

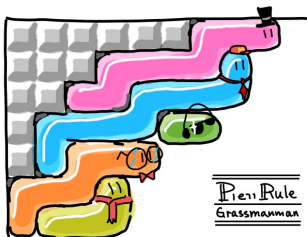
# Pieri Rules over Grassmannians

## — two more applications

arXiv:2402.04500

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# Ribbon Schubert operators

Fix  $0 \leq k \leq n$ . Let us consider the space

$$\bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q}[p, q] \cdot \lambda.$$

Let us define the **ribbon Schubert operator** to be the linear operator for  $1 \leq i \leq k$

$$\begin{aligned} |i] &\rightarrow \lambda = \sum p^{\text{ht}(\mu/\lambda)-1} q^{\text{wd}(\mu/\lambda)-1} \mu \\ [i| &\rightarrow \lambda = \sum p^{\text{ht}(\mu/\lambda)-1} q^{\text{wd}(\mu/\lambda)-1} \mu \end{aligned}$$

where the sum is taken over all  $\mu \subseteq (n-k)^k$  such that  $\mu/\lambda$  is a ribbon with **head/tail** in row  $i$ .



# Pieri rules

These operators naturally arise from the Pieri rule of **motivic Chern classes** over Grassmannian. Precisely, we have

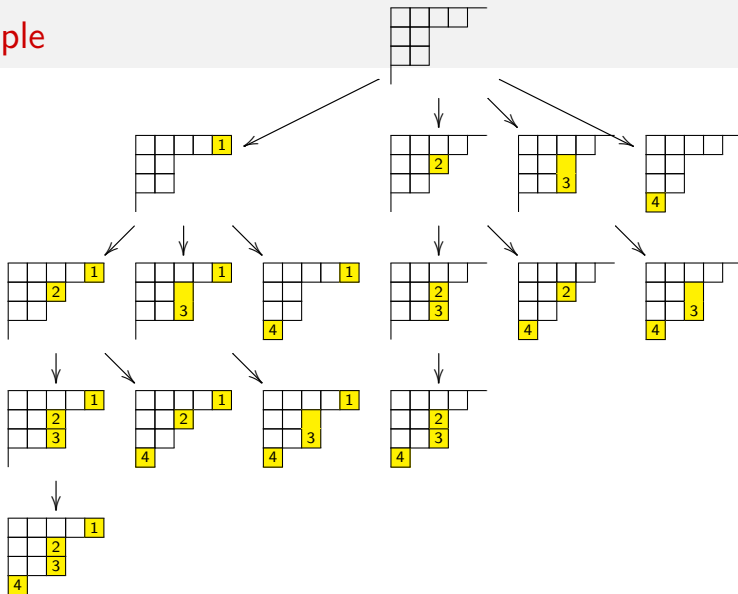
Theorem (Fan, Guo, Su, Xiong)

Set  $(p, q) = (1, -y)$ . Over  $K(\text{Gr}(k, n))[y]$ ,

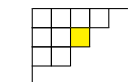
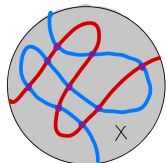
$$\begin{aligned} & c_r(\mathcal{V}^\vee) \cdot \text{MC}_y(\lambda) \\ &= (1+y)^r \sum_{1 \leq i_1 < \dots < i_r \leq k} [i_r] \rightarrow \dots [i_1] \rightarrow \text{MC}_y(\lambda) \\ &= (1+y)^r \sum_{1 \leq i_1 < \dots < i_r \leq k} [i_r | \rightarrow \dots [i_1 | \rightarrow \text{MC}_y(\lambda). \end{aligned}$$

*This is derived from the equivariant version.*

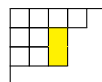
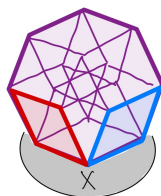
# Example



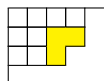
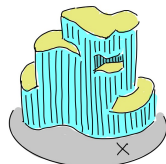
# Summary



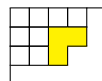
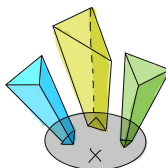
cohomology



K-theory



CSM/SSM classes



$-(1+y)y^{\text{width}}$   
MC/SMC classes

*Application*

— 21 —

# SMC classes

Motivic Chern classes admit a family of dual basis called **Segre motivic classes**. They specialize

$$\mathrm{MC}_Y(\lambda)|_{Y=0} = \text{ideal sheaf } \mathcal{I}_{Y(\lambda)} \in K(\mathrm{Gr}(k, n)),$$

$$\mathrm{SMC}_Y(\lambda)|_{Y=0} = \text{structure sheaf } \mathcal{O}_{Y(\lambda)} \in K(\mathrm{Gr}(k, n)).$$

Note that  $\mathcal{O}_{Y(\lambda)}$  is represented by the **symmetric Grothendieck polynomial**. We proved that

**Theorem (Fan, Guo, Su, Xiong)**

*The Pieri rule of  $\mathrm{SMC}_Y(\lambda)$  is the same as the rule of  $\mathrm{MC}_Y(\lambda)$ .*

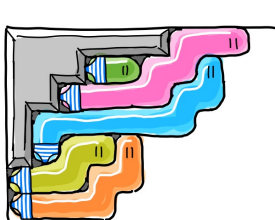


## Discussion of the proof

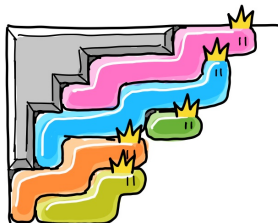
A priori, the Pieri rule for the opposite dual basis is given by  $|i]$ , the adjoint operator on the  $180^\circ$  rotated complement.

$[i|$  ...with its **tail**  
at the  $i$ -th row ...  $\longleftrightarrow$   $|i]$  ...with its **head**  
at the  $i$ -th row ...

But they are equivalent:



V.S.



# Relation between MC and SMC

The similarity between the Pieri formulas indicates that there should be some relation between motivic Chern classes and Segre motivic classes.

Theorem (Fan, Guo, Su and Xiong)

$$\lambda_y(\mathcal{T}_{\mathrm{Gr}(k,n)}^\vee) \cdot (1 - [\mathcal{O}_{Y(\square)}]) \cdot \mathrm{SMC}_y(Y(\lambda)^\circ) = \mathrm{MC}_y(Y(\lambda)^\circ).$$

*If we set  $y = 0$ , we will recover the result of Buch [1]*

$$(1 - [\mathcal{O}_{Y(\square)}]) \cdot [\mathcal{O}_{Y(\lambda)}] = [\mathcal{I}_{\partial Y(\lambda)}].$$

*Application*

— ℑ —

# Grothendieck polynomial

Recall the **symmetric Grothendieck polynomials** are defined using set-valued tableaux:

$$\tilde{G}_\lambda = \sum_{T \in \text{SVT}(\lambda)} x^T, \quad \text{e.g.}$$

1	123	35	6
234	46		
5			

$\left\{ \begin{array}{l} \text{filled by nonempty sets} \\ \text{strictly increasing in column} \\ \text{weakly increasing in row} \end{array} \right.$

Theorem (Buch [1])

$$(-1)^{|\lambda|} \tilde{G}_\lambda(-x_1, \dots, -x_k, 0, \dots) = [\mathcal{O}_{Y(\lambda)}] \in K(\text{Gr}(k, n)).$$

# Dualizing Sheaves

In Lam and Pylyavskyy [2], the omega involution of  $\tilde{G}_\lambda$  was studied. It is given by a sum over weak set-valued tableaux:

$$J_\lambda = \sum_{T \in \text{WSVT}(\lambda)} x^T, \quad \text{e.g.}$$

11	334	55	6
12	4		
223			

$$\left\{ \begin{array}{l} \text{filled by nonempty multi-sets} \\ \text{strictly increasing in row} \\ \text{weakly increasing in column} \end{array} \right.$$

Theorem (Fan, Guo, Su and Xiong)

$$((1 - G_\square)^n J_{\lambda'}) (x_1, \dots, x_k, 0, \dots) = [\omega_{Y(\lambda)}] \in K(\text{Gr}(k, n))$$

where  $\omega_{Y(\lambda)}$  is the dualizing sheaf of  $Y(\lambda)$ .

# Discussion of the proof

By [?],

$$\mathrm{MC}_y(Y(\lambda)^\circ) = y^{\dim}[\omega_{Y(\lambda)}] + (\text{lower } y\text{-degree}).$$

In the Pieri rule of motivic Chern classes, only the horizontal strip  $\square$  contributes the highest  $y$ -degree. Thus

Pieri rule of  $[\omega_{Y(\lambda)}] = \text{adding horizontal strips } \square$ .

Compare:

Pieri rule of  $[\mathcal{O}_{Y(\lambda)}] = \text{adding vertical strips } \sqcap$ .

The omega involution switches two kind of strips.

*Application*

—  $\mathfrak{e}$  —

# Hodge diamond

We get a fast algorithm for computing the **Hodge diamond** of the smooth Plücker hyperplane section of Grassmannian.

Note that

$$h^{pq}(X) = \dim H^{pq}(X) = \dim H^q(X, \Omega_X^p).$$

As a result, by definition,

$$\chi(X, \lambda_y(X)) = \sum_{p,q} y^p (-1)^q h^{pq}(X) := \chi_y(X).$$



# Algorithm

Now let us consider a smooth Plücker hyperplane  $Y \subset \text{Gr}(k, n)$ .  
Let us write

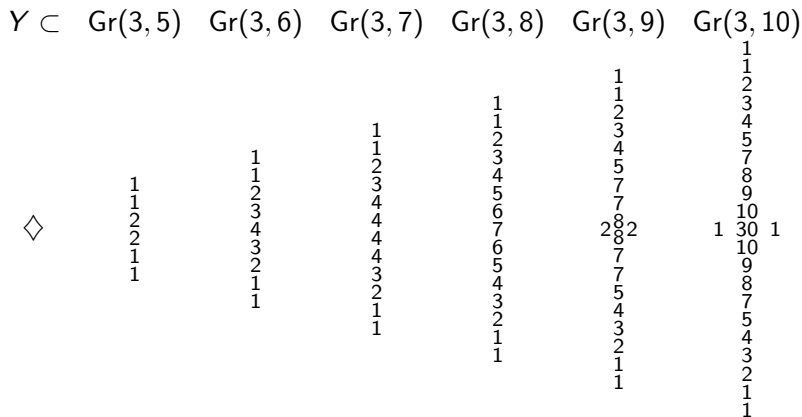
$$\lambda_y(\text{Gr}(k, n)) = \sum_{\lambda \subseteq (n-k)^k} \text{MC}_y(Y(\lambda)^\circ).$$

Using our Pieri rule, we can determine the expansion

$$\lambda_y(\text{Gr}(k, n)) \frac{1 - \det}{1 + y \det} = \sum_{\lambda \subseteq (n-k)^k} ? \text{MC}_y(Y(\lambda)^\circ).$$

Then we can compute  $\chi_y(Y)$ .

# Example: $k = 3$



The case of  $\text{Gr}(3,10)$  was studied in [3, Theorem 2.2] using Griffiths' description of the vanishing cohomology.

*Application*

—  $\mathfrak{D}$  —

# Ribbon operators

Consider the operator with parameter  $x$

$$\begin{aligned} v(x) &= \cdots (1 + x |2] \rightarrow) (1 + x |1] \rightarrow) \\ &= \sum_{r=0}^{\infty} x^r \sum_{1 \leq i_1 < \cdots < i_r} |i_r] \rightarrow \cdots |i_1] \rightarrow . \end{aligned}$$

By our Pieri rule, we have  $v(x)v(y) = v(y)v(x)$ . It would be more convenient to work with its omega involution  $u(x) = v(-x)^{-1}$ .

For a skew shape  $\lambda/\mu$ , we can define a symmetric function

$$c_{\lambda/\mu} = \text{coefficient of } \lambda \text{ in } \cdots u(x_2)u(x_1)\mu$$

We call it by **Chern polynomial**.



# Properties

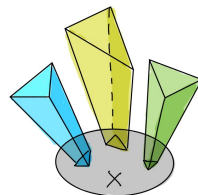
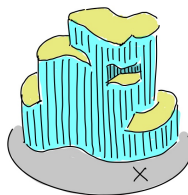
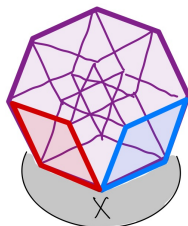
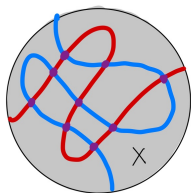
- ▶ We have  $c_{\lambda/\mu}|_{p=q=0} = s_{\lambda/\mu}$  the skew Schur function;
- ▶ We have  $c_{\lambda/\mu}|_{p=1,q=0} = g_{\lambda/\mu}$  the skew dual Grothendieck polynomial.
- ▶ The CSM class of open Richardson variety over  $\text{Gr}(k, n)$  is represented by




$$(x_1 \cdots x_k)^{n-k} c_{\lambda, \mu}(x_1^{-1}, \dots, x_k^{-1})|_{p=q=1}.$$

## Conjecture

*Chern polynomial  $c_{\lambda/\mu}$  is Schur positive.*

*Thank You!*



-  A. Buch, A Littlewood–Richardson rule for the K-theory of Grassmannians, Acta Math. 189 (2002), 37–78.
-  Lam T, Pylyavskyy P. Combinatorial Hopf algebras and K-homology of Grassmanians[J]. International Mathematics Research Notices, 2007, 2007(9): rnm125-rnm125.
-  Olivier Debarre, Claire Voisin. Hyper–Kähler fourfolds and Grassmann geometry.