BEHIND THE COMBINATORICS OF SYMMETRIC FUNCTIONS

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ABSTRACT. These are the lecture notes for "Online Learning Seminar in Algebraic Combinatorics" in 2025 Springer. The purpose of this course is to provide an algebraic introduction to Schur polynomials, Hall–Littlewood polynomials and Macdonald polynomials. I will minimize the involved combinatorics, and focus more on their connections to representation theory and algebraic geometry.

CONTENTS

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1. Schur polynomials

— **References.** Most of results can be found in

[24] I. G. Macdonald. Symmetric functions and Hall Polynomials, second version. Chapter I.
[7] W. Fulton. Young Tableaux: With Applications to Representation Theory and Geometry.

— **Assumption.**

1.1. Root system. Our definition will be given for a finite reduced **root system**. We take Λ to be the weight lattice, or any sub-lattice containing root lattice Q. In this note, a **polynomial** is an element of group ring

$$\mathbb{Q}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{Q} \cdot e^{\lambda}.$$

It is isomorphic to a Laurent polynomial ring after picking a basis of Λ.

1.2. Type A. After presenting the type-free definition, we will soon specialize to type A. However, type A is a little bit special. We prefer the theory for group $G = GL_n$, while root system only tells SL_n. For GL_n, we can take

$$\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n.$$

So

$$\mathbb{Q}[\Lambda] = \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

So we have a natural lifting by homogenization.

1.3. Symmetric functions. Usually, the symmetric polynomial is defined for a **dominant weight**. In type A, a weight

$$\lambda_1 e_1 + \cdots + \lambda_n e_n \in \Lambda$$

to be dominant means

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

When $\lambda_n \geq 0$, our polynomials are all honest polynomials of degree $|\lambda| = \lambda_1 + \cdots + \lambda_n$, i.e. in

$$\mathbb{Q}[x_1, \ldots, x_n].$$

It turns out all of our symmetric polynomials admit a limit

$$\Lambda = \varprojlim_{n \to \infty} \mathbb{Q}[x_1, \ldots, x_n].$$

Note that this is a projective limit, so we call an element of it by a **symmetric function**.

— Type-free definition.

1.4. Definition. For a dominant weight λ, we define the **Weyl character** as

$$\chi_\lambda = \sum_{w \in W} w \left(e^\lambda \prod_{\alpha > 0} \frac{1}{1 - e^{-\alpha}} \right).$$

Recall that $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. Note that

$$\prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2})$$

is anti-symmetric, we can write

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2})}.$$

1.5. Lemma. The rational function χ_λ is actually a polynomial.

Proof. Let

$$\Delta = \prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2}).$$

Note that

$$\mathbb{Q}[\Lambda]^{S_n\text{-alt}} = \mathbb{Q}[\Lambda]^{S_n} \cdot \Delta.$$

The inclusion "$\supseteq$" is trivial. The inclusion "$\subseteq$" follows from the fact that

$$s_\alpha f = -f \Longrightarrow (1 - e^\alpha) \mid f. \quad \square$$

1.6. Weyl denominator formula. We have $\chi_0 = 1$. That is,

$$\prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w\in W}(-1)^{\ell(w)}e^{w\rho}.$$

Proof. We know RHS/LHS is a polynomial. But comparing the support of LHS and RHS, the quotient can only be a constant. It is not hard to see this constant is 1. $\square$

1.7. Remark. If we set

$$a_\lambda = \sum_{w\in W}(-1)^{\ell(w)}e^{\lambda}.$$

Then we can rewrite the definition

$$\chi_\lambda = a_{\lambda+\rho}/a_\rho.$$

1.8. Lemma. We have

$$\chi_\lambda = m_\lambda + \sum_{\mu<_{\mathrm{dom}}\lambda} \mathbb{Z} \cdot m_\mu.$$

Proof. We can expand each term

$$e^{w\lambda}\prod_{\alpha>0}\frac{1}{1-e^{-w\alpha}} = e^{w\lambda}\prod_{\alpha>0}\begin{cases}\dfrac{1}{1-e^{-w\alpha}}, & w\alpha>0\\[2ex] \dfrac{-e^{w\alpha}}{1-e^{w\alpha}}, & w\alpha<0\end{cases}$$

in the Laurent series ring $\mathbb{Q}((e^{-\alpha}))_{\alpha>0}$. We see immediately that

$$\chi_\lambda = e^\lambda + \sum_{\mu<_{\mathrm{dom}}\lambda}\mathbb{Z}\cdot e^\mu.$$

Since χ_λ is symmetric, this implies the assertion. $\square$

1.9. Corollary. The Weyl character χ_λ forms a basis of symmetric polynomials.

1.10. Example. Here is an example in SL_3.

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

1 2 2 2 2 2 2 2 2 2 2 2 2 2 2 1

1 2 3 3 3 3 3 3 3 3 3 3 3 3 3 2 1

1 2 3 4 4 4 4 4 4 4 4 4 4 4 4 3 2 1

1 2 3 4 5 5 5 5 5 5 5 5 5 5 5 4 3 2 1

1 2 3 4 5 6 6 6 6 6 6 6 6 6 6 5 4 3 2 1

1 2 3 4 5 6 7 7 7 7 7 7 7 7 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 8 8 8 8 8 8 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 8 8 8 8 8 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 8 8 8 8 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 8 8 8 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 8 8 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 8 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 7 6 5 4 3 2 1

1 2 3 4 5 6 6 6 5 4 3 2 1

1 2 3 4 5 5 5 5 4 3 2 1

1 2 3 4 4 4 4 4 3 2 1

1 2 3 3 3 3 3 3 2 1

1 2 2 2 2 2 2 2 1

1 1 1 1 1 1 1 1

1.11. Example. If a weight λ such that no dominant weight $< \lambda$, then

$$s_\lambda = m_\lambda.$$

Such weight λ is said to be **minuscule**. Only a fundamental weight can be minuscule. In type A all fundamental weights are minuscule.

— TYPE-A. Let us restrict to type A. In this case, Weyl characters are called the Schur polynomials.

1.12. Definition. For a partition λ, the **Schur polynomial** is

$$s_\lambda = \sum_{w \in S_n} w \left(x^\lambda \prod_{1 \le i < j \le n} \frac{1}{1 - x_j/x_i} \right).$$

Similar as above, we can write

$$s_\lambda = \sum_{w \in S_n} w \left(x^{\lambda+\delta} \prod_{1 \le i < j \le n} \frac{1}{x_i - x_j} \right) = \frac{\sum_{w \in S_n} (-1)^{\ell(w)} x^{w(\lambda+\delta)}}{\prod_{1 \le i < j \le n} (x_i - x_j)}$$

where $\delta = (n-1, \ldots, 1, 0)$. Note that, $\delta \neq \rho$ but

$$\delta \equiv \rho \bmod e_1 + \cdots + e_n.$$

1.13. Remark. By a similar approach, we can show that s_λ is actually an honest polynomial, i.e. no negative power.

1.14. Determinant. We can also write the definition as a determinant. Say,

$$\sum_{w \in S_n} (-1)^{\ell(w)} x^{w(\lambda+\delta)} = \det(x_i^{\lambda_j + n - j})_{1 \le i,j \le n}.$$

For example,

$$s_{(2,1,0)} = \frac{\begin{vmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \end{vmatrix}}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} = \begin{matrix} x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 \\ + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 \\ + 2 x_1 x_2 x_3 \end{matrix}$$

In particular, the analogy of Weyl denominator formula gives the Vandermonde determinant.

1.15. Example. We have

$$s_{(1^k,0^{n-k})} = m_{(1^k,0^{n-k})} = e_k := \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}.$$

The elementary symmetric polynomial.

1.16. Example. We have

$$s_{(k,0^{n-1})} = h_k := \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k}.$$

The homogeneous symmetric polynomial.

1.17. Symmetric functions. It is not hard to check

$$s_\lambda(x_1, \ldots, x_k) = s_\lambda(x_1, \ldots, x_k, 0).$$

So s_λ can be upgraded to a symmetric function. Moreover, we have

$$\ker \left[\Lambda \to \mathbb{Q}[x_1, \ldots, x_n]^{S_n} \right] = \operatorname{span}(s_\lambda : \lambda_{n+1} \neq 0).$$

1.18. Example. Here is a SageMath script for computing Schur functions

```
Sym = SymmetricFunctions(QQ)
m = Sym.monomial(); p = Sym.power()
h = Sym.homogeneous(); e = Sym.elementary();
s = Sym.Schur()
print( s([2,1,1]).expand(3) )
```

```
print( e(s([2,1,1])) )
```

See the documentation.

— KEY POLYNOMIALS. Let us define the non-symmetric version of Weyl character.

1.19. Demazure operator. Let us define the **Demazure operator** for a polynomial f

$$\pi_i = \frac{1}{1-e^{-\alpha_i}}(\mathrm{id} - e^{-\alpha_i}s_i) = (1+s_i)\circ\frac{1}{1-e^{-\alpha}}.$$

That is,

$$\pi_i(f) = \frac{f - e^{-\alpha_i}s_i(f)}{1-e^{-\alpha_i}}.$$

In type A, it is written as

$$\pi_i = \frac{1}{x_i - x_{i+1}}(x_i \circ \mathrm{id} - x_{i+1}\circ s_i).$$

It is not hard to check π_i sends polynomial to polynomials.

1.20. Example. Let us compute the SL_2-case. For λ dominant,

$$\begin{matrix} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{matrix} \xmapsto{T} \begin{matrix} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ 1 & 1 & \cdots & 1 & 1 \\ 0 & -1 & \cdots & -1 & 0 \end{matrix}$$

We have

$$\pi(e^{\lambda} + e^{s_i\lambda}) = e^{\lambda} + e^{s_i\lambda}.$$

In general,

$$s_i f = f \Rightarrow \pi(f) = f.$$

1.21. Theorem. The Demazure operator satisfies

$$\pi_i^2 = \pi_i \qquad\qquad\qquad \textbf{(quadratic relations)}$$

$$\underbrace{\pi_i\pi_j\cdots}_{m_{ij}} = \underbrace{\pi_j\pi_i\cdots}_{m_{ij}} \qquad (i \neq j) \qquad\qquad \textbf{(braid relations)}$$

From the braid relation, it is well-defined to write

$$\pi_w = \pi_{i_1}\cdots\pi_{i_\ell}, \qquad w = s_{i_1}\cdots s_{i_\ell} \quad \text{(reduced)}.$$

1.22. Example.

$$\begin{matrix} 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ \mathrm{id} \end{matrix} \qquad \begin{matrix} 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ \pi_1 \end{matrix} \qquad \begin{matrix} 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ \pi_2 \end{matrix}$$

$$\begin{matrix} 1\ 1\ 1\ 1 \\ 0\ 1\ 1\ 1\ 1 \\ 0\ 0\ 1\ 1\ 1\ 1 \\ 0\ 0\ 1\ 1\ 1 \\ 0\ 0\ 1\ 1 \\ 0\ 0\ 1 \\ \pi_2\pi_1 \end{matrix} \qquad \begin{matrix} 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1 \\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \\ \pi_1\pi_2 \end{matrix} \qquad \begin{matrix} 1\ 1\ 1\ 1 \\ 1\ 2\ 2\ 2\ 1 \\ 1\ 2\ 3\ 3\ 2\ 1 \\ 1\ 2\ 3\ 2\ 1 \\ 1\ 2\ 2\ 1 \\ 1\ 1\ 1 \\ \pi_1\pi_2\pi_1 = \pi_2\pi_1\pi_2 \end{matrix}$$

1.23. Proposition. Let w_0 be the maximal element in W. We have

$$\pi_{w_0}(f) = \sum_{w\in W} w\left(f\prod_{\alpha>0}\frac{1}{1-e^{-\alpha}}\right).$$

Proof. As an operator, π_{w_0} must be of the form

$$\pi_{w_0} = \sum_{w\in W} w\circ a_w, \qquad a_w \in \mathbb{Q}(\Lambda).$$

We need to show

$$a_w = \prod_{\alpha>0}\frac{1}{1-e^{-\alpha}}.$$

Let us write

$$\pi_{w_0} = \pi_{i_1} \cdots \pi_{i_\ell}, \qquad w_0 = s_{i_1} \cdots s_{i_\ell}.$$

Firstly, the coefficient of w_0 can only be contributed by $s_i \circ \frac{1}{1-e^{\alpha_i}}$ in each π_i. So

$$s_{i_1} \circ \frac{1}{1-e^{-\alpha_{i_1}}} \cdots s_{i_\ell} \circ \frac{1}{1-e^{-\alpha_{i_\ell}}} = w_0 \circ a_{w_0}.$$

So

$$a_{w_0} = \frac{1}{1-e^{-s_{i_\ell}\cdots s_{i_2}\alpha_{i_1}}} \cdots \frac{1}{1-e^{-\alpha_{i_\ell}}} = \prod_{\alpha>0} \frac{1}{1-e^{-\alpha}}.$$

Since $w_0 = s_i(s_i w_0)$ is reduced, we have

$$\pi_i \pi_{w_0} = \pi_i \pi_i \pi_{s_i w_0} = \pi_i \pi_{s_i w_0} = \pi_{w_0}.$$

This implies $s_i \circ \pi_{w_0} = \pi_{w_0}$ for all $i \in I$. Then $w\pi_{w_0} = \pi_{w_0}$. This implies $a_w = a_{w_0}$. □

1.24. Demazure character. For a weight $\lambda = w\lambda^+$ for a dominant weight λ^+ and $w \in W$, the **Demazure character** is

$$\kappa_\lambda = \pi_w(e^{\lambda^+}).$$

That is,

$$\lambda \text{ dominant} \Rightarrow \kappa_\lambda = e^\lambda, \qquad \pi_i \kappa_\lambda = \begin{cases} \kappa_{s_i\lambda}, & s_i\lambda \le_{\mathrm{dom}} \lambda, \\ \kappa_\lambda, & s_i\lambda \ge_{\mathrm{dom}} \lambda. \end{cases}$$

In particular, for dominant λ,

$$\kappa_{w_0\lambda} = \chi_\lambda.$$

1.25. Example. In type A, this is called the **key polynomial**. Here is the code for computing them.

```
k = KeyPolynomials(QQ)
k([4,3,2,1]).expand()
```

See the documentation.

— **SCHUBERT POLYNOMIALS.** There is another non-symmetric version of Schur polynomials called Schubert polynomials. But this is only for type A.

1.26. Demazure operator. Let us define the **BGG Demazure operator** for a polynomial f

$$\partial_i = \frac{1}{x_i - x_{i+1}}(\mathrm{id} - s_i) = (1 + s_i) \circ \frac{1}{x_i - x_{i+1}}.$$

That is,

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

1.27. Example. We have

$$\begin{aligned} \partial_1(x_1^a) &= x_1^{a-1} + x_1^{a-2}x_2 + \cdots + x_1 x_2^{a-2} + x_2^{a-1} \\ \partial_1(x_2^a) &= -\partial_1(x_1^a). \end{aligned}$$

1.28. Remark. The BGG operators can be defined for general types, but we should work in the polynomial ring

$$\mathrm{Sym}(\Lambda_{\mathbb{C}}) = \text{symmetric algebra of } \Lambda \otimes \mathbb{C}.$$

The BGG operator is defined to be

$$\partial_i = \frac{1}{\alpha_i} \circ (1 - s_i) = (1 + s_i) \circ \frac{1}{\alpha_i}.$$

1.29. Theorem. The BGG Demazure operator satisfies

$$\begin{aligned} &\partial_i^2 = \partial_i && && (\textbf{quadratic relations}) \\ &\underbrace{\partial_i \partial_j \cdots}_{m_{ij}} = \underbrace{\partial_j \partial_i \cdots}_{m_{ij}} && (i \neq j) && (\textbf{braid relations}) \end{aligned}$$

From the braid relation, it is well-defined to write

$$\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}, \qquad w = s_{i_1} \cdots s_{i_\ell} \quad \text{(reduced)}.$$

1.30. Proposition. We have

$$\partial_{w_0}(f) = \sum_{w \in S_n} w\left(f \prod_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} \right).$$

In particular,

$$s_\lambda = \partial(x^{\lambda+\rho}).$$

1.31. Schubert polynomials. For a permutation $w \in S_n$, we define the **Schubert polynomial**

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}(x^\delta).$$

Recall that $\delta = (n-1, \cdots, 1, 0)$. That is,

$$\mathfrak{S}_{w_0\lambda} = x^\delta, \qquad \partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i}, & s_i w < w, \\ 0, & s_i w > w. \end{cases}$$

In particular, $\mathfrak{S}_{\mathrm{id}} = 1$.

1.32. Example. When $n = 3$, we have

$$\begin{aligned} \mathfrak{S}_{321} &= \mathfrak{S}_{w_0} = x_1^2 x_2 & \qquad \mathfrak{S}_{213} &= \mathfrak{S}_{s_1} = x_1 \\ \mathfrak{S}_{231} &= \mathfrak{S}_{s_1 s_2} = x_1 x_2 & \qquad \mathfrak{S}_{132} &= \mathfrak{S}_{s_2} = x_1 + x_2 \\ \mathfrak{S}_{312} &= \mathfrak{S}_{s_2 s_1} = x_1^2 & \qquad \mathfrak{S}_{123} &= \mathfrak{S}_{\mathrm{id}} = 1. \end{aligned}$$

1.33. Example. We can represent a reduced word of w_0 by a staircase. For example when $n = 5$,

$$w_0 = (s_4 s_3 s_2 s_1)(s_4 s_3 s_2)(s_4 s_3) s_4 \qquad \begin{array}{cccc} \multicolumn{4}{c}{\text{read } \nearrow} \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & \\ s_3 & s_4 & & \\ s_4 & & & \end{array}$$

Let me give some example.

$$\begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & \\ s_3 & s_4 & & \\ s_4 & & & \\ \multicolumn{4}{c}{x_1^4x_2^3x_3^2x_4} \end{array} \xrightarrow{\partial_2} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & & \\ s_3 & s_4 & & \\ s_4 & & & \\ \multicolumn{4}{c}{x_1^4x_2^2x_3^2x_4} \end{array} \xrightarrow{\partial_3} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & & \\ s_3 & & & \\ s_4 & & & \\ \multicolumn{4}{c}{x_1^4x_2^2x_3x_4} \end{array} \xrightarrow{\partial_2} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \\ s_2 & & & \\ s_3 & & & \\ s_4 & & & \\ \multicolumn{4}{c}{x_1^4x_2x_3x_4} \end{array}$$

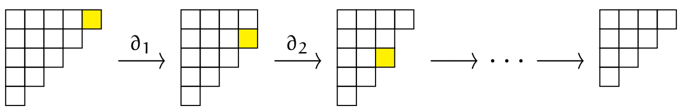
Note that we can do this kind of computation if the i-th row has exactly one more box than the $(i+1)$-th row. Actually, all the monomial Schubert polynomial is obtained in this way.

1.34. Stability. The Schubert polynomial does not depend on n, i.e. $\mathfrak{S}_w$ is defined for $w \in S_\infty = \bigcup_{n\geq 0} S_n$.

Sketch. We only need to show for the maximal element $w_0^{(n-1)} \in S_{n-1} \subset S_n$, we have

$$\mathfrak{S}_{w_0^{(n-1)}} = x_1^{n-2} \cdots x_{n-3}^2 x_{n-2}. \quad \square$$

The proof goes like this



1.35. Grassmannian permutation. Let λ be a partition of length n. Let us define the **Grassmannian permutation** $w_\lambda \in S_\infty$ such that

$$w_\lambda(1) < \cdots < w_\lambda(n), \qquad w_\lambda(n+1) < w_\lambda(n+2) < \cdots$$

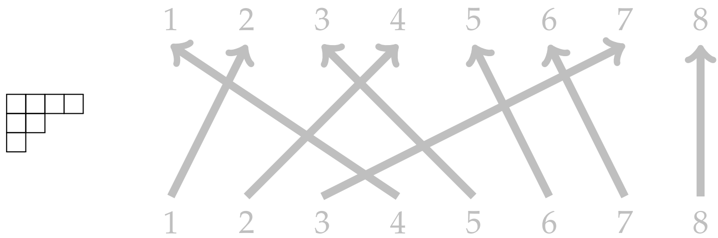
with

$$\{w_\lambda(i)\}_{i=1}^n = \{\lambda_i + n - i\}_{i=1}^n.$$

That is,

$$(w_\lambda(1), \ldots, w_\lambda(k)) = \mathrm{sort}(\lambda + \delta).$$

For instance



1.36. Theorem. Under the assumption above, we have

$$\mathfrak{S}_{w_\lambda} = s_\lambda(x_1, \ldots, x_n).$$

Sketch. Let $\mathfrak{w}_\lambda \in S_\infty$ be the permutation such that

$$\mathfrak{w}_\lambda(1) > \cdots > \mathfrak{w}_\lambda(n), \qquad \mathfrak{w}_\lambda(n+1) < \mathfrak{w}_\lambda(n+2) < \cdots$$

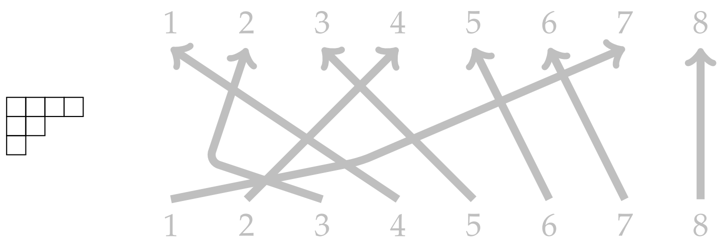
with

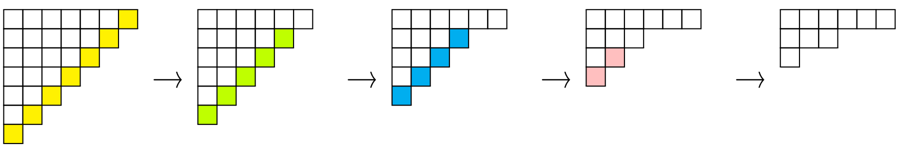
$$\{\mathfrak{w}_\lambda(i)\}_{i=1}^n = \{\lambda_i + n - i\}_{i=1}^n.$$

Then

$$w_\lambda = \mathfrak{w}_\lambda w_0, \qquad \ell(w_\lambda) = \ell(\mathfrak{w}_\lambda) - \ell(w_0).$$

Moreover, we have $\mathfrak{S}_{\mathfrak{w}_\lambda} = x^{\lambda+\delta}$. So by definition $\mathfrak{S}_{w_\lambda} = s_\lambda(x_1, \ldots, x_n)$. For example,





∎

1.37. Example. The following is the SageMath code for Schubert polynomials.

```
X = SchubertPolynomialRing(QQ)
X([4,2,3,5,1]).expand()
```

See the documentation.

2. Background and Applications

— Representation of Lie algebras. [7, Part II]

2.1. Semisimple Lie algebra. Given a root system, we have an associated semisimple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\text{root } \alpha} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the Cartan subalgebra, the Lie algebra of the maximal torus. We denote the standard Chevalley generators by

$$h_i, e_i, f_i \qquad i \in I.$$

2.2. Example. Consider $\mathfrak{g} = \mathfrak{sl}_n$. Then

$$\mathfrak{h} = \text{diagonal matrices}, \qquad \mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}.$$

We have

$$h_i = E_{ii} - E_{i+1,i+2}, \qquad e_i = E_{i,i+1}, \qquad f_i = E_{i+1,i}.$$

For example, when $n = 3$

$$e_1 = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \qquad f_1 = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix} \qquad f_2 = \begin{bmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{bmatrix}$$

We have a similar decomposition for $\mathfrak{gl}_n$.

2.3. Representation. Any finite-dimensional irreducible representation V of $\mathfrak{g}$ decompose into weight spaces

$$V = \bigoplus_{\lambda \in P} V_\lambda, \qquad V_\lambda = \{v \in V : h \cdot v = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}\}.$$

We define the **character** of V to be

$$\chi(V) = \sum_{\lambda \in \Lambda} e^{\lambda} \dim V_{\lambda} \in \mathbb{Z}[\Lambda].$$

2.4. Classification. The irreducible representations of $\mathfrak{g}$ are classified by dominant weights. Let $L(\lambda)$ denote the module of highest weight λ.

2.5. Remark. For a semi-simple Lie algebra $\mathfrak{g}$,

$$\begin{matrix}\text{finite-dim'l} \\ \text{reps of } \mathfrak{g}\end{matrix} = \begin{matrix}\text{finite-dim'l} \\ \text{algebraic reps of } G^{sc}\end{matrix} = \begin{matrix}\text{finite-dim'l} \\ \text{reps of } K^{sc}\end{matrix}$$

where G^{sc} is a simply-connected algebraic group with Lie algebra $\mathfrak{g}$ and K^{sc} a compact group with Lie algebra complexified to $\mathfrak{g}$. For example,

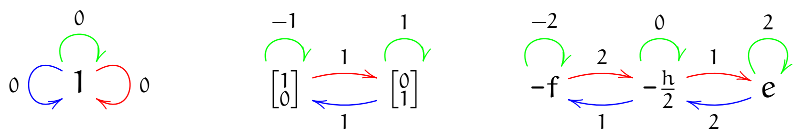
$$\mathfrak{g} = \mathfrak{sl}_n, \qquad G^{sc} = SL_n, \qquad K^{sc} = SU(n).$$

But for $\mathfrak{gl}_n$, we only have

$$\begin{matrix}\text{finite-dim'l} \\ \text{reps of } \mathfrak{gl}_n\end{matrix} \supseteq \begin{matrix}\text{finite-dim'l} \\ \text{algebraic reps of } GL_n\end{matrix} = \begin{matrix}\text{finite-dim'l} \\ \text{reps of } U(n)\end{matrix}$$

2.6. Example. Let ϖ be the fundamental weight of SL_2. Here are the representations $L(0\varpi), L(\varpi), L(2\varpi)$.

trivial	**natural**	**adjoint**



$x \in \mathfrak{sl}_2$ acts on $\mathbb{C}$ by 0	$x \in \mathfrak{sl}_2$ acts on $\mathbb{C}^2$ as a matrix	$x \in \mathfrak{sl}_2$ acts on $\mathfrak{sl}_2$ itself by $[x, \cdot]$

We remark that under the isomorphism $\mathfrak{sl}_2 \simeq \mathfrak{so}_3$, the adjoint representation becomes the natural representation of $\mathfrak{so}_3$.

2.7. Example. We have an $\mathfrak{sl}_n$-representation on

$$\Lambda^k\mathbb{C}^n = \bigoplus_{1\le a_1<\cdots<a_k\le n} \mathbb{C}u_a, \qquad u_a = \boldsymbol{e}_{a_1}\wedge\cdots\wedge\boldsymbol{e}_{a_k}.$$

We will only draw the action of f_i. For example, when $k=2$ and $n=5$, we can illustrate

$$\begin{tikzcd}
u_{15} \arrow[d, "f_1"] & u_{14} \arrow[l, "f_4"'] \arrow[d, "f_1"] & u_{13} \arrow[l, "f_3"'] \arrow[d, "f_1"] & u_{12} \arrow[l, "f_2"'] \\
u_{25} \arrow[d, "f_2"] & u_{24} \arrow[l, "f_4"'] \arrow[d, "f_2"] & u_{23} \arrow[l, "f_3"'] & \\
u_{35} \arrow[d, "f_3"] & u_{34} \arrow[l, "f_4"'] & & \\
u_{45} & & &
\end{tikzcd}$$

We have

$$L(\varpi_k) = \Lambda^k\mathbb{C}^n, \qquad \chi(L(\varpi_k)) = \sum_{1\le a_1<\cdots<a_k\le n} x_{a_1}\cdots x_{a_k}.$$

2.8. Example. We have a $\mathfrak{sl}_n$-representation on

$$S^k\mathbb{C}^n = \mathbb{C}[x_1,\ldots,x_n]_{\deg=k} = \bigoplus_{1\le a_1\le\cdots\le a_k\le n} \mathbb{C}x_{a_1}\cdots x_{a_k}.$$

The action can be given by a very similar formula

$$h = \sum_{i=1}^n h_i x_i \frac{\partial}{\partial x_i}, \qquad e_i = x_i\frac{\partial}{\partial x_{i+1}}, \qquad f_i = x_{i+1}\frac{\partial}{\partial x_i}.$$

For example, when $k = 2$ and $n = 5$, we can illustrate (we ignore scalars)

$$\begin{tikzcd}
 & & & & x_1^2 \arrow[d, "f_1"] \\
 & & & x_2^2 \arrow[d, "f_2"] & x_1x_2 \arrow[l, "f_1"'] \arrow[d, "f_2"] \\
 & & x_3^2 \arrow[d, "f_3"] & x_2x_3 \arrow[l, "f_2"'] \arrow[d, "f_3"] & x_1x_3 \arrow[l, "f_1"'] \arrow[d, "f_3"] \\
 & x_4^2 \arrow[d, "f_4"] & x_3x_4 \arrow[l, "f_3"'] \arrow[d, "f_4"] & x_2x_4 \arrow[l, "f_2"'] \arrow[d, "f_4"] & x_1x_4 \arrow[l, "f_1"'] \arrow[d, "f_4"] \\
x_5^2 & x_4x_5 \arrow[l, "f_4"'] & x_3x_5 \arrow[l, "f_3"'] & x_2x_5 \arrow[l, "f_2"'] & x_1x_5 \arrow[l, "f_1"']
\end{tikzcd}$$

We have

$$L(k\varpi_1) = S^k\mathbb{C}^n, \qquad \chi(L(\varpi_k)) = \sum_{1 \le a_1 \le \cdots \le a_k \le n} x_{a_1} \cdots x_{a_k}.$$

2.9. Remark. In general, for representations $V_1, \cdots, V_n$ of $\mathfrak{g}$, we have a representation of $\mathfrak{g}$ on $V_1 \otimes \cdots \otimes V_n$ with $x \in \mathfrak{g}$ acts by **Leibniz rule**

$$x \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes x.$$

Similarly, for a single representation V, the same formula equip a $\mathfrak{g}$-representation on the space $\Lambda^k V$ or $S^k V$.

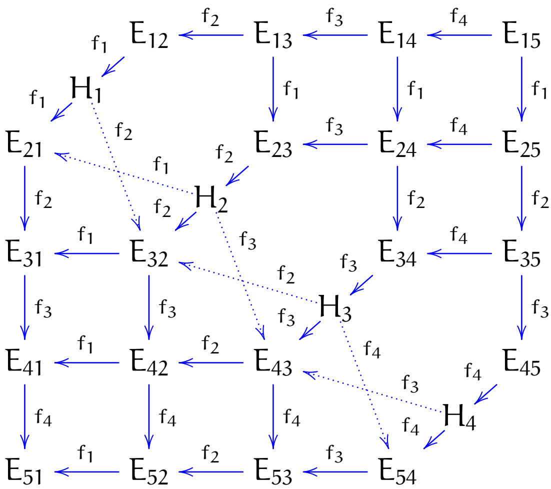
2.10. Example. The Lie algebra $\mathfrak{g}$ itself is an irreducible representation (when $\mathfrak{g}$ is simple). The action is

$$x \cdot y = [x, y].$$

We call this representation the **adjoint representation**. Then

$$L(\theta) = \text{adjoint representation}.$$

For example, when $\mathfrak{g} = \mathfrak{sl}_5$,



2.11. Weyl character formula. For a dominant weight λ,

$$\chi(L(\lambda)) = \chi_\lambda.$$

2.12. Demazure character formula. For a weight $\lambda = w\lambda^+$ for a dominant weight λ^+ and $w \in W$, we define

$$D(\lambda) = \begin{array}{c}\text{the } \mathfrak{b}^+\text{-submodule of } L(\Lambda) \\ \text{generated by a vector of weight } \lambda.\end{array}$$

Then

$$\chi(D(\lambda)) = \kappa_\lambda.$$

— **Representation of Symmetric groups.** [6, Chatper 4]

2.13. Examples. For $n \geq 1$, we denote the following representations of S_n.

$$\begin{aligned} \mathbf{tri} &= \text{trivial representation}, \\ \mathbf{alt} &= \text{alternative representation}, \\ \mathbf{std} &= \text{standard representation } \mathbb{C}^n/\mathbf{tri}. \end{aligned}$$

2.14. Grothendieck group. Let

$$\begin{aligned} K(S_n\text{-Rep}) &= \begin{matrix}\text{Grothendieck group of finite} \\ \text{dimensional representation of } S_n\end{matrix} \\ &= \bigoplus_{V \in S_n\text{-Rep}} \mathbb{Z}[V] \Big/ \left\langle \begin{matrix} [V_1]+[V_2]=[V] \text{ if} \\ 0 \to V_1 \to V \to V_2 \to 0 \end{matrix} \right\rangle \\ &= \bigoplus_{V \text{ irrep of } S_n} \mathbb{Z}\cdot[V]. \end{aligned}$$

Actually $K(S_n\text{-Rep})$ forms a ring under the tensor product. But this is **NOT** what we will mainly use.

2.15. Grothendieck ring. We consider all Grothendieck groups together

$$K = \bigoplus_{n\geq 0} K(S_n\text{-Rep}).$$

It forms a graded ring under the exterior product

$$[U]\cdot[V] = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}[U \boxtimes V].$$

Actually, it is a Hopf algebra with pairing.

2.16. Characters. The representation theory of a finite group G is controlled by its character table. By definition, the character of a representation V of G is

$$\chi_V : G \to \mathbb{C} \quad g \mapsto \mathrm{tr}(g|_V).$$

For example, for $G = S_n$

$$\chi_{\mathrm{tri}}(w) = 1, \qquad \chi_{\mathrm{alt}}(w) = (-1)^{\ell(w)}$$

$$\chi_{\mathrm{std}}(w) = \#\{i : w(i) = i\} - 1.$$

We know

$$K(G\text{-Rep})_{\mathbb{C}} = \{\text{class functions}\} := \mathrm{Fun}(G/\mathrm{Ad}G, \mathbb{C}).$$

2.17. Specht modules. For a parition $\lambda \vdash n$, we have an irreducible representation

$$\mathbf{Specht}_\lambda.$$

For example, when $n = 7$

$$\begin{aligned} \ydiagram{7} &= \mathbf{tri}, \\ \ydiagram{6,1} &= \mathbf{std}, \\ \ydiagram{5,1,1} &= \Lambda^2\mathbf{std}, \\ \cdots &= \cdots \end{aligned} \qquad \ydiagram{1,1,1,1,1,1,1} = \mathbf{alt}.$$

They are the full list of irreducible representations.

2.18. Characters.

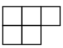
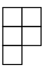
The following the character table of irreducible representations of S_n for $1 \le n \le 5$, see [6, Example 2.6, §2.3, §3.1].

S_1	id
#	1
tri	1

S_2	id	(12)
#	1	1
tri	1	1
alt	1	-1

S_3	id	(12)	(123)
#	1	3	2
tri	1	1	1
std	2	0	-1
alt	1	-1	1

S_4 #	id 1	(12) 6	(12)(34) 3	(123) 8	(1234) 6
tri	1	1	1	1	1
std	3	1	−1	0	−1
Λ^2**std**	2	0	2	−1	0
Λ^3**std**	3	1	−1	0	−1
alt	1	−1	1	1	−1

S_5 #	id 1	(12) 10	(12)(34) 15	(123) 20	(123)(45) 20	(1234) 30	(12345) 24
tri	1	1	1	1	1	1	1
std	4	2	0	1	−1	−1	1
Λ^2**std**	6	0	−2	0	0	0	1
Λ^3**std**	4	2	0	1	−1	−1	1
alt	1	−1	1	1	−1	−1	1
	5	1	1	−1	1	−1	0
	5	−1	1	−1	−1	−1	0

2.19. Frobenius character. Recall

$$p_k = x_1^k + x_2^k + \cdots, \qquad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

For a class function χ of S_n, we define the **Frobenius character** to be the symmetric function $\mathrm{Frob}(\chi)$ such that

$$\chi\begin{pmatrix}\text{any } w \in S_n \\ \text{of type } \lambda\end{pmatrix} = \langle \mathrm{Frob}(\chi), p_\lambda \rangle.$$

Explicitly,

$$\mathrm{Frob}(\chi) = \frac{1}{n!}\sum_{w\in S_n} \chi(w) p_{\mathrm{type}(w)} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda}\chi\begin{pmatrix}\text{any } w \in S_n \\ \text{of type } \lambda\end{pmatrix} p_\lambda.$$

The notations here:

- For $w \in S$, if the cycle type of w is $1^{m_1}2^{m_2}\cdots$, then $\mathrm{type}(w)$ is the partition with m_1 many 1's, m_2 many 2's etc.
- For λ with m_1 many 1's, m_2 many 2's etc., $z_\lambda = 1^{m_1}m_1!2^{m_2}m_2!\cdots$ which is the number of $w \in S_n$ with $\mathrm{type}(w) = \lambda$.

For a representation V, we define

$$\mathrm{Frob}(V) = \mathrm{Frob}(\chi_V) \in \Lambda.$$

2.20. Theorem. We have an isomorphism of Hopf algebras with pairing

$$\mathrm{Frob} : K_{\mathbb{Q}} \cong \Lambda \qquad \begin{aligned} \mathrm{Ind}_{S_\lambda}^{S_n} \mathbf{tri} &\longmapsto h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots \\ \mathrm{Ind}_{S_\lambda}^{S_n} \mathbf{sgn} &\longmapsto e_\lambda = e_{\lambda'_1}e_{\lambda'_2}\cdots \\ \mathbf{Specht}_\lambda &\longmapsto s_\lambda. \end{aligned}$$

The first condition can be rewritten as

$$\mathrm{Frob}(V) = \sum_{\lambda\vdash n} \dim(V^{S_\lambda}) m_\lambda.$$

2.21. Example. Here is an example of character table for $n = 5$.

```
Sym = SymmetricFunctions(QQ)
m = Sym.monomial(); p = Sym.power()
h = Sym.homogeneous(); e = Sym.elementary();
s = Sym.Schur()
ptt5 = list(Partitions(5)); p5 = len(ptt5)
```

```
M = p.transition_matrix(s,5)
print( table([[M[b][a] for b in range(p5)] for a in range(p5)],
             header_row=ptt5,
             header_column=["char-table"]+ptt5,
             frame = True) )
```

— SCHUBERT CALCULUS. [7, Part III]

2.22. Grassmannians. Let $0 \le k \le n$. We consider the Grassmannian variety

$$\mathrm{Gr}(k,n) = \{\text{subspace } V \le \mathbb{C}^n : \dim V = k\}.$$

Schubert calculus wants to understand the intersection theory of $\mathrm{Gr}(k,n)$.

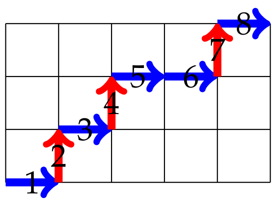
2.23. Schubert cells. We are going to decompose

$$\mathrm{Gr}(k,n) = \bigcup_{\lambda \subseteq (n-k)^k} X(\lambda)^\circ$$

where λ is a partition inside the rectangle $(n-k)^k$, bijection to k-subsets of $[n] = \{1,\ldots,n\}$ by

$$\lambda \mapsto \{\lambda_i + k - i + 1\} \subset [n].$$

Diagrammatically,



$\mapsto \{2,4,7\}.$

We will each V a k-subset of $[n]$, and thus assign a partition. We define

$$X(\lambda)^\circ = \{V : \text{the assigned partition is } \lambda\}.$$

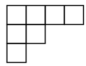
There are many equivalent way of defining the k-subsets.

Definition A. For a k-subspace V, we can find a basis

$$v_1, \dots, v_k \in V.$$

Write them as row vectors, and we can find its reduced echelon form. Then we can find a k-subset of $[n]$, with submatrix being the identity. For example, when $(k, n) = (3, 8)$,

$$\begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & * & * & 1 & 0 \end{bmatrix} \qquad \{2, 4, 7\}$$



Definition B. Let us denote the standard flag by

$$0 = F_0 < F_1 < \cdots < F_{n-1} < F_n = \mathbb{C}^n, \qquad F_i = \mathrm{span}(e_1, \dots, e_i).$$

Then the intersection with $V \in \mathrm{Gr}(k, n)$ gives another chain

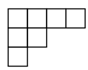
$$0 = F_0 \cap V \leq F_1 \cap V \leq \cdots \leq F_{n-1} \cap V \leq F_n \cap V = V.$$

Each step

$$? \leq ?? \Longrightarrow \begin{cases} ? = ??, \text{ or} \\ \dim ? + 1 = \dim ??. \end{cases}$$

The increasing indices form a k-subset of n. For example,

$$\begin{array}{cccc}
 & & & F_7\cap V = F_8\cap V \\
 & & & \cup \\
 & & F_4\cap V = F_5\cap V = F_6\cap V & \\
 & & \cup & \\
 & F_2\cap V = F_3\cap V & & \\
 & \cup & & \\
F_0\cap V = F_1\cap V & & &
\end{array} \qquad \{2, 4, 7\}$$



Definition C. For each k-subset $A \subset [n]$, we define

$$V_A = \mathrm{span}(e_a : a \in A) \in \mathrm{Gr}(k, n).$$

For the corresponding partition λ, we set $V_\lambda = V_A$. Then

$$X(\lambda)^\circ = B\text{-orbit of } V_A$$

where B is the Borel subgroup, i.e. the group of upper triangular matrices.

2.24. Schubert variety. We define **Schubert variety** to be the closure

$$X(\lambda) = \overline{X(\lambda)} = \bigcup_{\mu \subset \lambda} X(\mu)^\circ.$$

We define the **Schubert class**

$$\sigma_\lambda = [X(\lambda)] \in H^*(\mathrm{Gr}(k,n)).$$

We have

$$H^*(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Q} \cdot \sigma_\lambda.$$

2.25. Opposite Schubert class. Note that

$$\deg \sigma_\lambda = \operatorname{codim} X(\lambda) = k(n-k) - |\lambda|.$$

We define the **opposite Schubert class**

$$\sigma^\lambda = \text{dual basis of } \sigma_\lambda.$$

From the intersection theory,

$$\sigma^\lambda = [Y(\lambda)]$$

where $Y(\lambda)$ is defined similarly to $X(\lambda)$.

- In definition A, we should use the upper echelon form;
- In definition B, we should use the standard decreasing flag $F_\bullet^{op}$ with $F_i^{op} = \mathrm{span}(e_{i+1}, \dots, e_n)$.
- In definition C, we should use the opposite Borel (i.e. lower triangular matrices) orbit.

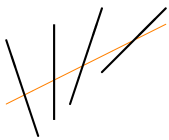
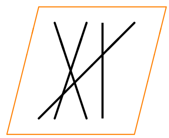
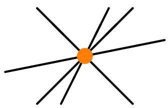
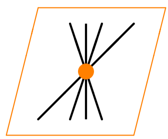
2.26. Example. Let us identify

$$\mathrm{Gr}(2,4) = \{2\text{-dimensional subspaces in } \mathbb{C}^4\} = \{\text{lines in } \mathbb{P}^3\}.$$

Then a flag is a choice of

$$\text{point} \subseteq \text{line} \subseteq \text{plane}.$$

The classes σ^λ can be described as follows. Here the colored point, line or plane is a member of the flag.

ALL LINES $\varnothing$	 $\ydiagram{1}$	 $\ydiagram{2}$
 $\ydiagram{1,1}$	 $\ydiagram{2,1}$	A LINE $\ydiagram{2,2}$

2.27. Theorem. The linear map defined by

$$\mathbb{Q}[x_1, \ldots, x_k]^{S_k} \to H^*(\mathrm{Gr}(k,n)), \qquad s_\lambda \mapsto \begin{cases} \sigma^\lambda, & \lambda \subseteq (n-k)^k, \\ 0, & \text{otherwise} \end{cases}$$

is a ring homomorphism.

2.28. Example. In $\mathrm{Gr}(2,4)$, we have

$$\square \cdot \square = \ydiagram{2} + \ydiagram{1,1}.$$

Compare:

$$s_\square \cdot s_\square = s_{\ydiagram{2}} + s_{\ydiagram{1,1}}.$$

The (transversal) intersection representing the cup product $\square \smile \square$ is

$$(*) = \{\text{the set of lines intersecting two given lines}\}.$$

Consider the special case when two lines intersect. Then any line in $(*)$ either

- going through the intersecing point;
- in the plane spanned by these two lines.

Thus we can decompose into irreducible components

$$(*) = \begin{smallmatrix}\square\\\square\end{smallmatrix} \cup \square\square.$$

By dimension reason, the intersection $(*)$ is generically transversal, so we have $\square \cdot \square = \square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$.

3. Hall–Littlewood polynomials

— **Reference.** Most of results can be found in

[26] I. G. Macdonald. Symmetric functions and Hall Polynomials, second version. Chapter III.
[29] K. Nelsen, A. Ram. Kostka–Foulkes polynomials and Macdonald spherical functions. [arXiv]

— **Type-free definition.** In this section, we will fix a formal variable t. The polynomials introduced in this section are defined for all Weyl groups. We take $\Lambda = P$ to be the weight lattice, or any sub-lattice containing root lattice Q.

3.1. The Poincaré polynomial. We define the **Poincaré polynomial** of a Weyl group W to be

$$W(t) := \sum_{w \in W} t^{\ell(w)}.$$

The special value $W(1) = |W|$ gives the order the Weyl group. We refer readers to [14, §3.20, §2.11, §4.9] for more formulæ.

3.2. Symmetrizer. Let us consider the following symmetrizer

$$\mathrm{Symm}(f) = \sum_{w \in W} w\left(f \prod_{\alpha > 0} \frac{1 - te^{-\alpha}}{1 - e^{-\alpha}}\right).$$

When $t = 1$, this is the naïve symmetrizer. When $t = 0$, this is the symmetrizer used to compute Weyl character.

3.3. Definition. For a dominant weight $\lambda \in \Lambda$, the **Hall–Littlewood polynomial** of it is

$$P_\lambda = \frac{1}{W_\lambda(t)} \mathrm{Symm}(e^\lambda).$$

The readers might wonder why there is a strange denominator. We will see soon that this is natural.

3.4. Specialization. We have

$$P_\lambda|_{t=0} = \chi_\lambda, \qquad P_\lambda|_{t=1} = m_\lambda.$$

This is the first explanation why we need the denominator $W_\lambda(t)$.

3.5. Example. Here is an example in SL_3

$$\begin{array}{ccccccc}
 & 1 & 1-t & 1-t & 1 & & \\
 1-t & (t-2)(t-1) & 2(t-1)^2 & (t-2)\cdot(t-1) & 1-t & & \\
1 & (t-2)\cdot(t-1) & (1-t)\cdot(t^2-2t+3) & (1-t)\cdot(t^2-2t+3) & (t-2)(t-1) & 1 & \\
 1-t & 2\cdot(t-1)^2 & (1-t)(t^2-2t+3) & 2(t-1)^2 & 1-t & & \\
 & 1-t & (t-2)(t-1) & (t-2)(t-1) & 1-t & & \\
 & & 1 & 1-t & 1 & &
\end{array}$$

3.6. Lemma. We have $P_0 = 1$, i.e.

$$\mathrm{Symm}(1) = \sum_{w \in W} w\left(\prod_{\alpha>0} \frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right) = W(t).$$

See [29, Coro 2.17].

First proof. By Weyl character formula, for any integer weight

$$\lambda \in \mathrm{Conv}(w\rho : w \in W)$$

we have

$$\sum w\left(\frac{e^{\lambda-\rho}}{1-e^{-\alpha}}\right) = \begin{cases} (-1)^{\ell(w)}, & \lambda = w\rho, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\prod_\alpha(1-te^{-\alpha})$ is supported over $\rho+\mathrm{Conv}(w\rho : w \in W)$. The expansion gives the formula. $\square$

Second proof. There is trick called **rigidity** [30, §2.4]

$$f(x) \in \mathbb{C}[x^{\pm 1}] \qquad \left.\begin{array}{r} \lim_{x\to 0} f(x) \ \exists \\ \lim_{x\to\infty} f(x) \ \exists \end{array}\right\} \Longrightarrow f(x) \text{ is a constant.}$$

Now we note that $\mathrm{Symm}(1)$ is a Laurent polynomial, and each term converges in any direction of limit:

$$\lim_{x\to 0}\frac{1-tx}{1-x} = 1, \quad \lim_{x\to\infty}\frac{1-tx}{1-x} = t.$$

As a result, $\mathrm{Symm}(1)$ only depends on t. If we take the limit $e^{\alpha_i} \to \infty$, then it is not hard to see, each term

$$\lim \prod_{\alpha>0} \frac{1-te^{-w\alpha}}{1-e^{-w\alpha}} = t^{\ell(w)}.$$

This finishes the proof. $\square$

3.7. Proposition. For any dominant weight $\lambda \in \Lambda$, we have

$$P_\lambda = \chi_\lambda + \sum_{\mu <_{\mathrm{dom}} \lambda} t\mathbb{Z}[t]\cdot \chi_\mu.$$

See [29, Coro 2.16].

Proof. We split

$$\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}} = \prod_{\langle\alpha,\lambda\rangle>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\prod_{\langle\alpha,\lambda\rangle=0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}},$$

$$\sum_{w\in W} w\,(\cdots) = \sum_{w\in W^\lambda} u\left(\sum_{v\in W_\lambda} v\,(\cdots)\right).$$

Applying the above Lemma to W_λ, it is not hard to obtain

$$P_\lambda = \sum_{u \in W^\lambda} u \left(x^\lambda \prod_{\substack{\alpha>0 \\ \langle \lambda, \alpha \rangle > 0}} \frac{1 - te^{-\alpha}}{1 - e^{-\alpha}} \right).$$

By expanding this in the completion with respect to anti-dominant cone, we have

$$P_\lambda = e^\lambda + \sum_{\mu < \lambda} \mathbb{Z}[t] e^\mu.$$

But P_λ itself is a symmetric polynomial. Since $P_\lambda|_{t=0} = \chi_\lambda$, so the proposition follows. □

3.8. Kostka–Foulkes polynomials. The **Kostka–Foulkes polynomial** $K_{\lambda\mu}$ for two dominant weights λ, μ is the coefficients of the expansion

$$\chi_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu.$$

From its geometric meaning, $K_{\lambda\mu}(t)$ is a Kazhdan–Lusztig polynomial, so it has non-negative coefficients. In type A, a combinatorial formula was found by Lascoux and Schützenberger, see [19], see also [29, §4].

— **Type A.** Let us restrict to type A. We use the standard **quantum number**

$$[n] = 1 + t + \cdots + t^{n-1} = \frac{1 - t^n}{1 - t}, \qquad [n]! = [n][n-1] \cdots [1].$$

3.9. Explicit formula. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition of length n. Then

$$W_\lambda = S_\alpha := S_{\alpha_1} \times \cdots \times S_{\alpha_m}$$

where $\alpha \models n$ (i.e. $\alpha_1 + \cdots + \alpha_m = n$) such that in λ

- λ_1 appears α_1 times;
- $\cdots$;

- λ_n appears α_m times.

We have
$$W_\lambda(t) = [\alpha]! = [\alpha_1]! \cdots [\alpha_m]!$$
For example,
$$\begin{aligned} \lambda &= (5,5,3,2,2,2,0) \\ \alpha &= (2,1,3,1) \\ W_\lambda &= S_2 \times S_1 \times S_3 \times S_1 \\ W_\lambda(t) &= (1+t)(1+t+t^2). \end{aligned}$$
The symmetrizer
$$\mathrm{Symm}(f) = \sum_{w\in W} w\left(f \prod_{1\le i<j\le n} \frac{x_i - tx_j}{x_i - x_j}\right).$$
So
$$P_\lambda = \frac{1}{[\alpha]!} \sum_{w\in W} w\left(x^\lambda \prod_{1\le i<j\le n} \frac{x_i - tx_j}{x_i - x_j}\right).$$

3.10. Example. Let us compute the case $n = 2$ by hand. We write $(x_1, x_2) = (x, y)$ and $\lambda = (a, b)$. Then
$$P_{(a,b)}(x,y) = \begin{cases} s_{(a,b)}(x,y), & a = b \text{ or } b+1. \\ s_{(a,b)}(x,y) - ts_{a-1,b+1}(x,y), & a > b+1. \end{cases}$$

Case A. When $a = b$, we have
$$W_\lambda = S_2, \qquad W_\lambda(t) = 1+t.$$
By definition,
$$\begin{aligned} P_\lambda(x,y) &= \frac{1}{1+t}\left(x^a y^a \frac{x-ty}{x-y} + y^a x^a \frac{y-tx}{y-x}\right) \\ &= \frac{x^a y^a}{1+t}\left(\frac{(x-ty)+(tx-y)}{x-y}\right) = \frac{x^a y^a}{1+t}(1+t) = x^a y^a. \end{aligned}$$
In particular, when $\lambda = \varnothing$, $P_\varnothing = 1$.

Case B. Let us compute for $a > b$. We see easily from the definition that

$$P_{(a,b)} = x^b y^b P_{(a-b,0)}.$$

So let us compute when $a > b = 0$. We have

$$S_\lambda = S_1 \times S_1, \qquad W_\lambda(t) = 1.$$

By definition,

$$\begin{aligned} P_\lambda(x,y) &= \left(x^a \frac{x - ty}{x - y} + y^a \frac{y - tx}{y - x} \right) = \frac{x^a(x - ty) + y^a(tx - y)}{x - y} \\ &= h_a(x,y) - t\,xy \begin{cases} h_{a-2}(x,y), & a \geq 2, \\ 0, & a = 1. \end{cases} \end{aligned}$$

3.11. Symmetric functions. One can check for each partition λ,

$$P_\lambda(x_1, \ldots, x_k) = P_\lambda(x_1, \ldots, x_k, 0).$$

In particular, P_λ can be upgraded to a symmetric function. Moreover, if $\lambda_{k+1} \neq 0$, then

$$P_\lambda(x_1, \ldots, x_k, 0) = 0.$$

This proves the stability of Kostka—Foulkes polynomials.

3.12. Example. One can compute the Hall–Littlewood function in SageMath

```
Sym = SymmetricFunctions(FractionField(QQ["t"]))
HLP = Sym.hall_littlewood().P();
HLP([3,1,1]).expand(3)
```

The expansion to Schur functions (similar to other basis)

```
Sym = SymmetricFunctions(FractionField(QQ["t"]))
HLP = Sym.hall_littlewood().P();
s = Sym.Schur();
s(HLP([3,1,1]))
```

See the documentation.

— **DEMAZURE–LUSZTIG OPERATORS.** As we seen from the theory of Schur polynomials, a family of symmetric polynomials could be studied by extending to its non-symmetric version. We could do the same for Hall–Littlewood polynomials. But in this section we only introduce the operators.

3.13. Demazure–Lustig operator. Let us denote the **Demazure–Lusztig operator** for a polynomial f and $i \in I$ by

$$T_i = ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - \mathrm{id}) = \frac{1-t}{e^{\alpha_i}-1}\mathrm{id} + \frac{te^{\alpha_i}-1}{e^{\alpha_i}-1}s_i$$

Then we have

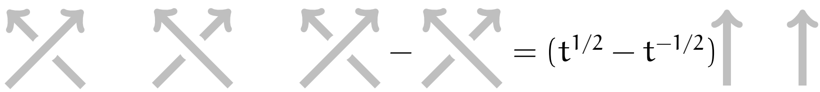
$$T_i^2 = (t-1)T_i + t \qquad \text{(quadratic relations)}$$

$$\underbrace{T_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_i\cdots}_{m_{ij}} \qquad (i \neq j) \qquad \text{(braid relations)}$$

From the braid relation, it is well-defined to write

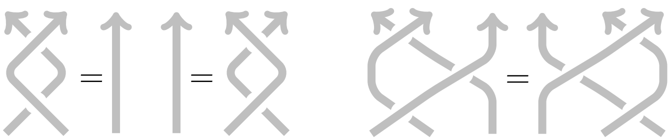
$$T_w = T_{i_1}\cdots T_{i_\ell}, \qquad w = s_{i_1}\cdots s_{i_\ell} \quad \text{(reduced)}.$$

3.14. Example. Consider the case $W = S_n$. We can represent the relations as diagrams. We use $H_i = t^{-1/2}T_i$. Let

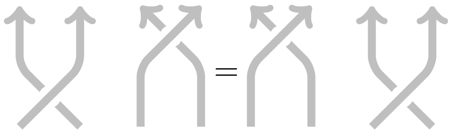


H_i $\qquad H_i^{-1} \qquad H_i - H_i^{-1} = t^{1/2} - t^{-1/2}$

The rest of relations are presented here:



$H_iH_i^{-1} = 1 = H_i^{-1}H_i \qquad H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1}$



$$H_iH_j = H_jH_i \qquad (|i-j|>1)$$

Actually, they are the relations of the **braid group** (i.e. the group generated without quadratic relations). The reader might find some analogy of skein relations in knot theory. Actually this diagram presentation motivates the construction of HOMFLYPY polynomials in knot theory, see [33] for an introduction.

3.15. Example. Let us compute the SL_2-case. For λ dominant,

$$\begin{array}{ccccc} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{array} \overset{T}{\longmapsto} \begin{array}{ccccc} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ 1 & 1-t & \cdots & 1-t & 0 \\ t-1 & t-1 & \cdots & t-1 & t \end{array}$$

We have

$$T(e^{\lambda}+e^{s_i\lambda}) = t(e^{\lambda}+e^{s_i\lambda}).$$

In general,

$$s_if = f \Rightarrow T(f) = tf.$$

3.16. Example. It is useful to understand the quadratic relations at the level of operators. In the computation, we omit the indices.

$$\begin{aligned}(T+1)(f) &= tsf + (t-1)\frac{sf-f}{e^{\alpha}-1} + f \\ &= \left(\frac{1-t}{e^{\alpha}-1}+1\right) f + \left(\frac{t-1}{e^{\alpha}-1}+t\right) sf \\ &= \left(\frac{e^{\alpha}-t}{e^{\alpha}-1}\right) f + \left(\frac{te^{\alpha}-1}{e^{\alpha}-1}\right) sf \\ &= (s+1)\left(f\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right).\end{aligned}$$

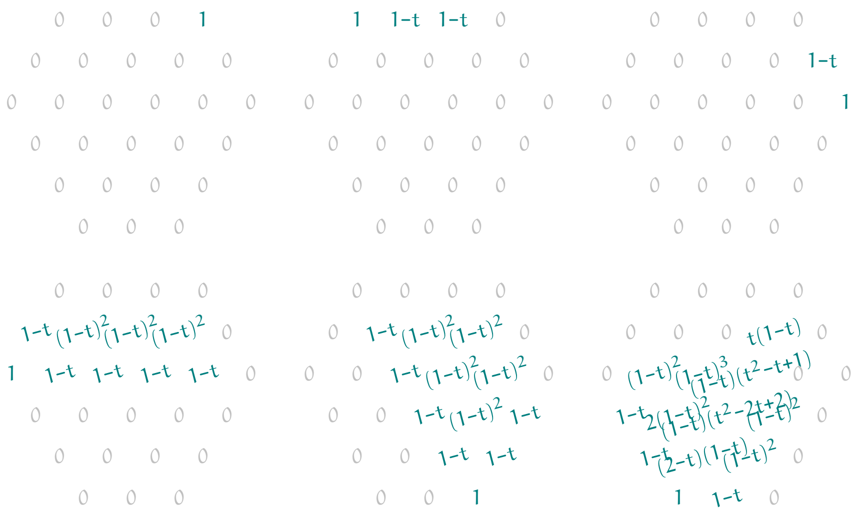
Similarly,

$$(T-t)(f) = \left(t + \frac{t-1}{e^\alpha - 1}\right)(s-1)(f)$$
$$= \frac{te^\alpha - 1}{e^\alpha - 1}(s-1)(f).$$

Since $(s-1)(s+1) = s^2 - \mathrm{id} = 0$, we have

$$(T-t)(T+1) = 0.$$

3.17. Example. Let us consider SL_3. We start from e^λ for $\lambda = (5,2,0)$.



3.18. Remark. For each $\lambda \in \Lambda$, we define the operator $X^\lambda(f) = e^\lambda f$. Then the affine Hecke algebra (for the dual system) is isomorphic to the algebra generated by

$$H_i = t^{-1/2}T_i \quad (i \in I); \qquad X^\lambda \quad (\lambda \in \text{root lattice}).$$

Actually, under the Coxeter presentation

$$\lambda \text{ anti-dominant} \Rightarrow H_{t_\lambda} = X^\lambda.$$

3.19. Symmetrizer. Notice that

$$T_iT_w = \begin{cases} T_{s_iw}, & s_iw > w, \\ tT_{s_iw} + (t-1)T_w, & s_iw < w. \end{cases}$$

In other word, if $w < s_iw$, we can represent the multiplication as

$$\begin{tikzcd} s_iw \arrow[r, "t-1"] \arrow[ddr, "t"' near end] & s_iw \\ \vee & \vee \\ w \arrow[uur, "1" near start] & w \end{tikzcd}$$
$$T_i$$

So we can represent

$$\begin{tikzcd} s_iw \arrow[r, "-1"] \arrow[ddr, "t" near end] & s_iw \\ \vee & \vee \\ w \arrow[uur, "1" near start] \arrow[r, "-t"] & w \end{tikzcd} \qquad \begin{tikzcd} s_iw \arrow[r, "t"] \arrow[ddr, "t" near end] & s_iw \\ \vee & \vee \\ w \arrow[uur, "1" near start] \arrow[r, "1"] & w \end{tikzcd}$$
$$T_i - t \qquad\qquad\qquad T_i + 1$$

If

$$C_{w_0} = \sum_{w\in W} T_w, \qquad C^-_{w_0} = \sum_{w\in W} (-t)^{\ell(w_0)-\ell(w)} T_w,$$

then

$$(T_i - t)C_{w_0} = 0 = (T_i + 1)C^-_{w_0}$$

Similarly,

$$C_{w_0}(T_i - t) = 0 = C^-_{w_0}(T_i + 1).$$

3.20. Theorem. We have

$$\sum_{w\in W} w\left(f\prod_{\alpha>0}\frac{1-te^{-\alpha}}{1-e^{-\alpha}}\right) =: \mathrm{Symm}(f) = C_{w_0}(f) := \sum_{w\in W} T_w(f).$$

Proof. As an operator, C_{w_0} must be of the form

$$C_{w_0} = \sum_{w \in W} w \circ a_w.$$

We need to show

$$a_w = \prod_{\alpha > 0} \frac{1 - te^{-\alpha}}{1 - e^{-\alpha}}.$$

- For $w = w_0$, this is similar as we did for ∂_{w_0}. Precisely, the term w_0 can be contributed only by the term $T_{w_0} = T_{i_1} \cdots T_{i_\ell}$; and the contribution is the product of $s_{i_1}(\cdots)$ in each factor T_{i_1}.
- Since $(T_i - t)C_{w_0} = 0$, and $T_i - t = (\cdots)(s_i - 1)$, so it implies $C_{w_0}(f)$ is symmetric for any f, i.e.

$$C_{w_0} = \left(\sum_{w \in W} w\right) \circ (\cdots).$$

  This shows $a_w = a_{w_0}$ for all w. $\square$

3.21. Remark. This gives another proof of $\mathrm{Symm}(1) = W(t)$.

3.22. Remark. There are other versions of Demazure–Lusztig operators. We will show uniqueness of such operators in certain sense. Assume we have an operator

$$\tilde{T}_i = p(\alpha_i)s_i - q(\alpha_i)\mathrm{id}$$

satisfying

$$(\tilde{T}_i - t)(\tilde{T}_i + 1) = 0, \qquad \tilde{T}_i\tilde{T}_j \cdots = \tilde{T}_j\tilde{T}_i \cdots .$$

Then

$$\tilde{T}_i = \pm \frac{g(\alpha_i)}{g(-\alpha_i)} \frac{te^{c\alpha_i} - 1}{e^{c\alpha_i} - 1} s_i + \frac{1 - t}{e^{c\alpha_i} - 1}\mathrm{id}$$

for a constant c, a function g. This is by explicit computation in SL_3. Here are two popular choices:

- when $c = 1$ and $g(\alpha) = t^{1/2}e^{\alpha_i/2} - t^{-1/2}e^{-\alpha_i/2}$, we get
$$\tilde{T}_i(f) = -s_i f + (t-1)\frac{s_i f - f}{e^{\alpha_i} - 1} = \frac{te^{\alpha_i} - 1}{e^{\alpha_i} - 1}s_i + \frac{1-t}{e^{\alpha_i} - 1}\mathrm{id}.$$
  This is the operator from another one-dimensional representation $\mathbb{k}^-$ with $T_w \mapsto v^{-\ell(w)}$.
- When $c = -1$ and $g(\alpha) = te^{\alpha} - 1$, we get
$$\tilde{T}_i(f) = \frac{te^{\alpha_i} - 1}{e^{-\alpha_i} - 1}s_i + \frac{1-t}{e^{-\alpha_i} - 1}\mathrm{id}.$$
  This was used to characterize **Whittacker functions** [1].

4. Background and Applications

4.1. Modified Hall–Littlewood. In this section, we write

$$\tilde{P} = t^{\langle \rho, \lambda \rangle} P_\lambda|_{t \mapsto t^{-1}}.$$

In type A, we need to fix a choice of ρ, our choice is

$$\tilde{P}_\lambda = t^{n(\lambda)} P_\lambda|_{t \mapsto t^{-1}}$$

where

$$n(\lambda) = \sum_{\square \in \lambda} \mathrm{row}(\square) - 1, \quad \text{e.g.} \quad n\left(\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \right) = 4.$$

In the following applications, all the role is played by $\tilde{P}_\lambda$.

— Hall polynomials. [26, §III.3]

4.2. Question. Assume we have three finite abelian p-groups A, B, C, we want to compute the number

$$c_{AB}^{C}(p) = \#\{M \leq C : M \cong B \text{ and } C/M \cong A\}.$$

Recall a finite abelian p-group must be of the form

$$\mathbb{A}_\lambda = \mathbb{Z}/(p^{\lambda_1}) \oplus \cdots \oplus \mathbb{Z}/(p^{\lambda_n})$$

for a partition λ. We say an abelian p-group isomorphic to $\mathbb{A}_\lambda$ of type λ. In other words,

$$\{\text{finite abelian } p\text{-groups}\}/_{\cong} = \{\text{partitions}\}.$$

Thus let us take

$$A = \mathbb{A}_\lambda, \quad B = \mathbb{A}_\mu, \quad C = \mathbb{A}_\nu, \qquad c_{\lambda\mu}^{\nu}(p) = c_{AB}^{C}(p).$$

4.3. Example. Consider

$$\ydiagram{2,1} = \mathbb{A}_{(2,1)} = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}.$$

Let us classify $M \subset \mathbb{A}_{(2,1)}$ isomorphic to

$$\ydiagram{1} = \mathbb{A}_{(1)} \cong \mathbb{Z}/p\mathbb{Z}.$$

Equivalently, we are classifying order p-elements. It is not hard to see

$$\{\text{order } p \text{ elements}\} = \{(ap, b) : a \neq 0 \text{ or } b \neq 0\}.$$

Let M be the subgroup generated by (ap, b).

Case A. If $b \neq 0$. Then $\mathbb{A}_{(2,1)}/M$ is generated by $(1, 0)$. Actually, M can be generated by $(a'p, 1)$ for some $a' \equiv b^{-1}a \bmod p$. So $(0, 1)$ can be generated $(1, 0)$ modulo M. This shows

$$\ydiagram{2} = \mathbb{A}_{(2)} \cong \mathbb{A}_{(2,1)}/M.$$

Case B. If $b = 0$. Then $\mathbb{A}_{(2,1)}/M$ is generated by two order-p elements $(1, 0)$ and $(0, 1)$. Actually, $M \cong p\mathbb{Z}/p^2\mathbb{Z} \oplus 0$. This shows

$$\ydiagram{1,1} = \mathbb{A}_{(1,1)} \cong \mathbb{A}_{(2,1)}/M.$$

Summary. As a result,

$$c^{(2,1)}_{(1),(2)}(p) = p, \qquad c^{(2,1)}_{(1),(1,1)}(p) = 1.$$

4.4. Example. Consider

$$\ydiagram{3} = \mathbb{A}_{(3)} = \mathbb{Z}/p^3\mathbb{Z}.$$

Similar as the computation above,

$$c^{(3)}_{(1),(2)}(p) = 1, \qquad c^{(3)}_{(1),(1,1)}(p) = 0.$$

4.5. Example. Consider

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array} = \mathbb{A}_{(1,1,1)} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}.$$

Similar as the computation above,

$$c^{(1,1,1)}_{(1),(1,1)}(p) = 1 + p + p^2, \qquad c^{(1,1,1)}_{(1),(2)}(p) = 0.$$

Another perspective is $\mathbb{A}_{(1,1,1)}$ is an $\mathbb{F}_p$-vector space, and a subgroup $M \subset \mathbb{A}_{(1,1,1)}$ isomorphic to $\mathbb{A}_{(1)}$ is nothing but a 1-dimensional subspace. So

$$c^{(1,1,1)}_{(1),(1,1)}(p) = \#\big(\mathbb{P}^2(\mathbb{F}_p)\big).$$

4.6. Hall algebra. The idea is the following. Consider

$$\mathbf{H} = \left\{ \left\{ \begin{array}{c} \text{finite abelian} \\ p\text{-groups} \end{array} \right\} \xrightarrow{f} \mathbb{Z} : \begin{array}{c} A \cong B \Rightarrow \\ f(A) = f(B) \end{array} \right\}.$$

We can define **Hall product**

$$(f * g)(A) = \sum_{M \leq A} f(M/A) g(A).$$

It is a good exercise to see $\mathcal{H}$ forms a ring with unit, called the **Hall algebra**. See [35] for a general introduction to Hall algebras.

4.7. Translation. Now, let $\mathbb{P}_\lambda$ denote the characteristic function of $\mathbb{A}_\lambda$. We see immediately that

$$\mathbb{P}_\lambda * \mathbb{P}_\mu = \sum_\nu c^\nu_{\lambda\mu}(p) \mathbb{P}_\nu.$$

This equips the original question an algebra structure. Let us restrict to the subalgebra

$$\mathcal{H} = \bigoplus_\lambda \mathbb{Z} \cdot \mathbb{P}_\lambda \subseteq \mathbf{H}.$$

It is the subalgebra of $\mathcal{H}$ of finite support.

4.8. Theorem. The linear map

$$\mathcal{H} \to \Lambda, \qquad \mathbb{P}_\lambda \mapsto \tilde{P}_\lambda|_{t\mapsto p}$$

is a ring homomorphism.

4.9. Example. We have

$$\begin{aligned} P_{(1,1)}P_{(1)} &= (t^2+t+1)P_{(1,1,1)} + P_{(2,1)}, \\ P_{(2)}P_{(1)} &= P_{(3)} + P_{(2,1)}. \end{aligned}$$

From the three examples above, we have

$$\begin{aligned} \mathbb{P}_{(1,1)}\mathbb{P}_{(1)} &= (1+p+p^2)\mathbb{P}_{(1,1,1)} + \mathbb{P}_{(2,1)} \\ \mathbb{P}_{(2)}\mathbb{P}_{(1)} &= \mathbb{P}_{(3)} + p\mathbb{P}_{(2,1)} \end{aligned}$$

Here is the value of $n(\cdot)$:

λ	(1)	$(1,1)$	(2)	$(1,1,1)$	$(2,1)$	(3)
$n(\lambda)$	0	1	0	3	1	0

4.10. Remark. The proof can be found in [26, III (3.4)]. A conceptual reason is a relation between Hall algebra $\mathcal{H}$ and spherical Hecke algebra of type A, see [26, V (2.6)].

— Spherical functions. [29, 13]

4.11. Assumption. Let $T \subset B \subset G$ be a maximal torus, a Borel subgroup and a reductive algebraic group Weyl group is W. For example, when $W = S_n$, then we can choose

$$T = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \subset B = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix} \subset G = GL_n\,.$$

Let us fix $\varpi \in \mathfrak{m} \subset \mathcal{O} \subset \mathcal{K}$ be the uniformizer, the maximal ideal, ring of integers and a local field with residue field $\mathbb{F}_q$. For example,

$$x \in x\mathbb{F}_q[[x]] \subset \mathbb{F}_q[[x]] \subset \mathbb{F}_q((x)), \qquad p \in p\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)} \subset \mathbb{Q}_{(p)}.$$

We will not distinguish the base change and its rational points.

4.12. Spherical Hecke algebra. The **spherical Hecke algebra** is defined to be

$$\mathcal{C}_c(G_{\mathcal{O}}\backslash G_{\mathcal{K}}/G_{\mathcal{O}}) = \left\{ \begin{array}{c} G_{\mathcal{K}} \xrightarrow{f} \mathbb{Q} \\ \text{of compact support} \end{array} : \begin{array}{c} g_1, g_2 \in G_{\mathcal{O}} \Rightarrow \\ f(g_1 x g_2) = f(x) \end{array} \right\}.$$

The algebraic structure is given by the convolution:

$$(f * g)(x) = \int_{G_{\mathcal{K}}} f(xy^{-1})g(y)dy, \qquad \mathrm{vol}(G_{\mathcal{O}}) = 1.$$

4.13. Cartain decomposition. Let $\Lambda = \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_m, T)$ is the **co**character (aka one-parameterized subgroups) lattice. We have **Cartan decomposition**

$$G_K = \bigsqcup_{\lambda \in \Lambda_{\mathrm{dom}}} G_O\, \varpi^\lambda G_O, \qquad \mathcal{C}_c(G_O\backslash G_K/G_O) \cong \bigoplus_{\lambda\in\Lambda_{\mathrm{dom}}} 1_\lambda..$$

4.14. Example. Let us consider $G = GL_n$. Firstly, we can identify

$$G_{\mathcal{K}}/G_{\mathcal{O}} = \{\mathcal{O}\text{-lattices } L \text{ in } \mathcal{K}^{\oplus n}\}.$$

Then for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have

$$\varpi^\lambda = \begin{bmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{bmatrix} \in G_{\mathcal{K}}.$$

Then ϖ^λ corresponds to the lattice

$$L_\lambda = \varpi^{\lambda_1}\mathcal{O} \oplus \cdots \oplus \varpi^{\lambda_n}\mathcal{O} \subset \mathcal{K}^{\oplus n}.$$

Then its $G_{\mathcal{O}}$ orbit can be described as

$$\{L \supseteq L_0 : L/L_0 \cong \mathbb{A}_\lambda\} \qquad \mathbb{A}_\lambda = \mathcal{O}/\mathfrak{m}^{\lambda_1} \oplus \cdots \oplus \mathcal{O}/\mathfrak{m}^{\lambda_n}$$

where $L_0 = \mathcal{O}^{\oplus n}$ is the standard lattice. It is not hard to imagine the structure constant for λ, μ, ν is

$$\# \left\{ L_0 \subseteq L \subseteq L_\nu : \begin{array}{c} L/L_0 \cong \mathbb{A}_\lambda \\ L_\nu/L \cong \mathbb{A}_\mu \end{array} \right\}$$

which coincide with that of the Hall product.

4.15. Affine Hecke algebra. We denote the **Iwahori subgroup** by

$$I = \{g(x) \in G_{\mathcal{O}} : g(x) \bmod \mathfrak{m} \in B_{\mathbb{F}_q}\} \subseteq G_{\mathcal{K}}.$$

Then the **(extended) affine Hecke algebra** is defined similarly

$$\mathcal{C}_c(I \backslash G_{\mathcal{K}}/I) \cong \widehat{\mathcal{H}}_G.$$

It can be described explicitly using generators and presentations, see [13].

4.16. Two presentations. There are two presentations of $\widehat{\mathcal{H}}_G$:

- Coxeter presentation, i.e. $\widehat{\mathcal{H}}_G$ is (almost) the Hecke algebra with respect to the affine Coxeter diagram;
- Bernstein presentation, i.e. $\widehat{\mathcal{H}}_G$ is the algebra generated by Demazure–Lusztig operators and multiplications.

A clear treatment can be found in [37].

4.17. Center of affine Hecke algebra. Using the Bernstein presentation, we see easily

$$Z(\widehat{\mathcal{H}}_G) = \mathbb{Q}[\Lambda]^W.$$

4.18. Satake isomorphism. We have the following diagram

$$\begin{tikzcd} & \mathcal{C}_c(G_{\mathcal{O}}\backslash G_{\mathcal{K}}/G_{\mathcal{O}}) \arrow[r, "\subset"] \arrow[d] & \mathcal{C}_c(I\backslash G_{\mathcal{K}}/I) \arrow[d] \\ \mathbb{Q}[\Lambda]^W \arrow[r, "\sim"] & Z(\widehat{\mathcal{H}}_G) \arrow[r, "\subset"] & \widehat{\mathcal{H}}_G \end{tikzcd}$$

4.19. Theorem. Under the isomorphism

$$\mathcal{C}_c(G_O\backslash G_K/G_O) \longrightarrow \mathbb{Q}[\Lambda]^W, \qquad \mathbf{1}_\lambda \longmapsto \tilde{P}_\lambda|_{t\mapsto q}.$$

Note that λ is a coweight, so the Hall–Littlewood polynomial is defined for the root system is dual to G.

— Springer fibers. [10, §3.4]

4.20. Springer theory. Let $\mathcal{N}$ be the set of nilpotent matrices in $\mathbb{M}_n(\mathbb{C})$. The **Springer resolution** is

$$\widetilde{\mathcal{N}} = \{(V_\bullet, x) \in \mathcal{F}\ell_n \times \mathcal{N} : xV_\bullet \subseteq V_\bullet\} \xrightarrow{\pi} \mathcal{N}.$$

For a nilpotent matrix x, the **Springer fiber** is

$$\mathcal{F}\ell_x = \{V_\bullet \in \mathcal{F}\ell_n : xV_\bullet \subseteq V_\bullet\}.$$

Since x itself is nilpotent, the equality never achieves, so the condition is equivalent to $xV_\bullet \subseteq V_{\bullet-1}$. Springer theory equips an

$$S_n \text{ action on } H_*(\mathcal{F}\ell_x), H^*(\mathcal{F}\ell_x).$$

We want to study the $H_*(\mathcal{F}\ell_x) \cong H^*(\mathcal{F}\ell_x)$ as graded S_n-representation.

Recall that the nilpotent matrices are classified by Jordan blocks

$$x_\lambda = \operatorname{diag}(J_{\lambda_1}, \ldots, J_{\lambda_m})$$

for a parititon $\lambda \vdash n$. We set $\mathcal{F}\ell_\lambda = \mathcal{F}\ell_{x_\lambda}$.

4.21. Example. Consider $n = 3$. We have three types of nilpotent matrices

$(1,1,1)$	$(2,1)$	(3)
$\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$

We have

- For $\lambda = (1,1,1)$, the matrix is 0. So $\mathcal{F}\ell_{(1,1,1)} = \mathcal{F}\ell_3$.
- For $\lambda = (3)$, the matrix is a single Jordan block, the only flag is the standard flag, so $F\ell_{(3)} = \{\text{standard flag}\} = \text{pt}$.
- For $\lambda = (2,1)$, $\mathcal{F}\ell_{(2,1)}$ is a union of two $\mathbb{P}^1$ intersecting at one point (justified below).

Assume $\lambda = (2,1)$. Let $(0 \subset V_1 \subset V_2 \subset \mathbb{C}^{\oplus 3}) \in \mathcal{F}\ell_{(2,1)}$. The condition says

$$xV_1 = 0, \qquad xV_2 \subseteq V_1, \qquad x\mathbb{C}^3 \subseteq V_2.$$

We can present basis of $\mathbb{C}^3$ as follows

$$\begin{array}{ccccc} 0 & \xleftarrow{x} & e_1 & \xleftarrow{x} & e_2 \\ 0 & \xleftarrow{x} & e_3 & & \end{array}$$

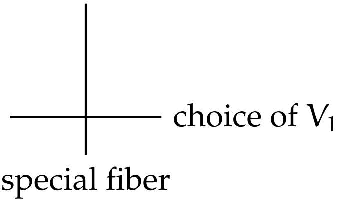
Then the conditions

$xV_1 = 0$	$xV_2 \subseteq V_1$	$x\mathbb{C}^3 \subseteq V_2$
$V_1 \subseteq \operatorname{span}(e_1, e_3)$	$\cdots$	$V_2 \supseteq \operatorname{span}(e_1)$

The choice of V_1 is

$$\{V_1 \subset \operatorname{span}(e_1, e_3)\} = \mathbb{P}^1.$$

If $V_1 = \operatorname{span}(e_1)$, then the choice V_2 is arbitrary, this gives a copy of $\mathbb{P}^1$. Otherwise, we have to take $V_2 = V_1 + \operatorname{span}(e_1)$, the choice of V_2 is a single point. Topologically, it is two $\mathbb{P}^1$'s intersecting at one point:



We can compute its homology group now

λ	$\mathcal{F}\ell_\lambda$	dim	$H_*(\mathcal{F}\ell_\lambda)$
(3)	pt	0	$\mathbf{tri}$
$(2,1)$	$\mathbb{P}^1 \cup_{\mathrm{pt}} \mathbb{P}^1$	1	$\mathbf{tri} \oplus \mathbf{std}$
$(1,1,1)$	$\mathcal{F}\ell_3$	3	$\mathbf{tri} \oplus \mathbf{std} \oplus \mathbf{std} \oplus \mathbf{alt}$

4.22. Theorem. We have

$$\dim(\text{any irreducible component of } \mathcal{F}\ell_\lambda) = n(\lambda).$$

Moreover, the graded character of

$$\mathrm{Frob}\left(H^*(\mathcal{F}\ell_\lambda)\right) = \tilde{H}_\lambda$$

where $\tilde{H}_\lambda$ is the dual basis of $\tilde{P}_\lambda$ under the Hall-pairing. In other word,

$$\langle \mathbf{Specht}_\mu, H^*(\mathcal{F}\ell_\lambda)\rangle = t^{n(\lambda)} K_{\mu\lambda}(t^{-1}),$$

the Kostka–Foulkes polynomial.

4.23. Remark. Equivalently,

$$\tilde{H}_\lambda = t^{n(\lambda)} H_\lambda|_{t\mapsto t^{-1}}$$

H_λ is the dual basis of P_λ under the Hall-pairing. There is another dual basis Q_λ, but it is dual with respect to the t-deformed Hall-pairing, see [26]. They are related by $H_\lambda = Q_\lambda[Z\frac{1}{1-t}]$.

4.24. Example. We have the following expansion.

expand $\rightarrow$	$P_{(1,1,1)}$	$P_{(2,1)}$	$P_{(3)}$
$s_{(1,1,1)}$	1		
$s_{(2,1)}$	t^2+t	1	
$s_{(3)}$	t^3	t	1

Compare:

expand $\downarrow$	$H^*(\mathcal{F}\ell_{(1,1,1)})$	$H^*(\mathcal{F}\ell_{(2,1)})$	$H^*(\mathcal{F}\ell_{(3)})$
$\mathtt{alt} = \mathtt{Specht}_{(1,1,1)}$	t^3		
$\mathtt{std} = \mathtt{Specht}_{(2,1)}$	t^2+t	1	
$\mathtt{alt} = \mathtt{Specht}_{(3)}$	1	1	1

4.25. Remark. We remark that Springer theory is not only for type A. But the relation with Hall–Littlewood polynomial is only in type A. A geometric explanation is, there is an open embedding

$$\mathcal{N} \longrightarrow \overline{G_{\mathcal{O}} t^{\mu} G_{\mathcal{O}}/G_{\mathcal{O}}} \subset G_{\mathcal{K}}/G_{\mathcal{O}}, \qquad \mu = (n, 0, \dots, 0)$$

for $G = GL_n$ where

$$\varpi = t \in \mathfrak{m} = t\mathcal{O} \subset \mathcal{O} = \mathbb{C}[[t]] \subset \mathcal{K} = \mathbb{C}((t)).$$

See [38, Example 2.1.8].

4.26. Remark. In type A, the pull-back $H^*(\mathcal{F}\ell_n) \rightarrow H^*(\mathcal{F}\ell_\lambda)$ is surjective, and the ideal can be explicitly computed. Moreover, the theorem can be proven directly from the description of the ideal, see [10, §3.4.6] and [3].

4.27. Parabolic Springer fibers. There is an equivalent way of stating the result via parabolic Springer fiber. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \models n$ be a composition. The partial flag variety is

$$\mathcal{F}\ell^\alpha = \{0 = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{C}^n : \dim V_i = \dim V_{i-1} + \alpha_i\}.$$

We can consider the parabolic version of the springer fiber

$$\mathcal{F}\ell_\lambda^\alpha = \{V_\bullet \in \mathcal{F}\ell^\alpha : x_\lambda V_\bullet \subseteq V_{\bullet-1}\}.$$

It is known

$$H_*(\mathcal{F}\ell_\lambda^\alpha) = H_*(\mathcal{F}\ell_\lambda)^{S_\alpha\text{-alt}}/\Delta_\alpha$$

where Δ_α is the discriminant of S_α. If $\mu = \mathrm{sort}(\alpha) \vdash n$, then

$$\mathbf{dim}\, H^*(\mathcal{F}\ell_\lambda^\alpha) = t^{-d}\langle e_\mu, \tilde{H}_\lambda\rangle$$

where $d = \dim \mathcal{F}\ell_n - \dim \mathcal{F}\ell_\alpha$. It is known $\mathcal{F}\ell_\lambda^\alpha$ admits an affine paving, so it coincides with the point-counting over finite fields, i.e.

$$\#(\mathcal{F}\ell_\lambda^\alpha(\mathbb{F}_q)) = \langle h_\mu, \tilde{H}_\lambda\rangle|_{t\mapsto q}.$$

4.28. Example. When $n = 3$,

$$\mathcal{F}\ell^{(1,1,1)} = \mathcal{F}\ell_3, \qquad \mathcal{F}\ell^{(1,2)} \cong \mathbb{P}^2 \cong \mathcal{F}\ell^{(2,1)}, \qquad \mathcal{F}\ell^{(3)} = \mathrm{pt}.$$

Similar as the computation above, we have

λ	$\mathcal{F}\ell_\lambda^{(3)}$	$\mathcal{F}\ell_\lambda^{(1,2)}$	$\mathcal{F}\ell_\lambda^{(2,1)}$	$\mathcal{F}\ell_\lambda^{(1,1,1)}$
(3)	$\varnothing$	$\varnothing$	$\varnothing$	pt
$(2,1)$	$\varnothing$	pt	pt	$\mathbb{P}^1 \cup_{\mathrm{pt}} \mathbb{P}^1$
$(1,1,1)$	pt	$\mathcal{F}\ell_3^{(1,2)}$	$\mathcal{F}\ell_3^{(2,1)}$	$\mathcal{F}\ell_3$

Compare

$\langle -, - \rangle$	$e_{(3)}$	$e_{(2,1)}$	$e_{(1,1,1)}$
$\tilde{H}_{(3)}$	0	0	1
$\tilde{H}_{(2,1)}$	0	t	$2t+1$
$\tilde{H}_{(1,1,1)}$	t^3	t^3+t^2+t	t^3+2t^2+2t+1

4.29. Generalized Springer fiber. There is another version of parabolic Springer fiber (known as **generalized (parabolic) Springer fiber**)

$$\mathcal{F}\ell^{\alpha}_{(\lambda)} = \{V_\bullet \in \mathcal{F}\ell^{\alpha} : x_\lambda V_\bullet \subseteq V_\bullet\}.$$

The difference is the indices. It is known

$$H_*(\mathcal{F}\ell^{\alpha}_{(\lambda)}) = H_*(\mathcal{F}\ell_\lambda)^{S_\alpha}.$$

If $\mu = \mathrm{sort}(\alpha) \vdash n$, then

$$\mathbf{dim}\, H^*(\mathcal{F}\ell^{\alpha}_{(\lambda)}) = \langle h_\mu, \tilde{H}_\lambda \rangle$$

the monomial expansion of $\tilde{H}_\lambda$.

4.30. Example. Similar as the computation above, we have

λ	$\mathcal{F}\ell^{(3)}_{(\lambda)}$	$\mathcal{F}\ell^{(1,2)}_{(\lambda)}$	$\mathcal{F}\ell^{(2,1)}_{(\lambda)}$	$\mathcal{F}\ell^{(1,1,1)}_{(\lambda)}$
(3)	pt	pt	pt	pt
$(2,1)$	pt	$\mathbb{P}^1$	$\mathbb{P}^1$	$\mathbb{P}^1 \cup_{\mathrm{pt}} \mathbb{P}^1$
$(1,1,1)$	pt	$\mathcal{F}\ell_3^{(1,2)}$	$\mathcal{F}\ell_3^{(2,1)}$	$\mathcal{F}\ell_3$

Compare

$\langle -,-\rangle$	$h_{(3)}$	$h_{(2,1)}$	$h_{(1,1,1)}$
$\tilde{H}_{(3)}$	1	1	1
$\tilde{H}_{(2,1)}$	1	$t+1$	$2t+1$
$\tilde{H}_{(1,1,1)}$	1	t^2+t+1	t^3+2t^2+2t+1

4.31. Remark. We can define generalized (parabolic) Springer fiber for any $x \in \mathbb{M}_n(\mathbb{C})$, not necessarily nilpotent. But by Jordan decomposition, the fiber is a product of nilpotent fibers. But when restricted to finite fields, which is not algebraically closed, this reduction is no longer available. The computation is more subtle, see [31][Prop. 2.11]. A nice Hall-algebraic treatment can be found in [28, §2].

4.32. Example. One can compute the transformed (resp., modified) Hall–Littlewood functions H_λ (resp., $\tilde{H}_\lambda$) in SageMath.

```
K = FractionField(QQ["t"]); t= K.gen()
Sym = SymmetricFunctions(K)
H = Sym.macdonald(0,t).H(); # H(t)=H(q=0,t)
Ht = Sym.macdonald(0,t).Ht(); # H~(t)=H~(q=0,t)
m = Sym.monomial(); h = Sym.homogeneous(); e = Sym.elementary();
     s = Sym.Schur()
ptt3 = list(Partitions(3))
print(table([[ Ht[a].scalar(e[b]) for b in ptt3] for a in ptt3],
             header_row=ptt3,
             header_column=["<H~,e>"]+ptt3,
             frame = True))
print(table([[ Ht[a].scalar(h[b]) for b in ptt3] for a in ptt3],
             header_row=ptt3,
             header_column=["<H~,h"]+ptt3,
             frame = True))
print(table([[ Ht[a].scalar(s[b]) for b in ptt3] for a in ptt3],
             header_row=ptt3,
```

```
            header_column=["<H~,s>"]+ptt3,
            frame = True))
```

5. MACDONALD POLYNOMIALS (I)

— **REFERENCE.** The main references are

[9] M. Haiman. Cherednik algebras, Macdonald polynomials and combinatorics.
[23] I. G. Macdonald. Affine Hecke algebras and orthogonal polynomials. Cambridge tracts in mathematics.

See also [22, 18]. For a historical introduction, see [8].

— **TYPE-FREE DEFINITION.** Again, we take $\Lambda = P$ to be the weight lattice, or any sub-lattice containing root lattice Q.

Recall that finite Hecke algebra $\mathcal{H}_W$ acts on the Laurent polynomial ring $\mathbb{Q}_t[\Lambda] = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}(t) \cdot e^{\lambda}$ via Demazure–Lusztig operators. We hope to define its affine version.

5.1. Demazure–Lusztig operators.

We define **Demazure–Lusztig operators** over the Laurent polynomial ring $\mathbb{Q}_{q,t}[\Lambda] = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}(q,t)\cdot e^{\lambda}$ for $i \in I \cup \{0\}$ by

$$T_i = ts_i + \frac{t-1}{e^{\alpha_i}-1}(s_i - \mathrm{id}) = \frac{1-t}{e^{\alpha_i}-1}\mathrm{id} + \frac{te^{\alpha_i}-1}{e^{\alpha_i}-1}s_i.$$

We need to justify the notations for $i = 0$.

Action. Recall the affine Weyl group $W_a = W \ltimes Q^\vee$ acts on $\Lambda \oplus \mathbb{Z}\delta$ by

$$wt_\beta(\alpha + k\delta) = w\alpha + (k - \langle \beta, \alpha\rangle)\delta.$$

By denoting $q = e^{\delta}$, this makes sense of

$$wt_\beta : q^k e^{\alpha} \mapsto q^{k-\langle\beta,\alpha\rangle} e^{w\alpha}.$$

Simple Roots. We have

$$\alpha_0 = -\theta + \delta, \qquad \theta = \text{highest root}.$$

This makes sense of

$$e^{\alpha_0} = qe^{-\theta}.$$

5.2. Example. Let us compute the SL_2-case. For λ dominant, with $\langle\lambda, \alpha^\vee\rangle = m$, we have

$$s_0\lambda = -\lambda + m\delta, \qquad s_0 e^\lambda = q^m e^{-\lambda} = q^m e^{\lambda - m\alpha}.$$

Then

$$\begin{array}{ccccc} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{array} \xmapsto{T_0} \begin{array}{ccccc} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ q^m t & q^{m-1}(t-1) & \cdots & q(t-1) & t-1 \\ 0 & q^{-1}(1-t) & \cdots & q^{-m+1}(1-t) & q^{-m} \end{array}$$

Compare with:

$$\begin{array}{ccccc} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{array} \xmapsto{T} \begin{array}{ccccc} e^{-\lambda} & e^{-\lambda+\alpha} & \cdots & e^{\lambda-\alpha} & e^{\lambda} \\ 1 & 1-t & \cdots & 1-t & 0 \\ t-1 & t-1 & \cdots & t-1 & t \end{array}$$

5.3. Example. For example, in SL_3:

$$\begin{array}{c} 0 \quad 0 \quad 0 \quad 1 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 1 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \end{array} \xmapsto{T_0} \begin{array}{c} 0 \quad q^{-3} \quad 0 \quad t-1 \\ 0 \quad -tq^{-2}+q^{-2} \quad 0 \quad qt-q \quad 0 \\ 0 \quad -tq^{-1}+q^{-1} \quad 0 \quad q^2t-q^2 \quad 0 \quad 0 \\ 0 \quad 0 \quad q^3t-q^3 \quad 0 \quad 0 \\ 0 \quad q^4t-q^4 \quad 0 \quad 0 \\ q^5t \quad 0 \quad 0 \end{array}$$

5.4. Bernstein elements. For a dominant coweight β, the length of its translation

$$\ell(t_\beta) = 2\langle\rho, \beta\rangle$$

which is linear in β. So for dominant β_1, β_2, we have

$$T_{t_{\beta_1}} T_{t_{\beta_2}} = T_{t_{\beta_2}} T_{t_{\beta_1}} = T_{t_{\beta_1+\beta_2}}.$$

We define Y^β for all $\beta \in Q^\vee$ such that

$$\beta \text{ dominant} \Rightarrow Y^\beta = t^{-\langle\rho,\beta\rangle} T_{t_\beta} \qquad Y^{\beta_1}(Y^{\beta_2})^{-1} = Y^{\beta_1-\beta_2}.$$

This defines an action of $\mathbb{Q}_{q,t}[Q^\vee]$-action on $\mathbb{Q}_{q,t}[P]$.

5.5. Example. In SL_2, we have $t_{\alpha^\vee} = s_0 s$, so
$$Y^{\alpha^\vee} = t^{-1} T_0 T.$$
Let simply denote by $x = e^{\varpi}$. We can compute
$$\begin{aligned}
1 &\mapsto t \\
x &\mapsto q^{-1}t^{-1}x \\
x^{-1} &\mapsto qt\,x^{-1} + \left(t - 1 + q^{-1} - q^{-1}t^{-1}\right) x \\
x^{2} &\mapsto (-t + 1 - q^{-1} + q^{-1}t^{-1}) + q^{-2}t^{-1}\,x^{2} \\
x^{-2} &\mapsto q^{2}t\,x^{-2} + (qt - q + t - 1 - q^{-1}t + 2q^{-1} - q^{-1}t^{-1}) \\
&\quad + \left(t - 1 + q^{-2} - q^{-2}t^{-1}\right) x^{2}
\end{aligned}$$
This can be checked directly:

5.6. Lemma. There is a certain order over Λ such that
$$Y^{\beta}(e^{\lambda}) = q^{-\langle \beta, \lambda \rangle} t^{-\langle \beta, \rho_\lambda \rangle} e^{\lambda} + (\text{lower terms}),$$
where
$$\rho_\lambda = \frac{1}{2} \sum_{\alpha > 0} \begin{cases} \alpha, & \langle \alpha, \lambda \rangle > 0, \\ -\alpha, & \langle \alpha, \lambda \rangle \leq 0. \end{cases}$$
Equivalently, if λ is anti-dominant,
$$w \in W^{\lambda} \implies \rho_{w\lambda} = -w\rho.$$

5.7. Remark. For a proof, we refer reader to [18, §6]. The order can be chosen to be

$$\mu \leq \lambda \iff \begin{cases} \mu^+ >_{\mathrm{dom}} \lambda^+, & W\mu \neq W\lambda, \\ \mu \leq_{\mathrm{dom}} \lambda, & W\mu = W\lambda. \end{cases}$$

where μ^+ is the unique dominant weight in $W\mu$. Here is an example

$$\begin{array}{ccccccccccc}
 & & 1 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
2 & & 0 & & 0 & & 0 & & 0 & & 1' \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 3 & & 0 & & 2' & & &
\end{array}
\qquad
\begin{array}{c}
\text{any gray } < \\
< 0 < \text{both } 1 \ \& \ 1' < \\
< \text{both } 2 \ \& \ 2' < 3
\end{array}$$

Actually, this choice is too strong, it suffices to take the affine Bruhat order, see [9, Prop 5.15 & Prop 6.9].

5.8. Example. In SL_2, we should choose order to be

$$1 < x^2 < x^{-2} < x^4 < x^{-4} < \cdots,$$

$$x < x^{-1} < x^3 < x^{-3} < \cdots.$$

The Lemma says, when $m > 0$

$$\begin{aligned}
Y^{\alpha^\vee} 1 &\mapsto t \\
Y^{\alpha^\vee} x^m &\mapsto q^{-m} t^{-1} x^m + (\text{lower terms}) \\
Y^{\alpha^\vee} x^{-m} &\mapsto q^m t x^{-m} + (\text{lower terms}).
\end{aligned}$$

Written as matrices (over $\mathbb{Q}_{q,t}[x^2]$)

$$\begin{bmatrix} 1 \\ x^2 \\ x^{-2} \\ x^4 \\ x^{-4} \\ \vdots \end{bmatrix} \quad \begin{array}{c} \begin{bmatrix} 1 & x^2 & x^{-2} & x^4 & x^{-4} & \cdots \end{bmatrix} \\ \begin{bmatrix} t & & & & & \\ * & q^{-2}t^{-1} & & & & \\ * & * & q^2t & & & \\ * & * & * & q^{-4}t^{-1} & & \\ * & * & * & * & q^4t & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

This tells that the spectrum (eigenvalues) of $Y^{\alpha^\vee}$ is

$$\{t, q^{-2}t^{-1}, q^2t, q^{-4}t^{-1}, q^4t, \ldots\}.$$

Each eigenvalue has multiplicity 1. This is true for general root systems, and the eigenfunction, is the **Macdonald polynomial**.

5.9. Macdonald polynomials. The **non-symmetric Macdonald polynomial** $E_\lambda \in \mathbb{Q}_{q,t}[\Lambda]$ for any weight $\lambda \in \Lambda$ is the unique polynomial such that

$$Y^\beta(E_\lambda) = q^{-\langle\beta,\lambda\rangle}t^{-\langle\beta,\rho_\lambda\rangle}E^\lambda, \qquad E_\lambda = e^\lambda + (\text{other terms}).$$

In particular, we can decompose as a $\mathbb{Q}_{q,t}[Q^\vee]$-modules

$$\mathbb{Q}_{q,t}[P] = \bigoplus_{\lambda\in P} \mathbb{Q}(q,t)\cdot E_\lambda.$$

5.10. Example. For SL_2, let us determine the constant term in

$$E_2 = x^2 + C, \qquad C = (\text{constant}).$$

Recall that

$$\begin{aligned} Y^{\alpha^\vee}(x^2) &= (-t + 1 - q^{-1} + q^{-1}t^{-1}) + q^{-2}t^{-1}x^2 \\ Y^{\alpha^\vee}(1) &= t \end{aligned}$$

So we need

$$q^{-2}t^{-1}C = (-t + 1 - q^{-1} + q^{-1}t^{-1}) + tC.$$

Thus $C = \frac{-qt+q}{-qt+1}$, i.e.

$$E_2 = x^2 + \frac{-qt+q}{-qt+1}.$$

Similarly,

$$E_{-1} = x^{-1} + \frac{-t+1}{-qt+1}x.$$

5.11. Remark. The similarity of the coefficients are not a coincidence. Actually, we can extend the action of $\mathbb{Q}_{q,t}[Q^\vee]$ to an action of $\mathbb{Q}_{q,t}[P^\vee]$. This will gives more automorphisms, from the theory of extended affine Hecke algebra.

In type A, explicitly

$$(qx_1)^{\mu_n} E_{(\mu_1,\dots,\mu_n)}(x_2,\dots,x_n,q^{-1}x_1) = E_{\mu_n+1,\mu_1,\dots,\mu_{n-1}}(x_1,\dots,x_n),$$

see [12, (10)]. For example, by homogenization, we get for $n=2$

$$E_{(0,1)} = \left(\frac{-t+1}{-qt+1}\right)x_1 + x_2, \qquad E_{(2,0)} = x_1^2 + \left(\frac{-qt+q}{-qt+1}\right)x_1x_2.$$

— Type A.

5.12. Explicit computation. In type A, the (extended) affine Weyl group is

$$\tilde{S}_n = \left\{\text{bijection } \mathbb{Z} \xrightarrow{f} \mathbb{Z} : f(i+n) = f(i)+n\right\}.$$

The action is given by $f : x_i \mapsto x_{f(i)}$ where

$\cdots$	x_{-n+1}	$\cdots$	x_0	x_1	$\cdots$	x_n	x_{n+1}	$\cdots$	x_{2n}	$\cdots$
$\cdots$	qx_1	$\cdots$	qx_n	x_1	$\cdots$	x_n	$q^{-1}x_1$	$\cdots$	$q^{-1}x_n$	$\cdots$

In particular

$$(s_0f)(x_1,\dots,x_n) = f(qx_n,x_2,\dots,x_{n-1},q^{-1}x_1)$$
$$(\pi f)(x_1,\dots,x_n) = f(x_2,x_3,\dots,x_{n-1},q^{-1}x_1)$$

Moreover,

α_1	α_2	$\cdots$	α_{n-1}	α_0
x_1/x_2	x_2/x_3	$\cdots$	x_{n-1}/x_n	$x_n/x_{n+1} = qx_n/x_1$

We can check easily that

$$\pi \circ T_i \circ \pi^{-1} = T_{i+1} \qquad i \in \mathbb{Z}/n\mathbb{Z}.$$

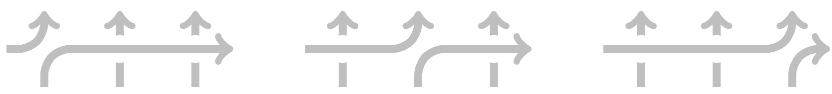
The element $Y_1, \dots, Y_n$ can be computed explicitly

$$\begin{aligned} Y_1 &= \pi H_{n-1} \cdots H_1, \\ Y_2 &= H_1^{-1} \pi H_{n-1} \cdots H_2, \\ \cdots &= \cdots \\ Y_n &= H_{n-1}^{-1} \cdots H_1^{-1} \pi. \end{aligned} \qquad \text{where } H_i = t^{1/2} T_i.$$

Note that

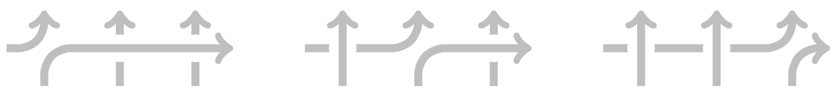
$$Y_1, Y_2, Y_3 \notin \mathbb{Q}_{q,t}[Q^\vee].$$

5.13. Example. Let $n = 3$. The following represents $H_{t_1}, H_{t_2}, H_{t_3}$



H_{t_1} H_{t_2} H_{t_3}

The following represents Y_1, Y_2, Y_3



Y_1 Y_2 Y_3

5.14. Example. In type A, we can compute E_α for a composition α via SageMath

```
K = LaurentPolynomialRing(QQ,["q","t"]).fraction_field();
q,t= K.gens();
from sage.combinat.sf.ns_macdonald import E
E([1,0,3],q,t)
```

See the documentation.

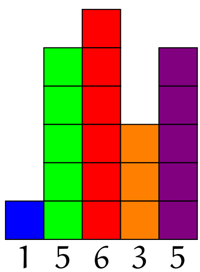
5.15. Explicit computation. In type A, explicitly,

$$(\rho_\lambda)_i = -\frac{n+1}{2} + 1 + \#\{j < i : \lambda_j \leq \lambda_i\} + \#\{j > i : \lambda_j < \lambda_i\}.$$

Define

$$\nu_\lambda(i) = 1 + \#\{j < i : \lambda_j \leq \lambda_i\} + \#\{j > i : \lambda_j < \lambda_i\}.$$

Actually ν_λ is a permutation. For example,



$$\nu_\lambda : \begin{array}{l} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 5 \\ 4 \mapsto 2 \\ 5 \mapsto 4 \end{array}$$

When λ is strictly dominant (increasing), $\nu_\lambda = \mathrm{id}$, when λ is anti-dominant (decreasing), $\nu_\lambda = w_0$. As a result,

$$Y_i E_\mu = q^{-\mu_i} t^{\frac{n+1}{2} - \nu_\lambda(i)} E_\mu$$

— Intertwine operators.

5.16. Remark. With some extra efforts, one can show

$$\boxed{\begin{array}{c}\text{algebra generated by}\\ T_i \quad (i \in I)\\ Y^\beta \quad (\beta \in Q^\vee)\end{array}} = \boxed{\begin{array}{c}\text{algebra generated by}\\ T_i \quad (i \in I \cup \{0\})\end{array}}$$

this is known as the affine Hecke algebra. Recall that the operator $X^\lambda : f \mapsto e^\lambda f$ for each $\lambda \in \Lambda$. From the algebra perspective, we get an algebra

$$\begin{array}{ccccc} & & \ddot{\mathcal{H}}_W & & \\ & \rotatebox{45}{$\subset$} & & \rotatebox{135}{$\supset$} & \\ & \widehat{\mathcal{H}}_W & & \widehat{\mathcal{H}}_{W^\vee} & \\ \rotatebox{45}{$\subset$} & & \rotatebox{135}{$\supset$}\ \rotatebox{45}{$\subset$} & & \rotatebox{135}{$\supset$} \\ \mathbb{Q}_{q,t}[Q^\vee] & & \mathcal{H}_W & & \mathbb{Q}_{q,t}[Q]. \end{array}$$

The algebra $\ddot{\mathcal{H}}_W$ is call the **double affine Hecke algebra**. We can replace $Q^\vee$ and Q by P and $P^\vee$. We have

$$H_i Y^\beta - Y^{s_i\beta} H_i = (t^{1/2} - t^{-1/2}) \frac{Y^{s_i\beta} - Y^\beta}{Y^{-\alpha_i^\vee} - 1},$$
$$H_i X^\lambda - X^{s_i\lambda} H_i = (t^{1/2} - t^{-1/2}) \frac{X^{s_i\lambda} - X^\lambda}{X^{\alpha_i^\vee} - 1}.$$

When $i \in I$, this is the Bernstein presentation. The case for $i = 0$ is highly non-trivial, it is equivalent to duality theorem, see [23, §3.5–3.7] and [9, §4.13]. In other word, we could change point of view

$$\boxed{\begin{array}{ll} X^\lambda: & \text{functions}\\ T_w: & \text{operators}\\ Y^\beta: & \text{operators}\end{array}} \xleftrightarrow{\text{duality}} \boxed{\begin{array}{ll} X^\lambda: & \text{operators}\\ T_w: & \text{operators}\\ Y^\beta: & \text{functions}\end{array}}$$

5.17. Intertwine operators. We denote the **intertwine operator** for $i \in I$ by

$$\tau_i = T_i + \frac{t-1}{Y^{-\alpha_i^\vee} - 1}.$$

This is a well-define operator since

$$\text{eigenvalues of } Y^{-\alpha_i^\vee} = \{q^{\langle \lambda, \alpha_i^\vee \rangle} t^{\langle \rho_\lambda, \alpha_i^\vee \rangle} : \lambda \in \Lambda\} \not\ni 1.$$

From the point of view of the above duality,

$$\tau_i = \frac{tY^{-\alpha_i^\vee} - 1}{Y^{-\alpha_i^\vee} - 1} \cdot s_i^Y \text{ viewed as an operator on } Y.$$

So we have

$$\tau_i Y^\lambda = Y^{s_i\lambda}\tau_i \qquad\qquad (\textbf{intertwine})$$

$$\underbrace{\tau_i\tau_j\cdots}_{m_{ij}} = \underbrace{\tau_j\tau_i\cdots}_{m_{ij}} \qquad (i \neq j) \qquad (\textbf{braid relations})$$

5.18. Theorem. If $s_i\lambda > \lambda$ for some $i \in I$, i.e. $s_i\lambda <_{\mathrm{dom}} \lambda$ equivalently $\langle \lambda, \alpha_i^\vee \rangle > 0$, then

$$E_{s_i\lambda} = \tau_i E_\lambda = \left(T_i + \frac{t-1}{q^{\langle \lambda, \alpha_i^\vee \rangle} t^{\langle \rho_\lambda, \alpha_i^\vee \rangle} - 1} \right) E_\lambda.$$

Actually, there is a version for $i = 0$. See [23, Coro 6.15]. One can take this as the definition of Macdonald polynomials, see [32, §3].

Sketch. Under the assumption, we have

$$\text{eigenvalue of } Y^\beta \text{ at } E_\lambda = \text{eigenvalue of } Y^{s_i\beta} \text{ at } E_{s_i\lambda}.$$

So intertwine operator τ_i must move E_λ to a scalar of $E_{s_i\lambda}$. Moreover, the behavior of the operator T_i tells the scalar is 1. $\square$

5.19. Example. Recall that $E_{(1,0)} = x_1$. We have $T_1(x_1) = x_2$. So

$$E_{(0,1)} = \left(T_1 + \frac{t-1}{qt-1}\right) E_{(1,0)} = x_2 + \frac{t-1}{qt-1} x_1.$$

In type A, with the automorphism, it is enough to determine all E_λ, see [12].

6. Macdonald Polynomials (II)

— **Symmetric Macdonald Polynomials.** Note that Y^β does not preserve the symmetric polynomials $\mathbb{Q}_{q,t}[P]^W$. But $\mathbb{Q}_{q,t}[Q^\vee]^W$ does.

6.1. Lemma. The action

$$\mathbb{Q}_{q,t}[Q^\vee] \quad \curvearrowright \quad \mathbb{Q}_{q,t}[P].$$

restricts to

$$\mathbb{Q}_{q,t}[Q^\vee]^W \quad \curvearrowright \quad \mathbb{Q}_{q,t}[P]^W.$$

Sketch. Recall the operator $C_{w_0} = \sum_{w \in W} T_w$. The lemma follows from the following two facts

$$\mathbb{Q}_{q,t}[P]^W = \text{image of } C_{w_0},$$
$$\mathbb{Q}_{q,t}[Q^\vee]^W \text{ commutes with } C_{w_0}. \quad \square$$

6.2. Macdonald polynomials. The **symmetric Macdonald polynomial** $P_\lambda \in \mathbb{Q}_{q,t}[\Lambda]^W$ for any dominant weight $\lambda \in \Lambda$ is the unique polynomial such that

$$f(Y)(P_\lambda) = f(q^{-\lambda}t^{-\rho})P^\lambda, \qquad P_\lambda = m_\lambda + (\text{other terms})$$

where $f(Y) \in \mathbb{Q}_{q,t}[Q^\vee]^W$ and $f(q^{-\lambda}t^{-\rho}) = f(Y)|_{Y^\beta \mapsto q^{-\langle\lambda,\beta\rangle}t^{-\langle\rho,\beta\rangle}}$.

6.3. Lemma. For a dominant λ, we have

$$P_\lambda \in \operatorname{span}(E_\mu : \mu \in W\lambda).$$

Moreover, up to a scalar, P_λ is the unique symmetric polynomial in the right-hand side.

Sketch. The right-hand side is exactly

$$\left\{g \in \mathbb{Q}_{q,t}[P] : f \in \mathbb{Q}_{q,t}[Q^\vee]^W \Rightarrow f(Y)g = f(q^{-\lambda}t^{-\rho})g\right\}.$$

Note that since f is W-invariant,

$$f(q^{-\lambda}t^{-\rho}) = f(q^{-w\lambda}t^{-\rho_{w\lambda}})$$

for any $w \in W$. The uniqueness follows from uniqueness in the definition. $\square$

6.4. Example. With this lemma, it is relatively easier to do computation. In SL_2, we have

$$E_{(1,0)} = x_1 \qquad E_{(0,1)} = \left(\frac{-t+1}{-qt+1}\right)x_1 + x_2.$$

Then obviously

$$P_{(1,0)} = x_1 + x_2.$$

Let us give another example.

$$E_{(2,0)} = x_1^2 + \left(\frac{-qt+q}{-qt+1}\right)x_1x_2$$
$$E_{(0,2)} = \left(\frac{-t+1}{-q^2t+1}\right)x_1^2 + \left(\frac{-qt+q-t+1}{-q^2t+1}\right)x_1x_2 + x_2^2.$$

As a result,

$$\begin{aligned} P_{(2,0)} &= \left(1 - \left(\frac{-t+1}{-q^2t+1}\right)\right)E_{(2,0)} + E_{(0,2)} \\ &= x_1^2 + \left(\frac{-qt+q-t+1}{-qt+1}\right)x_1x_2 + x_2^2. \end{aligned}$$

6.5. Remark. From two Lemmas above, it is not hard to show

$$P_\lambda = \frac{1}{W_\lambda(t)}\mathrm{Symm}(E_\lambda)$$

when λ dominant. The reader might ask the relation between P_λ and Hall–Littlewood polynomial. Actually, for dominant λ, we have

$$\begin{aligned} E_\lambda|_{q=0} &= e^\lambda, \\ P_\lambda|_{q=0} &= \text{Hall–Littlewood polynomial.} \end{aligned}$$

Our definition behavior very bad under the specialization $q = 0$. This specialization will make many eigenvalues to be the same, so the polynomials are no longer distinguished by $\mathbb{Q}_{q,t}[Q^\vee]$. While the characterization via the **Cherednik inner product** behavior better under specializations, see [9, Thm 6.6].

6.6. Remark. The algebra generated by

$$\begin{array}{ll} f(Y) & f \in \mathbb{Q}_{q,t}[Q^\vee]^W \\ g(X) & g \in \mathbb{Q}_{q,t}[Q]^W \end{array}$$

is called the spherical double affine Hecke algebra. We can replace $Q^\vee$ and Q by P and $P^\vee$. Since $\mathbb{Q}_{q,t}[Q]^W$ is the image of C_{w_0}, the algebra is isomorphic to $C_{w_0}\ddot{\mathcal{H}}_W C_{w_0}$.

— Type A. In type A a new feature is

$$Y_1 + \cdots + Y_n$$

already has distinct eigenvalues.

6.7. Explicit. We have

$$f(Y)P_\mu = (*)P_\mu$$

where

$$(*) = f\left(q^{-\mu_1}t^{-\frac{n+1}{2}+1}, q^{-\mu_2}t^{-\frac{n+1}{2}+2}, \cdots, q^{-\mu_n}t^{-\frac{n+1}{2}+n}.\right).$$

6.8. Macdonald operators. In type A, the action of

$$e_k(Y) = \sum_{1 \le i_1 < \cdots < i_k \le n} Y_{i_1} \cdots Y_{i_k}$$

can be computed explicitly as a difference operator

$$f(x_1, \ldots, x_n) \mapsto \sum_{I \in \binom{[n]}{k}} \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{t^{1/2} x_i - t^{-1/2} x_j}{x_i - x_j} \right) f|_{x_i \mapsto q^{-1} x_i \, \forall i \in I}.$$

6.9. Example. Let us illustrate the computation for $k = 1$ and $n = 2$. We use the following identity

$$(Y_1 + Y_2) C_{w_0} = C_{w_0} Y_2 C_{w_0} = t^{-1/2}\, C_{w_0} \pi C_{w_0}.$$

So for a symmetric polynomial f,

$$\begin{aligned}
(Y_1 + Y_2) f = t^{-1/2} C_{w_0} \pi f &= t^{-1/2}\, \mathrm{Symm}(f(x_2, q^{-1} x_1)) \\
&= t^{-1/2} \left(\frac{1 - t x_2/x_1}{1 - x_2/x_1} f(x_2, q^{-1} x_1) + \frac{1 - t x_1/x_2}{1 - x_1/x_2} f(x_1, q^{-1} x_2) \right) \\
&= \frac{t^{-1/2} x_1 - t^{1/2} x_2}{x_1 - x_2} f(q^{-1} x_1, x_2) + \frac{t^{-1/2} x_2 - t^{1/2} x_1}{x_2 - x_1} f(x_1, q^{-1} x_2).
\end{aligned}$$

For example, when $f = E_{(1,0)} = x_1 + x_2$,

$$\begin{aligned}
(Y_1 + Y_2)(x_1 + x_2) &= \frac{t^{-1/2} x_1 - t^{1/2} x_2}{x_1 - x_2} (q^{-1} x_1 + x_2) + \frac{t^{-1/2} x_2 - t^{1/2} x_1}{x_2 - x_1} (x_1 + q \\
&= (t^{-1/2} q^{-1} + t^{1/2})(x_1 + x_2).
\end{aligned}$$

6.10. Limit. It is not easy to show P_λ is stable from the definition. The trick is, we should stabilizer the operator

$$Y_1 + \cdots + Y_n.$$

Its eigenvalue at P_μ is

$$t^{-\frac{n-1}{2}} \left(q^{-\mu_1} t^0 + \cdots + q^{-\mu_n} t^{n-1} \right).$$

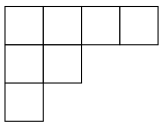
Then we modify the operator

$$\frac{1}{q^{-1}-1}\left(t^{\frac{n-1}{2}}Y_1+\cdots+Y_n-(t^1+\cdots+t^n)\right).$$

Its eigenvalue at P_μ is

$$\frac{q^{-\mu_1}-1}{q^{-1}-1}t^0+\cdots+\frac{q^{-\mu_n}-1}{q^{-1}-1}t^{n-1}=\sum_{(i,j)\in\mu}\bar{q}^{j-1}t^{i-1}.$$

For example



$$\sum\left\{\begin{matrix}1 & \bar{q} & \bar{q}^2 & \bar{q}^3\\ t & \bar{q}t & & \\ t^2 & & & \end{matrix}\right\}.$$

It is not hard to show the modified operator is stable, see [25]. In particular, P_λ is a symmetric function.

6.11. Remark. The algebra generated by the limit of the operators, and all symmetric functions are call the **elliptic Hall algebra**. The name comes from the Hall algebra of coherent sheaves over elliptic curves over finite fields. It is a coincidence (maybe not a coincidence) that they are the same algebra, see [36].

6.12. Example. Here is the SageMath code for computing P_λ in type A.

```
K = LaurentPolynomialRing(QQ,["q","t"]).fraction_field();
q,t= K.gens();
Sym = SymmetricFunctions(K)
P = Sym.macdonald().P()
P([3,1,0]).expand(3)
```

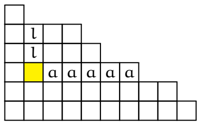
See the documentation.

6.13. Remark. An important characterization of P_λ is via the q,t-deformed Hall inner product, see [25, Chapter VI]. Note that it is different from the Cherednik inner product, see [27, §1.12.(e)].

6.14. Integral forms. There are different functions named by Macdonald polynomials. One defines

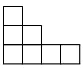
$$J_\lambda = P_\lambda \cdot \prod_{\square\in\lambda}(1 - q^{a(\square)}t^{l(\square)+1})$$

where a and l are the arm and length statistics.



$$a(\square) = 5$$
$$l(\square) = 2$$
$$1 - q^5t^3.$$

For example,

$\lambda =$ ,

$$\prod \left\{ \begin{array}{llll} 1-t & & & \\ 1-qt^2 & 1-t & & \\ 1-q^3t^3 & 1-q^2t^2 & 1-qt & 1-t \end{array} \right\}$$

The function J is called an integral form since the coefficients are polynomial in q,t.

6.15. Transformed. We define

$$H_\lambda = J_\lambda\left[\frac{Z}{1-t}\right] = J_\lambda\begin{pmatrix} x_1 & tx_1 & t^2x_1 & \cdots \\ x_2 & tx_2 & t^2x_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Note that

$$p_k\left[\frac{Z}{1-t}\right] = p_k + t^kp_k + t^{2k}p_k + \cdots = \frac{1}{1-t^k}p_k.$$

In particular, $f \mapsto f[Z/(1-t)]$ is invertible, its inverse is denoted by $f[Z(1-t)]$. More general, these are all special case of **plethysm**.

6.16. Modified. We denote **modified Macdonald polynomial** by

$$\tilde{H}_\lambda = t^{n(\lambda)} H_\lambda|_{t\mapsto t^{-1}}.$$

6.17. Example. Here are some examples. For $\lambda = (1,1)$,

$$\begin{aligned}
P_{(1,1)} &= m_{(1,1)},\\
J_{(1,1)} &= (t-1)(t^2-1)m_{(1,1)} = (t-1)(t^2-1)\tfrac{p_1^2-p_2}{2}\\
H_{(1,1)} &= \tfrac{t+1}{2}p_1^2 + \tfrac{t-1}{2}p_2 = (t+1)m_{(1,1)} + tm_{(2)}\\
\tilde{H}_{(1,1)} &= (t+1)m_{(1,1)} + m_{(2)} = ts_{(1,1)} + s_{(2)}.
\end{aligned}$$

For $\lambda = (2,0)$,

$$\begin{aligned}
P_{(2,0)} &= m_{(2)} + \frac{(q+1)(t-1)}{qt-1}m_{(1,1)},\\
J_{(2,0)} &= (t-1)(qt-1)m_{(2)} + (q+1)(t-1)^2 m_{(1,1)}\\
&= \tfrac{(q+1)(t-1)^2}{2}p_1^2 + \tfrac{(t^2-1)(q-1)}{2}p_2,\\
H_{(2,0)} &= \tfrac{q+1}{2}p_1^2 - \tfrac{q-1}{2}p_2 = (q+1)m_{(1,1)} + m_{(2)}\\
\tilde{H}_{(2,0)} &= (q+1)m_{(1,1)} + m_{(2)} = qs_{(1,1)} + s_{(2)}.
\end{aligned}$$

6.18. Example. The polynomials are all available in SageMath

```
K = LaurentPolynomialRing(QQ,["q","t"]).fraction_field();
q,t= K.gens();
Sym = SymmetricFunctions(K)
J = Sym.macdonald().J()
P = Sym.macdonald().P()
H = Sym.macdonald().H()
Ht = Sym.macdonald().Ht()
p = Sym.power(); m = Sym.monomial() ; s = Sym.Schur()
s(H([3,1]))
```

6.19. Remark. There are many reasons why we prefer $\tilde{H}_\lambda$. One reason is the q,t-symmetry

$$\tilde{H}_\lambda|_{q\leftrightarrow t} = \tilde{H}_{\lambda'}.$$

For example,

expand $\to$	$s_{(1,1,1)}$	$s_{(2,1)}$	$_{(3)}$
$\tilde{H}_{(1,1,1)}$	t^3	t^2+t	1
$\tilde{H}_{(2,1)}$	qt	$q+t$	1
$\tilde{H}_{(3)}$	q^3	q^2+q	1

This symmetry gives a new type of characterization of $\tilde{H}_\lambda$. For properties of $\tilde{H}_\lambda$, see [10, §3.5].

7. Background and Applications

— Affine Demazure character. [34] and [15]

7.1. Affine Lie algebra. For a simple Lie algebra $\mathfrak{g}$, there are many version of affine Lie algebras, summarized in the following diagram

$$\begin{tikzcd}[column sep=small] & \mathfrak{g}_{\mathrm{KM}} \arrow[dr] & \\ \hat{\mathfrak{g}} \arrow[ur, hook] \arrow[dr] & & L\mathfrak{g} \oplus \mathbb{C}d \\ & L\mathfrak{g} = \mathfrak{g}[t^{\pm 1}] \arrow[ur, hook] & \end{tikzcd} \quad \begin{array}{c}\text{with}\\ \text{Cartan}\\ \text{part}\end{array} \quad \begin{tikzcd}[column sep=small] & \mathfrak{h}_{\mathrm{KM}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d \arrow[dr] & \\ \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \arrow[ur, hook] \arrow[dr] & & h \oplus \mathbb{C}d \\ & \mathfrak{h} \arrow[ur, hook] & \end{tikzcd}$$

We have

$$\mathfrak{h}^*_{\mathrm{KM}} = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta.$$

Such that

$$\begin{array}{ccccccc} \mathfrak{h}_{\mathrm{KM}} & = & \overbrace{\mathbb{C}c \oplus \mathfrak{h}}^{\text{coroots}} & \oplus & \mathbb{C}\partial \\ \updownarrow & & \updownarrow \quad\quad \updownarrow & & \updownarrow \\ \mathfrak{h}^*_{\mathrm{KM}} & = & \underbrace{\mathbb{C}\Lambda_0 \oplus \mathfrak{h}^*}_{\text{weights}} & \oplus & \mathbb{C}\delta \end{array} \quad \text{dual} \quad \begin{array}{c} \langle \Lambda_0, c\rangle = m = \dfrac{\langle\theta,\theta\rangle}{2}, \\ \langle \delta, d\rangle = 1. \end{array}$$

$$\underbrace{\mathfrak{h}^* \oplus \mathbb{C}\delta}_{\text{roots}}$$

We can present $\mathfrak{g}_{\mathrm{KM}}$ by Chevalley generators

$$h_0 = h_{\theta^\vee} + \frac{c}{m}, \qquad e_0 = f_\theta \otimes t, \quad f_0 = e_\theta \otimes t^{-1}.$$

7.2. Example. For example, in SL_3,

$$e_1 = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 & & \\ & 0 & \\ t & & 0 \end{bmatrix}$$

7.3. Representation theory. The representation theory of them are different.

Loop algebra. We have an equivalence

$$\text{reps of } L\mathfrak{g} = \begin{array}{c}\text{quasi-coherent sheaves of}\\ \mathfrak{g}\text{-reps over } \operatorname{Spec}\mathbb{C}[t^{\pm 1}]) = \mathbb{C}^\times.\end{array}$$

For each $z \in \mathbb{C}^\times$ and a representation V, we can construct a representation

$$V(z) = V[t^{\pm 1}]/(t - z).$$

For example, the irreducible finite-dimensional representation of $L\mathfrak{g}$ takes the form

$$V_1(z_1) \otimes \cdots \otimes V_n(z_n),$$

where $V_1, \ldots, V_n$ are irreducible representation of $\mathfrak{g}$ and $z_1, \ldots, z_n$ are distinct points.

Introduce d. The d acts by the differential operator $z\frac{d}{dz}$

$$d : z^n \mapsto nz^n.$$

So

$$\text{reps of } L\mathfrak{g} \oplus \mathbb{C}d = \begin{array}{c}\text{D-module of } \mathfrak{g}\text{-reps}\\ \text{over } \operatorname{Spec}\mathbb{C}[t^{\pm 1}]) = \mathbb{C}^\times.\end{array}$$

In particular, there is no finite-dimensional representation except the trivial representation.

Introduce c. The element c is a central element to distinguish affine coroot in $L\mathfrak{g}$. For a representation V, if c acts as a constant ℓ, we call ℓ its **level**. Equivalently,

$$\text{level } \ell \text{ reps} = \text{resp of } \mathfrak{g}_{\mathrm{KM}}/\langle c - \ell\rangle.$$

As a Kac–Moody algebra, $\mathfrak{g}_{\mathrm{KM}}$ has highest weight module. There are some terminologies [16, §3.6,§9.2]

(D) diagonalizable module: we can decompose into weight space.

(I) integrable module: the action of Chevalley generators e_i, f_i ($i \in I \cup \{0\}$) are locally nilpotent.
(H) highest weight module: there exists a highest weight $\nu \in V$ generating V.

The irreducible diagonalizable integrable highest weight modules are classified by integral dominant weights [16, Lemma 10.1]. We denote $L(\Lambda)$ the module of highest weight Λ.

Affine Lie algebra. The affine Lie algebra $\hat{\mathfrak{g}}$ is the subalgebra of $\mathfrak{g}_{\mathrm{KM}}$ without d. We can similarly define **level**, **diagonalizable modules**, **integrable modules**. Note that a representation is finite-dimensional, then its level is 0.

7.4. Example. For $\mathfrak{g} = \mathfrak{gl}_n$, there is a famous representation called **Fock space**

$$\mathcal{F} = \Lambda^{\infty/2}\mathbb{C}^{\oplus n}[t^{\pm 1}] = \bigoplus_{\lambda \text{ partitions}} \mathbb{C} \cdot |\lambda\rangle.$$

A good survey of the Fock space can be found in [20]. The action of Chevalley operator can be described in terms of combinatorics of partitions. The submodule generated by $|\varnothing\rangle$ is the level-1 representation $L(\Lambda_0)$. The constriuction is the follwoing.

- We label the "n-residue" of each box on the partitions. For example,

$$n = 3 \qquad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

- Then we consider the graph of all partitions, with an arrow

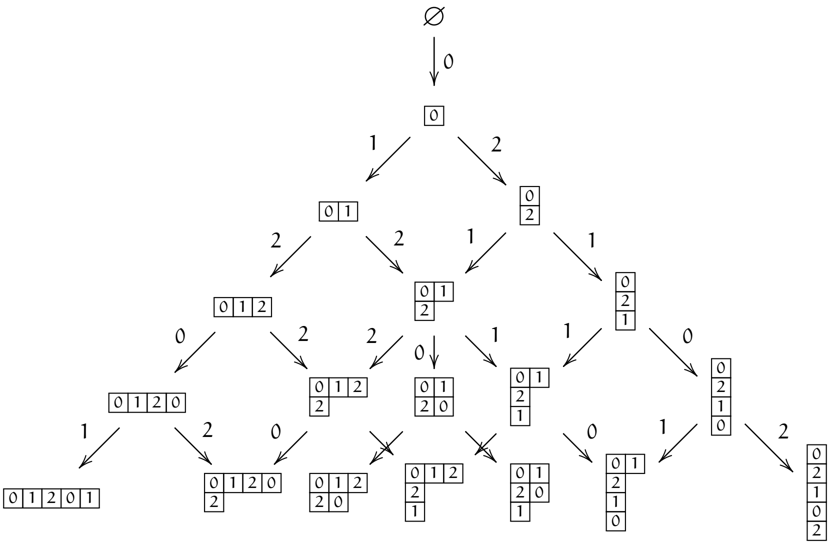
$$\lambda \xrightarrow{i} \mu, \qquad \mu/\lambda = \boxed{i}\ .$$

Then

$$f_i|\lambda\rangle = \sum_{\lambda \xrightarrow{i} \mu} |\mu\rangle, \qquad e_i|\mu\rangle = \sum_{\lambda \xrightarrow{i} \mu} |\lambda\rangle$$

$$h_i = \#\{\mu : \lambda \xrightarrow{i} \mu\} - \#\{\mu : \mu \xrightarrow{i} \lambda\}.$$

For example when $n = 3$



Actually, level 1 is saying that

$$\#\{\text{outer corner}\} = \#\{\text{inner corner}\} + 1.$$

7.5. Demazure module. For an integral dominant weight Λ and $\Lambda' = w\Lambda$ for $w \in W_a$, the affine Demazure module is

$$D(\Lambda') = \begin{array}{c}\text{the } \mathfrak{b}^+_{\mathrm{KM}}\text{-submodule of } L(\Lambda) \\ \text{generated by a vector of weight } \Lambda'.\end{array}$$

7.6. Theorem. Assume the Lie algebra is of type ADE. For $\Lambda \in W_a\Lambda_0$,

$$e^{-\Lambda_0}\chi(D(\Lambda)) = q^m E_\lambda|_{t\mapsto 0, q\mapsto q^{-1}}$$

if we can write

$$\Lambda = \Lambda_0 + \lambda + m\delta.$$

7.7. Example. Consider the case $\mathfrak{gl}_3$. Recall

$$e^{\alpha_i} = x_i/x_{i+1} \quad (i \in I), \qquad e^{\alpha_0} = qx_n/x_1.$$

- The trivial case, $\Lambda = \Lambda_0$,

$$D(\Lambda_0) = \mathrm{span}(|\varnothing\rangle), \qquad \chi(D(\Lambda_0)) = e^{\Lambda_0}.$$

This agrees with $E_0 = 1$.

- Consider $\Lambda = \Lambda_0 - \alpha_0$. Then $D(\Lambda)$ is spanned by

$$\begin{array}{cc} |\varnothing\rangle & \boxed{0} \\ \Lambda_0 & \Lambda_0 - \alpha_0 \end{array}$$

So

$$e^{-\Lambda_0}\chi(V(\Lambda)) = 1 + e^{-\alpha_0}$$
$$E_{-\alpha_0}(t=0) = E_{(1,0,-1)}(t=0) = x_0/x_2 + q$$

- Consider $\Lambda = \Lambda_0 - \alpha_0 - \alpha_1$. Then $D(\Lambda)$ is spanned by

$$\begin{array}{ccc} |\varnothing\rangle & \boxed{0} & \boxed{0}\boxed{1} \\ \Lambda_0 & \Lambda_0 - \alpha_0 & \Lambda_0 - \alpha_0 - \alpha_1 \end{array}$$

So

$$e^{-\Lambda_0}\chi(D(\Lambda)) = 1 + e^{-\alpha_0} + e^{-\alpha_0-\alpha_1}$$
$$E_{-\alpha_0-\alpha_1}(t=0) = E_{(0,1,-1)}(t=0) = x_0/x_2 + x_1/x_2 + q$$

- Consider $\Lambda = \Lambda_0 - \alpha_0 - \alpha_1 - 2\alpha_2$. Then $D(\Lambda)$ is spanned by

$$\begin{array}{cccc} |\varnothing\rangle & \boxed{0} & \boxed{0}\boxed{1} & \begin{array}{c}\boxed{0}\\\boxed{2}\end{array} \\ \Lambda_0 & \Lambda_0 - \alpha_0 & \Lambda_0 - \alpha_0 - \alpha_1 & \Lambda_0 - \alpha_0 - \alpha_2 \end{array}$$

$$\begin{array}{cc} \boxed{0}\boxed{1}\boxed{2} + \begin{array}{l}\boxed{0}\boxed{1}\\\boxed{2}\end{array} & \begin{array}{l}\boxed{1}\boxed{1}\boxed{2}\\\boxed{2}\end{array} \\ \Lambda_0 - \alpha_0 - \alpha_1 - \alpha_2 = \Lambda_0 - \delta & \Lambda_0 - \alpha_0 - \alpha_1 - 2\alpha_2 \end{array}$$

Compare:

$$\begin{aligned} E_{-\alpha_0-\alpha_1-2\alpha_2} &= E_{(0,-1,1)} \\ &= x_0/x_2 + x_1/x_2 + x_0/x_1 + (q+1) + x_2/x_1. \end{aligned}$$

Note: SageMath does not compute correctly the polynomial $E_{(1,0,-1)}$ since it contains negative indices. We should make use of

$$E_{(1,0,-1)} = E_{(2,1,0)}/(x_1x_2x_3)$$

to compute it.

7.8. Remark. There are other representation-theoretic meanings of E_λ with t specialized, see [2, 21, 5, 4, 17].

— Hilbert Schemes. [10]

7.9. Symmetric space. Let us consider the symmetric space for a space X

$$S^nX = X^n/S_n.$$

That is, a point of S^nX is an S_n-orbit of n-points over X, i.e. multi-set of order n of points of X. We mean,

$$\text{multi-set} = \text{set but multiplicity allowed.}$$

It is convenient to denote an element

$$S_n\text{-orbit of } (x_1, \ldots, x_n) = [x_1] + \cdots + [x_n] \in \mathbb{Z}^{\oplus X}.$$

7.10. Example. We have

$$S^n\mathbb{C} = \mathbb{C}^n$$

Precisely, we can consider

$$e : \mathbb{C}^n \to \mathbb{C}^n \qquad (x_1, \ldots, x_n) \mapsto (e_1(x), \ldots, e_n(x))$$

where e_i is the i-th elementary symmetric polynomial. This induces the isomorphism. We can understand this isomorphism by

$$S^n\mathbb{C} \xleftarrow{\text{roots}} \left\{ \begin{array}{c} \text{polynomials of the form} \\ x^n + (\text{lower terms}) \end{array} \right\} \xrightarrow{\text{coefficients}} \mathbb{C}^n.$$

From the algebraic geometry point of view, this says

$$\mathbb{C}[S^n\mathbb{C}] = \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[\mathbb{C}^n].$$

7.11. Diagonal invariant. Let us consider the space

$$S^n\mathbb{C}^2 = (\mathbb{C}^2)^n/S_n.$$

In other word,

$$\mathbb{C}[S^n\mathbb{C}^2] = \mathbb{C}[x_1, \cdots, x_n, y_1, \ldots, y_n]^{S_n}.$$

7.12. Example. When $n = 1$, obviously, $S^n\mathbb{C}^2 = \mathbb{C}^2$.

7.13. Example. When $n = 2$, let us study $\mathbb{C}[x_1, x_2, y_1, y_2]^{S_2}$. We first rewrite

$$\mathbb{C}[x_1, x_2, y_1, y_2] = R[x, y] \qquad \begin{cases} R = \mathbb{C}[x_1 + x_2, y_1 + y_2], \\ x = x_1 - x_2, \quad y = y_1 - y_2. \end{cases}$$

Then

$$\begin{aligned} \mathbb{C}[x_1, x_2, y_1, y_2]^{S_2} &= \{f(x, y) \in R[x, y] : f(-x, -y) = f(x, y)\} \\ &= \operatorname{span}(x^a y^b : a + b \in 2\mathbb{Z}). \end{aligned}$$

Let

$$a = xy, \qquad b = x^2, \qquad c = y^2.$$

Then

$$\mathbb{C}[x_1, x_2, y_1, y_2]^{S_2} = R[a, b, c]/(bc - a^2).$$

So

$$S^n\mathbb{C}^2 = \mathbb{C}^2 \times \{(a, b, c) \in \mathbb{C}^3 : bc = a^2\}$$

is a product of $\mathbb{C}^2$ and a a singular quadrics (a cone). Moreover, one can check the singular locus $\mathbb{C}^2 \times \{(0, 0, 0)\}$ corresponds to the diagonal

$$\{(p, p) : p \in \mathbb{C}^2\}/S_2.$$

7.14. Hilbert schemes. Let us define

$$H_n = \mathrm{Hilb}^n \mathbb{C}^2 = \{I \trianglelefteq \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n\}.$$

We have the **Hilbert–Chow morphism**

$$\pi : H_n \longrightarrow S^n\mathbb{C}^2, \qquad I \mapsto [\mathbb{C}[x,y]/I].$$

Algebraically, over the n-dimensional vector space $\mathbb{C}[x,y]/I$, the operators of multiplying x and y are commutative, thus can be upper triangulated simuteniously:

$$\begin{bmatrix} x_1 & \cdots & * \\ & \ddots & \vdots \\ & & x_n \end{bmatrix} \qquad \begin{bmatrix} y_1 & \cdots & * \\ & \ddots & \vdots \\ & & y_n \end{bmatrix}$$

The image of I is the multi-set of $\{\!\{(x_i, y_i)\}\!\}$.

7.15. Example. When $n = 1$. An ideal $I \in H_1$ must be a maximal ideal, so it corresponds to a point. We have $H_1 = \mathbb{C}^2$.

7.16. Example. When $n = 2$. Let us describe the fiber of π.

- If $\pi(I) = [p] + [q]$ for $p \neq q \in \mathbb{C}^2$, then the ideal I can only be the product $\mathfrak{m}_p\mathfrak{m}_q$. So the choice of I is a single point.
- If $\pi(I) = 2[p]$ for $p \in \mathbb{C}^2$, then the ideal $I \subseteq \mathfrak{m}_x^2$. Let us compute for $x = 0$ for simplicity. Then the choice of I is

  $$\{I \trianglelefteq R : \dim_{\mathbb{C}} R/I = 2\}$$

  where $R = \mathbb{C}[x,y]/\langle x^2, xy, y^2\rangle$ has dimension 3. So the ideal I must be generated by a nonzero element of R annihilated by x and y. That is

  $$I = \langle f \rangle = \mathbb{C}f, \qquad f \in \mathrm{span}(x,y).$$

  So the choice of I forms a projective line $\mathbb{P}^1$.

7.17. Theorem. The Hilbert scheme H_n is a smooth variety and the Hilbert–Chow morphism is a resolution of singularities.

7.18. Remark. In general, the Hilbert schemes is defined for a specific Hilbert polynomials, our definition is for the Hilbert polynomials of n-points. General Hilbert schemes are not smooth. The smoothness is truly a feature of dimension 2.

7.19. Punctured Hilbert scheme. We define the **punctured Hilbert scheme** X_n to be the reduced fiber product

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "\rho"'] & (\mathbb{C}^2)^n \arrow[d] \\ H_n \arrow[r, "\pi"] & S^n\mathbb{C}^2 \end{tikzcd} \qquad X_n = \big\{(I, p_1, \ldots, p_n) : \pi(I) = [p_1] + \cdots + [p_n]\big\}.$$

Note that X_n admits an action of S_n by permuting the points.

7.20. Garsia–Haiman algebra. Each partition $\mu \vdash n$ defines a monomial ideal I_μ in $\mathbb{C}[x, y]$, for example

$$\begin{array}{l} \square \\ \square\square \\ \square\square\square\square \end{array} \qquad \operatorname{span}\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ y^3 & xy^3 & x^2y^3 & x^3y^3 & x^4y^3 & \cdots \\ \square & xy^2 & x^2y^2 & x^3y^2 & x^4y^2 & \cdots \\ \square & \square & x^2y & x^3y & x^4y & \cdots \\ \square & \square & \square & \square & x^4 & \cdots \end{pmatrix}.$$

We define the bi-graded algebra with S_n-action

$$R_\mu = \mathbb{C}[\rho^{-1}(I_\mu)].$$

Note that here we view $\rho^{-1}(I_\mu)$ as a scheme, not a variety.

7.21. Example. Let us consider

$$I_{(n)} = \langle x^n, y\rangle \in H_n.$$

As a variety

$$\rho^{-1}(I_{(n)}) = \{(I_{(n)}, 0, \ldots, 0)\} \cong \mathrm{pt}.$$

But the key point is, we are considering the fiber as a scheme. So we need to describe the local structure near the point $I_{(n)}$. For $e = (e_1, \dots, e_n)$ and $c = (c_1, \dots, c_n)$, we denote

$$I_{e,c} = \left\langle \begin{matrix} x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n \\ y - (c_1 x^{n-1} + \cdots + c_n) \end{matrix} \right\rangle.$$

Note that $I_{e,c}$'s are different if (e, c)'s are different and $\dim H_n = 2n$, this could be viewed as a local chart near $I_{(n)}$ of H_n. Note that

$$\pi(I_{e,c}) = [(x_1, y_1)] + \cdots + [(x_n, y_n)]$$

with

$$\begin{aligned} \{\!\{x_1, \dots, x_n\}\!\} &= \text{solutions of } x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n \\ \text{each } y_i &= c_1 x_i^{n-1} + \cdots + c_n. \end{aligned}$$

The first equality is equivalent to

$$e_k(x_1, \dots, x_n) = e_k, \qquad k = 1, \dots, n.$$

The second coordinate y_i is determined by the first coordinate x_i. As a result, the variety X_n locally isomorphic to

$$\left\{ (e, c, x_1, \dots, x_n) : \begin{matrix} e_k(x_1, \dots, x_n) = e_k \\ \text{for } k = 1, \dots, n \end{matrix} \right\}.$$

So the coordinate ring of the fiber at (e, c) is

$$R_{(n)} = \mathbb{C}[x_1, \dots, x_n] / \langle e_k(x_1, \dots, x_n) : k = 1, \dots, n \rangle.$$

This is known as the **coinvariant algebra**.

7.22. Remark. There is a way of thinking the ideal J_μ such that

$$R_\mu = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] / J_\mu.$$

Let $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ be distinct numbers. Then J_μ is the $a \to 0, b \to 0$ limit of the ideal for

$$S_n\text{-orbit of } \{(a_i, b_j) : (i, j) \in \mu\} \subset (\mathbb{C}^2)^n.$$

Moreover, there is another way of of describing the ideal, expressed in certain determinants, called the **Garsia–Haiman modules**, see [11] and [10, §4.1].

7.23. Example. Let us consider

$$\mu = \begin{array}{|c|c}\hline \square & \\ \hline \square & \square \\ \hline\end{array}, \qquad I_\mu = \langle x^2, xy, y^2\rangle.$$

We need to consider the ideal for the 6 points

$$\left\{ (p_1, p_2, p_3) : \{p_1, p_2, p_3\} = \left\{ \begin{matrix} (a_1, b_2) & \\ (a_1, b_1) & (a_2, b_1) \end{matrix} \right\} \right\} \subset (\mathbb{C}^2)^3.$$

It is the radical of the product of 6 ideals. Using computer, we can compute

$$R_{(2,1)} = \mathrm{span}(1, x_2, x_3, y_2, y_3, x_2y_3).$$

Moreover,

$$\begin{aligned} x_1 &= -x_2 - x_3, & x_1y_2 &= -x_1y_3 = x_2y_3 \\ y_1 &= -y_2 - y_3 & &= -x_2y_1 = x_3y_1 = -x_3y_2. \end{aligned}$$

So, we can describe the S_3-action.

```
S = PolynomialRing(QQ,["x1","x2","x3","y1","y2","y3","a1","a2","
    b1","b2"])
x1, x2, x3, y1, y2, y3, a1, a2, b1, b2 = S.gens()
a = [a1,a2]; b = [b1,b2]
x = [x1,x2,x3]; y = [y1,y2,y3]

ideals = []
for w in Permutations(3):
    rel = []
    rel += [x[w(1)-1] - a1, y[w(1)-1] - b1]
    rel += [x[w(2)-1] - a1, y[w(2)-1] - b2]
    rel += [x[w(3)-1] - a2, y[w(3)-1] - b1]
    ideals.append(S.ideal(rel))
I = prod(ideals);
J = I.radical(); # compute the radical, slow
K = J + S.ideal(a1,a2,b1,b2); # taking limit a,b->0
print(K.normal_basis())
```

```
Rmu = S.quotient(K);
print(Rmu(x1*y2),Rmu(x1*y3),Rmu(x2*y3),Rmu(x2*y1),Rmu(x3*y1),Rmu
    (x3*y2))
```

7.24. Theorem. We have

$$\mathrm{Frob}(R_\mu) = \tilde{H}_\mu.$$

7.25. Example. Consider $n = 3$. We summarize our computation

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t & s & s & a & 0 \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s & a & 0 & 0 & 0 \\ t & s & 0 & 0 & 0 \end{array}$$

$$\ydiagram{3} \qquad\qquad \ydiagram{1,1,1} \qquad\qquad \ydiagram{1,2}$$

where

$$t = \mathbf{tri}, \qquad s = \mathbf{std}, \qquad a = \mathbf{alt}.$$

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