

# LECTURES ON AFFINE WEYL GROUPS

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## 1. REVIEW OF FINITE THEORY

*Coxeter groups.*

**1.1. Definition.** A **Coxeter system**  $(W, S)$  is a group  $W$  and  $S \subset W$  such that

$$W = \left\langle s \in S : \underbrace{st \cdots}_{m_{st}} = \underbrace{ts \cdots}_{m_{st}} \right\rangle \quad \text{where for each } s \neq t \in S \\ m_{st} \in \{2, 3, \dots\} \cup \{\infty\}.$$

We define **Coxeter diagram**

$\bullet$ $\bullet$	$\bullet$ — $\bullet$	$\bullet$ — $\overset{4}{\bullet}$	...	$\bullet$ — $\overset{\infty}{\bullet}$
$m_{st} = 2$	$m_{st} = 3$	$m_{st} = 4$	...	$m_{st} = \infty$
$st = ts$	$sts = tst$	$stst = tsts$	...	no relation

Usually, we reparametrize  $S$  by  $\{s_i : i \in I\}$  and  $m_{ij} = m_{s_i s_j}$ .

**1.2. Geometric representation.** We define

$$\mathfrak{h}_{\mathbb{R}}^* = \bigoplus_{i \in I} \mathbb{R} \alpha_i.$$

We equip a symmetric bilinear form such that

$$\text{length of } \alpha_i \neq 0, \quad \text{angle of } \alpha_i \text{ and } \alpha_j \text{ is } \pi - \frac{\pi}{m_{ij}}.$$

This form is unique up to a positive rescalar of  $\alpha_i$ . We define the geometric representation of  $W$  on  $\mathfrak{h}_{\mathbb{R}}^*$  by

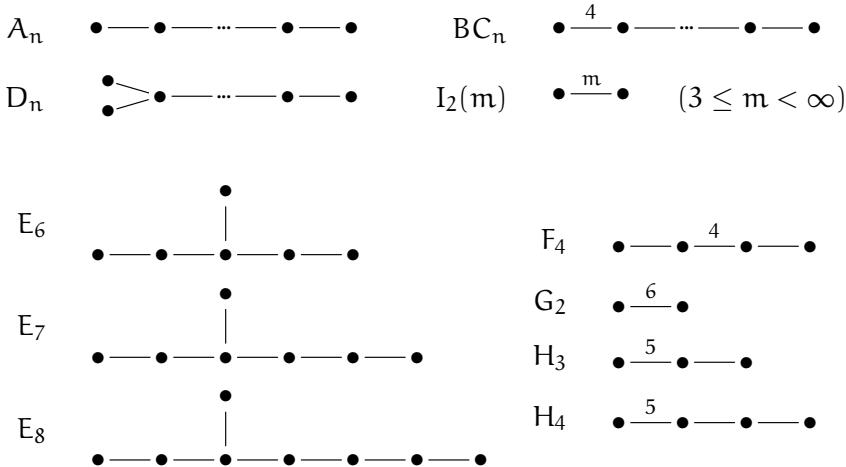
$$S \ni s_i \longmapsto (\text{reflection with respect to } \alpha_i^\perp) \in \text{GL}(\mathfrak{h}^*).$$

That is,

$$s_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda) \alpha_i, \quad \text{where } \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

For any Coxeter group, its geometric representation is faithful.

**1.3. Finite Coxeter groups.** A Coxeter group  $W$  is finite if and only if the bilinear form defined above is positive definite. The corresponding Coxeter diagram is a disjoint union of the following diagrams.

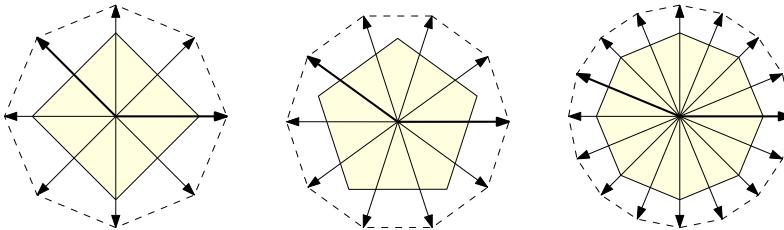


We have

$$D_2 = A_1 \times A_1, \quad D_3 = A_2, \quad A_2 = I_2(3), \quad BC_2 = I_2(4), \quad G_2 = I_2(6).$$

**1.4. Example.** The dihedral group  $D_m$  of order  $2m$  is the Coxeter group of type  $I_2(m)$ . We take  $\mathfrak{h}_\mathbb{R}^*$  to be the complex plane  $\mathbb{C}$ , and

$$\alpha_1 = 1, \quad \alpha_2 = -e^{-\frac{2\pi\sqrt{-1}}{m}}.$$



*Weyl groups.*

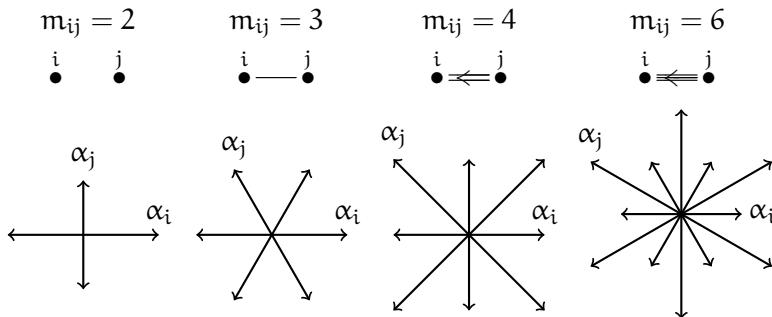
**1.5. Weyl group.** If a finite Coxeter group  $W$  stabilizes the root lattice

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}_{\mathbb{R}}^*,$$

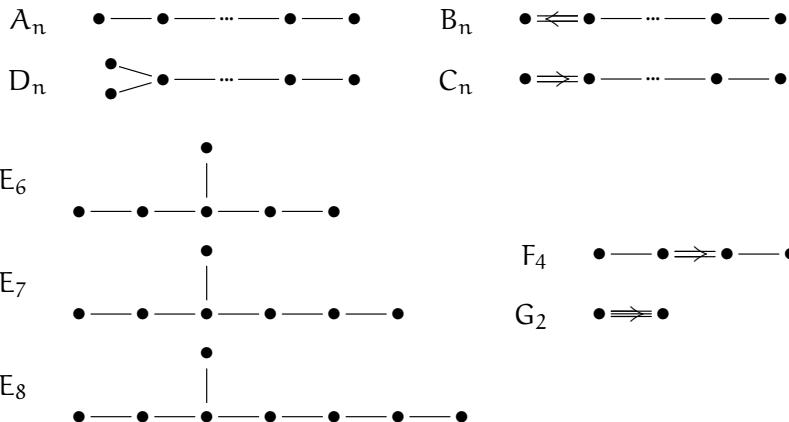
we call  $W$  a **Weyl group** and define the **root system**

$$R = \{w\alpha_i : w \in W, i \in I\}.$$

A Weyl group could only have  $m_{ij} \in \{2, 3, 4, 6\}$ . We define the **Dynkin diagram**



**1.6. Finite Weyl group.** Up to graph isomorphism, here is the classification of irreducible Weyl groups



We have

$$D_2 = A_1 \times A_1, \quad D_3 = A_3, \quad B_2 = C_2.$$

### 1.7. Example.

The symmetric group

$$\mathfrak{S}_n = \left\{ \text{bijections } \{1, \dots, n\} \xrightarrow{w} \{1, \dots, n\} \right\}$$

is the Coxeter group of type  $A_{n-1}$ . The Coxeter generator

$$s_i = \left[ \begin{array}{c} \text{the permutation exchanging } i \text{ and} \\ i+1 \text{ with other numbers fixed} \end{array} \right] \in \mathfrak{S}_n$$

labeled as

$$\bullet_1 — \bullet_2 — \cdots — \bullet_{n-2} — \bullet_{n-1}$$

The geometric representation

$$\mathfrak{h}_{\mathbb{R}}^* = \{(a_1, \dots, a_n) : a_1 + \cdots + a_n = 0\} \subset \mathbb{R}^n.$$

The natural pairing over  $\mathbb{R}^n$  restricts to  $\mathfrak{h}_{\mathbb{R}}^*$ . We define

$$\alpha_i = e_i - e_{i+1}, \quad 1 \leq i \leq n-1.$$

We have a diagram notation

$$\begin{aligned} & \text{Diagram notation for } s_i: \text{A horizontal line with dots at } i \text{ and } i+1. \text{ A red } X \text{ is drawn between them.} \\ & \text{Below the line: } i \quad i+1 \\ & \text{e.g. } 42|3 = \\ & \text{Diagram of } 1, 2, 3, 4: \text{Red lines connect } 1 \text{ to } 2, 2 \text{ to } 3, \text{ and } 3 \text{ to } 4. \text{ A green line connects } 1 \text{ to } 4. \\ & \text{Followed by: } = \text{ (yellow circle)} \text{ (blue circle)} \text{ (green circle)} \text{ (orange circle)} \\ & \text{Below:} \\ & s_i^2 = \text{id} \quad s_i s_j = s_j s_i \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{aligned}$$

### 1.8. Example.

The Coxeter group of type  $BC_n$  is known as the **signed symmetric group**

$$\mathfrak{BC}_n = \left\{ \text{bijections } \{\pm 1, \dots, \pm n\} \xrightarrow{w} \{\pm 1, \dots, \pm n\} : w(-i) = -w(i) \right\}.$$

Using the monotone bijection

$$\{\pm 1, \dots, \pm n\} \cong \{1, \dots, 2n\}$$

We can describe it as the subgroup of  $S_{2n}$  generated by

$$s_0 = s_n, \quad s_i = s_{n-i}s_{n+i} \quad (1 \leq i \leq n-1).$$

That is,

$$s_0 = \left[ \begin{array}{c} \text{the permutation exchanging 1 and } -1 \\ \text{with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

$$s_i = \left[ \begin{array}{c} \text{the permutation exchanging } \pm i \text{ and } \pm(i+1) \\ \text{with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

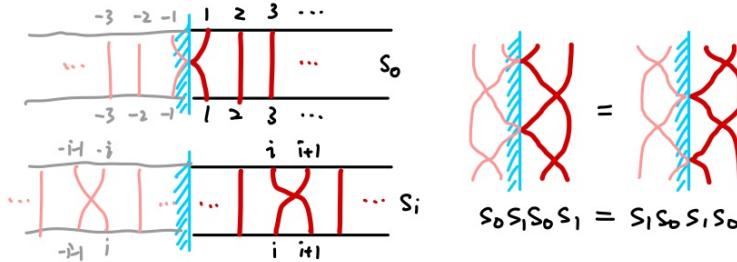
The label is like this

$$\bullet \overset{4}{\overbrace{\cdots}} \bullet \overset{n-2}{\overbrace{\cdots}} \bullet \overset{n-1}{\overbrace{\cdots}} \bullet$$

The geometric representation  $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$  with natural pairing and

$$\alpha_0 = \begin{cases} e_1, & \text{type B,} \\ 2e_1, & \text{type C,} \end{cases} \quad \alpha_i = e_{i+1} - e_i \quad (1 \leq i \leq n-1).$$

We have a diagram notation



**1.9. Example.** The Coxeter group of type  $D_n$  is known as the **even-signed symmetric group**.

$$\mathfrak{D}_n = \left\{ w \in \mathfrak{BC}_n : \#\{i < 0 : w(i) > 0\} \text{ is even} \right\}.$$

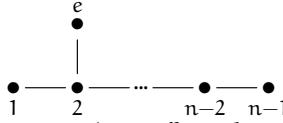
Note that the  $s_0 \notin \mathfrak{D}_n$  while the  $s_i \in \mathfrak{D}_n$  for  $1 \leq i \leq n-1$ . We define

$$s_e = s_0 s_1 s_0 \in \mathfrak{D}_n.$$

That is,

$$s_e = \left[ \begin{array}{c} \text{the permutation exchanging } \pm 1 \text{ and} \\ \mp 2 \text{ with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

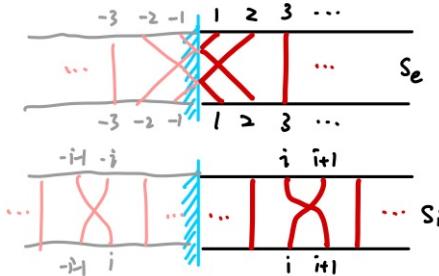
The label is



The geometric representation  $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$  with natural pairing and

$$\alpha_e = e_1 + e_2 \quad \alpha_i = e_{i+1} - e_i \quad (1 \leq i \leq n-1).$$

We have a diagram notation



### Miscellany.

**1.10. Remark.** From now, we will assume  $W$  is a Weyl group, i.e. we are equipped with a underlying root system. The same result holds for any Coxeter group if we replace  $R$  by the set of **root directions**

$$\vec{R} = \left\{ \frac{w\alpha_i}{\|w\alpha_i\|} : w \in W, i \in I \right\} \subset \mathfrak{h}_{\mathbb{R}}^*.$$

**1.11. Reflections.** For  $\alpha \in R$ , denote

$r_\alpha$  = the reflection with respect to  $\alpha \in W$ .

If  $\alpha = w\alpha_i$ , then  $r_\alpha = ws_iw^{-1}$ . We define **reflections** by

$$\{\text{reflections}\} = \{ws_iw^{-1} : w \in W, i \in I\} = \{r_\alpha : \alpha \in R\}.$$

We call  $s_i$  ( $i \in I$ ) a **simple reflection**.

### 1.12. Positive roots.

The set of **positive/negative roots**

$$\mathbb{R}^\pm = \{\alpha \in \mathbb{R} : \pm \alpha \in \text{span}_{\geq 0}(\alpha_i)_{i \in I}\}.$$

We have  $\mathbb{R} = \mathbb{R}^+ \sqcup \mathbb{R}^-$ . For  $\alpha \in \mathbb{R}$ , we denote  $\alpha > 0$  if  $\alpha \in \mathbb{R}^+$  and  $\alpha < 0$  otherwise. We call  $\alpha_i$  ( $i \in I$ ) a **simple root**.

### 1.13. Hyperplanes.

Let us consider

$$\mathfrak{h}_\mathbb{R} = \text{dual space of } \mathfrak{h}_\mathbb{R}^* \cong \mathfrak{h}_\mathbb{R}^*.$$

For any  $\alpha \in \mathbb{R}$ , we denote

$$H_\alpha = \{x \in \mathfrak{h}_\mathbb{R} : \langle x, \alpha \rangle = 0\} \subset \mathfrak{h}_\mathbb{R}.$$

### 1.14. Fundamental coweights.

Denote **fundamental (co)weight**  $\varpi_i \in \mathfrak{h}_\mathbb{R}^*$  ( $\varpi_i^\vee \in \mathfrak{h}_\mathbb{R}$ ) be such that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}.$$

### 1.15. Chamber.

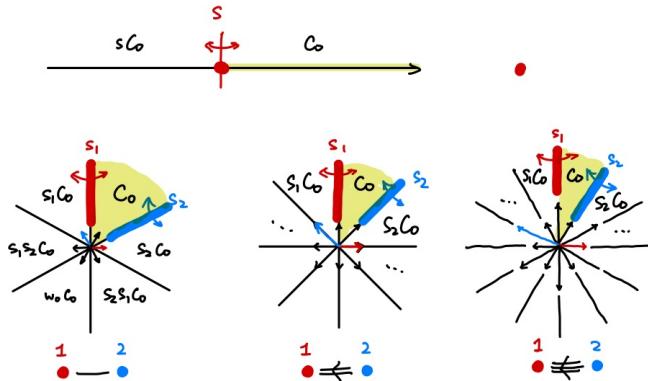
We define **chambers** by

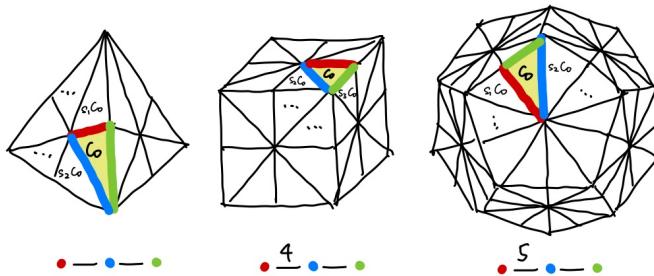
$$\{\text{chambers}\} = \text{connected components of } \left( \mathfrak{h}_\mathbb{R} \setminus \bigcup_{\alpha \in \mathbb{R}} H_\alpha \right).$$

We define the **dominant chamber** to be the cone

$$C_0 = \{x \in \mathfrak{h}_\mathbb{R} : \langle \alpha_i, x \rangle > 0\} = \text{span}_{\geq 0}(\varpi_i^\vee : i \in I).$$

Here we collect example in small dimensions.





**1.16. Theorem.** We have a bijection

$$W \longrightarrow \{\text{chambers}\}, \quad w \longmapsto wC_0.$$

Under this bijection,

$$\begin{aligned} \text{the chamber of } s_i w &= \text{reflection of the chamber} \\ &\quad \text{of } w \text{ with respect to } \alpha_i \\ \text{the chamber of } ws_i &= \text{the chamber sharing the wall} \\ &\quad wH_{\alpha_i} \text{ with the chamber of } w \end{aligned}$$

**1.17. Length.** For any  $w \in W$ , we define

$$\ell(w) = \text{minimal length of writing } w \text{ as} \\ \text{a product of simple reflections}$$

If

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \quad \ell = \ell(w).$$

We call  $(i_1, i_2, \dots, i_\ell)$  is a **reduced word** of  $w$ .

**1.18. Length formula.** In terms of chambers,

$$\ell(w) = \#\{\text{hyperplanes separating } C_0 \text{ and } wC_0\}$$

In terms of roots,

$$\ell(w) = \# \text{Inv}(w), \quad \text{Inv}(w) = \{\alpha \in R^+ : w\alpha \in R^-\}.$$

There is a bijection between hyperplanes and  $\text{Inv}(w^{-1})$ .

**1.19. Bruhat order.** We define the **Bruhat order** over  $W$  to be the following equivalent order

- the order generated by

$$u < w \text{ if } w = ur_\alpha \text{ and } \ell(w) = \ell(u) + 1.$$

- the order generated by

$$u < w \text{ if } w = ur_\alpha \text{ and } \ell(w) > \ell(u).$$

- $u \leq w$  if there is a subword of  $u$  in a reduced word of  $w$ .
- $u \leq w$  if there is a subword of  $u$  in any reduced word of  $w$ .

We remark that for  $\alpha \in R^+$ ,

$$ur_\alpha > u \iff u\alpha > 0 \iff \alpha \in \text{Inv}(u).$$

## 2. TWO REALIZATIONS

**Realization A.** Let  $W$  be a finite Weyl group with root system  $R$ . Let  $\{\alpha_i : i \in I\} \subset R$  be the set of simple roots.

**2.1. Root lattice.** Recall the definition of  $\alpha^\vee$  for  $\alpha \in R$ . Let us denote the (co)root lattice

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}_\mathbb{R}^* \quad Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_\mathbb{R}.$$

**2.2. Definition.** The affine Weyl group is

$$W_a = W \ltimes Q^\vee.$$

For  $\lambda \in Q^\vee$ , we define  $t_\lambda \in W_a$  the corresponding element. That is,

$$t_\lambda t_\mu = t_{\lambda+\mu}, \quad t_\lambda^{-1} = t_{-\lambda}, \quad t_0 = \text{id}, \quad w t_\lambda w^{-1} = t_{w(\lambda)}.$$

**2.3. Example.** For type  $A_1$ ,

$$\begin{aligned} &\text{the Weyl group } W = \{\text{id}, s\} = \mathfrak{S}_2 \\ &\text{the coroot lattice } Q^\vee = \mathbb{Z}\alpha^\vee. \end{aligned}$$

Let us denote  $t = t_{\alpha^\vee}$ . Then we have

$$\begin{aligned} W_a &= \left\langle s, t : \begin{array}{c} s^2 = \text{id} \\ sts = t^{-1} \end{array} \right\rangle \xrightarrow{s_0 = ts} \left\langle s, s_0 : s^2 = s_0^2 = \text{id} \right\rangle \\ &= \text{the Coxeter group of } \left[ \begin{array}{c|cc} s & \bullet & \infty \\ \bullet & \hline & s_0 \end{array} \right] \end{aligned}$$

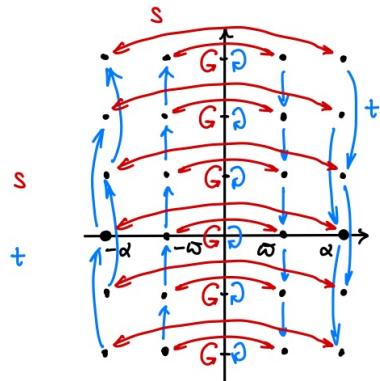
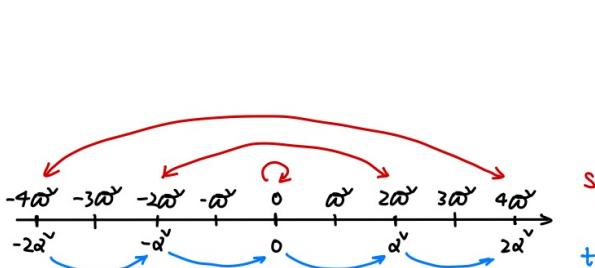
**2.4. Two Actions.** The affine Weyl group acts

on $Q^\vee$ <b>affinely</b> : $(wt_\lambda) \cdot \mu = w(\lambda + \mu).$	on $Q \oplus \mathbb{Z}\delta$ <b>linearly</b> : $(wt_\lambda) \cdot (\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta.$
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Here  $\delta$  is a formal variable, called the **null root**. Note that the same formula defines an action on

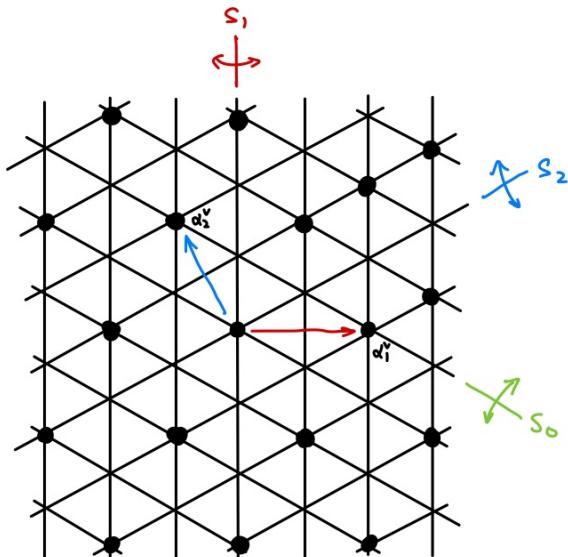
$$\mathfrak{h}_\mathbb{R} = \mathbb{R} \otimes_{\mathbb{Z}} Q^\vee, \quad \mathfrak{h}_\mathbb{R}^* \oplus \mathbb{R}\delta = \mathbb{R} \otimes_{\mathbb{Z}} (Q \oplus \mathbb{Z}\delta).$$

**2.5. Example.** Here are the example of type  $A_1$ . We denote  $\omega^\vee = \frac{1}{2}\alpha^\vee$  and  $\omega = \frac{1}{2}\alpha$ .



**2.6. Exercise.** Find the action of  $s_0 = ts$  in the above example.

**2.7. Example.** Let us consider  $A_2$ . Let  $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$ . Consider  $s_0 = t_{\theta^\vee} s_1 s_2 s_1$ . The following figure shows the action of  $W_\alpha$  on  $Q^\vee$ .



With more efforts, we can see

$$W_a = \text{the Coxeter group of } \left[ \begin{array}{c} s_0 \\ \diagdown \quad \diagup \\ s_1 \quad s_2 \end{array} \right]$$

**2.8. Roots.** We define the set of **real affine roots**

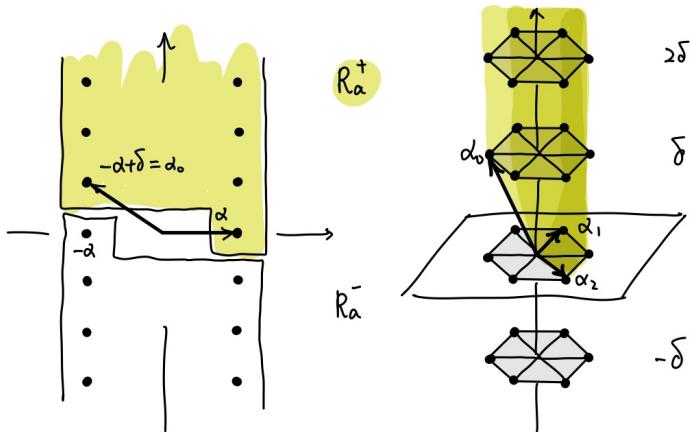
$$R_a = \{\alpha + k\delta : \alpha \in R, k \in \mathbb{Z}\} \subset Q \oplus \mathbb{Z}\delta.$$

We define the set of **positive real roots**

$$R_a^+ = \{\alpha + k\delta : k > 0 \text{ or } (k = 0 \text{ and } \alpha \in R^+)\} \subset R_a.$$

We similarly define the set of **negative roots**  $R_a^- = -R_a^+$ .

**2.9. Examples.** Here is the illustration of affine root systems of type  $A_1$  and  $A_2$



**2.10. Reflections.** For each root  $\alpha + k\delta \in R_a$ , we define the **reflection**

$$r_{\alpha+k\delta} = r_\alpha t_{k\alpha^\vee} \in W_a.$$

The action of  $r_{\alpha+k\delta}$  on  $Q \oplus \mathbb{Z}\delta$  is given by a linear reflection

$$r_{\alpha+k\delta}(\beta + n\delta) = \beta + n\delta - \langle \alpha^\vee, \beta \rangle (\alpha + k\delta).$$

The action of  $r_{\alpha+k\delta}$  on  $Q^\vee$  is given by the affine reflection along the hyperplane

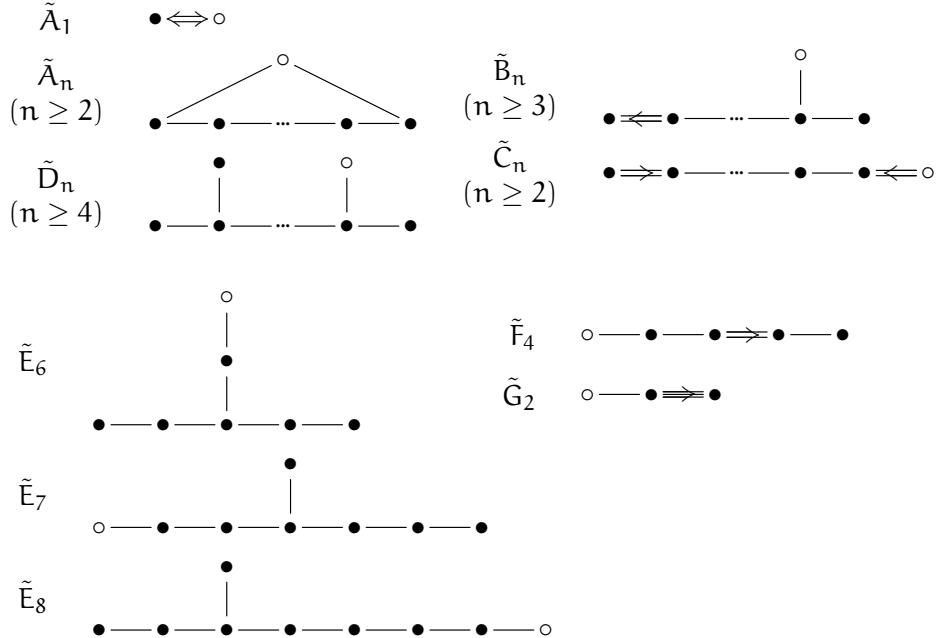
$$H_{\alpha+k\delta} = H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

**2.11. Simple roots.** Let  $\theta \in R^+$  be the unique highest root. We denote

$$\alpha_0 = -\theta + \delta \in R_a^+, \quad s_0 = r_{\alpha_0} = t_{\theta^\vee} r_\theta \in W_a, \quad I_a = I \cup \{0\}.$$

*Realization B.*

**2.12. Affine Dynkin diagram.** The following are **untwisted affine Dynkin diagrams**



The **twisted affine Dynkin diagrams** are their dual.

**2.13. Theorem.** The affine Weyl group  $W_a$  constructed above is a Coxeter group with Coxeter generator  $\{s_i : i \in I_a\}$ .

**2.14. Example.** Let  $n \geq 2$ . For type  $A_{n-1}$ , the Weyl group is  $S_n$  and the coroot lattice

$$Q^\vee = \{(a_1, \dots, a_n) : a_1 + \dots + a_n = 0\} \subset \mathbb{Z}^n.$$

The affine Weyl group admits the following realization

$$\tilde{S}_n^0 = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{l} f(i+n) = f(i) + n \\ \sum_{i=1}^n (f(i) - i) = 0. \end{array} \right\}.$$

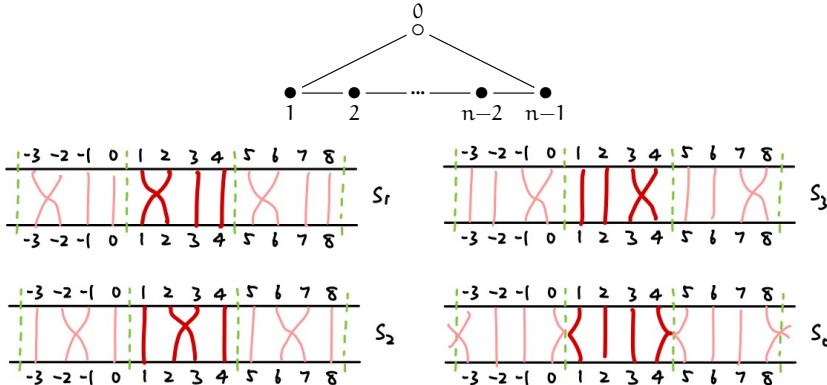
An element in  $\tilde{S}_n$  is determined by its values at  $1 \leq i \leq n$ . The identification is given by

$$\text{wt}_\lambda(i) = w(i) + \lambda_i n \quad (1 \leq i \leq n).$$

Denote  $s_i$  for  $i \in \mathbb{Z}/n\mathbb{Z}$  by

$$s_i = \begin{array}{l} \text{the affine permutation exchanging } j \text{ and } j+1 \\ \text{when } i \equiv j \pmod{n} \text{ with other numbers fixed} \end{array} \in \tilde{S}_n^0.$$

This equips the Coxeter group structure over  $\tilde{S}_n^0$ , where the Coxeter diagram is ( $n \geq 3$ )



**2.15. Example.** Let  $n \geq 2$ . For type  $C_n$ , the Weyl group is  $\mathfrak{BC}_n$ , and the coroot lattice is

$$\begin{aligned} Q^\vee &= \mathbb{Z}e_1 \oplus \mathbb{Z}(e_2 - e_1) \oplus \cdots \oplus \mathbb{Z}(e_n - e_{n-1}) \\ &= \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n. \end{aligned}$$

We can realize the affine Weyl group as

$$\tilde{\mathfrak{C}}_n = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{l} f(-i) = -f(i) \\ f(2n+2+i) = f(i) \end{array} \right\}.$$

Note that for any  $a \in \mathbb{Z}(n+1) = \{\dots, -(n+1), 0, n+1, 2n+2, \dots\}$ , we have

$$f(a+i) + f(a-i) = 2a.$$

An element of  $\tilde{\mathfrak{C}}_n$  is determined by its value at  $1 \leq i \leq n$ . The identification is given by

$$wt_\lambda = w(i) - \lambda_i(2n+2).$$

The Coxeter generators are

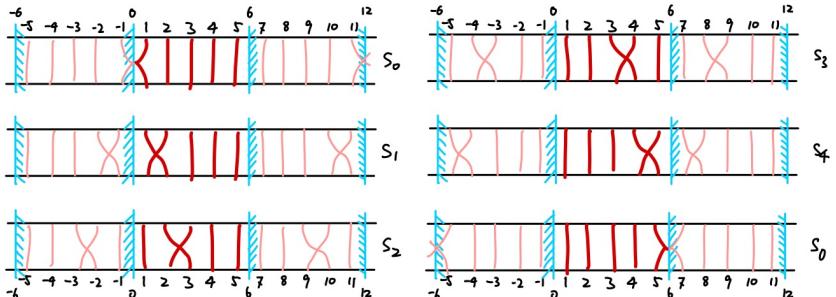
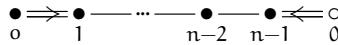
$$s_0 = \left[ \begin{array}{l} \text{the permutation exchanging } a+1 \text{ and } a-1 \text{ for} \\ a \in \mathbb{Z}(2n+2) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{C}}_n$$

$$s_0 = \left[ \begin{array}{l} \text{the permutation exchanging } a+1 \text{ and } a-1 \text{ for} \\ a \in (n+1) + \mathbb{Z}(2n+2) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{C}}_n$$

and  $1 \leq i \leq n-1$ ,

$$s_i = \left[ \begin{array}{l} \text{the permutation exchanging } a \pm i \text{ and } a \pm (i+1) \\ (a \in \mathbb{Z}(2n+2)) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{C}}_n.$$

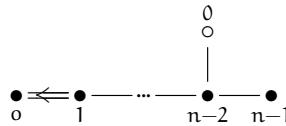
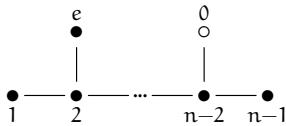
The Dynkin diagram is



**2.16. Example.** We will not go into details of affine type B/D. But we mention that

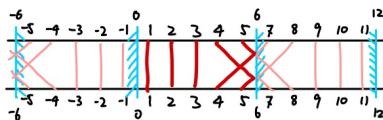
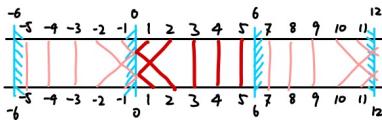
$$\tilde{\mathfrak{D}}_n \subset \tilde{\mathfrak{B}}_n \subset \tilde{\mathfrak{C}}_n$$

with generators described by



$$\begin{aligned}s_e^D &= s_0 s_1 s_0 \\s_0^D &= s_0 s_{n-1} s_0\end{aligned}$$

$$s_0^B = s_0 s_{n-1} s_0$$



### Alcoves.

**2.17. Alcove.** For each root  $\alpha + k\delta \in R_a$ , we defined a **hyperplane**

$$H_{\alpha+k\delta} = H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

We define **alcoves** by

$$\{\text{alcoves}\} = \text{connected components of } \mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha,k} H_{\alpha,k}.$$

Let us consider the **fundamental alcove**, i.e. the unique alcove  $A_0$  with

$$A_0 \subset C_0, \quad 0 \in \text{closure of } A_0.$$

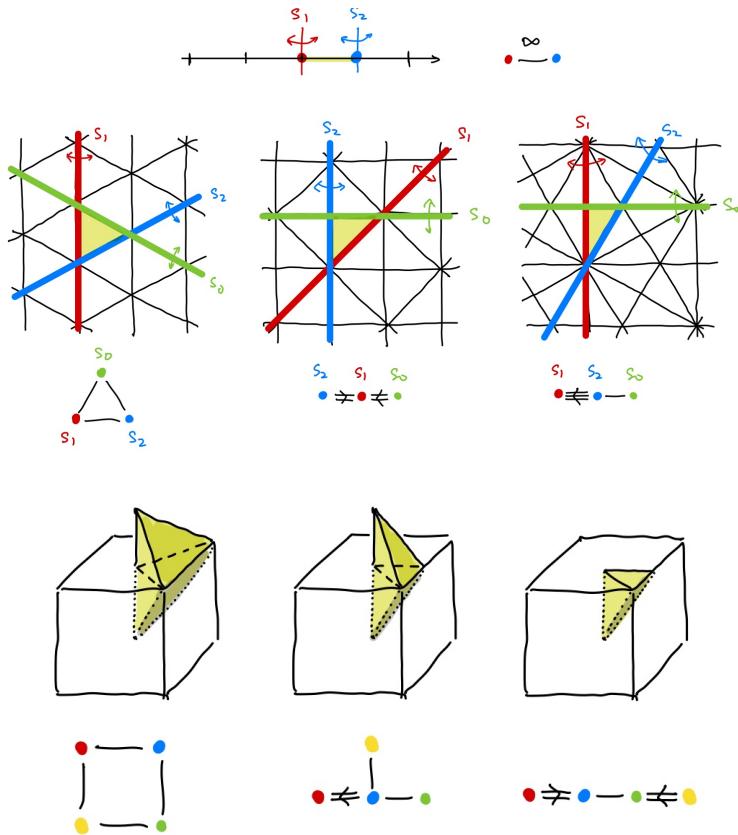
It can be described as

$$\begin{aligned}A_0 &= \{x \in \mathfrak{h}_{\mathbb{R}} : 0 < \langle x, \alpha \rangle < 1 \text{ for all } \alpha > 0\} \\&= \{x \in \mathfrak{h}_{\mathbb{R}} : 0 < \langle x, \alpha_i \rangle \text{ for } i \in I \text{ and } \langle x, \theta \rangle < 1\} \\&= \text{bounded convex set cut by } H_{\alpha_i} \text{ for } i \in I_a \\&= \text{interior of Conv} \left( \{0\} \cup \left\{ \frac{1}{(\omega_i^\vee, \theta)} \omega_i^\vee : i \in I \right\} \right).\end{aligned}$$

Here  $\theta$  is the highest root. Note that

$$\langle \omega_i^\vee, \theta \rangle = \text{coefficient of } \alpha_i \text{ in } \theta$$

Here we collect some example in small dimensions



**2.18. Theorem.** We have a bijection

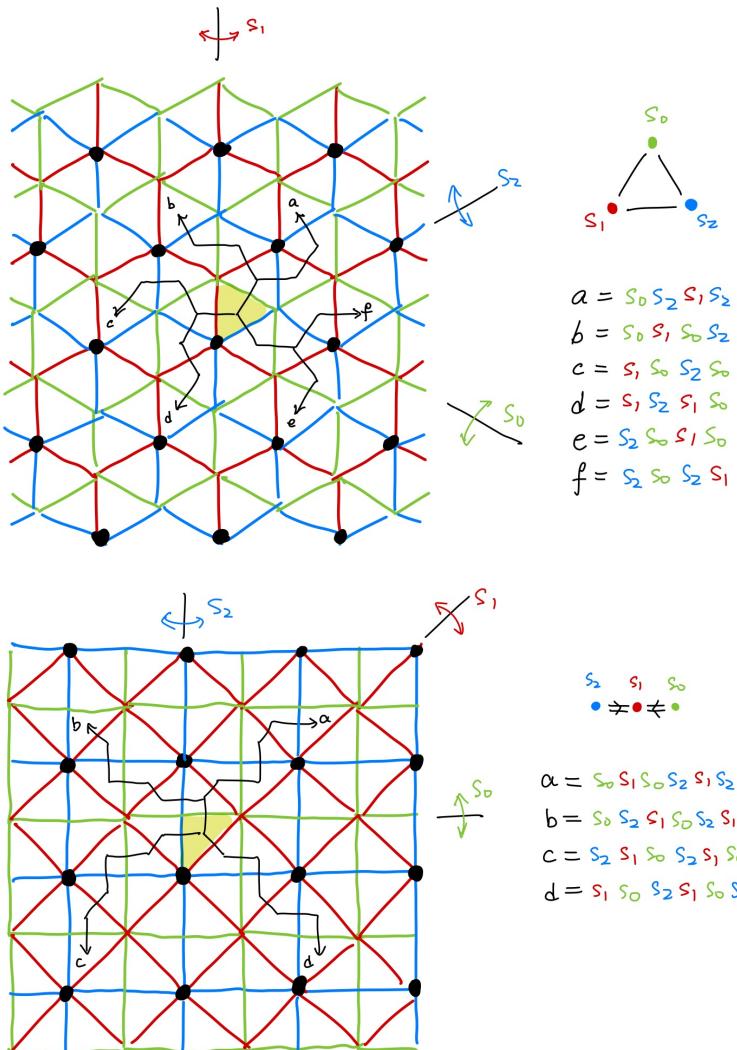
$$W_a \longrightarrow \{\text{alcoves}\}, \quad \text{wt}_\lambda \longmapsto \text{wt}_\lambda(A_0).$$

Under this bijection,

the alcove of  $s_i \text{wt}_\lambda$  = reflection of the alcove of  $w$  with respect to  $H_{\alpha_i}$

the alcove of  $\text{wt}_\lambda s_i$  = the alcove sharing the wall  $wH_{\alpha_i}$  with the alcove of  $w$

**2.19. Example.** The following shows how “alcove move” corresponds to a reduced word



### 3. LENGTH FORMULA

*Iwahori–Matsumoto Formula.*

**3.1. Inversion.** For any  $\text{wt}_\lambda \in W_a$ , we define set of **inversions**

$$\text{Inv}(\text{wt}_\lambda) = \{\alpha + k\delta \in R_a^+ : \text{wt}_\lambda(\alpha + k\delta) \in R_a^-\}.$$

Then the length function is given by

$$\begin{aligned}\ell(\text{wt}_\lambda) &= \begin{matrix} \text{minimal length of writing } \text{wt}_\lambda \text{ as} \\ \text{a product of simple reflections} \end{matrix} \\ &= \#\{\text{hyperplanes separating } A_0 \text{ and } \text{wt}_\lambda A_0\} \\ &= \# \text{Inv}(\text{wt}_\lambda)\end{aligned}$$

There is a bijection between hyperplanes and  $\text{Inv}((\text{wt}_\lambda)^{-1})$ .

**3.2. Left inversions.** Let us denote the set of **left inversions**

$$\begin{aligned}\text{LInv}(\text{wt}_\lambda) &= \text{Inv}((\text{wt}_\lambda)^{-1}) = \{-\text{wt}_\lambda(\alpha + k\delta) : \alpha + k\delta \in \text{Inv}(\text{wt}_\lambda)\} \\ &= R_a^+ \setminus \text{wt}_\lambda R_a^+.\end{aligned}$$

There is a bijection between hyperplanes and left inversions.

**3.3. Example.** Let us consider  $A_1$ . The fundamental alcove  $A_0$  is the interval  $(0, \varpi)$  and

$$sA_0 = (-\varpi, 0), \quad s_0A_0 = (\varpi, 2\varpi), \quad tA_0 = (2\varpi, 3\varpi).$$

So we have

$$\ell(s) = \ell(s_0) = 1, \quad \ell(t) = 2.$$

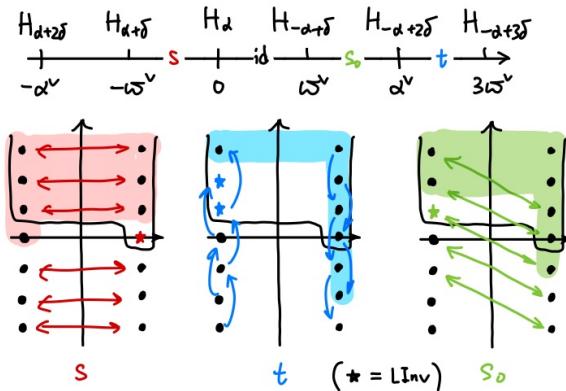
Alternatively, it is not hard to compute

$$\text{Inv}(s) = \{\alpha\}, \quad \text{Inv}(s_0) = \{-\alpha + \delta\}, \quad \text{Inv}(t) = \{\alpha, \alpha + \delta\}.$$

This confirms the computation of the lengths.

$$\begin{array}{ccccccc} \cdots & H_{\alpha+2\delta} & H_{\alpha+\delta} & H_\alpha & H_{-\alpha+\delta} & H_{-\alpha+2\delta} & \cdots \\ \cdots & -2\varpi^\vee & -\varpi^\vee & 0 & \varpi^\vee & 2\varpi^\vee & \cdots \end{array}$$

Here is the diagram



**3.4. Theorem.** We have

$$\ell(wt_\lambda) = \sum_{\alpha > 0} |\langle \alpha, \lambda \rangle + \delta_{w\alpha < 0}|.$$

Here  $\delta_p = 1$  if a statement  $p$  is true and equals 0 otherwise.

**Proof.** Fix a positive root  $\alpha \in R^+$ . We want to compute the contribution of

$$\pm\alpha + k\delta \in \text{Inv}(wt_\lambda)$$

Note that

$$wt_\lambda(\pm\alpha + k\delta) = \pm w\alpha + (k \mp \langle \lambda, \alpha \rangle)\delta.$$

For this vector in  $R_a^-$ , we summarize four cases in the following table

	$w\alpha > 0$	$w\alpha < 0$
$\pm = +$ i.e. $k \geq 0$	$k - \langle \lambda, \alpha \rangle < 0$ i.e. $0 \leq k < \langle \lambda, \alpha \rangle$	$k - \langle \lambda, \alpha \rangle \leq 0$ i.e. $0 \leq k \leq \langle \lambda, \alpha \rangle$
$\pm = -$ i.e. $k > 0$	$k + \langle \lambda, \alpha \rangle \leq 0$ i.e. $0 < k \leq -\langle \lambda, \alpha \rangle$	$k + \langle \lambda, \alpha \rangle > 0$ i.e. $0 < k < -\langle \lambda, \alpha \rangle$
Total #	$ \langle \lambda, \alpha \rangle $	$ \langle \lambda, \alpha \rangle + 1 $

This completes the proof. □

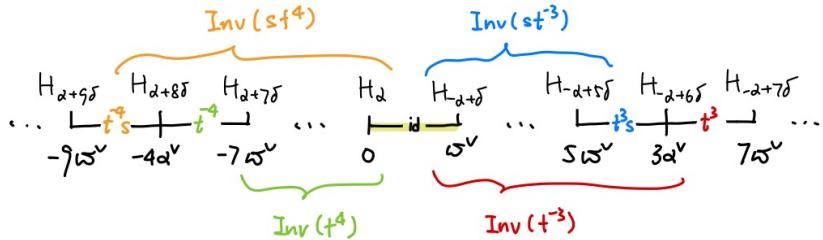
**3.5. Corollary.** Let us record the set of inversions for future references. For  $\alpha \in R^+$ , we denote

$$\text{Inv}_\alpha(\text{wt}_\lambda) = \{\pm\alpha + k\delta \in \text{Inv}(\text{wt}_\lambda)\}$$

the contribution of the affine positive roots as in the proof. Then the above table shows

$$\text{Inv}_\alpha(\text{wt}_\lambda) = \begin{cases} \{\alpha + k\delta : 0 \leq k < \langle \lambda, \alpha \rangle + \delta_{w\alpha < 0}\}, & \langle \lambda, \alpha \rangle \geq 0, \\ \{-\alpha + k\delta : 0 < k \leq -\langle \lambda, \alpha \rangle - \delta_{w\alpha < 0}\}, & \langle \lambda, \alpha \rangle < 0. \end{cases}$$

**3.6. Example.** Consider the case  $A_1$ . Recall the hyperplanes are in bijection with left inversions.



**3.7. Exercise.** Note that  $(\text{wt}_\lambda)^{-1} = t_{-\lambda} w^{-1} = w^{-1} t_{-w\lambda}$ . Check that

$$\ell(\text{wt}_\lambda) = \ell((\text{wt}_\lambda)^{-1}).$$

**3.8. Example.** In type  $\tilde{A}_{n-1}$ , we realized the affine Weyl group as  $\tilde{\mathfrak{S}}_n^0$ . For  $f \in \tilde{\mathfrak{S}}_n^0$ , we can compute the length

$$\ell(f) = \#\left\{(i, j) : \begin{array}{l} 1 \leq i \leq n \\ i < j, f(i) > f(j) \end{array}\right\}.$$

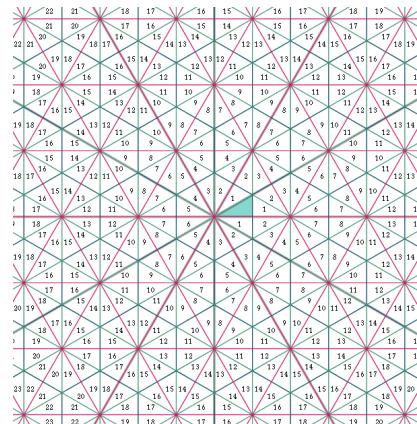
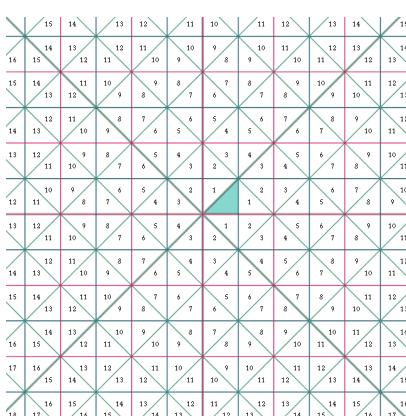
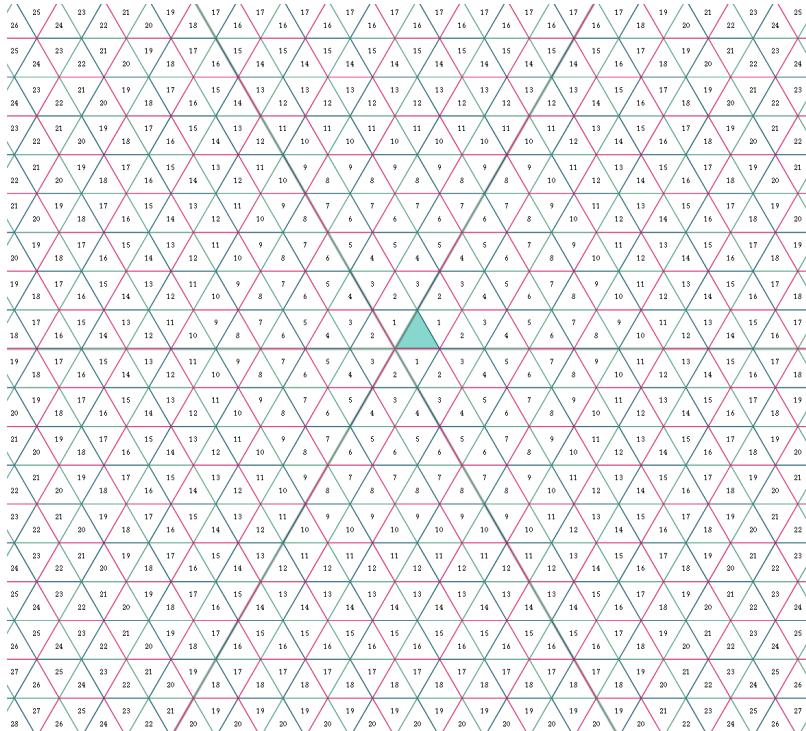
In terms of Iwahori–Matsumoto formula 3.4,

$$\ell(\text{wt}_\lambda) = \sum_{i < j} |\lambda_i - \lambda_j + \delta_{w(i) > w(j)}|.$$

Actually,  $(i, j + nk)$  corresponds to  $e_i - e_j + k\delta$ .

**3.9. Rank 2 cases.** You can visualize alcoves in rank 2 here

[https://www.jgibson.id.au/lievis/affine\\_weyl/](https://www.jgibson.id.au/lievis/affine_weyl/)



*Examples.*

**3.10.** In this paragraph, we will use a lot of facts about parabolic subgroups, which is summarized at the appendix of this section.

— Length of translations.

**3.11. Cartan vector.** Let us denote

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{i \in I} \omega_i \in \mathfrak{h}_{\mathbb{R}}^*.$$

It satisfies

$$\rho - w\rho = \sum_{\alpha \in \text{Inv}(w^{-1})} \alpha.$$

**3.12. Dominant case.** Let  $\lambda \in Q^\vee$  be dominant. Then

$$\ell(t_\lambda) = \sum_{\alpha > 0} |\langle \alpha, \lambda \rangle + \delta_{\alpha < 0}| = \sum_{\alpha > 0} \langle \alpha, \lambda \rangle = 2\langle \rho, \lambda \rangle.$$

**3.13. General case.** For general  $\lambda \in Q^\vee$ , we can always find  $w \in W$  such that

$$w\lambda_0 = \lambda, \quad \lambda_0 \text{ is dominant.}$$

Then

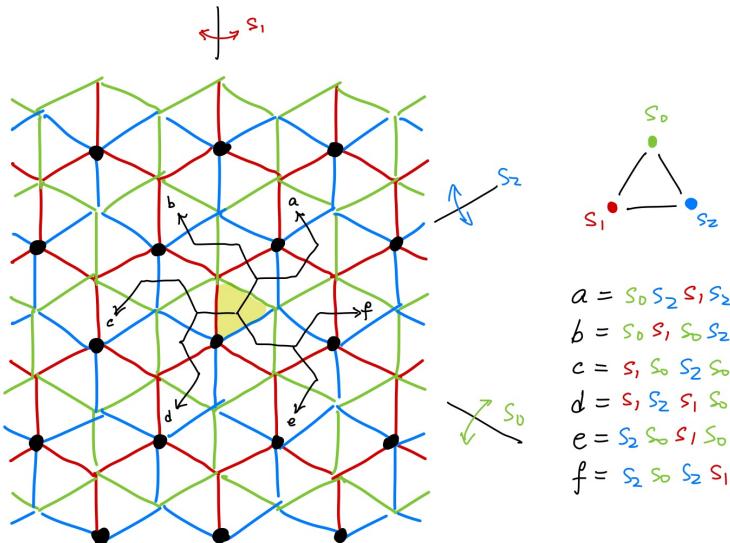
$$\begin{aligned} \ell(t_\lambda) &= \sum_{\alpha > 0} |\langle \alpha, \lambda \rangle| = \sum_{\alpha > 0} |\langle \alpha, w\lambda_0 \rangle| \\ &= \sum_{\alpha' > 0} |\langle w^{-1}\alpha, \lambda_0 \rangle| = \sum_{\alpha' > 0} |\langle \alpha', \lambda_0 \rangle| \\ &= \sum_{\alpha' > 0} \langle \alpha', \lambda_0 \rangle = 2\langle \rho, \lambda_0 \rangle = \ell(t_{\lambda_0}). \end{aligned}$$

Here  $\alpha' = \pm w^{-1}\alpha > 0$ .

**3.14. Example.** Consider the case  $A_2$ . Recall that  $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$ . Then  $\rho = \theta$ . So

$$\ell(t_{\theta^\vee}) = \ell(t_{w\theta^\vee}) = 2\langle \rho, \theta \rangle = 4.$$

This can be seen from the first diagram of Example 2.19.



**3.15. Inversion set.** It would be useful to compute the set of inversions. We have

$$\text{Inv}_\alpha(t_\lambda) = \begin{cases} \{\alpha + k\delta : 0 \leq k < \langle \lambda, \alpha \rangle\}, & \langle \lambda, \alpha \rangle > 0, \\ \emptyset, & \langle \lambda, \alpha \rangle = 0, \\ \{-\alpha + k\delta : 0 < k \leq -\langle \lambda, \alpha \rangle\}, & \langle \lambda, \alpha \rangle < 0. \end{cases}$$

### — Minimal representatives.

**3.16. Formulation.** We have a bijection

$$Q^\vee \xrightarrow{1:1} W_a/W, \quad \lambda \mapsto t_\lambda W.$$

We will describe the parabolic decomposition

$$t_\lambda = u_\lambda v_\lambda, \quad u_\lambda = \min(t_\lambda W) \text{ and } v_\lambda \in W.$$

**3.17. Example.** Let us consider type  $A_1$ . The set of minimal representative is

$$\begin{array}{ccccccc} \lambda & 0 & \alpha^\vee & -\alpha^\vee & 2\alpha^\vee & -2\alpha^\vee & \dots \\ t_\lambda & \text{id} & s_0s & ss_0 & (s_0s)^2 & (ss_0)^2 & \dots \\ u_\lambda & \text{id} & s_0 & ss_0 & s_0ss_0 & ss_0ss_0 & \dots \\ v_\lambda & \text{id} & s & \text{id} & s & \text{id} & \dots \end{array}$$

Equivalently, we want to find  $v \in W$  such that  $u^{-1} = vt_{-\lambda}$  has minimal length

$$\ell(u) = \ell(u^{-1}) = \sum_{\alpha > 0} | -\langle \alpha, \lambda \rangle + \delta_{v\alpha < 0} |.$$

To minimize  $\ell(u)$ , we wish that each summand is minimal, i.e.

$$\begin{aligned} \langle \alpha, \lambda \rangle \leq 0 &\implies v\alpha > 0, \\ \langle \alpha, \lambda \rangle > 0 &\implies v\alpha < 0. \end{aligned}$$

We will see, this is achievable.

**3.18. Antidominant case.** Let  $\lambda \in Q^\vee$  be antidominant, i.e.  $-\lambda$  is dominant. To minimize  $\ell(u^{-1})$ , it suffices to take  $v = \text{id}$ .

**3.19. General case.** Let us pick  $w \in W$  such that

$$\lambda = w\lambda_0, \quad \lambda \text{ is anti-dominant.}$$

Such  $w$ 's form a coset of  $W/W_P$  for  $W_P$  the stabilizer of  $\lambda_0$ . Let us pick the minimal one, i.e.  $w \in W^P$ . Then

$$\langle \alpha, \lambda \rangle = \langle w^{-1}\alpha, \lambda_0 \rangle < 0 \implies w^{-1}\alpha > 0$$

$$\langle \alpha, \lambda \rangle = \langle w^{-1}\alpha, \lambda_0 \rangle = 0 \implies w^{-1}\alpha \in R_P \xrightarrow{w \in W^P} w^{-1}\alpha \in R_P^+,$$

$$\langle \alpha, \lambda \rangle = \langle w^{-1}\alpha, \lambda_0 \rangle > 0 \implies w^{-1}\alpha < 0.$$

It suffices to take  $v = w^{-1}$ .

**3.20. Dominant case.** Let  $\lambda \in Q^\vee$  be dominant. Let  $W_P = w_0 W_\lambda w_0$  be the stabilizer of  $w_0\lambda$ . By above computation,  $v_\lambda = (w_0^P)^{-1}$  for  $w_0^P = \max(W^P)$  the maximal element of  $W^P$ . Actually,

$$v_\lambda = \max(W^\lambda).$$

This is because  $w_0^P = \max(W^P) = w_0 \cdot w_{0,P}$ , so

$$v_\lambda = w^{-1} = w_{0,P} \cdot w_0 = w_0 \cdot w_{0,\lambda} = \max(W^\lambda).$$

**3.21. Summary v1.** In the parabolic decomposition

$$t_\lambda = u_\lambda v_\lambda,$$

the element  $v_\lambda$  is the minimal element  $v \in W$  such that  $v\lambda$  is anti-dominant.

**3.22. Summary v2.** Let  $\lambda \in Q^\vee$  be anti-dominant. Denote  $W_P$  the stabilizer of  $-\lambda$ . Then for  $w \in W^P$ , the expression

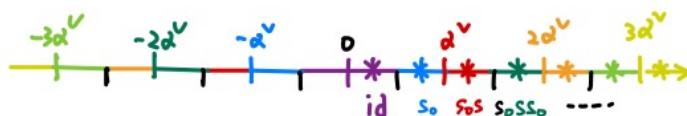
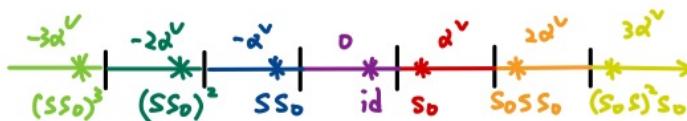
$$t_{w\lambda} = (wt_\lambda) \cdot w^{-1}$$

gives the parabolic decomposition. In particular,

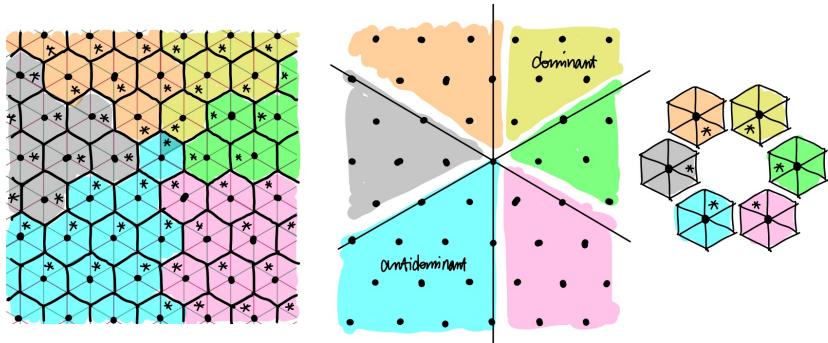
$$wt_\lambda = \min(wt_\lambda W) \iff \lambda \text{ is anti-dominant and } w \in W^\lambda.$$

**3.23. Example.** Consider type  $A_1$ . We have

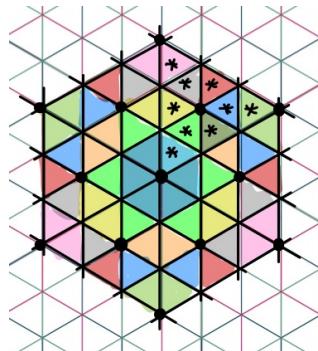
$$\begin{array}{ccccccccc} \lambda & 0 & \alpha^\vee & -\alpha^\vee & 2\alpha^\vee & -2\alpha^\vee & \dots \\ \min(t_\lambda W) & id & s_0 & ss_0 & s_0ss_0 & ss_0ss_0 & \dots \\ \lambda & 0 & -\alpha^\vee & \alpha^\vee & -2\alpha^\vee & 2\alpha^\vee & \dots \\ \min(Wt_\lambda) & id & s_0 & s_0s & s_0ss_0 & s_0ss_0s & \dots \end{array}$$



**3.24. Example.** The case  $A_2$ . We mark the minimal element in the cosets. Each right coset corresponds to



Each left coset corresponds to a  $W$ -orbit



**3.25. Exercise.** Prove that

$$\text{wt}_\lambda = \min(W\text{t}_\lambda) \iff \text{wt}_\lambda A_0 \subset C_0.$$

This gives a bijection.

**3.26. Example.** For  $\theta^\vee \in Q^\vee$ , recall that

$$s_0 = t_{\theta^\vee} r_\theta.$$

We actually have

$$u_{\theta^\vee} = s_0, \quad v_{\theta^\vee} = r_\theta \in W.$$

This implies

$$2\langle \rho, \theta^\vee \rangle = \ell(r_\theta) + 1.$$

**3.27. Example.** Let us consider the case of  $A_{n-1}$ . For any  $f \in \tilde{\mathfrak{S}}_n^0$ , it is clear that in the decomposition

$$f = uv, \quad u = \min(fW) \text{ and } v \in \mathfrak{S}_n$$

we have

$$\begin{aligned} u(1) &= \min(f(1), \dots, f(n)), \\ &\dots = \dots \\ u(n) &= \max(f(1), \dots, f(n)). \end{aligned}$$

### — Double cosets.

**3.28. Double cosets.** We have a bijection

$$Q_{\text{dom}}^\vee \xrightarrow{1:1} W \backslash W_a / W, \quad t_\lambda \longmapsto Wt_\lambda W.$$

Recall that  $wt_\lambda w^{-1} = t_{w\lambda}$ . We actually have

$$Wt_\lambda W = \bigcup_{w \in W} t_{w\lambda} W.$$

Similar to one-side case, there is also a unique minimal element in each double coset.

**3.29. Summary v3.** Let  $\lambda \in Q^\vee$  be anti-dominant. Denote  $W_P$  the stabilizer of  $-\lambda$ . By Summary 3.22 above,

$$W^P \times W \xrightarrow{1:1} Wt_\lambda W, \quad (w, u) \longmapsto wt_\lambda u,$$

with

$$\ell(wt_\lambda u) = -\ell(w) + \ell(t_\lambda) + \ell(u).$$

By taking inverse, we have a dominant version.

**3.30. Summary v4.** Let  $\lambda \in Q^\vee$  be dominant, with stabilizer  $W_P$ . We have a bijection

$$W \times W^P \longrightarrow W t_\lambda W, \quad (u, w) \longmapsto u t_\lambda w^{-1}$$

with

$$\ell(u t_\lambda w^{-1}) = \ell(u) + \ell(t_\lambda) - \ell(w).$$

As a result,

$$\min(W t_\lambda W) = \min(t_\lambda W).$$

### Appendix: Parabolic subgroups.

**3.31. Defintion.** Let  $I_P$  be a subset of  $I$ . Then we denote the **parabolic subgroup**

$$W_P = (\text{subgroup generated by } s_i \text{ with } i \in I_P) \subset W$$

and  $R_P \subset R$  the root system of  $W_P$ .

**3.32. Minimal representative.** For any  $w \in W$ , there is a minimial element, called the **minimal representative**, in the right coset  $wW_P$  under the Bruhat order. Let us denote the set of **minimal representative**

$$W^P = \{\min(wW_P) : w \in W\}.$$

We have a length-additive bijection

$$W^P \times W_P \longrightarrow W, \quad (u, v) \longmapsto uv.$$

Note that

$$w \in W_P \iff \text{Inv}(w) \subset R_P^+$$

$$w \in W^P \iff \text{Inv}(w) \subset R^+ \setminus R_P^+.$$

**3.33. Parabolic Bruhat order.** For two cosets  $uW_P, wW_P$ , we define

$$uW_P \leq wW_P \iff uv \leq wv' \text{ for some } v, v' \in W_P.$$

The the bijection

$$W^P \longrightarrow W/W_P, \quad w \longmapsto wW_P$$

is an isomorphism of posets.

**3.34. Stabilizer.** For a dominant  $\lambda \in \mathfrak{h}_{\mathbb{R}}$ , the stabilizer

$$W_{\lambda} = W_P = \{w \in W : w\lambda = \lambda\}$$

is a parabolic subgroup with

$$I_P = \{i \in I : \langle x, \alpha_i \rangle = 0\}.$$

We denote

$$W^{\lambda} = W^P.$$

Then we have a bijection

$$W^{\lambda} \longrightarrow W\lambda, \quad w \longmapsto w\lambda.$$

## 4. EXTENDED AFFINE WEYL GROUPS

***Definition.***

**4.1. Weight lattice.** Recall the definition of  $\omega_i^\vee$  for  $i \in I$ . Let us denote the **(co)weight lattice**

$$P = \bigoplus_{i \in I} \mathbb{Z}\omega_i \subset \mathfrak{h}_\mathbb{R}^*, \quad P^\vee = \bigoplus_{i \in I} \mathbb{Z}\omega_i^\vee \subset \mathfrak{h}_\mathbb{R}.$$

From the axiom of root system, we have

$$Q \subseteq P, \quad Q^\vee \subseteq P^\vee.$$

In general, they are not equal.

**4.2. Definition.** The **extended affine Weyl group** is

$$W_e = W \ltimes P^\vee.$$

For  $\lambda \in P^\vee$ , we define  $t_\lambda \in W_a$  the corresponding element.

**4.3. Two Actions.** The extended affine Weyl group acts

on $P^\vee$ <b>affinely</b> : $(wt_\lambda) \cdot \mu = w(\lambda + \mu).$	$ $	on $Q \oplus \mathbb{Z}\delta$ <b>linearly</b> : $(wt_\lambda) \cdot (\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta.$
---	-----	--

It is not hard to see  $R_a$  is stable under  $W_e$ , so the **set of inversions** also makes sense. We define the length

$$\ell(wt_\lambda) = \# \text{Inv}(wt_\lambda).$$

It is also computed by Iwahori–Matsumoto formula [3.4](#).

**4.4. Remark.** However,  $W_e$  is not a Coxeter group in general. Actually, there would be many elements in  $W_e$  of length 0. The main purpose of this section is to study them.

#### 4.5. The group $\Omega$ .

Let us denote

$$\Omega = \{\pi \in W_e : \ell(\pi) = 0\} = \{\pi \in W_e : \pi A_0 = A_0\}.$$

Note that the norm vector of facets of  $\overline{A_0}$  are simple roots. So we have

$$\begin{aligned}\Omega &\hookrightarrow \text{Aut}(A_0) = \text{Aut}(\text{affine Coxeter diagram}) \\ &= \text{Aut}(\text{affine Dynkin diagram}).\end{aligned}$$

The last equality follows from the classification, i.e. any automorphism of affine Coxeter group preserving length. Thus  $\Omega$  acts on  $W_a$ , and

$$W_e = \Omega \ltimes W_a.$$

#### 4.6. Fundamental group.

In particular the composition is an isomorphism

$$\Omega \subset W_e \twoheadrightarrow W_a / W_a = P^\vee / Q^\vee.$$

The group  $P^\vee / Q^\vee$  is known to be the fundamental group of the adjoint algebraic group. Here is the table

$A_n$	$\mathbb{Z}/(n+1)\mathbb{Z}$
$B_n$	$\mathbb{Z}/2\mathbb{Z}$
$C_n$	$\mathbb{Z}/2\mathbb{Z}$
$D_n$	$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & (n \text{ even}) \\ \mathbb{Z}/4\mathbb{Z} & (n \text{ odd}) \end{cases}$
$E_6$	$\mathbb{Z}/3\mathbb{Z}$
$E_7$	$\mathbb{Z}/2\mathbb{Z}$
$E_8, F_4, G_2$	trivial

#### 4.7. Example.

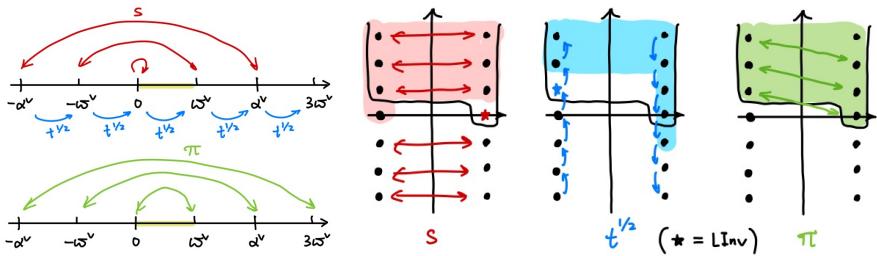
Consider type  $A_1$ .

$$P^\vee = \mathbb{Z}\omega^\vee \subset Q^\vee = \mathbb{Z}\alpha^\vee.$$

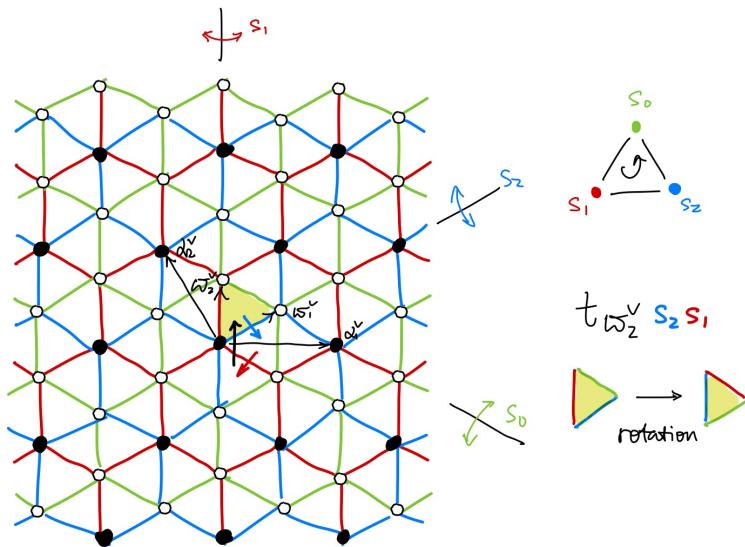
The index is 2. Let  $t^{1/2} = t_{\omega^\vee} \in W_e$ . We see

$$\pi := t^{1/2}s \in \Omega.$$

It acts on  $P^\vee$  by reflection with respect to  $\frac{\omega^\vee}{2}$ . It acts on  $Q \oplus \mathbb{Z}\delta$  by interchanging  $\alpha_1$  and  $\alpha_0$ .

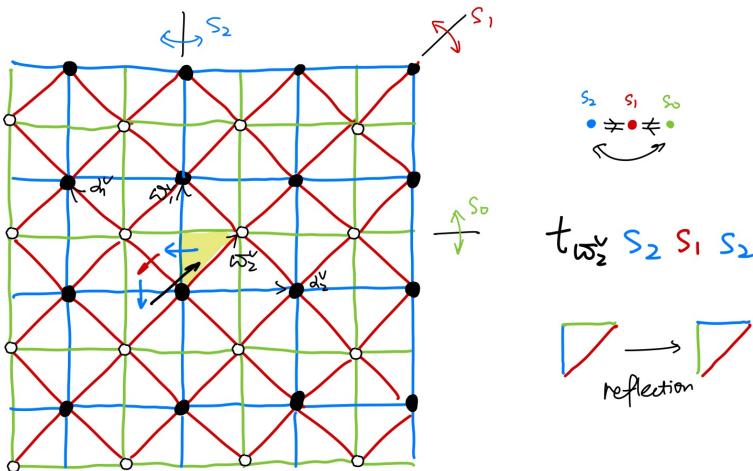


**4.8. Example.** Let us consider type  $A_2$ .



**4.9. Exercise.** Prove that  $Q^v$  has index 3 in  $P^v$ .

**4.10. Example.** Let us consider type  $B_2$ .



**4.11. Example.** Consider type  $A_{n-1}$ . Let us first give some remark on the geometric representation. The geometric representation  $\mathfrak{h}_{\mathbb{R}}^*$  can be chosen to be one of two isomorphic spaces (the subspace/quotient space realization)

$$\{(a_1, \dots, a_n) : a_1 + \dots + a_n = 0\} \subset \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n / \mathbb{R}(1, \dots, 1)$$

Then we can realize

$$\begin{array}{ccc} Q^\vee & = & \{(a_1, \dots, a_n) \in \mathbb{Z}^n : a_1 + \dots + a_n = 0\} \\ \downarrow & & \downarrow \\ \mathbb{Z}^n & & \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1). \\ \downarrow & & \downarrow \\ P^\vee & = & \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1). \end{array}$$

We thus have

$$W_a = \mathfrak{S}_n \ltimes Q^\vee \subset \tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes \mathbb{Z}^n \rightarrow \mathfrak{S}_n \ltimes P^\vee = W_e.$$

Here

$$\tilde{\mathfrak{S}}_n = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : f(i+n) = f(i) + n. \right\}.$$

For any  $\lambda \in \mathbb{Z}^n$ , the corresponding translation  $t_\lambda \in \tilde{\mathfrak{S}}_n$  by

$$t_\lambda(i) = i + \lambda_i n, \quad 1 \leq i \leq n-1.$$

Then the extended Weyl group

$$W_e = \tilde{\mathfrak{S}}_n / \langle t_{(1,\dots,1)} \rangle.$$

Actually, all theory of extended Weyl groups can be lifted to  $\tilde{\mathfrak{S}}_n$ . So  $\tilde{\mathfrak{S}}_n$  is also called the **extended Weyl group** of type A.

Denote  $\pi \in \tilde{\mathfrak{S}}_n$  by

$$\pi(i) = i+1.$$

Note that  $\pi \notin \tilde{\mathfrak{S}}_n^0$  and  $\pi^n = t_{(1,\dots,1)}$ . For  $i \in \mathbb{Z}/n\mathbb{Z}$ , we have

$$\pi s_i \pi^{-1} = s_{i+1}.$$

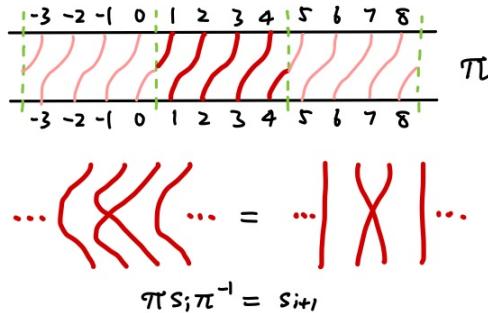
We have

$$\tilde{\mathfrak{S}}_n = \left\langle \begin{array}{c} s_0, s_1, \dots, s_{n-1} \\ \pi \end{array} : \begin{array}{l} s_i^2 = \text{id}, \text{ braid relations} \\ \pi s_i \pi^{-1} = s_{i+1} \end{array} \right\rangle$$

This shows

$$\tilde{\mathfrak{S}}_n = \pi^\mathbb{Z} \ltimes \tilde{\mathfrak{S}}_n^0, \quad W_e = \pi^\mathbb{Z} / \langle \pi^n \rangle \ltimes \tilde{\mathfrak{S}}_n^0.$$

The diagram notation



*Cominuscule node.*

**4.12. Cominuscule node.** We say a node  $k \in I$  is **cominuscule** if

$$\langle \omega_k^\vee, \theta \rangle = 1.$$

Equivalently, for any positive roots  $\alpha > 0$ ,

$$\langle \omega_k^\vee, \alpha \rangle \in \{0, 1\}.$$

Let us denote  $W_P$  the stabilizer of  $\omega_k^\vee$ .

**4.13. Elements in  $\Omega$ .** For any cominuscule  $k \in I$ , by 3.20 or 3.30, the parabolic decomposition is given by

$$t_{\omega_k^\vee} = \pi_k \cdot w_0^P, \quad w_0^P = \max(W^P) \text{ and } \pi_k = \min(t_{\omega_k^\vee} W).$$

We have

$$\begin{aligned} \ell(\pi_k) &= \ell(t_{\omega_k^\vee}(w_0^P)^{-1}) = \ell(w_0^P t_{-\omega_k^\vee}) \\ &= \sum_{\alpha > 0} | -\langle \alpha, \omega_k^\vee \rangle + \delta_{w_0^P \alpha < 0} |. \end{aligned}$$

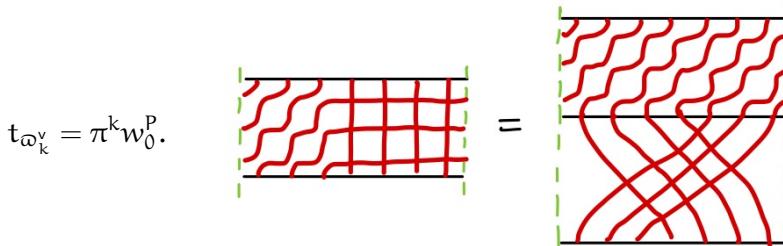
Note that

$$\begin{aligned} \alpha \in R_P^+ &\implies \langle \alpha, \omega_k^\vee \rangle = 0, & w_0^P \alpha > 0 \\ \alpha \in R^+ \setminus R_P^+ &\implies \langle \alpha, \omega_k^\vee \rangle = 1, & w_0^P \alpha < 0. \end{aligned}$$

Each term is zero. Thus

$$\pi_k \in \Omega.$$

**4.14. Example.** In type  $A_{n-1}$ , each node is cominuscule. We have  $\omega_k^\vee = e_1 + \cdots + e_k$  in the quotient space realization. The following diagram reads



**4.15. Theorem.** Denote  $\pi_0 = \text{id}$ , and call 0 cominuscule. We have

$$\Omega = \{\pi_k : k \in I_a \text{ is cominuscule}\}.$$

**4.16. Description of the automorphism.** Note that for any  $\pi \in \Omega$ , we have

$$\pi\alpha_i = \alpha_{\pi(i)}.$$

So if

$$w_0^P \alpha_i = \alpha_j \pmod{\theta}$$

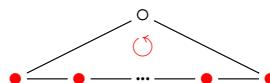
we must have

$$\pi_k \alpha_j = t_{\omega_k^v}(\alpha_i + (\dots) \delta) = \alpha_i.$$

That is,  $\pi(j) = i$ . In particular, since  $w_0^P \alpha_k < 0$  we must have  $\pi(0) = k$ .

**4.17. Type A.** For type  $\tilde{A}_{n-1}$ , every node is cominuscule. The automorphism

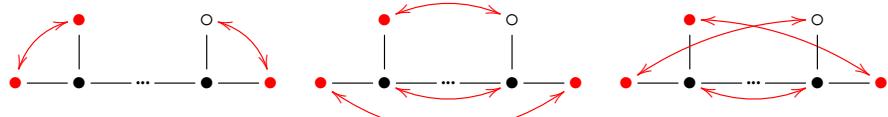
$$\pi_k(i) = i + k \pmod{n},$$



**4.18. Type B and type C.** For type  $\tilde{B}_n$  and  $\tilde{C}_n$ , there is one cominuscule node



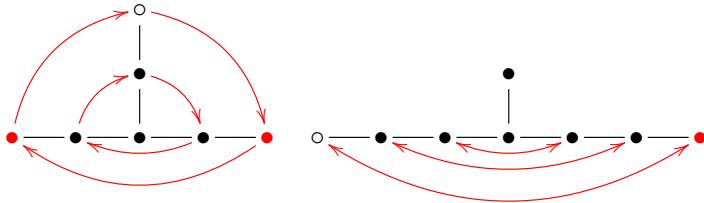
**4.19. Type D.** For type  $\tilde{D}_n$ , there are three cominuscule nodes. When  $n$  is even,



When  $n$  is odd,



**4.20. Type E.** For type  $\tilde{E}_6$  and  $\tilde{E}_7$ , there are 2 and 1 cominuscule node respectively.



**4.21. Corollary.** A node  $k \in I$  is cominuscule if and only if  $k$  is in the orbit of affine node under automorphism of affine Dynkin diagram. Moreover,

$$\text{Aut}(\text{Finite Dynkin diagram}) \ltimes \Omega = \text{Aut}(\text{Affine Dynkin diagram}).$$

*Bruhat order.*

**4.22. Extending Bruhat order.** We can define Bruhat order over  $W_e$ , and extend it to  $W_a$  by the disjoint union of ordering over

$$W_e = \bigcup_{\pi \in \Omega} \pi W_a.$$

Note that for any  $\pi \in \Omega$ ,

$$ut_\mu \leq wt_\lambda \iff \pi(ut_\mu)\pi^{-1} \leq \pi(wt_\lambda)\pi^{-1}.$$

So it gives the same order if we use the left cosets.

**4.23. Bruhat order.** Let us describe the **Bruhat order** over

$$W_e/W \xrightarrow{1:1} P^\vee.$$

We first mention that the above map is an isomorphism of  $W_e$ -sets. We denote the Bruhat order

$$\lambda \leq \mu \iff t_\lambda W \leq t_\mu W.$$

Note that a general fact of parabolic Bruhat order tells

$$\begin{aligned} t_\lambda W \leq t_\mu W : &\iff u_\lambda \leq u_\mu \\ &\iff \exists x \in t_\lambda W \text{ and } y \in t_\mu W \text{ such that } x \leq y. \end{aligned}$$

Here  $u_\lambda = \min(t_\lambda W)$  the minimal representative.

The Bruhat order is generated by

$$\lambda < \mu \text{ when } \mu = r_{\hat{\alpha}}\lambda \text{ for some } \hat{\alpha} \in R_a^+.$$

Note that

$$\mu < r_{\hat{\alpha}}\mu \iff \hat{\alpha} \in LInv(t_\mu).$$

As we computed in [3.15](#) the inversion set of  $t_\lambda$ , it is not hard to conclude

- When  $\langle \lambda, \alpha \rangle < 0$ ,  $\alpha \in LInv(t_\lambda)$ , i.e.  $r_\alpha t_\lambda < t_\lambda$ . We have

$$r_\alpha \lambda < \lambda.$$

- When  $\langle \lambda, \alpha \rangle > 0$ ,  $-\alpha + \delta \in LInv(t_\lambda)$ , i.e.  $r_{-\alpha+\delta} t_\lambda < t_\lambda$ . Recall that  $r_{-\alpha+\delta} = t_{\alpha^\vee} r_\alpha$ , we have

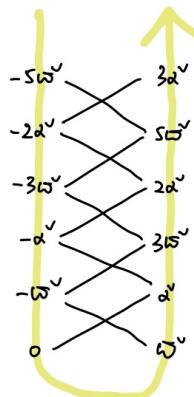
$$r_\alpha \lambda + \alpha^\vee < \lambda.$$

This can also be seen from the alcove. As a result, the Bruhat order is generated by

$$\lambda < \lambda + \alpha < \alpha - \alpha < \alpha + 2\alpha < \dots \quad \langle \lambda, \alpha \rangle = 0$$

$$\lambda < \lambda - \alpha < \alpha + \alpha < \alpha - 2\alpha < \dots \quad \langle \lambda, \alpha \rangle = 1$$

**4.24. Example.** Consider type  $A_1$ .



**4.25. Exercise.** For  $\alpha \in R^+$ , denote

$$\ell_\alpha(wt_\lambda) = \# \text{Inv}_\alpha(wt_\lambda).$$

Prove that

$\langle \lambda + k\alpha^\vee, \alpha \rangle$	...	-8	-6	-4	-2	0	2	4	6	8	...
$\ell_\alpha(u_{\lambda+k\alpha^\vee})$	...	8	6	4	2	0	1	3	5	7	...
$\langle \lambda + k\alpha^\vee, \alpha \rangle$	...	-7	-5	-3	-1	1	3	5	7	9	...
$\ell_\alpha(u_{\lambda+k\alpha^\vee})$	...	7	5	3	1	0	2	4	6	8	...

## 5. SEMI-INFINITY

*Semi-infinite length.*

**5.1. Length.** Recall for  $x \in W_e$ , we defined

$$\text{Inv}(x) = \{\alpha + k\delta \in R_a^+ : x(\alpha + k\delta) \in R_a^-\}.$$

Then the length function is given by

$$\begin{aligned}\ell(x) &= \#\{\text{hyperplanes separating } A_0 \text{ and } x^{-1}A_0\} \\ &= \#\text{Inv}(x)\end{aligned}$$

There is a bijection between hyperplanes and inversions.

**5.2. Semi-infinite length.** For  $x \in W_e$ , we define

$$\ell^\% (x) = \ell(xt_\mu) - \ell(t_\mu)$$

for  $\mu$  sufficiently dominant. Here, sufficiently dominant means,

$$\langle \mu, \alpha_i \rangle \gg 0 \quad \text{for each } i \in I.$$

In particular,  $\ell^\%(\pi x) = \ell^\%(x)$  for  $\pi \in \Omega$ . Note that unlike the usual length,  $\ell^\%(x)$  might be negative and  $\ell^\%(x) \neq \ell^\%(x^{-1})$  in general.

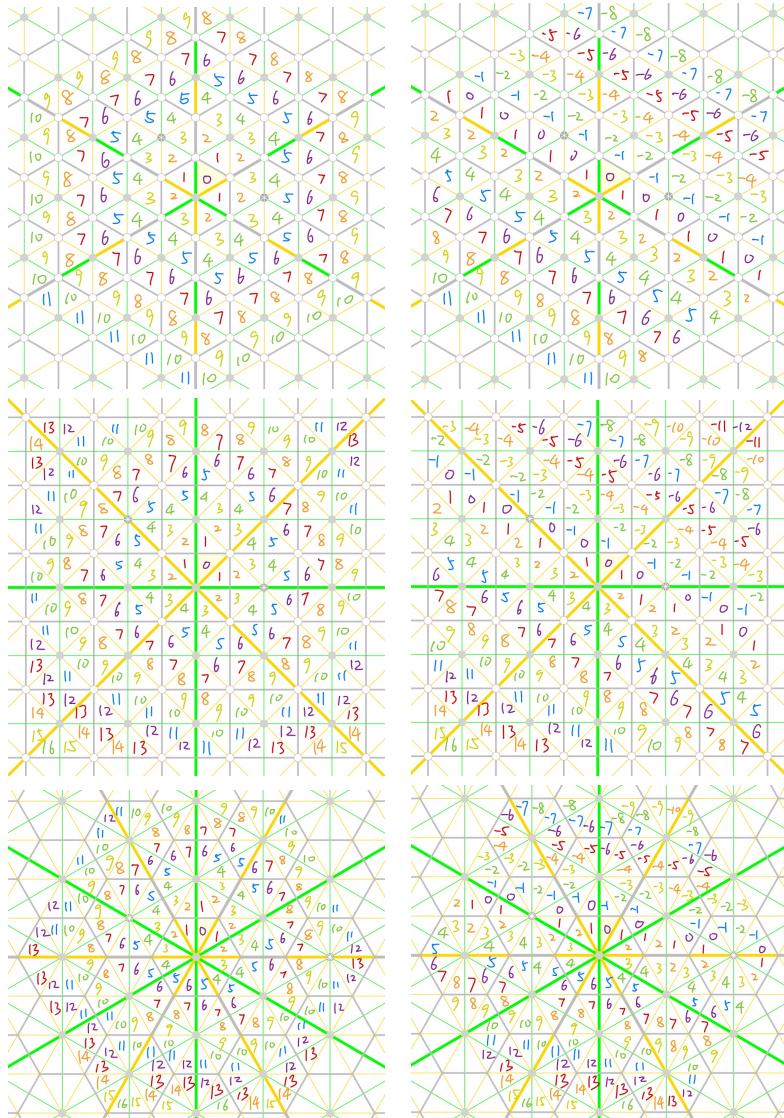
**5.3. Computation.** If we write  $x = wt_\lambda$ , then Iwahori–Matsumoto formula 3.4 implies

$$\begin{aligned}\ell^\%(wt_\lambda) &= \sum_{\alpha > 0} |\langle \alpha, \lambda + \mu \rangle + \delta_{w\alpha < 0}| - |\langle \mu, \alpha \rangle| \\ &= \sum_{\alpha > 0} (\langle \alpha, \lambda \rangle + \delta_{w\alpha < 0}) = \ell(w) + 2\langle \rho, \lambda \rangle.\end{aligned}$$

**5.4. Example.** Let us consider  $A_1$  case. We have

$$\begin{array}{ccccccccccccc} x & \cdots & s_0ss_0 & ss_0 & s_0 & \text{id} & s & s_0s & ss_0s & (ss_0)^2 & \cdots \\ & \cdots & st^{-2} & t^{-2} & st^{-1} & \text{id} & s & t & st & t^2 & \cdots \\ \ell^\%(x) & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \end{array}$$

**5.5. Example.** Compare length and semi-infinite length for  $W_a$ .



$\ell(x)$  on  $x^{-1}A_0$

$\ell\%_0(x)$  on  $x^{-1}A_0$

**5.6. Example.** Let us compute

$$\ell\%(\mathbf{r}_{\alpha+k\delta}) = \ell\%(\mathbf{r}_\alpha \mathbf{t}_{k\alpha^\vee}) = \ell(\mathbf{r}_\alpha) + 2k\langle \rho, \alpha^\vee \rangle.$$

We proved in 3.26 that

$$2\langle \rho, \theta^\vee \rangle = \ell(\mathbf{r}_\theta) + 1.$$

So

$$\ell\%(s_0) = \ell(\mathbf{r}_\theta) - 2\langle \rho, \theta^\vee \rangle = -1.$$

**5.7. A trick.** Let  $\alpha + k\delta \in R_a$ . Note that

$$\alpha + (k + \langle \mu, \alpha \rangle)\delta$$

We have

$$\mathbf{t}_{-\mu}(\alpha + k\delta) \in R_a^\pm \text{ for sufficiently dominant } \mu \iff \alpha \in R^\pm.$$

**5.8. Semi-infinite Inversion.** Note that for  $\alpha + k\delta \in R_a^+$ ,

$$\begin{aligned} & \alpha + k\delta \in LInv(x\mathbf{t}_\mu) \text{ for sufficiently dominant } \mu \\ \iff & \mathbf{t}_{-\mu}(x^{-1}(\alpha + k\delta)) \in R_a^- \text{ for sufficiently dominant } \mu \\ \iff & x^{-1}(\alpha + k\delta) \bmod \delta \in R^-. \end{aligned}$$

Let us denote

$$R_{\%}^\pm = \{\alpha + k\delta : \alpha \in R^\pm \text{ and } k \in \mathbb{Z}\}.$$

We denote

$$LInv_{\%}(x) = \{\alpha + k\delta \in R_a^+ : x^{-1}(\alpha + k\delta) \in R_{\%}^-\} \subset R_a^+.$$

We denote

$$Inv_{\%}(x) = \{a + k\delta \in R_{\%}^+ : x(a + k\delta) \in R_a^-\} \subset R_{\%}^+.$$

**5.9. Computation.** We have

$$\text{wt}_\lambda(\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta.$$

So  $\alpha + k\delta \in \text{Inv}_{\%}(x)$  if and only if

$$k - \langle \lambda, \alpha \rangle < \delta_{w\alpha < 0}.$$

We see

$$\text{Inv}_{\%}(\text{wt}_\lambda) = \{\alpha + k\delta : \alpha \in R^+ \text{ and } k < \langle \lambda, \alpha \rangle + \delta_{w\alpha < 0}\}.$$

Compare with 3.5.

**5.10. Theorem.** We have

$$\ell^{\%}(x) = \#(\text{Inv}_{\%}(\text{wt}_\lambda) \setminus \text{Inv}_{\%}(\text{id})) - \#(\text{Inv}_{\%}(\text{id}) \setminus \text{Inv}_{\%}(\text{wt}_\lambda)).$$

We can write it as

$$\ell^{\%}(x) = \sum_{\alpha+k\delta \in \text{Inv}(x)} \begin{cases} 1, & \alpha > 0 \\ -1, & \alpha < 0 \end{cases}$$

**5.11. Half-space.** For  $\alpha + k\delta \in R_a$ , we defined

$$H_{\alpha+k\delta} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\}.$$

We define the **half-space**

$$H_{\alpha+k\delta}^{>0} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k > 0\}.$$

Similarly, we define  $H_{\alpha+k\delta}^{<0}$  etc. One can check

$$\text{wt}_\lambda \cdot H_{\alpha+k\delta}^{>0} = H_{\text{wt}_\lambda(\alpha+k\delta)}^{>0}.$$

**5.12. Alcove.** Let  $x \in W_e$ . Let us justify the bijection

$$\text{Inv}(x) \xrightarrow{1:1} \#\{\text{hyperplanes separating } x^{-1}A_0 \text{ and } A_0\}.$$

The key observation is

$$\alpha + k\delta \in R_a^+ \iff A_0 \subset H_{\alpha+k\delta}^{>0}.$$

As a result, for  $\alpha + k\delta \in R_a^+$ ,

$$\begin{aligned} & \text{the hyperplane } H_{\alpha+k\delta} \text{ separates } A_0 \text{ and } x^{-1}A_0 \\ \iff & x^{-1}A_0 \subset H_{\alpha+k\delta}^{<0} \iff A_0 \subset H_{x(\alpha+k\delta)}^{<0} \iff x(\alpha + k\delta) \in R_a^- \\ \iff & \alpha + k\delta \in \text{Inv}(x). \end{aligned}$$

The semi-infinite analogy is

$$\begin{aligned} \alpha + k\delta \in R_{\%}^+ & \iff A_0 + \mu \subset H_{\alpha+k\delta}^{>0} \text{ for sufficiently dominant } \mu \\ & \iff C_0 \cap H_{\alpha+k\delta}^{>0} \neq \emptyset. \end{aligned}$$

Here  $C_0$  is the fundamental chamber. As a result, for  $\alpha > 0$ ,

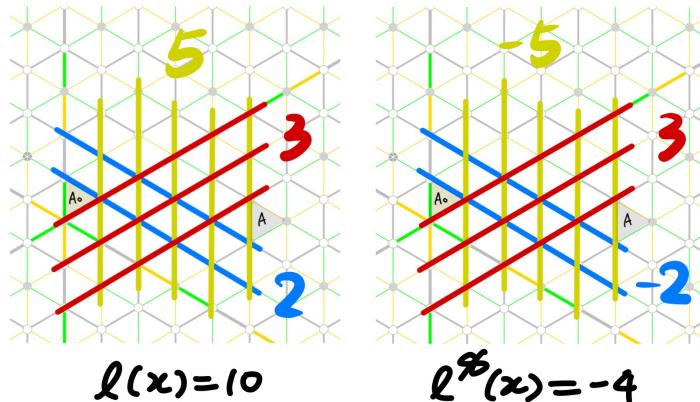
$$\begin{aligned} & \text{the hyperplane } H_{\alpha+k\delta} \text{ separates } A_0 + \mu \text{ and } x^{-1}A_0 \text{ for } \mu \text{ sufficiently dom} \\ \iff & x^{-1}A_0 \subset H_{\alpha+k\delta}^{<0} \iff A_0 \subset H_{x(\alpha+k\delta)}^{<0} \iff x(\alpha + k\delta) \in R_a^- \\ \iff & \alpha + k\delta \in \text{Inv}_{\%}(x). \end{aligned}$$

As a result,

$$\ell_{\%}(x) = \sum_H \begin{cases} 1, & A_0 \subset H + C_0, \\ -1, & A_0 \subset H - C_0, \end{cases}$$

with the sum over hyperplanes  $H$  separating  $x^{-1}A_0$  and  $A_0$ .

**5.13. Example.** Consider the case  $A_2$ .



**5.14. Exercise.** For  $x \in W_e$ , prove that

$$-\ell(x) \leq \ell\%_e(x) \leq \ell(x).$$

*Semi-infinite Bruhat order.*

**5.15. Bruhat order.** Recall the Bruhat order can be equivalently described by

- the order generated by

$$x < xr_{\hat{\alpha}} \quad \text{when} \quad \ell(xr_{\hat{\alpha}}) = \ell(x) + 1.$$

- the order generated by

$$x < xr_{\hat{\alpha}} \quad \text{when} \quad \ell(xr_{\hat{\alpha}}) > \ell(x).$$

That is,  $\hat{\alpha} \in R_a^+$  and  $x\hat{\alpha} \in R_a^+$ .

- $x \leq y$  if there is a subword of  $x$  in a reduced word of  $y$ .
- $x \leq y$  if there is a subword of  $x$  in any reduced word of  $y$ .

**5.16. Semi-infinite Bruhat order.** For  $x, y \in W_e$ , we define the **semi-infinite Bruhat order**

$$x \leq\%_e y \iff xt_\mu \leq yt_\mu \text{ for } \mu \in Q^\vee \text{ sufficiently dominant.}$$

The well-definedness follows from the description below. Note that unlike Bruhat order,  $x \leq\%_e y$  does not imply  $x^{-1} \leq\%_e y^{-1}$ .

**5.17. Description.** The semi-infinite Bruhat order can be equivalently described by

- the order generated by

$$x <\%_e xr_{\hat{\alpha}} \quad \text{when} \quad \ell\%_e(xr_{\hat{\alpha}}) = \ell\%_e(x) + 1.$$

- the order generated by

$$x <\%_e xr_{\hat{\alpha}} \quad \text{when} \quad \ell\%_e(xr_{\hat{\alpha}}) > \ell\%_e(x).$$

That is,  $\hat{\alpha} \in R_a^+$  and  $x\hat{\alpha} \in R_a^+$ .

- $x \leq y$  if there is a subword of  $xt_\mu$  in a reduced word of  $yt_\mu$  for sufficiently dominant  $\mu$ .
- $x \leq y$  if there is a subword of  $xt_\mu$  in a reduced word of  $yt_\mu$  for sufficiently dominant  $\mu$ .

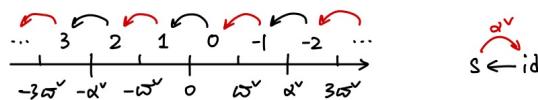
**5.18. Exercise.** Prove that

$$x \leq_{\%} y \iff xw_0 \geq_{\%} yw_0$$

where  $w_0 = \max(W)$  the longest element in finite Weyl group.

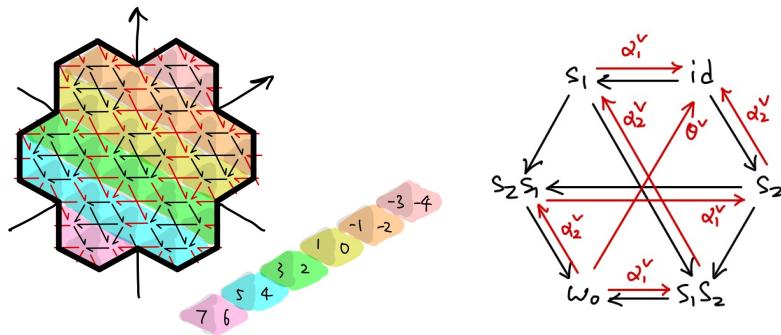
**5.19. Remark.** This order is also known as the **quantum Bruhat order**.

**5.20. Example.** Consider the case  $A_1$ .



As usual, we mark the semi-infinite length  $\ell^{\%}(x)$  on  $x^{-1}A_0$ .

**5.21. Example.** Consider the case  $A_2$ .



Note that, we use  $x^{-1}A_0$  to represent  $x$ , so

left multiplication by  $s_i$  = wall-crossing

**5.22. Lemma.** We have

$$\ell(r_\alpha) \leq 2\langle \rho, \alpha^\vee \rangle - 1.$$

**Proof.** Generally,

$$\rho = w\rho + \sum_{\beta \in L\text{Inv}(w)} \beta.$$

Substituting  $w = r_\alpha$ , we get

$$\langle \rho, \alpha^\vee \rangle \alpha = \sum_{\beta \in L\text{Inv}(r_\alpha)} \beta.$$

Note that  $\beta \in \text{Inv}(r_\alpha)$  implies

$$\beta - \langle \alpha^\vee, \beta \rangle \alpha < 0.$$

We must have  $\langle \alpha^\vee, \beta \rangle \geq 1$ . Note that  $\alpha \in \text{Inv}(r_\alpha)$ , with  $\langle \alpha^\vee, \alpha \rangle = 2$ . Thus we get

$$2\langle \rho, \alpha^\vee \rangle = \sum_{\beta \in L\text{Inv}(r_\alpha)} \langle \beta, \alpha^\vee \rangle \geq \ell(r_\alpha) + 1.$$

This proves the inequality.  $\square$

**5.23. Corollary.** From the proof, for  $\alpha \in R^+$ , it is easy to see the inequality achieves

$$\ell(r_\alpha) = 2\langle \rho, \alpha^\vee \rangle - 1$$

exactly when

- $\alpha$  is long;
- the coefficient of each long simple root of  $\alpha$  is 0.

In particular, it is always true for simply-laced types.

**5.24. Computation.** Let us give a more precise description of

$$x <_x r_\alpha \quad \text{when} \quad \ell^{\%}(xr_\alpha) = \ell^{\%}(x) + 1.$$

Firstly, the order the translation invariant, i.e.

$$x \leq \% y \iff xt_\mu \leq \% yt_\mu, \quad \forall \mu \in P^\vee.$$

Let us assume  $x = w \in W$ ,  $\hat{\alpha} = \alpha + k\delta$  for  $\alpha > 0$ . Recall  $r_{\alpha+k\delta} = r_\alpha t_{k\alpha^\vee}$ . We have

$$\ell^{\%}(xr_{\alpha+k\delta}) - \ell^{\%}(x) = \ell(wr_\alpha) + 2k\langle \rho, \alpha^\vee \rangle - \ell(w) = 1.$$

So

$$2k\langle \rho, \alpha^\vee \rangle - 1 = \ell(w) - \ell(wr_\alpha).$$

We have

$$-2\langle \rho, \alpha^\vee \rangle + 1 \leq -\ell(r_\alpha) \leq 2k\langle \rho, \alpha^\vee \rangle - 1 \leq \ell(r_\alpha) \leq 2\langle \rho, \alpha^\vee \rangle - 1.$$

Thus  $-1 < k \leq 1$ . When  $k = 0$ , this is a cover relation in the finite Bruhat order

$$w <_{\%} wr_\alpha \quad \text{when } \ell(xr_\alpha) = \ell(x) + 1.$$

When  $k = 1$ , the equality must be achieved, i.e.

$$w <_{\%} wr_\alpha t_{k\alpha^\vee} \quad \text{when } \ell(xr_\alpha) = \ell(x) - \ell(r_\alpha) \text{ for } \alpha \text{ in 5.23.}$$

**5.25. Theorem.** The semi-infinite Bruhat order is generated by

$$wt_\lambda <_{\%} wr_\alpha t_\alpha \quad \ell(wr_\alpha) = \ell(w) + 1$$

$$wt_\lambda <_{\%} wr_\alpha t_{\alpha+\alpha^\vee} \quad \ell(wr_\alpha) = \ell(w) - \ell(r_\alpha) \text{ for } \alpha \text{ in 5.23.}$$

### Grassmannian elements.

**5.26. Minimal representative.** Let  $x = wt_\lambda$ . Recall that in 3.22, we get

$$x = \min(xW) \iff \lambda \text{ is anti-dominant and } w \in W^\lambda.$$

This is also true for extended Weyl group. A general facts of Weyl groups tells

$$x = \min(xW) \iff \text{Inv}(x) \cap R^+ = \emptyset.$$

From the computation of 3.5, we see that  $x \in \min(xW)$  if and only if

$$\text{Inv}(x) \subset R^-_\%.$$

**5.27. Proposition.** By 5.10, we have

$$\ell(x) = -\ell\%_-(x) \iff x = \min(xW).$$

**5.28. Example.** Recall that  $x = \min(xW)$  if and only if  $x^{-1}A_0 \subset C_0$ . The above examples give examples of this theorem.

**5.29. Lemma.** When  $x = \min(xW)$ , any anti-dominant  $\lambda \in P^\vee$ , we have

$$xt_\mu = \min(xt_\lambda W), \quad \ell(x) + \ell(t_\lambda) = \ell(xt_\lambda).$$

This is obvious from above description.

**5.30. Theorem.** When  $x = \min(xW)$ , for any  $y \in W_e$

$$\begin{aligned} y \leq x &\implies y \geq_{\%} x, \\ y \leq_{\%} x &\implies y \geq x. \end{aligned}$$

**Proof.** We have

$$\begin{aligned} y \leq x &\implies yt_{\lambda} \leq xt_{\lambda} && \text{for sufficiently anti-dominant } \lambda \\ &\implies yt_{\lambda}w_0 \leq xt_{\lambda}w_0 && \text{for sufficiently anti-dominant } \lambda \\ &\implies yw_0t_{\mu} \leq xw_0t_{\mu} && \text{for sufficiently dominant } \mu = w_0\lambda \\ &\implies yw_0 \leq_{\%} xw_0 \\ &\implies y \geq_{\%} x && (\text{by 5.18}). \end{aligned}$$

$$\begin{aligned} y \leq_{\%} x &\implies yw_0 \geq_{\%} xw_0 && (\text{by 5.18}) \\ &\implies yw_0t_{\mu} \geq xw_0t_{\mu} && \text{for sufficiently dominant } \mu \\ &\implies yt_{\lambda}w_0 \geq xt_{\lambda}w_0 && \text{for sufficiently anti-dominant } \lambda = w_0\mu \\ &\implies y \geq x && (\text{Lifting property below}). \end{aligned}$$

We are done.  $\square$

**5.31. Lifting property.** When  $\ell(uv) = \ell(u) + \ell(v)$ , we have

$$uv \leq wv \implies u \leq w.$$

**Proof.** It suffices to show when  $v = s_i$ . When  $ws_i < w$ , then  $u \leq us_i \leq ws_i \leq w$  it is obvious. When  $ws_i > w$ , then

$$(a \text{ reduced word of } w) \oplus s_i$$

is a reduced word of  $ws_i$ . Since  $us_i \leq ws_i$ , we can find a subword of  $us_i$  inside. If the last  $s_i$  is chosen, then drop it, we get  $u \leq w$ . If the last  $s_i$  is not chosen, then  $us_i \leq w$ , we also have  $u \leq w$ .  $\square$

**5.32. Corollary.** For  $x = \min(xW)$  and  $y = \min(yW)$ ,

$$x \leq y \iff x \geq_{\%} y.$$

**5.33. Exercise.** Prove that

$xw_0 \leq_{\%} yw_0 \iff xt_{\lambda} \leq yt_{\lambda}$  for  $\lambda \in Q^{\vee}$  sufficiently anti-dominant.

When  $\lambda$  is sufficiently anti-dominant,  $xt_{\mu} = \min(xt_{\mu}W)$  by above. So it is equivalent to say  $xt_{\lambda}w_0 \leq yt_{\lambda}w_0$ .

## 6. COMBINATORICS IN TYPE A

*Remind.*

**6.1. Two presentations.** Recall that

$$\tilde{\mathfrak{S}}_n = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : f(i+n) = f(i) + n \right\}.$$

Any  $\lambda \in \mathbb{Z}^n$  defines a translation  $t_\lambda(i) \in \tilde{\mathfrak{S}}_n$  by

$$t_\lambda(i) = i + \lambda_i n \quad 1 \leq i \leq n-1.$$

This gives the first presentation

$$\tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes \mathbb{Z}^n.$$

Denote  $s_i$  for  $i \in \mathbb{Z}/n\mathbb{Z}$  by

$$s_i = \begin{cases} \text{the affine permutation exchanging } j \text{ and } j+1 \\ \text{when } i \equiv j \pmod{n} \text{ with other numbers fixed} \end{cases} \in \tilde{\mathfrak{S}}_n^0.$$

They generate the subgroup

$$\tilde{\mathfrak{S}}_n^0 = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{l} f(i+n) = f(i) + n \\ \frac{1}{n} \sum_{i=1}^n (f(i) - i) = 0 \end{array} \right\}.$$

Recall the element

$$\pi \in \tilde{\mathfrak{S}}_n, \quad \text{given by } \pi(i) = i + 1.$$

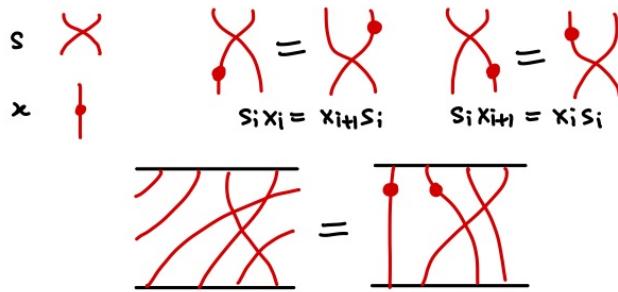
We have the second presentation

$$\tilde{\mathfrak{S}}_n = \pi^{\mathbb{Z}} \ltimes \tilde{\mathfrak{S}}_n^0.$$

**6.2. Dot notation.** If we denote  $x_i = t_{e_i}$ , we have very explicit formula

$$x_i = (s_{i-1} \cdots s_1) \pi (s_{n-1} \cdots s_i).$$

There is another diagram notation for  $\tilde{\mathfrak{S}}_n$ .



**6.3. Exercise.** For any  $f \in \tilde{\mathfrak{S}}_n$ , prove the **average**

$$\text{av}(f) = \frac{1}{n} \sum_{i=1}^n (f(i) - i) \in \mathbb{Z}, \quad \text{av}(fg) = \text{av}(f) + \text{av}(g).$$

This proves  $\text{av} : \tilde{\mathfrak{S}}_n \rightarrow \mathbb{Z}$  defines a group homomorphism. Actually  $\ker \text{av} = \tilde{\mathfrak{S}}_n^0$  the affine Weyl group.

**6.4. Length function.** For  $f \in \tilde{\mathfrak{S}}_n$ , the length

$$\ell(f) = \# \left\{ (i, j) : \begin{array}{l} 1 \leq i \leq n-1 \\ i < j, f(i) > f(j) \end{array} \right\}.$$

Assume  $f = wt_\lambda$ , then

$$\ell(f) = \sum_{i < j} |\lambda_i - \lambda_j + \delta_{w(j) > w(i)}|.$$

In particular,

$$\ell(s_i) = 1, \quad \ell(\pi) = 0, \quad \ell(x_i) = n-1$$

*Actions of  $\tilde{\mathfrak{S}}_n$ .*

**6.5. Exercise.** Prove that  $P^\vee \cong W_e/W$  is an isomorphism of  $W_e$ -sets.

**6.6. Remark.** By the very definition, as a subgroup of  $\mathfrak{S}_{\mathbb{Z}}$ , the group  $\tilde{\mathfrak{S}}_n$  acts on any objects indexed by  $\mathbb{Z}$ . Precisely, for any set  $X$ ,

$$X^{\mathbb{Z}} = \{(\dots, a_{-1}, a_0, a_1, \dots), a_i \in X\},$$

the group  $\tilde{\mathfrak{S}}_n$  acts by

$$f(\dots, a_{-1}, a_0, a_1, \dots) = (\dots, a_{f^{-1}(-1)}, a_{f^{-1}(0)}, a_{f^{-1}(1)}).$$

That is,  $a_i$  is moves to the  $f(i)$ -th position, so the  $j$ -th entry is supposed to be  $a_{f^{-1}(j)}$ .

**6.7. Action on  $\mathbb{Z}^n$ .** The group  $\mathfrak{S}_n$  acts on  $\mathbb{Z}^n$  linearly by

$$w(a_1, \dots, a_n) = (a_{w^{-1}(1)}, \dots, a_{w^{-1}(n)}).$$

We can extend this action non-linearly to  $\tilde{\mathfrak{S}}_n$  by

$$wt_{\lambda}(a_1, \dots, a_n) = (a_1 + \lambda_1, \dots, a_n + \lambda_n).$$

This induces an isomorphism of  $\tilde{\mathfrak{S}}_n$ -set  $\mathbb{Z}^n \xrightarrow{1:1} \tilde{\mathfrak{S}}_n / \mathfrak{S}_n$ . In particular,

- since  $s_0 = t_1 t_n^{-1} s_{1n}$ ,

$$s_0(a_1, a_2, \dots, a_{n-1}, a_n) = (a_n + 1, a_2, \dots, a_{n-1}, a_1 - 1),$$

- since  $\pi = t_1 s_1 \cdots s_{n-1}$ ,

$$\pi(a_1, a_2, \dots, a_{n-1}, a_n) = (a_n + 1, a_1, \dots, a_{n-2}, a_{n-1}).$$

**6.8. Example.** Take  $n = 3$ . for simplicity, we denote  $-m = \bar{m}$ .

$$000 \xrightarrow{\pi} 100 \xrightarrow{s_1} 010 \xrightarrow{s_0} 11\bar{1} \xrightarrow{s_2} 1\bar{1}1 \xrightarrow{s_0} 2\bar{1}0 \xrightarrow{s_2} 20\bar{1}$$

**6.9. Identification A.** Actually, we can extend any  $n$ -tuple  $(a_1, \dots, a_n)$

$$\text{to } (a_i)_{i \in \mathbb{Z}} \text{ by } a_{kn+i} = a_i - k.$$

That is, we can embedding

$$\mathbb{Z}^n \xrightarrow{1:1} \left\{ (a_i)_{i \in \mathbb{Z}} : \begin{array}{l} a_i \in \mathbb{Z} \\ a_{i+n} = a_i - 1 \end{array} \right\} \subset \mathbb{Z}^{\mathbb{Z}}.$$

Then this can be included in Remark 6.6 above. For example, ( $\bar{k} = -k$ )

$$\begin{array}{rcl} 20\bar{1} & \longmapsto & \cdots | 310 | 20\bar{1} | 1\bar{1}\bar{2} | \cdots \\ 2\bar{1}0 & \longmapsto & \cdots | 301 | 2\bar{1}0 | 1\bar{2}\bar{1} | \cdots \\ 1\bar{1}1 & \longmapsto & \cdots | 212 | 1\bar{1}1 | 0\bar{2}0 | \cdots \\ 11\bar{1} & \longmapsto & \cdots | 221 | 11\bar{1} | 00\bar{2} | \cdots \\ 010 & \longmapsto & \cdots | 121 | 010 | \bar{1}0\bar{1} | \cdots \\ 100 & \longmapsto & \cdots | 211 | 100 | 0\bar{1}\bar{1} | \cdots \\ 000 & \longmapsto & \cdots | 111 | 000 | \bar{1}\bar{1}\bar{1} | \cdots \end{array}$$

Compare with the example above.

**6.10. Identification B.** For any  $n$ -tuple  $(a_1, \dots, a_n)$ , we can associate a subset

$$A = t_a \mathbb{Z}_{<0} = \{i + (a_i - k)n : 1 \leq i \leq n, k < 0\} \subset \mathbb{Z}$$

Equivalently, we split  $\mathbb{Z}$  into  $n$  copies of  $\mathbb{Z}$  by

$$\mathbb{Z} \xrightarrow{1:1} (i + n\mathbb{Z}), \quad d \mapsto i + nd.$$

Then  $A$  is the union of the image of lower ideal  $\{j < a_i\}$ . This defines a bijection

$$\mathbb{Z}^n \xrightarrow{1:1} \left\{ A \subset \mathbb{Z} : \begin{array}{l} i \in A \Rightarrow i - n \in A \\ i \ll 0 \Rightarrow i \in A \\ i \gg 0 \Rightarrow i \in A \end{array} \right\} \subset 2^\mathbb{Z}.$$

Then this can be included in Remark 6.6 above. For example

$$\begin{array}{rcl} 20\bar{1} & \longmapsto & \{\dots, 4, \bar{1}, \bar{3}\} \\ 2\bar{1}0 & \longmapsto & \{\dots, 4, \bar{4}, 0\} \\ 1\bar{1}1 & \longmapsto & \{\dots, 1, 4, 3\} \\ 11\bar{1} & \longmapsto & \{\dots, 1, 2, \bar{3}\} \\ 010 & \longmapsto & \{\dots, \bar{2}, 2, 0\} \\ 100 & \longmapsto & \{\dots, 1, \bar{1}, 0\} \\ 000 & \longmapsto & \{\dots, \bar{2}, \bar{1}, 0\} \end{array} \quad \begin{array}{ccccccccc} \dots & \bar{2} & \bar{1} & 0 & 1 & 2 & 3 & \dots \\ \dots & \bar{8} & \bar{5} & \bar{2} & 1 & 4 & 7 & \dots \\ \dots & \bar{7} & \bar{4} & \bar{1} & 2 & 5 & 8 & \dots \\ \dots & \bar{6} & \bar{3} & 0 & 3 & 6 & 9 & \dots \end{array}$$

**6.11. Maya diagram.** We represent any subset of  $\mathbb{Z}$  by a  $\mathbb{Z}$ -tuple of  $\{\oplus, \ominus\}$ . That is, the  $a$ -th position is  $\oplus$  if and only if  $a \in A$ . For example,

$$\begin{aligned} 20\bar{1} &\longmapsto \cdots \oplus\oplus\oplus\ominus | \oplus\ominus\oplus\cdots \\ 2\bar{1}0 &\longmapsto \cdots \oplus\oplus\ominus\oplus | \oplus\ominus\oplus\oplus\cdots \\ 1\bar{1}1 &\longmapsto \cdots \oplus\oplus\ominus\oplus | \oplus\oplus\oplus\ominus\cdots \\ 1\bar{1}\bar{1} &\longmapsto \cdots \oplus\oplus\ominus\ominus | \oplus\oplus\ominus\cdots \\ 010 &\longmapsto \cdots \oplus\oplus\oplus\oplus | \ominus\oplus\ominus\cdots \\ 100 &\longmapsto \cdots \oplus\oplus\oplus\oplus | \oplus\ominus\ominus\cdots \\ 000 &\longmapsto \cdots \oplus\oplus\oplus\oplus | \ominus\ominus\ominus\cdots \end{aligned}$$

**6.12. Partitions.** The set

$$\left\{ A \subset \mathbb{Z}: \begin{array}{l} i \ll 0 \Rightarrow i \in A \\ i \gg 0 \Rightarrow i \in A \end{array} \right\} \subset 2^{\mathbb{Z}}$$

can be identified with the set of partitions with charges.

$$\{(\lambda, m) : \lambda \text{ is a partition, } m \in \mathbb{Z}\}.$$

We say  $(\lambda, m)$  is of center charge if  $m = 0$ . We identify usual partitions by a partition of center charge.

The identification is given by

$$(\lambda, m) \longmapsto \{m + 1 + \lambda_i - i : i = 1, 2, 3, \dots\}.$$

For example,

$$(\emptyset, m) \longmapsto (\cdots, \overset{m-1}{\oplus}, \overset{m}{\ominus}, \overset{m+1}{\oplus}, \overset{m+2}{\ominus}, \cdots).$$

$$(\square, m) \longmapsto (\cdots, \overset{m-1}{\oplus}, \overset{m}{\ominus}, \overset{m+1}{\oplus}, \overset{m+2}{\ominus}, \cdots).$$

**6.13. Residue.** For a partition with charge  $(\lambda, m)$  and a box  $(i, j)$  in  $\lambda$ , we define

$$\text{res}(\square) = j - i + m \in \mathbb{Z}$$

Then the action of  $\tilde{S}_n$  translated to operators on partitions. By

$$\pi(\lambda, m) = (\lambda, m + 1), \quad s_i(\lambda, m) = (s_i \lambda, m)$$

where

$$s_i \lambda = \lambda \cup \left\{ \square : \begin{array}{l} \square \text{ is addable} \\ \text{res}(\square) \equiv i \pmod{n} \end{array} \right\} \setminus \left\{ \square : \begin{array}{l} \square \text{ is removable} \\ \text{res}(\square) \equiv i \pmod{n} \end{array} \right\}.$$

**6.14. Example.** Take  $n = 3$ . We label the residues on the boxes.

$$\begin{array}{c} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{array} \xrightarrow{\pi} \begin{array}{c} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{array} \xrightarrow{s_1} \begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \xrightarrow{s_0} \begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ \boxed{1} & 0 & 1 \end{array} \xrightarrow{s_2} \begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ \boxed{1} & 0 & 1 \end{array} \xrightarrow{s_0} \begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ \boxed{1} & 0 & 1 \end{array} \xrightarrow{s_2} \begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ \boxed{1} & 0 & 1 \end{array}$$

**6.15. Identification C.** Under the above discussion, we can identify

$$\mathbb{Z}^n \xrightarrow{1:1} \tilde{\mathfrak{S}}_n \text{ orbit of } (\emptyset, 0) = \{(\lambda, m) : \lambda \text{ is } n\text{-core}\}.$$

Moreover, if we restrict to

$$Q^\vee = \{(a_i)_{i \in \mathbb{Z}} : a_1 + \cdots + a_n = 0\} \subset \mathbb{Z}^n,$$

it gives

$$Q^\vee \xrightarrow{1:1} \tilde{\mathfrak{S}}_n^0 \text{ orbit of } (\emptyset, 0) = \{\lambda : \lambda \text{ is } n\text{-core}\}.$$

For example,

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\oplus} \color{red}{\oplus} \color{blue}{\ominus} | \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\ominus} \color{blue}{\oplus} \dots$$

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\oplus} \color{blue}{\oplus} | \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\ominus} \color{blue}{\oplus} \dots$$

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\oplus} | \color{red}{\ominus} \color{blue}{\oplus} \color{red}{\oplus} \color{blue}{\ominus} \dots$$

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\ominus} | \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\ominus} \color{blue}{\ominus} \dots$$

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\oplus} \color{red}{\ominus} | \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\ominus} \color{blue}{\ominus} \dots$$

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\oplus} \color{red}{\oplus} | \color{red}{\ominus} \color{blue}{\oplus} \color{red}{\ominus} \color{blue}{\ominus} \dots$$

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\oplus} \color{red}{\oplus} | \color{red}{\oplus} \color{blue}{\ominus} \color{red}{\ominus} \color{blue}{\ominus} \dots$$

$$\begin{array}{c} \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 \\ -1 & 0 & 1 \end{array} \dots \oplus \color{red}{\oplus} \color{blue}{\oplus} \color{red}{\oplus} | \color{red}{\ominus} \color{blue}{\oplus} \color{red}{\ominus} \color{blue}{\ominus} \dots$$

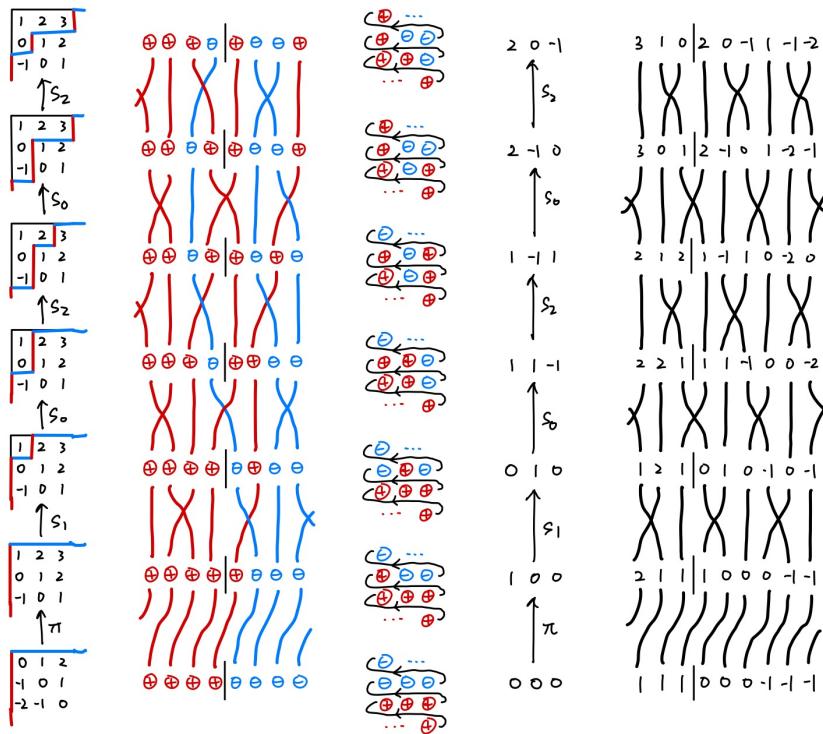
**6.16.  $n$ -core partition.** A partition  $\lambda$  is an  $n$ -core partition if there exists no  $\mu \subseteq \lambda$  such that  $\lambda/\mu$  is a ribbon of length  $n$ . Removing a ribbon of length  $n$  corresponds to the exchange

$$(\dots \overbrace{\ominus \dots \oplus}^{\text{distance} = n} \dots) \longmapsto (\dots \overbrace{\oplus \dots \ominus}^{\text{distance} = n} \dots).$$

As a result, if we cannot exchange, then the corresponding subset satisfies

$$i \in A \implies i - n \in A.$$

**6.17. Summary.** Here is the summary of all three identification of  $\mathbb{Z}^n$ .



*Minimal representatives.*

**6.18. Description.** For any  $f \in \tilde{\mathfrak{S}}_n^0$ , it is clear that in the decomposition

$$f = uv, \quad u = \min(fW) \text{ and } v \in \mathfrak{S}_n$$

we have

$$u(1) = \min(f(1), \dots, f(n)),$$

$$\dots = \dots$$

$$u(n) = \max(f(1), \dots, f(n)).$$

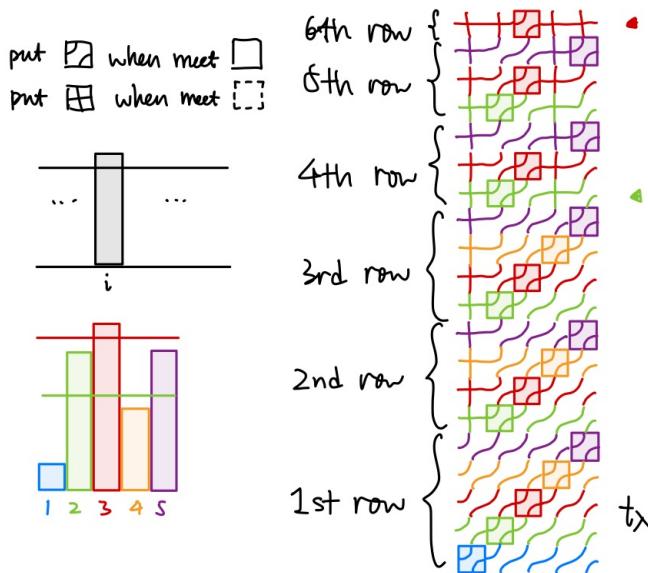
and

$$\begin{aligned} v(i) &= \text{the position of } f(i) \text{ in } \{f(1), \dots, f(n)\} \\ &= 1 + \#\{j : f(j) < f(i)\}. \end{aligned}$$

Now, let us give a combinatorial description of

$$t_\lambda = u_\lambda v_\lambda, \quad u_\lambda = \min(t_\lambda W) \text{ and } v_\lambda \in W.$$

**6.19. Description of  $t_\lambda$ .** There is a combinatorial way of constructing  $t_\lambda$  as follows.



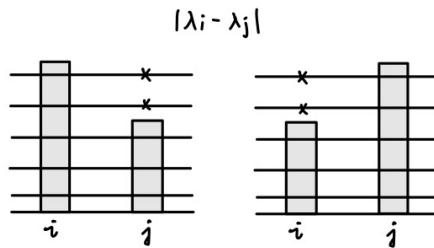
Actually, each row is a reduced word of  $t_\mu$  with  $\mu \in \{0, 1\}^n$ . For example, the above example is

$$t_{(1, 5, 6, 3, 5)} = t_{(0, 0, 1, 0, 0)} t_{(0, 1, 1, 0, 1)}^2 t_{(0, 1, 1, 1, 1)}^2 t_{(1, 1, 1, 1, 1)}.$$

Note that

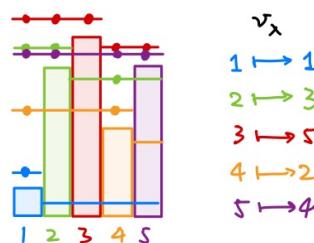
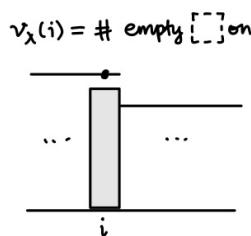
$$\ell(t_\lambda) = \sum_{i < j} |\lambda_i - \lambda_j|.$$

This is compatible:

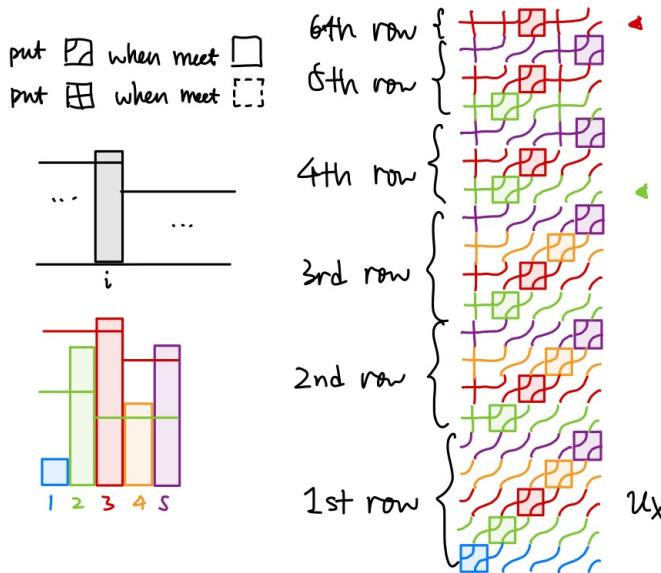


**6.20. Description of  $v_\lambda$ .** When  $f = t_\lambda$ , then

$$v_\lambda(i) = 1 + \#\{j < i : \lambda_i \leq \lambda_j\} + \#\{j > i : \lambda_i < \lambda_j\}.$$



**6.21. Description of  $u_\lambda$ .** There is a combinatorial way of constructing  $u_\lambda$  as follows.



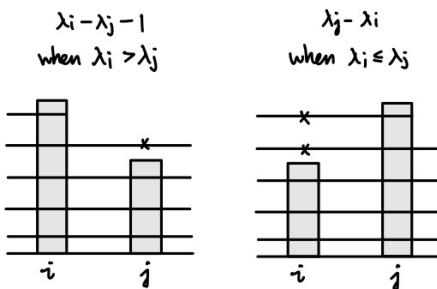
**6.22. Compatible with length.** Recall that the minimal representative minimizes each summand of

$$\ell(u_\lambda) = \ell(u_\lambda^{-1}) = \ell(v_\lambda t_{-\lambda}) = \sum_{i < j} | -\lambda_i + \lambda_j + \delta_{v_\lambda(j) > v_\lambda(i)} |.$$

That is,

$$\ell(u_\lambda) = \sum_{i < j} \begin{cases} \lambda_j - \lambda_i, & \lambda_i \leq \lambda_j, \\ \lambda_i - \lambda_j - 1, & \lambda_i > \lambda_j. \end{cases}$$

This is also compatible:

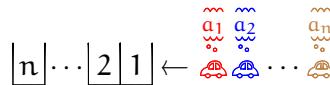


**6.23. Remark.** Let us identify compositions by a subset of  $\mathbb{Z}^n$ . Then each box  $(i, j)$  corresponds to the minimal affine permutation by “creating this box”, i.e. change the  $i$ -th component  $j-1 \mapsto j$ . Using our identification A, the  $(i-n)$ -th component is  $j$ , we can just move it to the  $i$ -th component. When moving, we need  $n - 1$  simple reflections, but once  $j$  meets another  $j$ , we do not need to exchange them, so it saves a simple reflection.

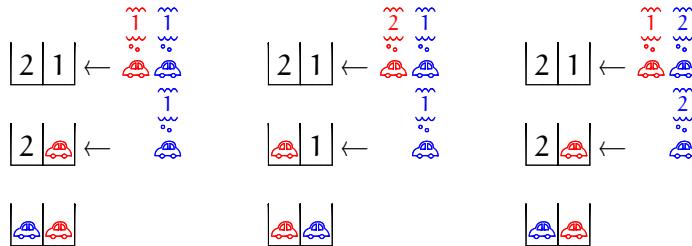
## 7. FUNNY BIJECTIONS

*Stable affine permutations.*

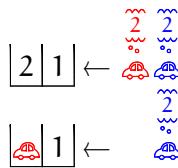
**7.1. Stable affine permutations.** The  $i$ -th car prefers the space  $a_i$ . If  $a_i$  is occupied, then the  $i$ -th takes the next available space. We call  $(a_1, \dots, a_n)$  a **parking function** (of length  $n$ ) if all cars can park.



**7.2. Example.** For example, when  $n = 2$ , all parking functions 11, 21, 12 are



While 22 is not:

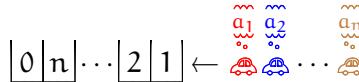


**7.3. Example.** For example, when  $n = 3$ ,

111	112	121	211	113	131	311	122
212	221	123	132	213	231	312	321

**7.4. Theorem.** There are exactly  $(n + 1)^{n-1}$  parking functions.

**Proof.** Consider the following parking procedure:



on a circular road (i.e. the next space of  $\boxed{0}$  is  $\boxed{1}$ ). Then we see all cars

can park for any preferences. It is a parking function if  $\boxed{0}$  is left empty. The rotation symmetry tells that there are exactly  $\frac{(n+1)^n}{n+1}$  many parking functions.  $\square$

**7.5. Equivalent condition.** We see  $(a_1, \dots, a_n)$  is a parking function if and only if

- at most 1 car prefers position  $n$ ;
- at most 2 cars prefer position  $\geq n - 1$ ;
- at most 3 cars prefer position  $\geq n - 2$ ;
- etc.

That is,

$$\{i : a_i \geq n - k + 1\} \leq k.$$

In particular, does not depend on the order of cars.

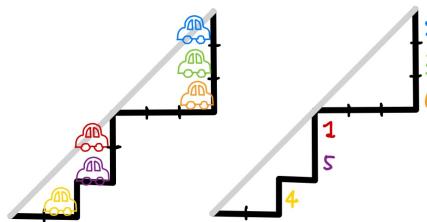
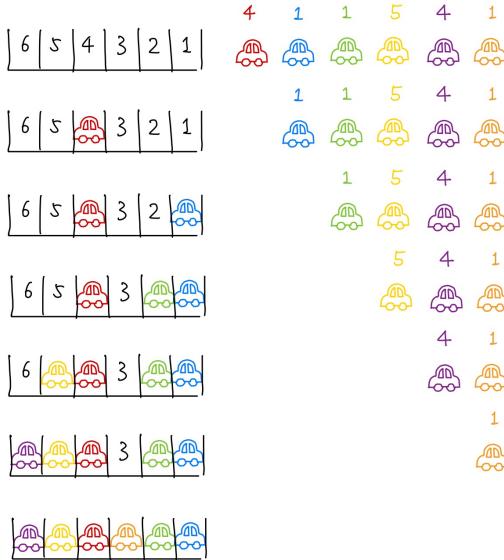
**7.6. Dyck path.** Dyck path (of length  $n$ ) is a lattice path from  $(0, 0)$  to  $(n, n)$  below the diagonal. It is well-known that the number of Dyck path is Catalan number

$$\frac{1}{n+1} \binom{2n}{n} = \frac{1}{2n+1} \binom{2n+1}{n}.$$

**7.7. Labelled Dyck path.** A labeling of Dyck path is a labeling on vertical steps by  $[n]$  which is increasing for consecutive vertical steps. It is not hard to find a bijection

$$\{\text{parking functions}\} \xleftrightarrow{1:1} \{\text{labelled Dyck paths}\}.$$

That is, the labels on the vertical steps over  $x = i$  gives  $a_{n+1-i}$ .



**7.8. Stable affine permutation.** We say  $f \in \tilde{\mathfrak{S}}_n^0$  is **stable** if

$$f(i+n+1) = f(i+1) + n > f(i).$$

**7.9. Enumeration.** Note that every  $(x_1, \dots, x_n) \in \mathbb{R}^n$  can be translated into  $Q^\vee$

$$(x_1, \dots, x_n) - \frac{x_1 + \dots + x_n}{n} (1, \dots, 1).$$

It would be convenient to work with

$$\mathfrak{h}^* = \{(x_1, \dots, x_n) : x_1 + \dots + x_n = 0\} \cong \mathbb{R}^{\oplus n}/\mathbb{R}(1, \dots, 1).$$

The fundamental alcove is

$$A_0 = \{(x_1, \dots, x_n) \in \mathfrak{h}^* : x_1 < x_2 < \dots \geq x_n < x_1 - 1\}.$$

Its centroid is

$$\mathbf{c} = -\frac{1}{n}(1, 2, \dots, n) \bmod (1, \dots, 1) \in A_0.$$

Explicit computation shows for  $f \in \tilde{\mathfrak{S}}_n$ ,

$$f\mathbf{c} = -\frac{1}{n}(f(1), f(2), \dots, f(n)) \bmod (1, \dots, 1) \in A_0.$$

So  $f$  is stable if and only if

$$\begin{aligned} f\mathbf{c} &\in \left\{ -\frac{1}{n}(x_1, \dots, x_n) : \begin{array}{l} x_2 + n < x_1 \\ \vdots \\ x_n + n < x_{n-2} \\ x_1 + 2n < x_n \end{array} \right\} \\ &= \left\{ (x_1, \dots, x_n) : \begin{array}{l} x_1 < x_2 - 1 \\ \vdots \\ x_{n-1} < x_n - 1 \\ x_n < x_1 - 2 \end{array} \right\} \\ &= \left\{ (x_1, \dots, x_n) : x_1 < x_2 - 1 < x_3 - 2 < \dots < x_n - (n-1) < x_1 - (n+1) \right\} \end{aligned}$$

The set is a union of alcoves, since it is bounded by hyperplanes. Moreover, its volume is the the volume of

$$(n+1)A_0 = \{(x_1, \dots, x_n) : x_1 < x_2 < \dots < x_n < x_1 - (n+1)\}$$

(by a translation). This concludes

$$\#\text{[stable permutations]} = (n+1)^{n-1}.$$

**7.10. Table of numbers.** Let us fill all integers into  $[n] \times \mathbb{Z}$  such that

- 0 is filled in  $(0, 0)$  position;
- locally

$$\boxed{a \mid b} \quad \text{then } b = a + n$$

$$\boxed{\begin{matrix} a \\ b \end{matrix}} \quad \text{then } b = a + n+1.$$

For example, when  $n = 3$ ,

...	-9	-6	-3	0	3	6	...
...	-5	-2	1	4	7	10	...
...	-1	2	5	8	11	14	...

**7.11. A Dyck path.** Let  $f$  be stable. Consider the set

$$\Delta = \{i \in \mathbb{Z} : f(i) > 0\}.$$

By periodicity and stability,

$$i \in \Delta \implies i + n \in \Delta, \quad i + n+1 \in \Delta.$$

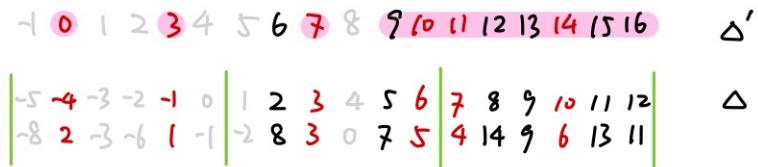
Let us shift  $\Delta$  such that it has minimum 0:

$$\Delta' = \Delta - \min(\Delta).$$

Then coloring elements of  $\Delta'$  in the table above, we will get a Dyck path. We label  $f(i + \min \Delta)$  on the leftmost colored box  $i$  in each row. Note that the labels are exactly from  $[n]$ . This defines a bijection

$$\{\text{stable permutations}\} \xleftrightarrow{1:1} \{\text{labelled Dyck paths}\}$$

**7.12. Example.**



### 7.13. References.

- Eugene Gorsky, Mikhail Mazin, Monica Vazirani. Affine permutations and rational slope parking functions.
- T. Hikita. Affine Springer fibers of type A and combinatorics of diagonal coinvariants.

*Bounded affine permutations.*

**7.14. Setup.** We call a permutation  $w \in \mathfrak{S}_n$  an **k-Grassmannian permutation** if

$$w(1) < \cdots < w(k), \quad w(k+1) < \cdots < w(n).$$

The set of k-Grassmannian permutation is in bijection with k-subset of  $[n]$ . In terms of Weyl group, we have

$$w = \min(wW_P), \quad W_P = \mathfrak{S}_k \times \mathfrak{S}_{n-k} \subset \mathfrak{S}_n,$$

i.e. the set of k-Grassmannian permutations is  $W^P$ .

**7.15. Description of Bruhat order in type A.** For two k-subsets A, B of  $[n]$ , we define **Bruhat order**

$$A < B \iff \begin{cases} \min(A) < \min(B) \\ \dots \\ \max(A) < \max(B) \end{cases}$$

Then obviously

$$A \leq B \iff [n] \setminus A \geq [n] \setminus B$$

**7.16. Theorem.** For two permutations  $u, w \in \mathfrak{S}_n$ ,

$$u \leq w \iff u[k] \leq w[k] \text{ for all } 1 \leq k \leq n-1,$$

where  $w[k] = \{w(1), \dots, w(k)\}$ .

**7.17. Theorem.** When  $w$  is Grassmannian,

$$u \leq w \iff \begin{cases} u(1) \leq w(1) \\ \dots \\ u(k) \leq w(k) \end{cases} \quad \text{and} \quad \begin{cases} u(k+1) > w(k+1) \\ \dots \\ u(n) > w(n) \end{cases}$$

**Proof.** Firstly

$$u[1] \leq w[1] \iff u(1) \leq w(1)$$

Since  $w(1) < w(2)$ ,

$$u[2] \leq w[2] \xrightarrow{u(1) \leq w(1)} u(2) \leq w(2).$$

Keep using this argument, we conclude the first set of condition is equivalent to

$$u[1] \leq w[1], \dots, u[k] \leq w[k].$$

Here is the diagram:

$$\begin{array}{ccccccc} u(1) & u(2) & \cdots & u(k) \\ | \wedge & | \wedge & \cdots & | \wedge \\ w(1) < w(2) < \cdots < w(k) \end{array}$$

Similarly, the the second set is equivalent to

$$u[k] \leq w[k], \dots, u[n-1] \leq w[n-1].$$

One needs to notice that  $[n] \setminus u[i] = u\{i+1, \dots, n\}$ , and use the similar argument.  $\square$

**7.18. Affine bounded permutation.** We say  $f \in \tilde{\mathfrak{S}}_n$  is a **bounded affine permutation** if

$$i \leq f(i) \leq i + n.$$

We say it is **k-affine permutation** if

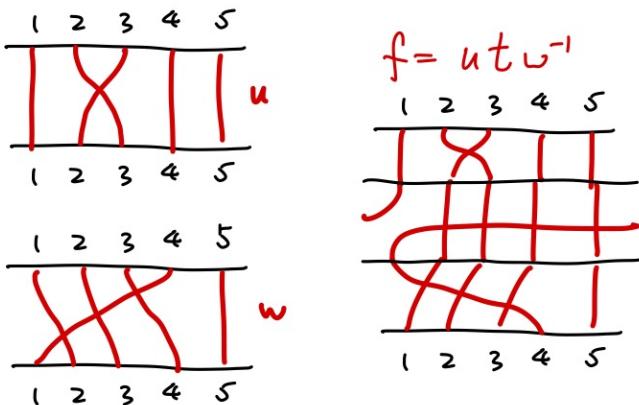
$$av(f) = \frac{1}{n} \sum_{i=1}^n (f(i) - i) = k.$$

**7.19. The map.** Let us denote

$$\omega_k^\vee = e_1 + \cdots + e_k = (\underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{n-k}) \in \mathbb{Z}^n.$$

This is a lifting of the k-th fundamental coweight. We denote  $t = t_{\omega_k^\vee}$  for simplicity. For  $(u, w) \in \mathfrak{S}_n \times \mathfrak{S}_n$ , we define

$$f_{u,w} = utw^{-1} \in \tilde{\mathfrak{S}}_n.$$



**7.20. Bijection.** The map  $(u, w) \mapsto f_{u,w}$  restricts to a bijection

$$\left\{ (u, w) : \begin{array}{l} u \leq w, \\ w \text{ is } k\text{-Grassmannian} \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{l} k\text{-affine bounded} \\ \text{permutations} \end{array} \right\}.$$

**Proof.** The injectivity is not hard. Note that  $vt = tv$  for any  $v \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$ . We can decompose

$$\mathfrak{S}_n t \mathfrak{S}_n = \bigcup_{\lambda \in \mathfrak{S}_n \cdot \omega_k^{\vee}} \mathfrak{S}_n t_{\lambda} = \bigcup_{w \text{ is } k\text{-Grassmannian}} \mathfrak{S}_n t w^{-1},$$

i.e. any element in  $WtW$  can be uniquely written as  $utw^{-1}$  where  $u, w \in \mathfrak{S}_n$  and  $w$  is  $k$ -Grassmannian.

Let us check this is well-defined. Firstly, since  $\text{av}(t) = k$ , so  $\text{av}(f_{u,w}) = k$ . For  $1 \leq i \leq n$ , we have

$$f_{u,w}(i) = utw^{-1}(i) = \begin{cases} u(w^{-1}(i)) + n, & 1 \leq w^{-1}(i) \leq k, \\ u(w^{-1}(i)), & k+1 \leq w^{-1}(i) \leq n. \end{cases}$$

In the first case, write  $a = w^{-1}(i)$ , we have

$$f_{u,w}(i) = u(a) + n \begin{cases} \leq w(a) + n = i + n, \\ \geq 1 + n \geq i. \end{cases}$$

In the second case, write  $b = w^{-1}(i)$ , we have

$$f_{u,w}(i) = u(b) \begin{cases} \leq n \leq i+n, \\ \geq w(b) = i. \end{cases}$$

This proves the boundedness.

Lastly, let us check the surjectivity. Let  $f$  be a  $k$ -bounded affine permutation. Consider

$$A = \{i \in [n] : f(i) > n\}.$$

This must be an  $k$ -subset. The reason is the following: the bounded condition implies

$$\text{either } f(i) \in [n] \text{ or } f(i) - n \in [n].$$

The condition  $av(f) = k$  implies there are exactly  $k$  many  $i$ 's satisfying  $f(i) - n \in [n]$ .

Let  $w$  be the  $k$ -Grassmannian permutation such that

$$\{w(1), \dots, w(k)\} = A.$$

Then we define  $u = fwt^{-1}$ , so it rests to show  $u \in S_n$  and  $u \leq w$ . Explicitly,

$$u(i) = \begin{cases} f(w(i)) - n, & 1 \leq i \leq k, \\ f(w(i)), & k+1 \leq i \leq n. \end{cases}$$

In other case the value is in  $[n]$ , so  $u \in S_n$ . In the first case,

$$u(i) = f(w(i)) - n \leq w(i) + n - n = w(i)$$

In the second case

$$u(i) = f(w(i)) \geq w(i).$$

Thus  $u \leq w$ . □

**7.21. Length formula.** We proved that when  $w$  is  $k$ -Grassmannian,

$$\ell(f_{u,w}) = \ell(u) + \ell(t) - \ell(w).$$

In our case,  $\ell(t) = k(n - k)$ , i.e.

$$\ell(f_{u,w}) = \ell(u) + k(n - k) - \ell(w).$$

In particular, for length to be maximal, it can only be

$$f_{w,w} = t_\lambda, \quad \lambda = w\varpi_k^\vee.$$

**7.22. Explicitly.** Recall  $\pi \in \tilde{S}_n$  the permutation of length 0:

$$\pi(i) = i + 1.$$

Let  $w_0^P$  be the maximal  $k$ -Grassmannian permutation, i.e.

$$\begin{aligned} w_0^P(1) &= n - k + 1, \dots, w_0^P(k) = n, \\ w_0^P(k+1) &= 1, \dots, w_0^P(n) = n - k. \end{aligned}$$

We can write  $t = \pi^k w_0^P$ . Then

$$f_{u,w} = u \cdot \pi^k \cdot (w_0^P \cdot w^{-1})$$

is length additive:

$$\ell(f_{u,w}) = \ell(u) + 0 + \ell(w_0^P \cdot w^{-1}).$$

Note that

$$w \mapsto w_0^P w^{-1}$$

defines a order-reversed bijection between  $k$ -Grassmannian permutations and  $(n-k)$ -Grassmannian permutations.

**7.23. Inversions.** But it is still useful to compute the inversions. There are three types of inversions ( $i < j$ ). Denote  $a = w^{-1}(i)$  and  $b = w^{-1}(j)$ .

- $a \leq k < b \leq n$ . In this case  $f(i) > n \geq f(j)$ . They form the set

$$\left\{ (a, b) : \begin{array}{l} 1 \leq a \leq k < b \leq n \\ w(a) < w(b) \end{array} \right\}$$

- $a, b \leq k$  or  $k < a, b$ . In this case we must have  $a < b$  and  $u(a) > u(b)$ . They form the set

$$\left\{ (a, b) : \begin{array}{l} 1 \leq a, b \leq k \text{ or } k < a, b \leq n \\ u(a) > u(b) \end{array} \right\}$$

- $a \leq k \leq n+k < b$ . In this case we must have  $u(b) > u(a+n)$ . They form the set

$$\left\{ (a, b+n) : \begin{array}{l} 1 \leq a \leq k < b \leq n \\ u(a) > u(b) \end{array} \right\}.$$

The contribution of the first type is  $k(n-k) - \ell(w)$ . The rest contributes  $\ell(u)$ .

### 7.24. Theorem.

The set

$$\left\{ \begin{array}{l} \text{k-affine bounded} \\ \text{permutations} \end{array} \right\} = \left\{ f \in S_n : \begin{array}{l} f \leq t_\lambda \text{ for some} \\ \lambda \in S_n \varpi_k^v. \end{array} \right\}$$

**Proof.** It suffices to show the left-hand-side is a lower ideal. Let  $f$  be an affine bounded permutation. We can pick a reduced word of  $f$  to be

$$(a \text{ reduced word for } u)\pi^k(a \text{ reduced word for } w_0^P w^{-1}).$$

Let  $f'$  be the affine permutation obtained by deleting a simple reflection and such that  $\ell(f') = \ell(f) - 1$ . We need to show  $f'$  is still an affine bounded permutation.

- If we delete from  $u$ . Then  $f' = f_{u',w} = u'tw^{-1}$  for some  $u' \leq u \leq w$ .
- If we delete from  $w_0^P w^{-1}$ . Then we get  $f' = f_{u,w'}$ , where  $w' = ws_{ab} > w$ ,  $a \leq k < b$ ,  $\ell(ws_{ab}) = \ell(w) + 1$ . This implies  $w' = ws_{ab}$  is also  $k$ -Grassmannian. We have  $u \leq w \leq w'$ .

In both case,  $f'$  is a bounded affine permutation.  $\square$

### 7.25. References.

- A. Knutson, T. Lam, D. E. Speyer. Positroid Varieties: Juggling and Geometry.
- A. Knutson, T. Lam, D. E. Speyer. Projections of Richardson varieties.

