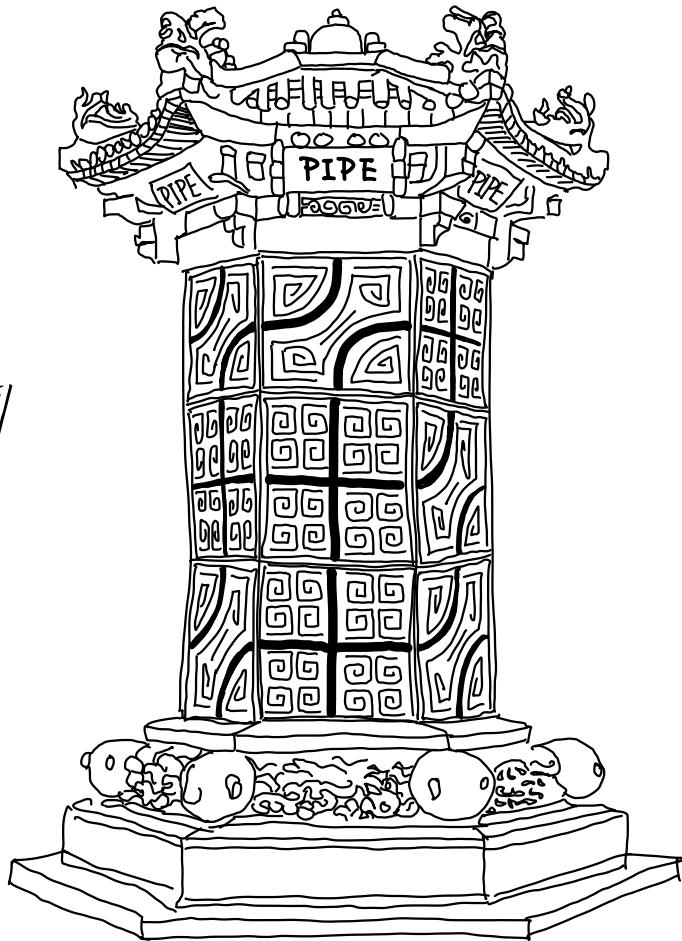


Fun with Positroid Varieties

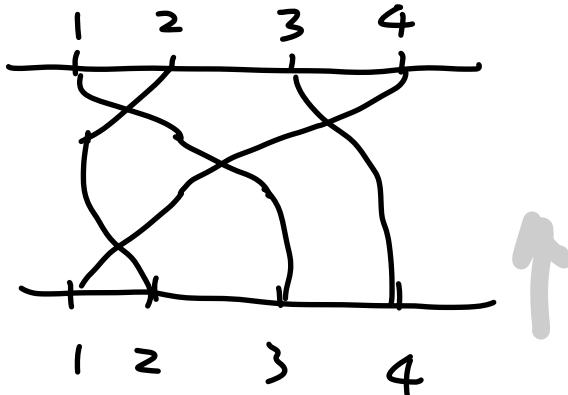
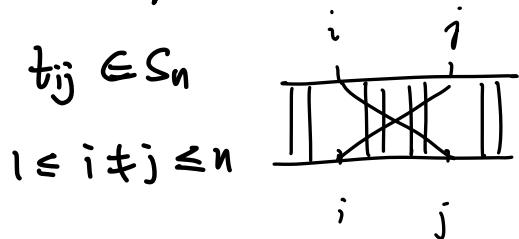


Symmetric Groups

$S_n = \{ [n] \text{ bijective} \}$

\cup
w

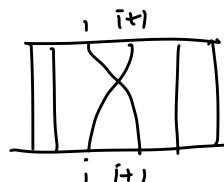
Reflection/transposition



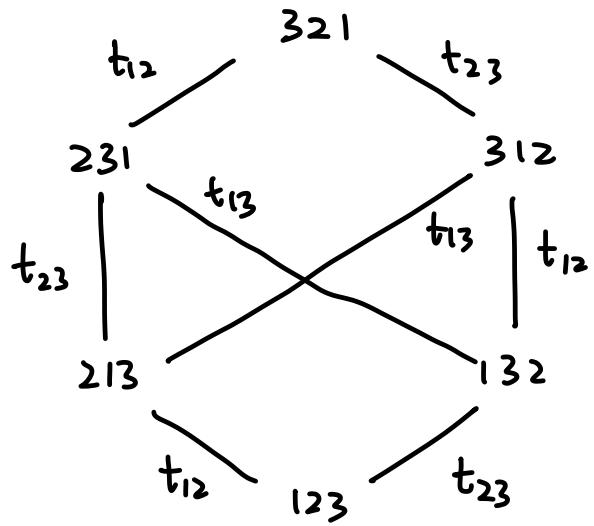
Simple reflection

$$s_i = t_{i,i+1}$$

$$1 \leq i \leq n-1$$

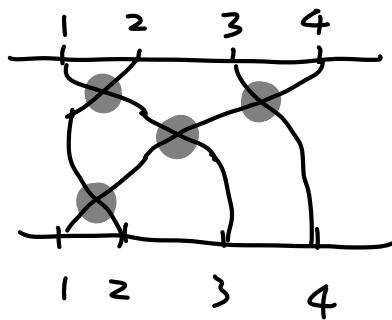
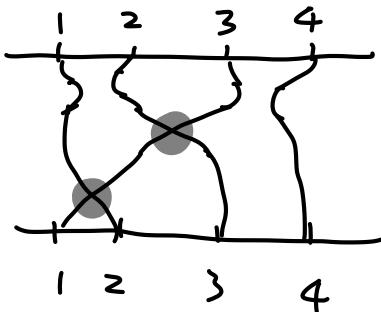


Bruhat Order

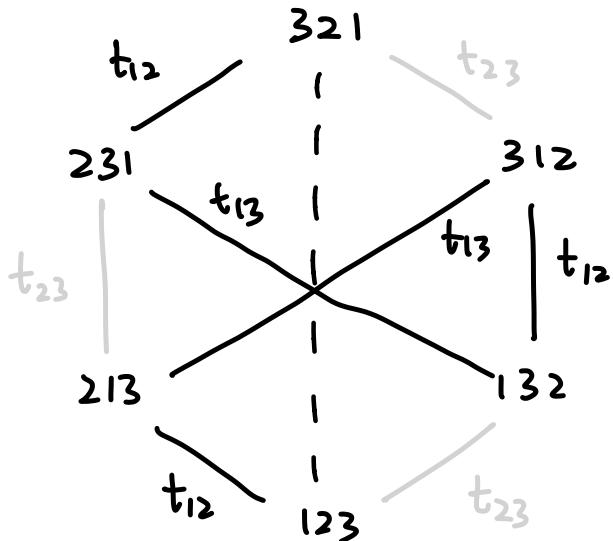


$u < w \iff w = ut_{ij}, l(w) = l(u) + 1$

$u < w \iff \exists u < u_1 < u_2 < \dots < w$



(Extended) k -Bruhat Order



$$n=3, \quad k=1$$

$$u <_k w \iff w = ut_{ij} \quad l(u)+1 = l(w)$$

$$u \leq_k w \iff \exists u \subset^{\circ} u_1 \subset u_2 \subset \dots \subset w$$

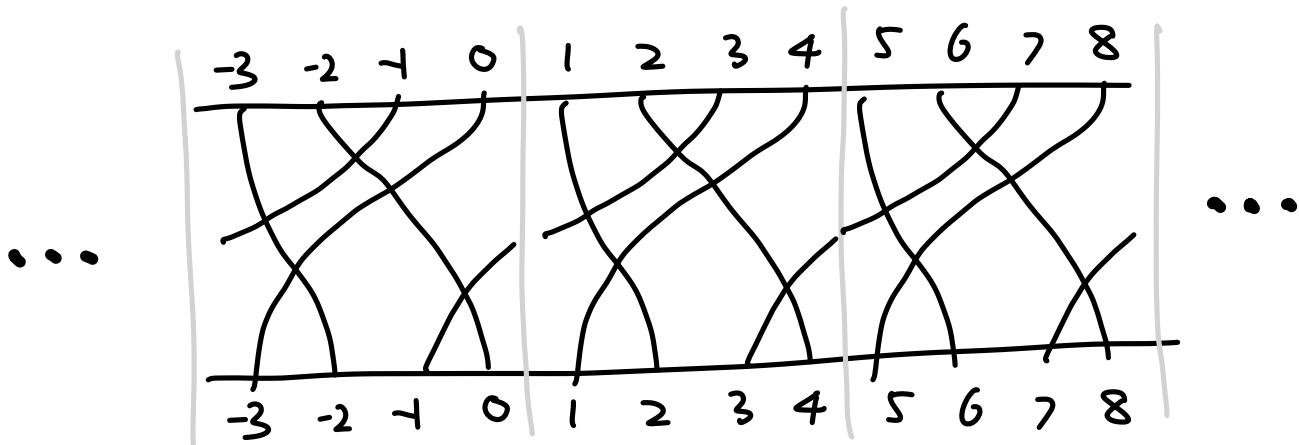
i $\leq k < j$

$$u <'_k w \iff w = ut_{ij} \quad l(w) > l(u)$$

$$u \leq'_k w \iff \exists u \subset^{\circ} u_1 \subset^{\circ} u_2 \subset^{\circ} \dots \subset^{\circ} w$$

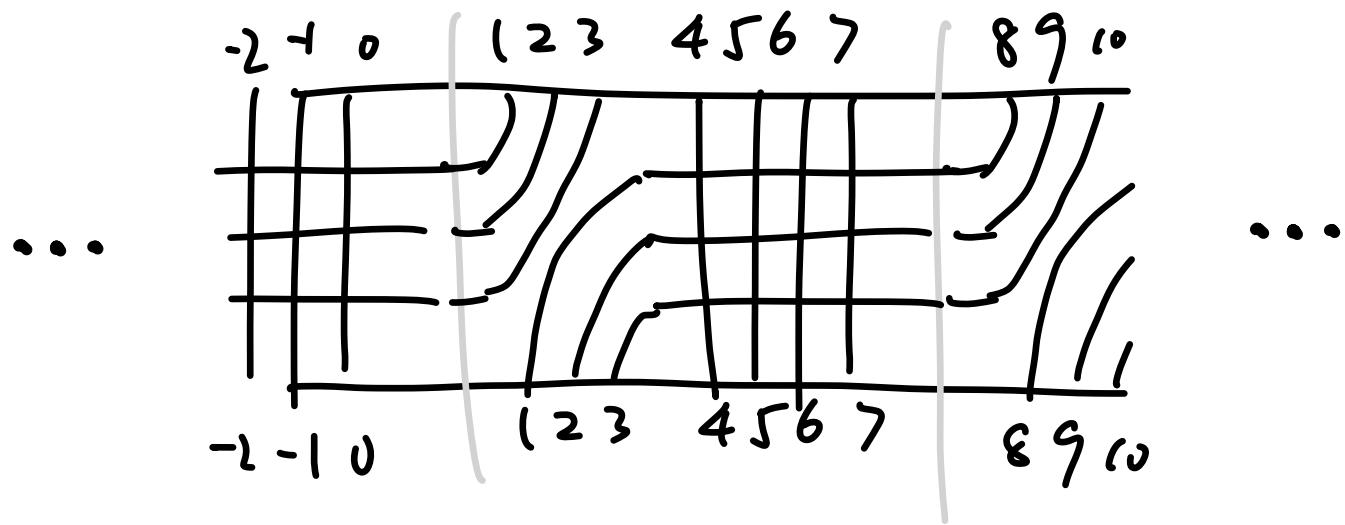
Affine Permutations

$$\tilde{S}_n = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{\text{bijection}} \mathbb{Z} \\ f \end{array} \mid f(i+n) = f(i) + n \right\}$$



Translation $\omega_k \in \tilde{S}_n$

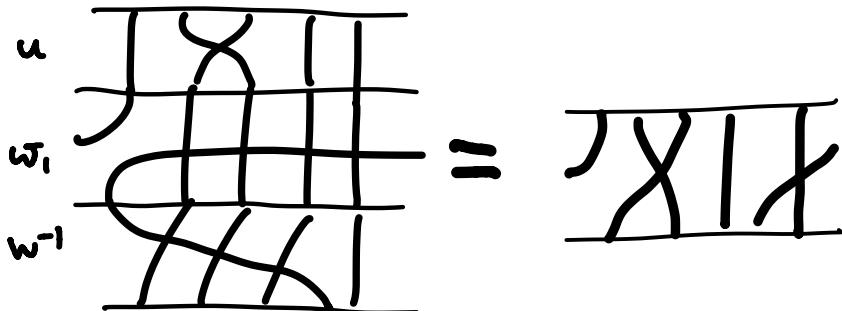
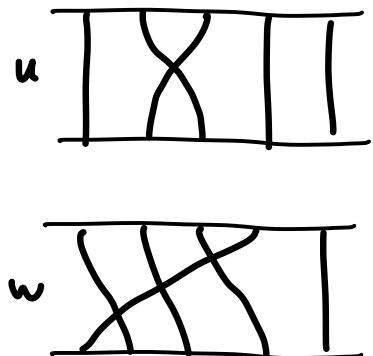
$$\omega_k(i) = \begin{cases} i+n, & 1 \leq i \leq k, \\ i, & k < i \leq n \end{cases}.$$



Extended k -Bruhat Order Again

For $u \leq_k^l w$, define $f_{u,w} = uw_k w^{-1} \in \tilde{S}_n$

$$\begin{array}{c} \uparrow \\ S_n \\ \uparrow \\ S_n \end{array}$$



Affine Bounded Permutations

extended
k-Bruhat
intervals

$$\{(u, w) : u \leq_k w\}$$

$$(uv, wv) \sim (u, w)$$

$$v \in S_k \times S_{n-k}$$



affine
bounded
permutations

$$\left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{\text{f}} \mathbb{Z} \\ \text{bijection} \end{array} \middle| \begin{array}{l} f(i+n) = f(i) + n \\ \frac{1}{n} \sum_{i=1}^n (f(i) - i) = k \\ i \leq f(i) \leq i+n \end{array} \right\} f_{u,w}$$

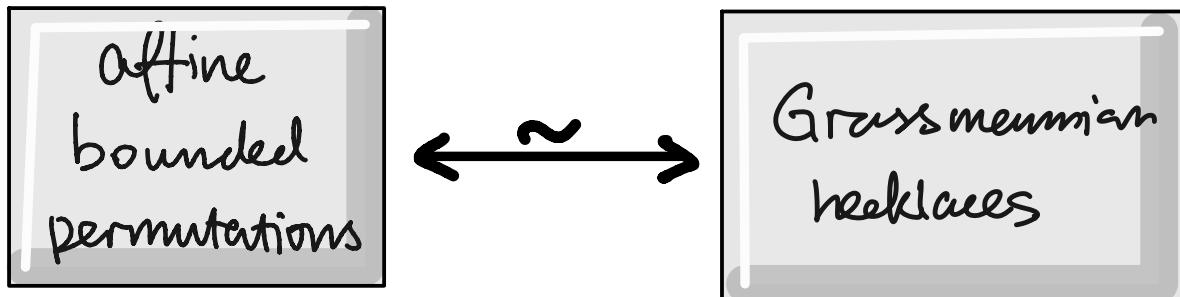
Grassmannian Necklaces

$$I = (I_1, \dots, I_n) , \quad I_i \in \binom{[n]}{k}$$

$$(1) \quad I_{a+1} = I_a \quad \text{if } a \notin I_a$$

$$(2) \quad I_{a+1} = I_a \setminus \{a\} \cup \{a'\} \quad \text{if } a \in I_a$$

(convention $n+1=1$, i.e. indexed by $\mathbb{Z}/n\mathbb{Z}$).

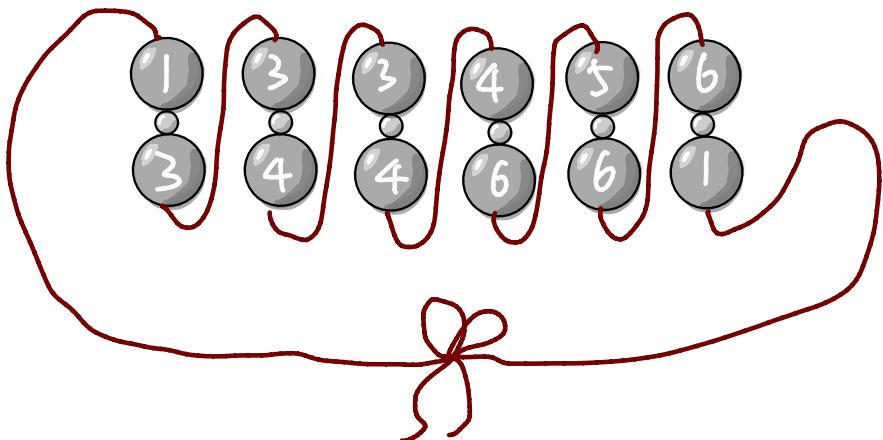
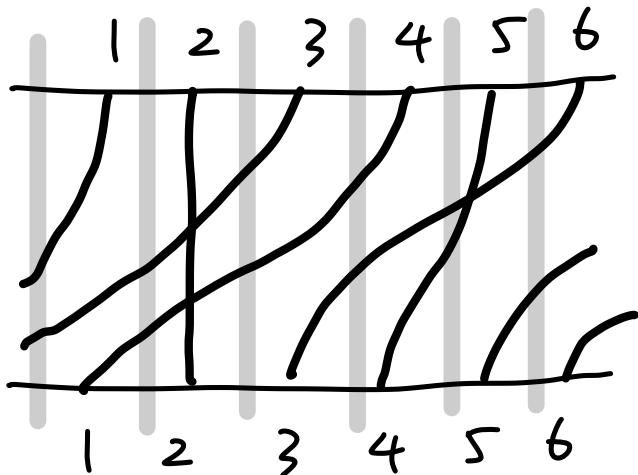




Example

$f \mapsto I = (I_1, \dots, I_n)$

$I_a = \{f(b) \mid b < a, f(b) \geq a\}$
 $\text{mod } n$



Grassmannians

$$\text{Gr}(k, n) = \left\{ V \leq \mathbb{C}^n : \dim V = k \right\}$$

We find $v_1, \dots, v_k \in \mathbb{C}^n$ spanning $V \in \text{Gr}(k, n)$

$$\begin{bmatrix} \overbrace{v_1} \\ \overbrace{v_2} \\ \vdots \\ \overbrace{v_k} \end{bmatrix} \underbrace{\quad}_{n} \quad \left\} k \right. \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & * & 0 & 0 & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & & & \\ 0 & 0 & 0 & 1 & * & * & & \\ 0 & 0 & 0 & 0 & 1 & * & & \\ 0 & 0 & 0 & 0 & 0 & 1 & * & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

echelon form

$\rightsquigarrow k\text{-subset}$

Associated Affine Permutations

We find $v_1, \dots, v_k \in \mathbb{C}^n$ spanning $V \in \text{Gr}(k, n)$

$$\left[\begin{array}{c} \overbrace{v_1} \\ \overbrace{v_2} \\ \vdots \\ \overbrace{v_k} \end{array} \right] \}^k = \left[\begin{array}{cccc} u_1 & u_2 & \dots & u_n \\ | & | & & | \end{array} \right]$$

$$\rightsquigarrow f_v \in \tilde{S}_n$$

Convention

$$u_{n+1} = u_1, \quad u_{n+2} = u_2 \quad \text{etc.}$$

$$f_v(i) = \min \{ j \geq i : u_i \in \text{span}(u_{i+1}, \dots, u_j) \}$$

Schubert and Positroid Varieties

- For an k -subset $A \subseteq [n]$,

$$\Sigma_A^0 = \{v : v \rightsquigarrow A\} \quad \Sigma_A = \overline{\Sigma_A^0}$$

- For an affine bounded permutation f

$$\Pi_f^0 = \{v : v \rightsquigarrow f\} \quad \Pi_f = \overline{\Pi_f^0}$$

Examples

$$f(1) = 1+k$$

$$u_1 \notin \text{Span}(u_2, \dots, u_k)$$

$$u_1 \in \text{Span}(u_2, \dots, u_{k+1}).$$

$$f(2) = 2+k$$

$$u_2 \notin \text{Span}(u_3, \dots, u_{k+1})$$

$$\pi^k(i) = i+k$$

$$u_2 \in \text{Span}(u_3, \dots, u_{k+2})$$

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ u_1 & u_2 & \dots & u_n \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \dots & & & \\ & 1 & 1 & \dots \\ & \dots & u_{k+1} & \dots & u_{n+k} & \dots \\ & | & | & \dots \\ & \underbrace{\quad}_{\text{linearly independent}} & \underbrace{\quad} & \dots \end{bmatrix}$$

linearly independent

$$\text{general } v \in \Pi_{\pi^k}^0 \Rightarrow \Pi_{\pi^k} = \text{Gr}(k, n)$$

Examples

$$w_k(i) = \begin{cases} i+n & 1 \leq i \leq k \\ i & k < i \leq n \end{cases}$$

$$f(1) = 1+n$$

$$u_1 \notin \text{Span}(u_2, \dots, u_n)$$

$$f(2) = 2+n$$

$$u_1 \in \text{Span}(u_2, \dots, u_n, u_1)$$

...

$$f(k) = k+n$$

$$u_k \notin \text{Span}(u_{k+1}, \dots, u_n, u_1, \dots, u_{k-1})$$

$$\left[\begin{array}{c|c|c|c|c|c} | & | & | & | & | & | \\ u_1 & u_2 & \cdots & u_k & u_{k+1} & \cdots & u_n \\ | & | & | & | & | & \cdots & | \end{array} \right]$$

$$f(k+1) = k+1$$

$$u_{k+1} = 0$$

...

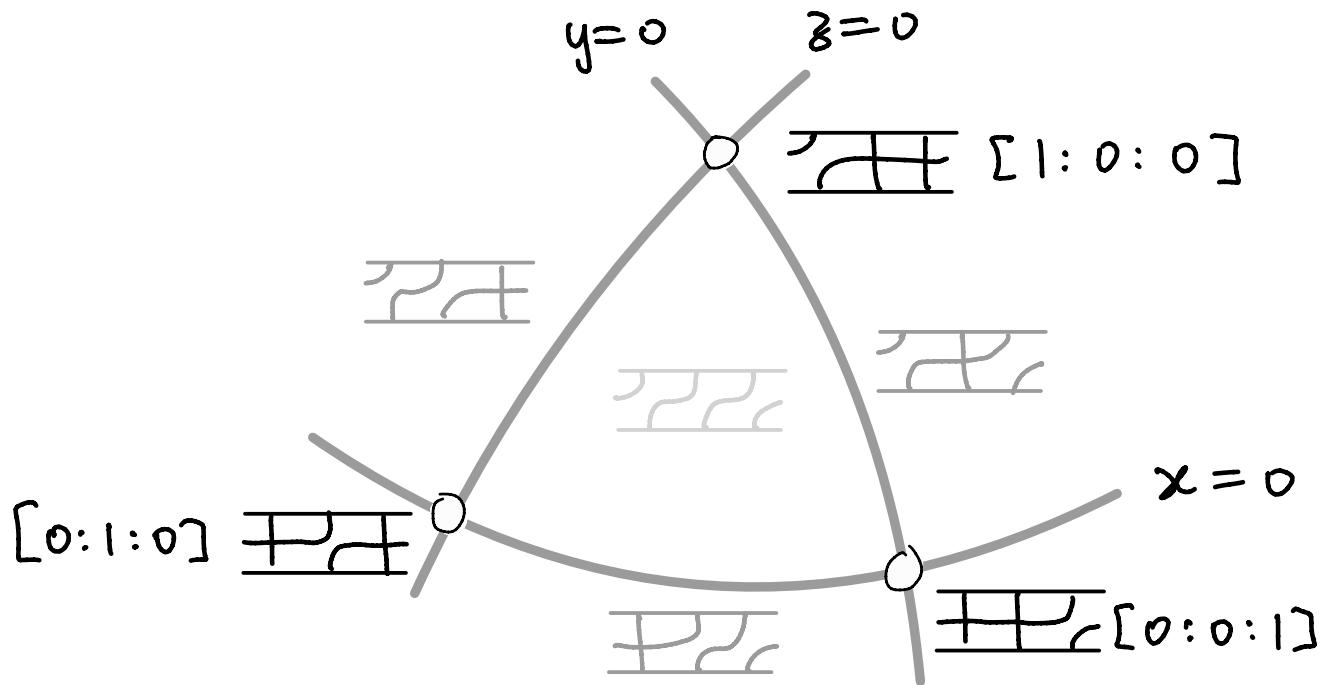
$$f(n) = n$$

$$u_n = 0$$

$$\left[\underbrace{\text{basis}}_k \quad \mid \quad 0 \quad \right] \rightsquigarrow \begin{matrix} \text{unique} \\ V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k \\ \sum w_k = \{V\} \text{ a point} \end{matrix}$$

Examples

$$\text{Gr}(1, 3) \equiv \mathbb{P}^2$$



Cohomology

Λ_k = ring of symmetric polynomials in k variables.

$$\Lambda_k \longrightarrow H^*(Gr(k,n))$$

$$f \longmapsto f \begin{pmatrix} \text{Chern roots of dual} \\ \text{tautological bundle} \end{pmatrix}$$

Then:

$$\begin{matrix} \text{Schur} \\ \text{polynomials} \end{matrix} \longmapsto [\Sigma_A] \text{ or } 0.$$

Classes of Positroid Varieties

affine Stanley
symmetric polynomial $\mapsto [\pi_f]$

no double crossing

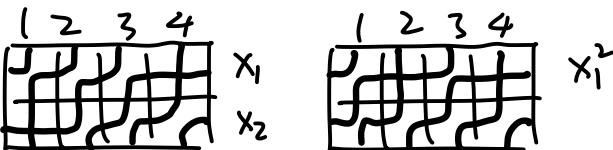
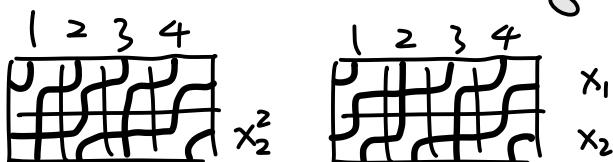
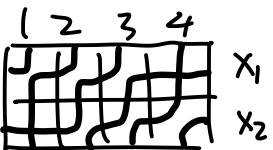
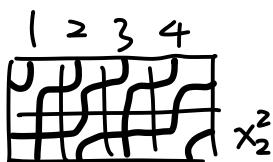
$$\text{Ex } (k, n) = (2, 4)$$

$$f(1) = 2$$

$$f(2) = 5$$

$$f(3) = 4$$

$$f(4) = 7$$



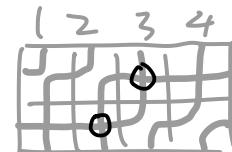
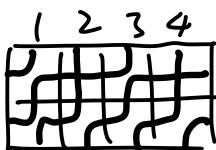
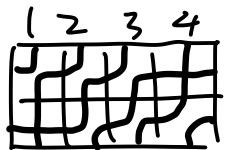
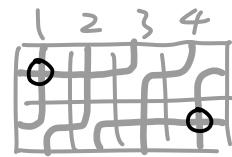
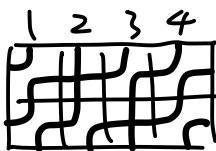
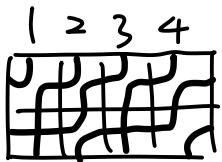
$$[\pi_f] = x_1^2 + x_1 x_2 + x_1 x_2 + x_2^2 = (x_1 + x_2)^2 \in H^*(Gr(2,4))$$

Allowing double crossings

What happens if we remove

no double
crossing

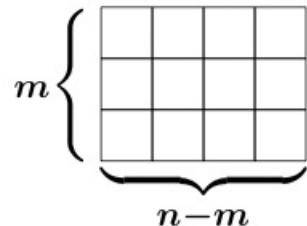
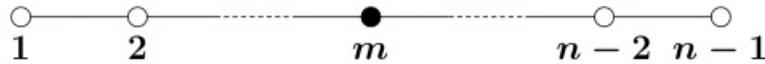
?



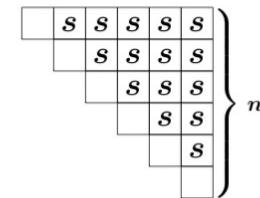
Thm (FGSX) They give the Chern class of π_f^* .

Generalization to Other Types

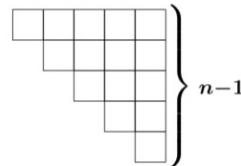
Grassmannian $\text{Gr}(m, n)$ of type A_{n-1}



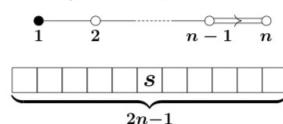
Lagrangian Grassmannian $\text{LG}(n, 2n)$



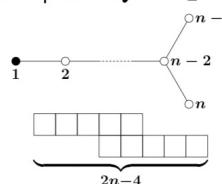
Max. orthog. Grassmannian $\text{OG}(n, 2n)$



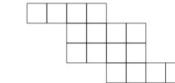
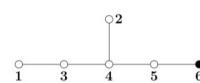
Odd quadric $Q^{2n-1} \subset \mathbb{P}^{2n}$



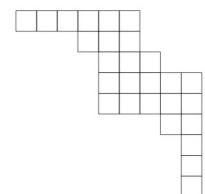
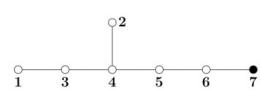
Even quadric $Q^{2n-2} \subset \mathbb{P}^{2n-1}$



Cayley Plane E_6/P_6



Freudenthal variety E_7/P_7



Generalization to Other Types

$$\{(u, w) : u \leq_k' w\}$$

(u, w)

$$(uv, wv) \sim (u, v)$$

if $v \in w_p$



$$\{w \in \widetilde{W} : \begin{array}{l} w \leq t_\lambda \\ \lambda \in W \omega_k^\vee \end{array}\}$$

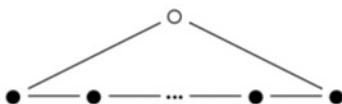
$w t_{w_k} w^{-1}$

Generalization to Other Types

\tilde{A}_1

$\bullet \Leftrightarrow \circ$

\tilde{A}_n
($n \geq 2$)



\tilde{B}_n
($n \geq 3$)



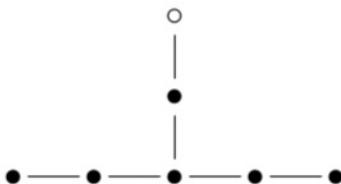
\tilde{D}_n
($n \geq 4$)



\tilde{C}_n
($n \geq 2$)



\tilde{E}_6



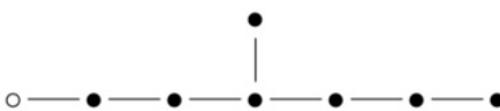
\tilde{F}_4



\tilde{G}_2



\tilde{E}_7



\tilde{E}_8



Generalization to Other Types

Thm (FGSX)

$$\text{Chern class of } \pi_f^\circ \text{ (generalized)} \quad \approx \quad \text{Chern class of Schubert cell of } f$$

over

affine flag variety

Grassmannian

over

Cominuscule variety

Thank
You

