1-\xi^{-1}}+\frac{1}{1-\xi}=0.$$

Thus

$$\chi(\mathbb{P}^1,\mathfrak{m}^d)=\frac{\xi^{-d}}{1-\xi^{-1}}+\frac{1}{1-\xi}=\frac{\xi^{-d}-\xi^{-1}}{1-\xi^{-1}}=\left(\sum_{a\le -d}-\sum_{a<0}\right)\xi^a.$$

As a result, for a divisor D,

$$\mathrm{Eu}(\cdots\otimes\mathcal{O}(D))=\frac{\prod_{a\le -d}(D+az)}{\prod_{a<0}(D+az)}.$$

7.7. Push forward. As a result,

$$\mathscr{T}_{HQ}|_\Phi=\mathrm{pr}_2*(\mathcal{H}om(\mathcal{V},\mathcal{Q}))|_\Phi=\chi\left(\mathbb{P}^1,\mathcal{H}om(\mathcal{V},\mathcal{Q})|_{\mathcal{P}^1\times\{\Phi\}}\right).$$

Its Euler class

$$\mathrm{Eu}(\mathscr{T}_{HQ}|_\Phi)=\prod_{i=1}^{n}\prod_{j=1}^{k}\frac{\prod_{a\le d_j}(t_i-t_j+az)}{\prod_{a<0}(t_i-t_j+az)}\prod_{i=1}^{k}\prod_{j=1}^{k}\frac{\prod_{a<0}(t_i-t_j+az)}{\prod_{a\le d_j-d_i}(t_i-t_j+az)}$$

Note that

$$\mathrm{Eu}(\mathscr{T}_X|_\phi)=\mathrm{Eu}\left(\mathrm{Hom}_X(\mathrm{Taut},\mathrm{Quot})|_\phi\right)=\prod_{i=1}^{k}\prod_{j=k+1}^{n}(t_i-t_j)$$

coincides with $\mathrm{Eu}(\mathscr{T}_{HQ}|_\Phi)$ when all $d_i=0$. As a result,

$$\frac{\mathrm{Eu}(\mathscr{T}_X|_\phi)}{\mathrm{Eu}(\mathscr{T}_{HQ}|_\Phi)}=\prod_{i=1}^{n}\prod_{j=1}^{k}\frac{\prod_{a\le 0}(t_i-t_j+az)}{\prod_{a\le d_j}(t_i-t_j+az)}\prod_{i=1}^{k}\prod_{j=1}^{k}\frac{\prod_{a\le d_j-d_i}(t_i-t_j+az)}{\prod_{a\le 0}(t_i-t_j+az)}.$$

Then if sum over all fixed point above ϕ, we get

$$j_*\left(\frac{1}{\mathrm{Eu}(q)}\right)\bigg|_{\phi} = \sum_{d_1+\cdots+d_k=d}(\cdots).$$

Global expression. The above computation works for any fixed points

$$\mathbb{C}e_{a_1}\oplus\cdots\oplus\mathbb{C}e_{a_k}\in \mathrm{Gr}(k,n).$$

. The computation gives the same expression, but with $t_1,\ldots,t_k$ replaced by $t_{a_1},\ldots,t_{a_k}$. Let $x_1,\ldots,x_n$ be the Chern roots of the tautological bundle. The computation shows

$$\begin{aligned}
j_*\left(\frac{1}{\mathrm{Eu}(q)}\right) &= \sum_{d_1+\cdots+d_k=d}\prod_{i=1}^{n}\prod_{j=1}^{k}\frac{\prod_{a\leq 0}(t_i-x_j+az)}{\prod_{a\leq d_j}(t_i-x_j+az)}\prod_{i,j=1}^{k}\frac{\prod_{a\leq 0}(x_i-x_j+az)}{\prod_{a\leq d_j-d_i}(x_i-x_j+az)}\\
&= \sum_{d_1+\cdots+d_k=d}\prod_{j=1}^{k}\prod_{i=1}^{n}\frac{1}{\prod_{a=1}^{d_j}(t_i-x_j+az)}\prod_{i,j=1}^{k}\frac{\prod_{a\leq 0}(x_i-x_j+az)}{\prod_{a\leq d_j-d_i}(x_i-x_j+az)}.
\end{aligned}$$

Nonequivariantly, it is given by

$$\sum_{d_1+\cdots+d_k=d}\frac{1}{\prod_{j=1}^{k}\prod_{a=1}^{d_j}(az-x_j)^n}\prod_{i,j=1}^{k}\frac{\prod_{a\leq 0}(x_i-x_j+az)}{\prod_{a\leq d_j-d_i}(x_i-x_j+az)}.$$

As a result, the J-function is

$$\sum_{d_1+\cdots+d_k}\frac{Q^{d_1}\cdots Q^{d_k}}{\prod_{j=1}^{k}\prod_{a=1}^{d_j}(az-x_j)^n}\prod_{i,j=1}^{k}\frac{\prod_{a\leq 0}(x_i-x_j+az)}{\prod_{a\leq d_j-d_i}(x_i-x_j+az)}.$$

Vector bundles over $\mathbb{P}^1$.

7.8. Decomposition. Any vector bundle $\mathcal{V}$ over $\mathbb{P}^1$ is a direct sum of line bundles.

Let us denote $\deg(\mathcal{V}) = c_1(\mathcal{V}) \in H^2(\mathbb{P}^1) \simeq \mathbb{Z}$. In particular, for a line bundle $\mathcal{L}$, $\deg \mathcal{L} = d$ if $\mathcal{L} \simeq \mathcal{O}(d)$. Moreover, $\deg(\mathcal{V}_1 \oplus \mathcal{V}_2) = \deg(\mathcal{V}_1) + \deg(\mathcal{V}_2)$.

7.9. Proof. Over $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ and $\mathbb{P}^1 \setminus \{0\}$, the vector bundle is trivial. It only depends on how we glue them along $\mathbb{C}^\times$. So

$$\{\text{vector bundles over } \mathbb{P}^1\}/_{\cong} = GL[z^{-1}]\backslash GL[z^{\pm 1}]/GL[z].$$

Linear algebra tells each class can be diagonalized to be $\mathrm{diag}(z^{k_1}, \dots, z^{k_n})$. This gives the decomposition into line bundles.

7.10. Uniqueness. Let $\mathcal{V}$ be a vector bundle over $\mathbb{P}^1$. Then in any decomposition $\mathcal{V} = \bigoplus \mathcal{L}_i$ the summand $\sum_{\deg \mathcal{L}_i \geq 0} \mathcal{L}_i$ does not depends on the decomposition. This follows from the fact

$$\mathrm{Hom}_{\mathbb{P}^1}\left(\begin{matrix}\text{line bundle of}\\ \text{degree} \geq 0\end{matrix}, \begin{matrix}\text{line bundle of}\\ \text{degree} < 0\end{matrix} \right) = 0.$$

This can be seen from the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \binom{\text{summand of}}{\text{degree} \geq 0} \arrow[r] \arrow[drr, "0"] & \mathcal{V} \arrow[r] \arrow[d, equal] & \binom{\text{summand of}}{\text{degree} > 0} \arrow[r] & 0 \\
0 \arrow[r] & \binom{\text{summand of}}{\text{degree} \geq 0} \arrow[r] & \mathcal{V} \arrow[r] & \binom{\text{summand of}}{\text{degree} > 0} \arrow[r] & 0
\end{tikzcd}$$

In fancy language, this defines a torsion pair over the category of vector bundles over $\mathbb{P}^1$.

7.11. Filtration. We have a natural (i.e. functorial) split flag (called the Harder–Narasimhan filtration.)

$$\mathcal{V} \supseteq \cdots \supseteq \mathcal{V}_{\deg \geq -1} \supseteq \mathcal{V}_{\deg \geq 0} \supseteq \mathcal{V}_{\deg \geq 1} \supseteq \cdots \supseteq 0.$$

7.12. Ext vanishing. By Serre duality,

$$\mathrm{Ext}_{\mathbb{P}^1}\left(\begin{matrix}\text{line bundle of}\\ \text{degree} \leq 1\end{matrix}, \begin{matrix}\text{line bundle of}\\ \text{degree} \geq 0\end{matrix} \right) = 0.$$

Hyperquot scheme.

7.13. Definition. Let us consider the case $X = \mathrm{Gr}(k,n)$ the Grassmannian variety. Then

$$\mathrm{Mor}_{\deg=d}(\mathbb{P}^1, \mathrm{Gr}(k,n)) = \{\text{locally split } \mathcal{V} \subset \mathcal{O}_{\mathbb{P}^1}^n \text{ of rank } k \text{ of degree } -d\}.$$

Let us denote

$$\mathrm{HQ}_d = \{\mathcal{V} \subset \mathcal{O}_{\mathbb{P}^1}^n \text{ of rank } k \text{ of degree } -d\}.$$

This is anther compactification of $\mathrm{Mor}_{\deg=d}(\mathbb{P}^1, \mathrm{Gr}(k,n))$. Denote $\mathrm{HQ} = \bigcup_{d\geq 0} \mathrm{HQ}_d$.

7.14. Remark. Since there is no nonzero morphism from $\mathcal{O}(n)$ to $\mathcal{O}_{\mathbb{P}^1}$ for $n > 0$, each line bundle in any decomposition of $\mathcal{V}$ has degree ≤ 0. It would be useful to consider the quotient bundle $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^1}^n/\mathcal{V}$. By the similar reason, each line bundle in any decomposition of $\mathcal{Q}$ has degree ≥ 0.

7.15. Universal bundle and tangent bundle. We have the universal exact sequence over $\mathbb{P}^1 \times \mathrm{HQ}$

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathrm{HQ}}^n \longrightarrow \mathcal{Q} \longrightarrow 0.$$

It is well-known that the tangent bundle

$$\mathscr{T}_{\mathrm{HQ}} = \mathrm{pr}_{2*}\,\mathcal{H}om_{\mathbb{P}^1\times\mathrm{HQ}}(\mathcal{V}, \mathcal{Q}).$$

We have

$$0 \longrightarrow \mathcal{V}\otimes\mathcal{V}^\vee \longrightarrow (\mathcal{V}^\vee)^{\oplus n} \longrightarrow \mathcal{H}om_{\mathbb{P}^1\times\mathrm{HQ}}(\mathcal{V}, \mathcal{Q}) \longrightarrow 0.$$

As $R^1\pi_{2*}\mathcal{V}^\vee = 0$, we have

$$R^1\,\mathrm{pr}_{2*}\,\mathcal{H}om_{\mathbb{P}^1\times\mathrm{HQ}}(\mathcal{V}, \mathcal{Q}) = 0.$$

This proves

$$[\mathcal{T}_{\mathrm{HQ}}] = \mathrm{pr}_{2*}(\mathcal{Q}\otimes\mathcal{V}^\vee) \in K_{\mathbb{C}^\times}(\mathrm{HQ}).$$

7.16. Remark. Since there is no nonzero morphism from $\mathcal{O}(\text{positive})$ to $\mathcal{O}_{\mathbb{P}^1}$, the filtration of a subsheaf of $\mathcal{O}^n$ takes the form

$$\mathcal{O}_{\mathbb{P}^1}^n \supseteq \mathcal{V} \supseteq \cdots \supseteq \mathcal{V}_{\deg\geq -2} \supseteq \mathcal{V}_{\deg\geq -1} \supseteq \mathcal{V}_{\deg\geq 0} \supseteq 0 = \cdots = 0.$$

7.17. degenerate loci. Let $\mathcal{V} \subset \mathbb{O}^n$ be a subsheaf of rank k. The inclusion induces

$$\Lambda^k\mathcal{V} \longrightarrow \Lambda^k\mathcal{O}_{\mathbb{P}^1}^n = \mathcal{O}_{\mathbb{P}^1}^{\binom{n}{k}}.$$

We denote the zero of this morphism by $|\mathcal{V}|$ (a 0-dimensional subscheme). Actually, this is the locus of rank degeneration of the inclusion $\mathcal{V} \subset \mathcal{O}^n$.

7.18. Fixed points. Let us compute the $\mathbb{C}^\times$-fixed points of HQ_d.

$$HQ^{\mathbb{C}^\times} = \{\mathbb{C}^\times\text{-equivariant } \mathcal{V} \subset \mathcal{O}^n_{\mathbb{P}^1} \text{ of rank } k\}.$$

Let $\mathcal{V}$ be an equivariant bundle of rank k. Note that $|\mathcal{V}|$ can only be supported over $(\mathbb{P}^1)^{\mathbb{C}^\times} = \{0, \infty\}$. For simplitICTY, assume

$$|\mathcal{V}| \text{ is supported over } \{0\}, \text{ i.e. } \mathcal{V} \text{ is locally split at } \infty. \tag{$*$}$$

Over $\mathbb{C}^\times \subset \mathbb{P}^1$ the subsheaf $\mathcal{V}|_{\mathbb{C}^\times}$ is a subbundle (i.e. locally split subsheaf) of $\mathcal{O}^n_{\mathbb{C}^\times}$ and it must be $\mathcal{O}_{\mathbb{C}^\times} \otimes V$ for some subspace $V \subset \mathbb{C}^n$ of dimension k.

Since the filtration is natural, each member of the filtration is also equivariant. This defines a flag

$$\mathbb{C}^n \supseteq \varphi = \varphi_m \supseteq \cdots \supseteq \varphi_2 \supseteq \varphi_1 \supseteq \varphi_0 \supseteq 0, \qquad \dim \varphi = k.$$

Then we can reconstruct

$$\mathcal{V} = \sum_{r \geq 0} \mathfrak{m}_0^r \otimes \varphi_r \subset \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^n = \mathcal{O}^{n+1}_{\mathbb{P}^1}.$$

Recall that the ideal sheaf $\mathfrak{m}_0 \subset \mathcal{O}_{\mathbb{P}^1}$ is isomorphic to $\mathcal{O}(-1)$. Its degree is

$$\sum_{r \geq 0} (\dim \varphi_r - \dim \varphi_{r-1}) \cdot r.$$

This defines an embedding

$$k : \mathcal{F}\ell \longrightarrow HQ^{\mathbb{C}^\times}$$

to the component with property $(*)$.

7.19. Translating J-function. We have the following diagram.

$$\begin{tikzcd}
G(X,d) \arrow[r, dashed, "\text{birational}"] & QM(X,d) & HQ_d \arrow[l, dashed, "\text{birational}"'] \\
\overline{\mathcal{M}}_1(X,d) \arrow[u, "i"] \arrow[r, "ev"] & X \arrow[u, "j"] & \mathcal{F}\ell . \arrow[u, "k"] \arrow[l, "\pi"']
\end{tikzcd}$$

We see that

$$ev_*\left(\frac{1}{z-\psi}\right) = \pi_*\left(\frac{1}{\mathrm{Eu}(\mathrm{Nm}(k))}\right).$$

7.20. Euler class. When restricting to $\mathcal{F}\ell$ via k, we have

$$[\mathcal{V}] = \sum_{r \geq 0} \mathfrak{m}_0^r \boxtimes (\Phi_r - \Phi_{r-1}) \in K(\mathbb{P}^1 \times \mathcal{F}\ell).$$

So

$$\mathcal{H}om(\mathcal{V}, \mathcal{Q}) = \overline{[\mathcal{V}]}^n - [\mathcal{V}]\overline{[\mathcal{V}]}$$