Pieri Rules over Grassmannian and Applications

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1 Introduction

We prove a Pieri formula for motivic Chern classes of Schubert cells in the equivariant K-theory of Grassmannians, which is described in terms of ribbon operators on partitions. Our approach is to convert the Schubert calculus over Grassmannians into the calculation in a certain affine Hecke algebra. As a consequence, we derive a Pieri formula for Segre motivic classes of Schubert cells in Grassmannians. We apply the Pieri formulas to discover a relation between motivic Chern classes and Segre motivic classes, extending a well-known relation between the classes of structure sheaves and ideal sheaves. As another application, we find a symmetric power series representative for the class of the dualizing sheaf of a Schubert variety.

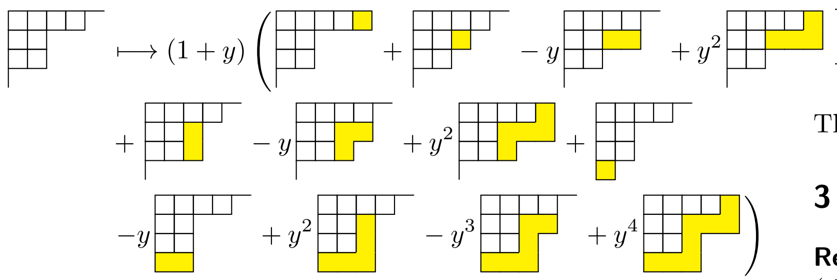
2 The Pieri rules

Our result is a Pieri rule for motivic Chern classes, a common generalization of Grothendieck polynomial and Chern–Schwartz–MacPherson classes over Grassmannians.

Chevalley formula The Chevalley formula for motivic Chern classes is given by adding a ribbon and counting width

$$c_1(\mathcal{V}^\vee)\cdot \mathrm{MC}_y(Y(\lambda)^\circ) = (1+y)\sum_{\mu=\lambda+ribbon}(-y)^{\mathrm{wd}(\mu/\lambda)-1}\,\mathrm{MC}_y(Y(\mu)^\circ).$$

Example:



Pieri formula Let us denote ribbon Schubert operators

$$[i\,| \to \mathrm{MC}_y(Y(\lambda)^\circ) = (1+y)\sum_{\mu}(-y)^{\mathrm{wd}(\mu/\lambda)-1}\,\mathrm{MC}_y(Y(\mu)^\circ)$$

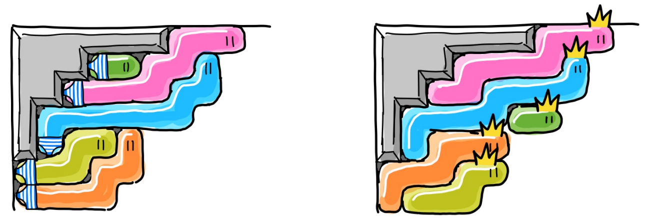
where the sum over $\mu=\lambda+$ a ribbon strip with its tail at the i-th row. Then our Pieri formula can be stated as follows.

$$c_r(\mathcal{V}^\vee)\cdot\mathrm{MC}_y(Y(\lambda)^\circ) = \sum_{1\le i_1<\cdots<i_r\le k}[i_r\,|\to\cdots[i_1\,|\to \mathrm{MC}_y(Y(\lambda)^\circ).$$

We also proved the equivalence of the following two operators

$[i\,|$ ...with its tail at the i-th row ... $\longleftrightarrow$ $|\,i]$ ...with its head at the i-th row ...

Example:



Affine Hecke Algebra Our approach is by introducing a version of affine Hecke algebra of three parameters. It turns out that $p,q,\hbar$ control the following ribbon statistics

$$p:\text{height}-1,\qquad q:\text{width}-1,\qquad \hbar:\text{number of ribbons}.$$

We have the following table

classes	$(p,q,\hbar)$	Pieri rule
$[Y(\lambda)]$	$(0,0,1)$	adding boxes $\square$
$[\mathcal{O}_{Y(\lambda)}]$	$(1,0,1)$	adding vertical strips
$c_{\mathsf{SM}}(Y(\lambda)^\circ)$	$(1,1,1)$	adding ribbons
$\mathrm{MC}_y(Y(\lambda)^\circ)$	$(1,-y,1+y)$	adding ribbons & width

This unifies many results [1–3].

3 Applications

Relations with SMC classes We proved the Segre motivic class (the opposite dual basis) has the same Pieri rule.

Since they have the same Pieri rule, we arrive a surprizing result on their relations

$$\lambda_y(\mathscr{T}^\vee_{\mathrm{Gr}(k,n)})\cdot(1-[\mathcal{O}_{Y(\square)}])\cdot\mathsf{SMC}_y(Y(\lambda)^\circ)=\mathrm{MC}_y(Y(\lambda)^\circ).$$

This generalizes the famous relation between ideal sheaves and structure sheaves $(1-[\mathcal{O}_{Y(\square)}])\cdot[\mathcal{O}_{Y(\lambda)}]=[\mathcal{I}_{\partial Y(\lambda)}]$ by Buch [4].

Representatives for dualizing sheaves By [5],

$$\mathrm{MC}_y(Y(\lambda)^\circ)=y^{\dim}[\omega_{Y(\lambda)}]+(\text{lower }y\text{-degree})$$

where $\omega_{Y(\lambda)}$ is the dualizing sheaf of the Schubert variety. In the Pieri rule of motivic Chern classes, only the horizontal strip contributes the highest y-degree.

Using this fact and Pieri rule, we proved

$$\big((1-G_\square)^n J_{\lambda'}\big)(x_1,\ldots,x_k,0,\ldots)=[\omega_{Y(\lambda)}]\in K(\mathrm{Gr}(k,n))$$

where J_λ be its omega involution of the stable grothendieck polynomial (without sign). By Lam and Pylyavskyy [6], J_λ is given by a sum over weak set-valued tableaux:

$$J_\lambda=\sum_{T\in\mathrm{WSVT}(\lambda)}x^T,\quad \text{e.g.}$$

11	334	55	6
12	4		
223			

filled by nonempty multi-sets strictly increasing in row weakly increasing in column

Hodge diamond of smooth Plücker surface Using our Pieri rule, we get a fast algorithm of computing the Hodge diamond of a smooth Plücker surface in Grassmannian. For example

$Y\subset$	Gr(3,5)	Gr(3,6)	Gr(3,7)	Gr(3,8)	Gr(3,9)	Gr(3,10)
$\diamondsuit$	1 1 2 2 2 1 1	1 1 2 3 4 3 2 1 1	1 1 2 3 4 4 4 4 3 2 1 1	1 1 2 3 4 5 6 7 6 5 4 3 2 1 1	1 1 2 3 4 5 7 7 8 2 8 2 7 7 5 4 3 2 1 1	1 1 2 3 4 5 7 8 9 10 1 30 1 10 9 8 7 5 4 3 2 1 1

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