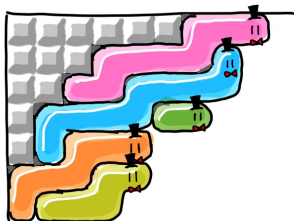


Pieri Rules over Grassmannian and Applications

arXiv:2402.04500

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Rui Xiong



Grassmannian

Recall that **Grassmannian manifold**

$$\mathrm{Gr}(k, n) = \{ V \subseteq \mathbb{C}^n : \dim V = k \}.$$

We have the following **Bruhat decomposition**

$$\mathrm{Gr}(k, n) = \bigcup_{\lambda \subseteq (n-k)^k} Y(\lambda)^\circ \quad (\text{disjoint}),$$

where $Y(\lambda)^\circ$ is the **opposite Schubert cell**.

Description

For example, $V \in Y \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \right)^\circ \iff$

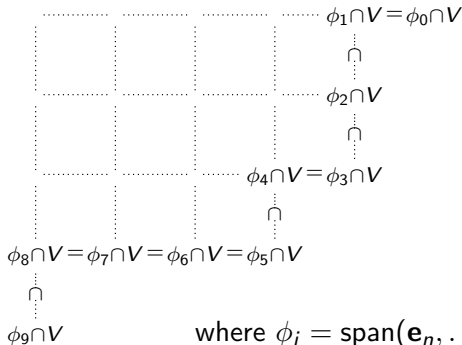
codim=4

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codim=1

codim=0



where $\phi_i = \text{span}(\mathbf{e}_n, \dots, \mathbf{e}_{n-i+1})$.

Bruhat Decomposition

Denote **Schubert variety**

$$Y(\lambda) = \overline{Y(\lambda)^\circ} = \bigcup_{\mu \supseteq \lambda} Y(\mu)^\circ.$$

The **cohomology group** and **K-group** can be computed to be

$$H^*(\mathrm{Gr}(k, n)) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q} \cdot [Y(\lambda)].$$

$$K(\mathrm{Gr}(k, n)) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q} \cdot [\mathcal{O}_{Y(\lambda)}]$$

Chern Classes

Let \mathcal{V} be the **tautological bundle** over $\mathrm{Gr}(k, n)$. We denote

$c_r = c_r(\mathcal{V}^\vee)$ = the r -th equivariant **Chern classes** of \mathcal{V}^\vee .

It is known that

$$H^*(\mathrm{Gr}(k, n)) = \mathbb{Q}[c_1, \dots, c_k] / \text{some ideal}.$$

$$K(\mathrm{Gr}(k, n)) = \mathbb{Q}[c_1, \dots, c_k] / \text{some ideal}.$$

Geometry of cohomology

Roughly speaking **cohomology**

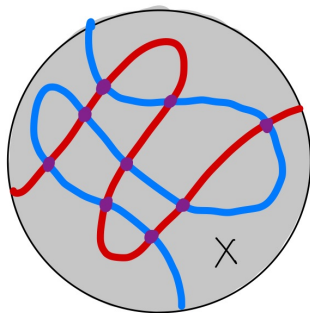
$$H^\bullet(X) = \bigoplus_{Y \text{ closed } \subseteq X} \mathbb{Q} \cdot [Y] \Big/ \text{HOMOTOPY EQUIVALENCE}$$

with product **the transversal intersection**

$$[Y_1] \cdot [Y_2] = [Y_1 \pitchfork Y_2].$$

Over $\text{Gr}(k, n)$, we have **Schubert class**

$$[Y(\lambda)] \in H^{2\ell(\lambda)}(\text{Gr}(k, n)).$$



Example: $\text{Gr}(1, 2) = \mathbb{P}^1$

We have

$Y(\square) = \text{the point } \infty,$

$Y(\emptyset) = \text{the entire } \mathbb{P}^1.$

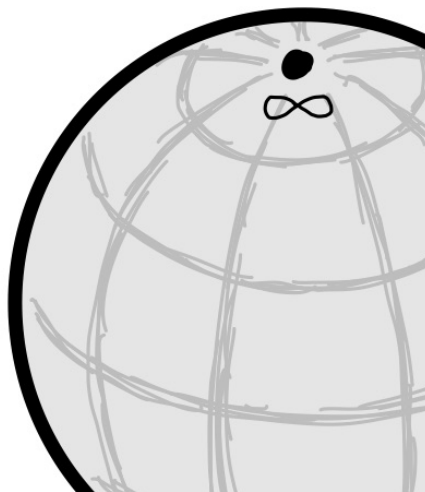
The intersection

| \cap | pt | \mathbb{P}^1 |
|----------------|-------------|----------------|
| pt | \emptyset | pt |
| \mathbb{P}^1 | pt | \mathbb{P}^1 |

The cohomology

$$H^*(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2)$$

where $x = [Y(\square)]$.

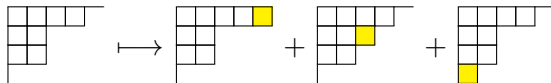


Classical Chevalley Formula

Theorem (Chevalley Formula)

$$c_1(\mathcal{V}^\vee) \cdot [Y(\lambda)] = \sum_{\mu=\lambda+\square} [Y(\mu)].$$

Example:



Classical Pieri Rule

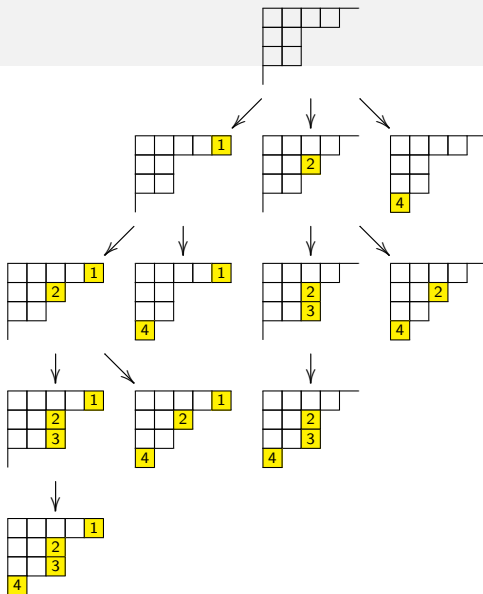
Let us denote **Schur operators**

$$[i] \rightarrow [Y(\lambda)] = \begin{cases} [Y(\mu)], & \mu = \lambda + \square \text{ in the } i\text{-th row,} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (Pieri Rule)

$$c_r(\mathcal{V}^\vee) \cdot [Y(\lambda)] = \sum_{1 \leq i_1 < \dots < i_r \leq k} [i_r] \rightarrow \dots \rightarrow [i_1] \rightarrow [Y(\lambda)].$$

Example



Geometry of K-theory

Roughly speaking, the **K-theory**

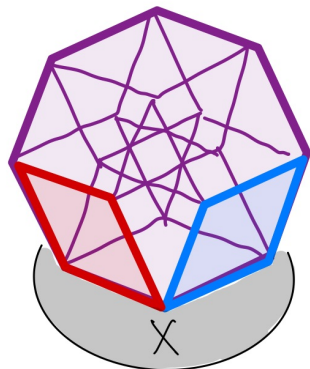
$$K(X) = \bigoplus_{\mathcal{F} \in \text{Coh } X} \mathbb{Q} \cdot [\mathcal{F}] \Big/ \text{EXACT SEQUENCES}$$

with product **the tensor product**

$$[\mathcal{F}_1] \cdot [\mathcal{F}_2] = [\mathcal{F}_1 \otimes \mathcal{F}_2].$$

Over $\text{Gr}(k, n)$, we have **structure sheaves**

$$[\mathcal{O}_{Y(\lambda)}] \in K(\text{Gr}(k, n)).$$



Example: $\mathrm{Gr}(1, 2) = \mathbb{P}^1$

We have

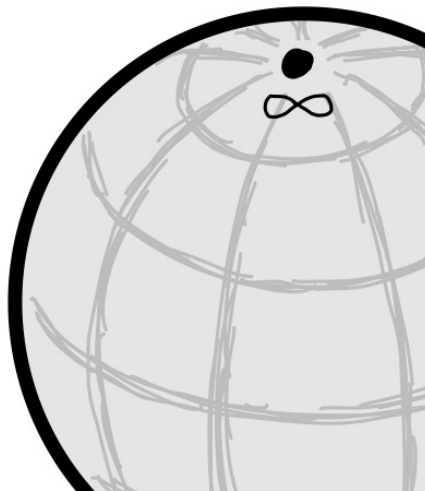
$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_\infty \rightarrow 0.$$

Thus $[\mathcal{O}_\infty] = 1 - \mathcal{O}(-1)$.

The K-theory

$$K(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2)$$

where $x = 1 - \mathcal{O}(-1) = [\mathcal{O}_{Y(\square)}]$.

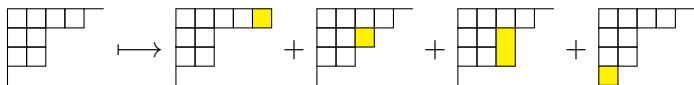


K-theory Chevalley Formula

Theorem (Lenart [1])

$$c_1(\mathcal{V}^\vee) \cdot [\mathcal{O}_{Y(\lambda)}] = \sum_{\mu=\lambda+\square} [\mathcal{O}_{Y(\mu)}].$$

Example:



K-theory Pieri Rule

Let us denote **Schur operators**

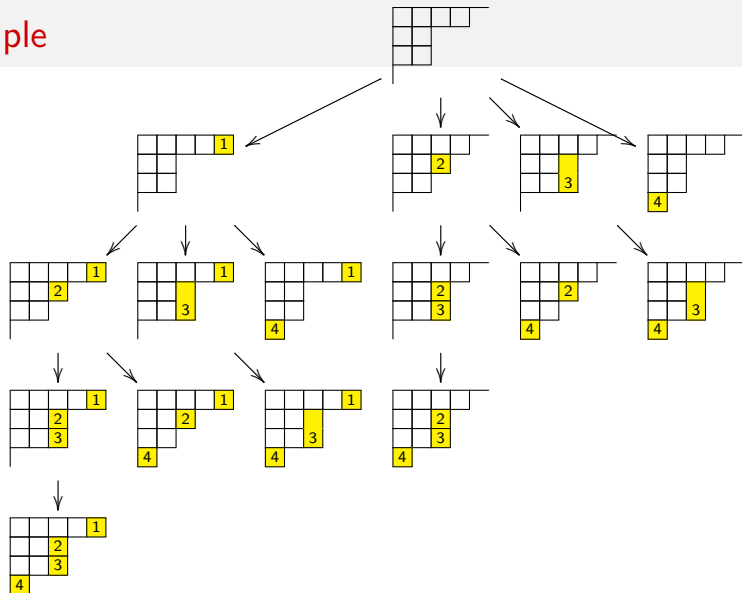
$$[i] \rightarrow [\mathcal{O}_{Y(\lambda)}] = [\mathcal{O}_{Y(\mu)}]$$

where $\mu = \lambda +$ a vertical strip with its tail at the i -th row.

Theorem (Lenart [1])

$$c_r(\mathcal{V}^\vee) \cdot [\mathcal{O}_{Y(\lambda)}] = \sum_{1 \leq i_1 < \dots < i_r \leq k} [i_r] \rightarrow \dots [i_1] \rightarrow [\mathcal{O}_{Y(\lambda)}].$$

Example



Constructible Functions

Consider

$$\begin{aligned}\mathrm{Fun}(X) &= \{\text{constructible functions over } X\} \\ &= \mathrm{span}(\mathbf{1}_A : A \subseteq X \text{ closed}).\end{aligned}$$

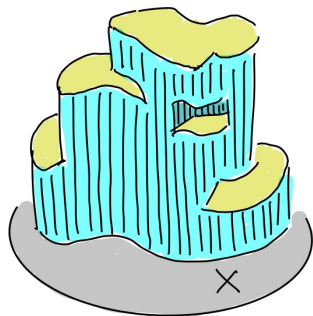
For any proper map $f : X \rightarrow Y$, we have a **push-forward**

$$f_* : \mathrm{Fun}(X) \rightarrow \mathrm{Fun}(Y)$$

defined such that

$$(f_*(\mathbf{1}_A))(y) = \chi_c(A_y)$$

the Euler characteristic of fibre A_y .



CSM classes

By MacPherson [2], there is a natural transform (wrt push-forward) called **Chern–Schwartz–MacPherson classes**

$$c_{\text{SM}} : \text{Fun}(-) \rightarrow H_{\bullet}(-),$$

such that when X is smooth

$$c_{\text{SM}}(X) = \text{total Chern class of the tangent bundle of } X.$$

Over $\text{Gr}(k, n)$, we have **CSM classes**

$$c_{\text{SM}}(Y(\lambda)^{\circ}) := c_{\text{SM}}(\mathbf{1}_{Y(\lambda)^{\circ}}) \in H^*(\text{Gr}(k, n)).$$

Example: $\text{Gr}(1, 2) = \mathbb{P}^1$

Recall

$$Y(\square)^\circ = \text{the point } \infty,$$

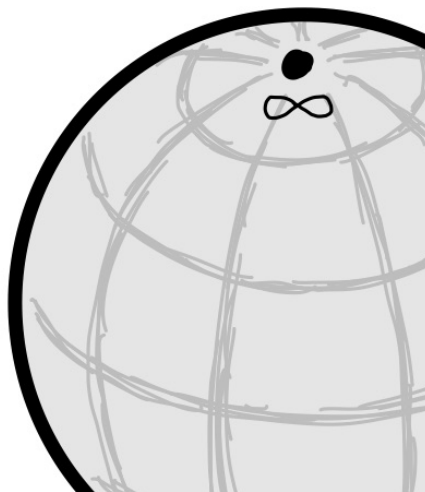
$$Y(\emptyset)^\circ = \mathbb{P}^1 \setminus \{\infty\}.$$

So by definition,

$$c_{\text{SM}}(Y(\square)^\circ) = [Y(\square)] = x.$$

Since $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}(2)$, we have

$$\begin{array}{rcl} \text{total Chern class} & = & 1 + 2x \\ \hline c_{\text{SM}}(Y(\square)^\circ) & = & x \\ c_{\text{SM}}(Y(\emptyset)^\circ) & = & 1 + x \end{array}$$

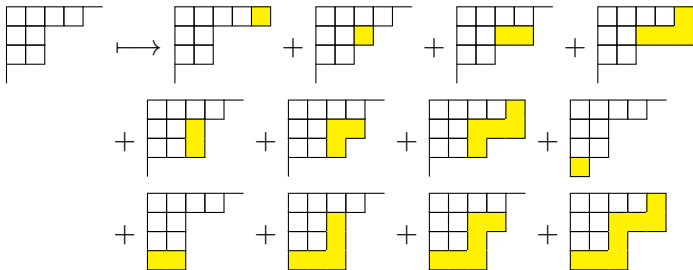


CSM Chevalley formula

Theorem (Aluffi, Mihalcea, Schürmann and Su [3])

$$c_1(\mathcal{V}^\vee) \cdot c_{\text{SM}}(Y(\lambda)^\circ) = \sum_{\mu=\lambda+\begin{smallmatrix} \square \\ \square \end{smallmatrix}} c_{\text{SM}}(Y(\mu)^\circ).$$

Example:



CSM Pieri Rule

Let us denote **ribbon Schubert operators**

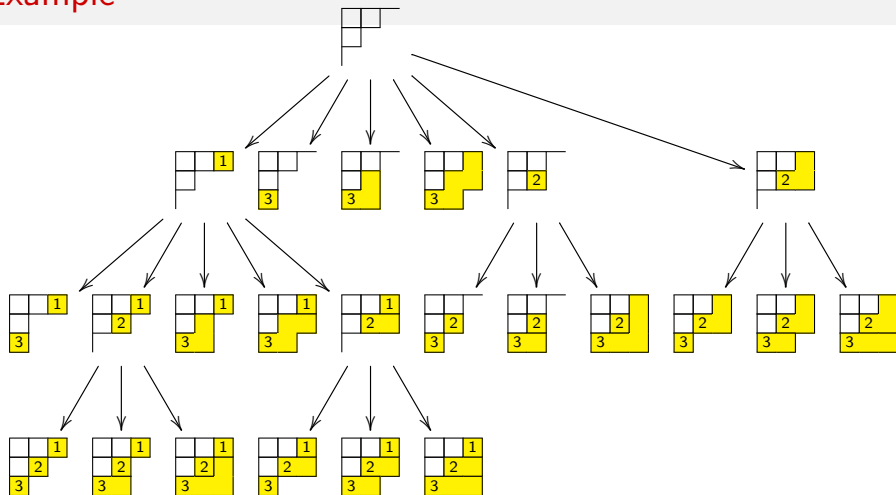
$$[i] \rightarrow c_{\text{SM}}(Y(\lambda)^\circ) = \sum_{\mu} c_{\text{SM}}(Y(\mu)^\circ)$$

where the sum over $\mu = \lambda +$ a ribbon strip with its tail at the i -th row.

Theorem (Fan, Guo and Xiong [4])

$$c_r(\mathcal{V}^\vee) \cdot c_{\text{SM}}(Y(\lambda)^\circ) = \sum_{1 \leq i_1 < \dots < i_r \leq k} [i_r] \rightarrow \dots [i_1] \rightarrow c_{\text{SM}}(Y(\lambda)^\circ).$$

Example



Grothendieck group

Consider the **Grothendieck group of varieties**

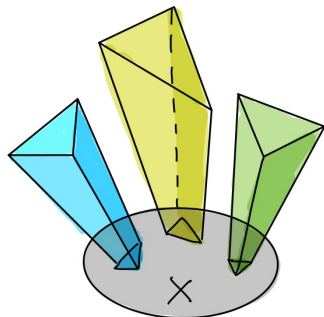
$$G(X) = \bigoplus_{\text{variety } Z \rightarrow X} \mathbb{Z} \cdot [Z \rightarrow X] \Big/ [U \rightarrow X] + [Z \setminus U \rightarrow X] = [Z \rightarrow X].$$

For any proper map $f : X \rightarrow Y$, we have a **push-forward**

$$f_* : G(X) \rightarrow G(Y)$$

with

$$f_*[Z \rightarrow X] = [Z \rightarrow X \rightarrow Y].$$



Motivic Chern classes

By Brasselet, Schürmann and Yokura [5], there is a natural transform (wrt push-forward) called **motivic Chern classes**

$$\mathrm{MC}_y : G(-) \rightarrow K(-)[y],$$

such that when X is smooth,

$$\mathrm{MC}_y(X) = \lambda\text{-class} = \sum_{k=1}^{\dim X} y^k [\Lambda^k \mathcal{T}_X^\vee].$$

Over $\mathrm{Gr}(k, n)$, we have **motivic Chern classes**

$$\mathrm{MC}_y(Y(\lambda)^\circ) := \mathrm{MC}_y([Y(\lambda)^\circ \rightarrow \mathrm{Gr}(k, n)]) \in K(\mathrm{Gr}(k, n)).$$

Example: $\text{Gr}(1, 2) = \mathbb{P}^1$

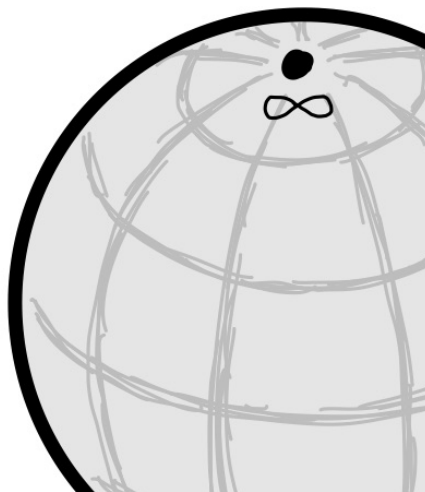
Similarly,

$$\text{MC}_y(Y(\square)^\circ) = [\mathcal{O}_{Y(\square)}] = x.$$

Recall that $x = 1 - \mathcal{O}(-1)$.

Since $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}(2)$, we have

$$\begin{array}{l} \lambda\text{-class} = 1 + y\mathcal{O}(-2) \\ \quad = (1 + y) - 2yx \\ \hline \text{MC}_y(Y(\emptyset)^\circ) = (1 + y) - (2y + 1)x \\ \text{MC}_y(Y(\square)^\circ) = \quad \quad \quad x \end{array}$$



MC Chevalley formula

Theorem (Fan, Guo, Su and Xiong)

$$c_1(\mathcal{V}^\vee) \cdot \text{MC}_y(Y(\lambda)^\circ) = (1+y) \sum_{\mu=\lambda+\begin{smallmatrix} \square \\ \square \end{smallmatrix}} (-y)^{\text{wd}(\mu/\lambda)-1} \text{MC}_y(Y(\mu)^\circ).$$

Example:

$$\begin{aligned} & \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} \mapsto (1+y) \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \textcolor{yellow}{\square} \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} - y \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} + y^2 \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & & & \\ \hline \end{array} \right. \\ & + \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} - y \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & & & \\ \hline \end{array} + y^2 \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \textcolor{yellow}{\square} & & & \\ \hline \end{array} \\ & \left. - y \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \textcolor{yellow}{\square} & & & \\ \hline \end{array} + y^2 \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \end{array} - y^3 \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \end{array} + y^4 \begin{array}{|c|c|c|c|} \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \square & \square & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} & \textcolor{yellow}{\square} \\ \hline \end{array} \right) \end{aligned}$$

MC Pieri Rule

Let us denote **ribbon Schubert operators**

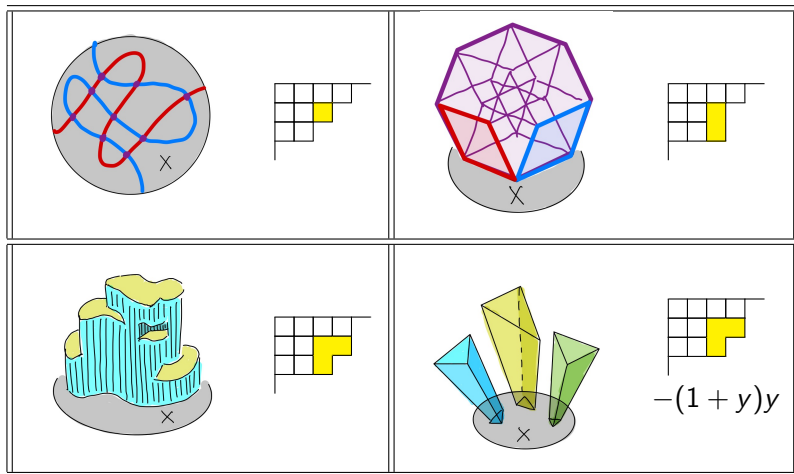
$$[i | \rightarrow \text{MC}_y(Y(\lambda)^\circ) = (1 + y) \sum_{\mu} (-y)^{\text{wd}(\mu/\lambda)-1} \text{MC}_y(Y(\mu)^\circ)$$

where the sum over $\mu = \lambda +$ a ribbon strip with its tail at the i -th row.

Theorem (Fan, Guo, Su and Xiong)

$$c_r(\mathcal{V}^\vee) \cdot \text{MC}_y(Y(\lambda)^\circ) = \sum_{1 \leq i_1 < \dots < i_r \leq k} [i_r | \rightarrow \dots [i_1 | \rightarrow \text{MC}_y(Y(\lambda)^\circ).$$

Summary



Affine Hecke algebra

Our approach is by introducing a version of affine Hecke algebra of three parameters

$$T_i^2 = -(p - q)T_i + pq$$

$$T_i T_j = T_j T_i, \quad |i - j| > 1,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$x_i x_j = x_j x_i,$$

$$T_i x_j = x_j T_i, \quad j \neq i, i + 1,$$

$$T_i x_i = x_{i+1} T_i + (\hbar - (p - q)x_i),$$

$$T_i x_{i+1} = x_i T_i - (\hbar - (p - q)x_i).$$

Rôles of p, q, \hbar

It turns out that p, q, \hbar control the following ribbon statistics

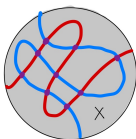
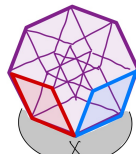
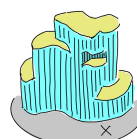
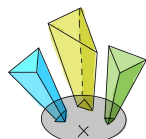
p : height $- 1$, q : width $- 1$, \hbar : number of ribbons.

We have the following table

| classes | (p, q, \hbar) | Pieri rule |
|------------------------------|------------------|---------------------------------------------------------------------------------------------|
| $[Y(\lambda)]$ | $(0, 0, 1)$ | adding boxes \square |
| $[\mathcal{O}_{Y(\lambda)}]$ | $(1, 0, 1)$ | adding vertical strips $\begin{array}{ c } \hline \square \\ \hline \end{array}$ |
| $c_{SM}(Y(\lambda)^\circ)$ | $(1, 1, 1)$ | adding ribbons $\begin{array}{ c } \hline \square \\ \hline \end{array}$ |
| $MC_y(Y(\lambda)^\circ)$ | $(1, -y, 1 + y)$ | adding ribbons $\begin{array}{ c } \hline \square \\ \hline \end{array}$ and counting width |

Dual Basis

In all four cases, we have another choice of basis

| | | | | |
|---------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
| Theory |  |  |  |  |
| basis | $[Y(\lambda)]$ | $[\mathcal{I}_{\partial Y(\lambda)}]$ | $c_{SM}(Y(\lambda)^\circ)$ | $MC_y(Y(\lambda)^\circ)$ |
| opposite dual basis | $[Y(\lambda)]$ | $[\mathcal{O}_{Y(\lambda)}]$ | $s_{SM}(Y(\lambda)^\circ)$ | $SMC_y(Y(\lambda)^\circ)$ |

Theorem (Fan, Guo, Su and Xiong)

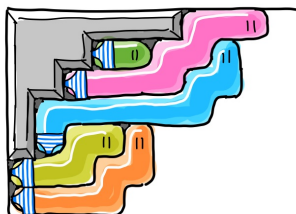
The **opposite dual basis** has the same Pieri rule as **basis**.

Discussion of the proof

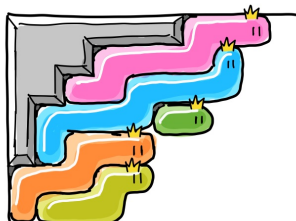
A priori, the Pieri rule for opposite dual basis is given by $|i\rangle$, the adjoint operators on the 180° rotated of the complement.

$[i] \dots$ with its **tail** at the i -th row \longleftrightarrow $|i\rangle \dots$ with its **head** at the i -th row \dots

But they are equivalent:



V.S.



Equivariant version

All the basis are defined in equivariant cohomology/K-theory.

Theorem (Fan, Guo, Su and Xiong)

The equivariant classes

$$[Y(\lambda)], \quad [\mathcal{I}_{\partial Y(\lambda)}], \quad c_{SM}(Y(\lambda)^\circ), \quad MC_Y(Y(\lambda)^\circ)$$

satisfy the **head-valued Pieri rule**, i.e. $|i\rangle$ or equivalently $[i|$.

Theorem (Fan, Guo, Su and Xiong)

The equivariant classes

$$[Y(\lambda)], \quad [\mathcal{O}_{Y(\lambda)}], \quad s_{SM}(Y(\lambda)^\circ), \quad SMC_Y(Y(\lambda)^\circ)$$

satisfy the **tail-valued Pieri rule**, i.e. $\dagger i]$ or equivalently $\{i|$.

$$|2\rangle \rightarrow \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} = t_3 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_4) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_5) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\ + (\hbar - (p - q)t_4)pq \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_5)pq^2 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array},$$

$$\begin{aligned} \downarrow 2] \rightarrow & \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} = t_3 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_3) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_3) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\ & + (\hbar - (p - q)t_1)pq \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_1)pq^2 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \end{aligned}$$

$$\begin{aligned}
[2\vdash \rightarrow \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} &= t_3 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_4) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & 2 & \\ \hline \end{array} + (\hbar - (p - q)t_5) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & 2 & \\ \hline \end{array} \\
&+ (\hbar - (p - q)t_5)q \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & 2 & \\ \hline \end{array} + (\hbar - (p - q)t_7)pq^2 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & 2 & \\ \hline \end{array},
\end{aligned}$$

$$\begin{aligned} \{2| \rightarrow & \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} = t_3 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_3) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_3) \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\ & + (\hbar - (p - q)t_3)q \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + (\hbar - (p - q)t_3)pq^2 \cdot \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}. \end{aligned}$$

Application A

There is a classic relation between ideal sheaves and structure sheaves over Grassmannian.

Theorem (Buch [6], see also [7, Prop. 4.2])

$$(1 - [\mathcal{O}_{Y(\square)}]) \cdot [\mathcal{O}_{Y(\lambda)}] = [\mathcal{I}_{\partial Y(\lambda)}].$$

This can be generalized to equivariant K-theory.

$$\frac{(1 - [\mathcal{O}_{Y(\square)}]) \cdot [\mathcal{O}_{Y(\lambda)}]}{1 - [\mathcal{O}_{Y(\square)}]|_{\lambda}} = [\mathcal{I}_{\partial Y(\lambda)}] \in K_T(\mathrm{Gr}(k, n)).$$

Relation between MC and SMC

Using our Pieri rule, we can prove the following analogy for MC and SMC classes.

Theorem (Fan, Guo, Su and Xiong)

$$\lambda_y(\mathcal{T}_{\text{Gr}(k,n)}^\vee) \cdot (1 - [\mathcal{O}_{Y(\square)}]) \cdot \text{SMC}_y(Y(\lambda)^\circ) = \text{MC}_y(Y(\lambda)^\circ).$$

This can be generalized to equivariant K-theory.

$$\lambda_y(\mathcal{T}_{\text{Gr}(k,n)}^\vee) \cdot \frac{(1 - [\mathcal{O}_{Y(\square)}]) \cdot \text{SMC}_y(Y(\lambda)^\circ)}{1 - [\mathcal{O}_{Y(\square)}]_\lambda} = \text{MC}_y(Y(\lambda)^\circ).$$

If we set $y = 0$, we will recover the result in the previous page.

Discussion of the proof

The proof is one sentence:

both sides have the same Pieri rule and agrees after certain specialization.

Precisely:

- ▶ the factor

$$1 - [\mathcal{O}_{Y(\square)}]_{\lambda}$$

intertwines $\{i\}$ and $[i\vdash]$;

- ▶ the factor rest gives normalization.

Example: $\text{Gr}(1, 2) = \mathbb{P}^1$

Recall

$$K(\mathbb{P}^1) = \mathbb{Q}[x]/(x^2), \quad x = 1 - [\mathcal{O}(-1)].$$

We have

$$\begin{array}{ccc}
 & (1-x)((1+y)-2yx) & \\
 & \curvearrowright & \\
 1 + \frac{y}{1+y}x = \text{SMC}_y(Y(\emptyset)^\circ) & & \text{MC}_y(Y(\emptyset)^\circ) = (1+y) - (2y+1)x \\
 & \swarrow \text{dual basis} \searrow & \\
 \frac{1}{1+y}x = \text{SMC}_y(Y(\square)^\circ) & & \text{MC}_y(Y(\square)^\circ) = x \\
 & \curvearrowleft (1-x)((1+y)-2yx) \curvearrowright &
 \end{array}$$

Application B

Recall the **stable grothendieck polynomial** is defined using set-valued tableaux:

$$\tilde{G}_\lambda = \sum_{T \in \text{SVT}(\lambda)} x^T, \quad \text{e.g.}$$

| | | | |
|-----|-----|----|---|
| 1 | 123 | 35 | 6 |
| 234 | 46 | | |
| 5 | | | |

{ filled by sets
strictly increasing in column
weakly increasing in row

Theorem (Buch [6])

$$(-1)^{|\lambda|} \tilde{G}_\lambda(-x_1, \dots, -x_k, 0, \dots) = [\mathcal{O}_{Y(\lambda)}] \in K(\text{Gr}(k, n)).$$

Dualizing Sheaves

In Lam and Pylyavskyy [8], the omega involution of \tilde{G}_λ was studied. It is given by a sum over weak set-valued tableaux:

$$J_\lambda = \sum_{T \in \text{WSVT}(\lambda)} x^T, \quad \text{e.g.}$$

| | | | |
|-----|-----|----|---|
| 11 | 334 | 55 | 6 |
| 12 | 4 | | |
| 223 | | | |

{ filled by multi-sets
strictly increasing in row
weakly increasing in column

Theorem (Fan, Guo, Su and Xiong)

$$((1 - G_\square)^n J_{\lambda'}) (x_1, \dots, x_k, 0, \dots) = [\omega_{Y(\lambda)}] \in K(\text{Gr}(k, n))$$

where $\omega_{Y(\lambda)}$ is the dualizing sheaf of $Y(\lambda)$.

Recall Schubert variety is always Cohen–Macaulay, thus $\omega_{Y(\lambda)}$ is a coherent sheaf.

Discussion of the proof

By [9],

$$\mathrm{MC}_y(Y(\lambda)^\circ) = y^{\dim}[\omega_{Y(\lambda)}] + (\text{lower } y\text{-degree}).$$

In the Pieri rule of motivic Chern classes, only the horizontal strip \square contributes the highest y -degree. Thus

Pieri rule of $[\omega_{Y(\lambda)}] = \text{adding horizontal strips } \square$.

Compare:

Pieri rule of $[\mathcal{O}_{Y(\lambda)}] = \text{adding vertical strips } \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$.

The omega involution switches two kind of strips.

Example: $\text{Gr}(1, 2) = \mathbb{P}^1$

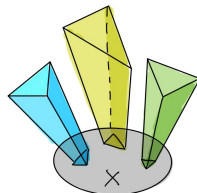
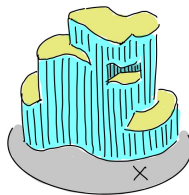
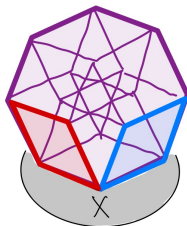
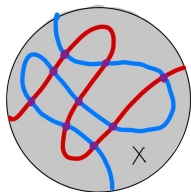
From the (weak) set-tableaux model, $\tilde{G}_{\square} = J_{\square} = 1$, and






$$\begin{aligned}\tilde{G}_{\square} &= \sum_A x^A = -1 + \prod_{i=1}^{\infty} (1 + x_i), & \boxed{A} & \text{nonempty sets } A \\ & & & \text{of positive integers,} \\ J_{\square} &= \sum_B x^B = -1 + \prod_{i=1}^{\infty} \frac{1}{1 - x_i}, & \boxed{B} & \text{nonempty multisets } B \\ & & & \text{of positive integers.}\end{aligned}$$





Thus for \mathbb{P}^1

$$\begin{aligned}(-1)^0 \tilde{G}_{\emptyset}(-x, 0, \dots) &= 1 = [\mathcal{O}_{Y(\emptyset)}], \\ (-1)^1 \tilde{G}_{\square}(-x, 0, \dots) &= x = [\mathcal{O}_{Y(\square)}], \\ ((1 - G_{\square})^2 \cdot J_{\emptyset})(x, 0, \dots) &= 1 - 2x = [\omega_{Y(\emptyset)}], \\ ((1 - G_{\square})^2 \cdot J_{\square})(x, 0, \dots) &= x = [\omega_{Y(\square)}].\end{aligned}$$

Thank You!



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