BRAID VARIETIES (II)

2026/01/09

1. Parametrization

We fix a "pin" $x_i, y_i : \mathbb{C} \to G$ by

$$x_i(z) = \text{image of } \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}, \qquad y_i(z) = \text{image of } \begin{bmatrix} 1 & \\ z & 1 \end{bmatrix}$$

under the map $SL_2 \to G$ determined by simple root α_i. We pick a lifting of $s_i \in W = N_G(T)/T$

$$\dot{s} = \text{image of } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in G.$$

Note that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin SL_2$. Let $B_i : \mathbb{C} \to G$ be

$$B_i(z) = x_i(z)\dot{s}_i = \text{image of } \begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix} \in G.$$

Example. When $G = SL_n$, the map determined by simple root α_i is

$$SL_2 \to SL_n, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & a & b & & & \\ & & & c & d & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$

where the entry "a" is at the (i,i)-entry. Then

$$y_i(z) = \begin{bmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&& z & 1 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{bmatrix}, \qquad B_i(z) = \begin{bmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& z & -1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{bmatrix}.$$

Lemma. For a flag g_0B,

$$\{xB : g_0B \xrightarrow{s_i} xB\} = g_0Bs_iB/B = \{g_0B_i(z)B/B : z \in \mathbb{C}\} \cong \mathbb{F}.$$

That is, once a representative of g_0B is chosen, there is a canonical choice of representative for xB by $x = g_0B_i(z)$.

Proof. This follows from $\mathbb{C} \xrightarrow{\sim} Bs_iB/B$ given by $x_i(z)s_iB/B = B_i(z)B/B$. □

Recall the definition of braid variety:

$$X(\beta, u) = \{(g_kB) : B \xrightarrow{s_{i_1}} g_1B \xrightarrow{s_{i_2}} \cdots g_{\ell-1}B \xrightarrow{s_{i_\ell}} uB\}$$

where $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell}$ is a braid word and $u \in W$. We parametrize all the flags as follows

- the flag $g_0B = B$ can be parametrized by any element of B, let us choose 1;
- the flag g_1B is parametrized by $1 \cdot B_{i_1}(z_1) = B_{i_1}(z_1)$;
- the flag g_2B is parametrized by $B_{i_1}(z_1)B_{i_2}(z_2)$;
- $\cdots$;
- finally, the flat $g_\ell B$ is parametrized by $B_{i_1}(z_1)B_{i_2}(z_2)\cdots B_{i_\ell}(z_\ell)$.
- We need to require it lies in uB.

As a result,

$$X(\beta, u) = \{(z_k) : B_{i_1}(z_1)\cdots B_{i_\ell}(z_\ell) \in uB\} \subset \mathbb{C}^\ell.$$

Example. When $G = SL_2$, the flag variety $G/B = \mathbb{P}^1$ with B, sB corresponds to $0, \infty$. Recall

$$P_1 \xrightarrow{\mathsf{id}} P_2 \iff P_1 = P_2, \qquad\qquad P_1 \xrightarrow{s_1} P_2 \iff P_1 \neq P_2.$$

Consider $\beta = \sigma_1^2$ and $u = s_1$. Then

$$X(\beta, u) = \{P_1 : 0 \neq P_1 \neq \infty\} \cong \mathbb{C}^\times.$$

On the other hand,

$$\begin{aligned} X(\beta, u) &= \left\{(z_1, z_2) : \begin{bmatrix} z_1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_2 & -1 \\ 1 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}\right\} \\ &= \{(z_1, z_2) : z_1z_2 - 1 = 0\} \cong \mathbb{C}^\times. \end{aligned}$$

Consider $\beta = \sigma_1^3$ and $u = s_1$. Then

$$\begin{aligned} X(\beta,u) &= \{(P_1,P_2) : 0 \neq P_1 \neq P_2 \neq \infty\} \\ &= (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \big((0 \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \infty) \cup (\mathrm{diag}(\mathbb{P}^1) \big). \end{aligned}$$

Assume $P_1 = x = [1:x]$ and $P_2 = y^{-1} = [y:1]$ for $x, y \in \mathbb{C}$. Then

$$X(\beta,u) = \{(x,y) : y \neq 0 \Rightarrow x \neq y^{-1}\} = \{(x,y) : xy \neq 1\}.$$

Its coordinate ring is $\mathbb{C}\big[x, y, \frac{1}{xy-1}\big]$. Note that this is the cluster algebra of

$$\begin{array}{ccc} \boxed{xy-1} & & \boxed{xy-1} \\ \downarrow & \overset{\text{mutate}}{\longleftrightarrow} & \downarrow \\ x & & y \end{array}$$

On the other hand,

$$\begin{aligned} X(\beta,u) &= \left\{ (z_1,z_2,z_3) : \begin{bmatrix} z_1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_3 & -1 \\ 1 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \right\} \\ &= \{(z_1,z_2,z_3) : z_1z_2z_3 - z_1 - z_3 = 0\}. \end{aligned}$$

Its coordinate ring is

$$\mathbb{C}[z_1,z_2,z_3]/\langle z_1z_2z_3 - z_1 - z_3 \rangle \cong \mathbb{C}\big[z_2, z_3, \tfrac{1}{z_2z_3-1}\big].$$

The isomorphism is given by

$$z_1 \longmapsto \frac{z_3}{z_2z_3-1}, \qquad z_1z_2 - 1 \longleftarrow\!\shortmid \frac{1}{z_2z_3-1}.$$

2. Mellit Stratification

Following [1] and [2], we stratify

$$X(\beta,u) = \bigsqcup_{\underline{\mathbf{u}}=(u_0,\dots,u_\ell)} X(\beta,\underline{\mathbf{u}}), \qquad X(\beta,\underline{\mathbf{u}}) = \left\{ (g_kB) \in X(\beta,u) : \begin{array}{c} uB \\ {\scriptstyle u_k}\big\downarrow \\ g_kB \end{array} \right\}$$

The condition can be summarized in the following diagram

$$\begin{tikzcd}
 & & uB \arrow[dll, "u_0"'] \arrow[dl, "u_1"] \arrow[d, phantom, "\cdots"] \arrow[dr, "u_{\ell-1}"'] \arrow[drr, "u_\ell"] & & \\
B \arrow[r, "s_{i_1}"'] & g_1B \arrow[r, "s_{i_2}"'] & \cdots \arrow[r, "s_{\ell-1}"'] & g_{\ell-1}B \arrow[r, "s_\ell"'] & uB.
\end{tikzcd}$$

To have non-empty piece, we require $\underline{\mathbf{u}}$ to be **distinguished**

- $u_0 = u^{-1}$, $u_\ell = \mathsf{id} \in W$;

- $u_k \in \{u_{k-1}, u_{k-1}s_{i_k}\}$;
- $u_k \geq u_{k-1}s_{i_k}$.

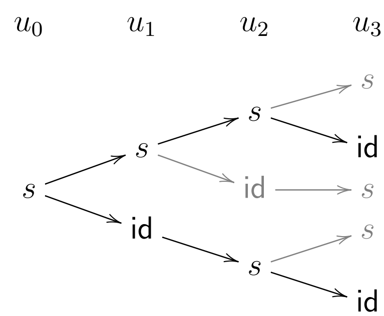
The last condition is equivalent to

	skip $u_k = u_{k-1}$	choose $u_k = u_{k-1}s_{i_k}$
increase $\nearrow$ $u_{k-1}s_{i_k} > u_{k-1}$	NO	OK
decrease $\searrow$ $u_{k-1}s_{i_k} < u_{k-1}$	OK	OK

That is, the sequence never lose any chance of incresing its length.

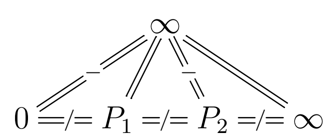
Example. Let us consider the case $G = SL_2$, $\beta = \sigma^3$ and $u = s$. Then there are two distinguished sequences

$$\underline{\mathbf{u}} = (s, \mathsf{id}, s, \mathsf{id}), \qquad \underline{\mathbf{u}} = (s, s, s, \mathsf{id}).$$



For $\underline{\mathbf{u}} = (s, \mathsf{id}, s, \mathsf{id})$,

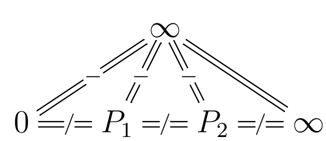
$$X(\beta, \underline{\mathbf{u}}) = \left\{ (P_1, P_2) : \right.$$



$$\left. \right\} = \{P_2 : P_2 \neq \infty\} \cong \mathbb{C}.$$

For $\underline{\mathbf{u}} = (s, s, s, \mathsf{id})$,

$$X(\beta, \underline{\mathbf{u}}) = \left\{ (P_1, P_2) : \right.$$



$$\left. \right\} = \{(x, y) : xy - 1 \neq 0, x \neq 0\} \cong (\mathbb{C}^\times)^2.$$

Theorem. One can parametrize $X(\beta, \underline{\mathbf{u}})$ and it is isomorphic to

$$\mathbb{C}^{\#(\text{chosen increasing steps})} \times (\mathbb{C}^{\times})^{\#(\text{skipped decreasing steps})}.$$

Proof. Let $g_1B \xleftarrow{u} g_2B$ for $u \in W$.

Case 1. If $us_i > u$. We have

$$\left\{ gB : \begin{tikzcd} & g_2B \arrow[dl, "u"'] \arrow[dr, "us_i"] & \\ g_1B \arrow[rr, leftrightarrow, "s_i"] & & gB \end{tikzcd} \right\} \begin{aligned} &= g_1Bs_iB/B \\ &= \{g_1B_i(z)B/B : z \in \mathbb{F}\} \cong \mathbb{F}, \end{aligned}$$

We can parametrize $g = g_1B_i(z)$ for $z \in \mathbb{F}$. It is easy to check that under this parametrization $g_1 \in g_1Bu \Longrightarrow g \in g_1Bus_i$.

Case 2. When $us_i < u$, we assume $g_1 \in g_2Bu$.

Case 2a. We have

$$\left\{ gB : \begin{tikzcd} & g_2B \arrow[dl, "u"'] \arrow[dr, "us_i"] & \\ g_1B \arrow[rr, leftrightarrow, "s_i"] & & gB \end{tikzcd} \right\} = g_1s_iB/B = \{\mathsf{pt}\},$$

We can parametrize $g = g_1\dot{s}_i^{-1}$. It is easy to check that under this parametrization $g \in g_1Bus_i$.

Case 2b. We have

$$\left\{ gB : \begin{tikzcd} & g_2B \arrow[dl, "u"'] \arrow[dr, "u"] & \\ g_1B \arrow[rr, leftrightarrow, "s_i"] & & gB \end{tikzcd} \right\} \begin{aligned} &= g_1Bs_iB/B \setminus \{g_1s_iB/B\} \\ &= \{g_1y_i(t)B/B : t \in \mathbb{F}^{\times}\} \cong \mathbb{F}^{\times} \end{aligned}$$

We can parametrize $g = g_1y_i(t)$ for $t \in \mathbb{F}^{\times}$. It is easy to check that under this parametrization $g \in g_1Bu$. □

Remark of the proof. (1) In the Case 2b, we can also parametrize $g = g_1B_i(z)$ for $z \in \mathbb{F}^{\times}$, but it does not satisfy $g \in g_1Bu$. (2) We have a better control of the parametrization. If we further require $g_1 \in g_2U\dot{u}$, then $g \in g_2U\dot{u}'$ where $g_2 \xrightarrow{u'} gB$. That is the reason we use $\dot{s}_i^{-1}$ in Case 2a.

It is useful to read $\underline{\mathbf{u}}$ from right to left. Compare:

	skip $u_k = u_{k-1}$	choose $u_k = u_{k-1}s_{i_k}$
increase $\nearrow$ $u_{k-1}s_{i_k} > u_{k-1}$	♠	♣
decrease $\searrow$ $u_{k-1}s_{i_k} < u_{k-1}$	♢	♡

	skip $u_{k-1} = u_k$	choose $u_{k-1} = u_ks_{i_k}$
increase $\nwarrow$ $u_ks_{i_k} > u_k$	♠	♡
decrease $\swarrow$ $u_ks_{i_k} < u_k$	♢	♣

Example. Recall that Richardson variety

$$\mathring{R}_{u,w} = X(\sigma_w \sigma_{u^{-1}w_0}, w_0).$$

Assume $w = s_{i_1} \cdots s_{i_\ell}$ and fix a choice of $u^{-1}w_0$. Then a distinguished sequence will looks like

$$(w_0, \underbrace{\ldots, \ldots,}_{\text{(i)}} u^{-1}w_0, \underbrace{\ldots, \ldots,}_{\text{(ii)}} e).$$

Since when read from right to left, each step is increasing, and the simple reflection has to choose. The sequence (ii) is the uniquely determined, i.e. sequence of partial product of the fixed reduced word of $u^{-1}w_0$. The sequence (i) is in bijection with distinguished subwords for u in a fixed reduced word of w. This gives the classical Deodhar piece.

It is useful to define the following properties of sequences

distinguished	skip	choose
$\nearrow$	$\times$	$\bigcirc$
$\searrow$	$\bigcirc$	$\bigcirc$

positive	skip	choose
$\nearrow$	$\times$	$\times$
$\searrow$	$\bigcirc$	$\bigcirc$

reduced	skip	choose
$\nearrow$	$\bigcirc$	$\times$
$\searrow$	$\bigcirc$	$\bigcirc$

Equivalently,

distinguished	skip	choose
$\nwarrow$	$\times$	$\bigcirc$
$\swarrow$	$\bigcirc$	$\bigcirc$

positive	skip	choose
$\nwarrow$	$\times$	$\bigcirc$
$\swarrow$	$\bigcirc$	$\times$

reduced	skip	choose
$\nwarrow$	$\bigcirc$	$\bigcirc$
$\swarrow$	$\bigcirc$	$\times$

Now that a sequence is positive if and only if

$$u_{k-1} = \max(u_k s_{i_k}, u_k) = u_k * s_{i_k}.$$

By induction, $u_k = s_{i_\ell} * \cdots * s_{i_{k+1}}$, and thus we get

$$u^{-1} = u_0 = s_{i_\ell} * \cdots * s_{i_1}$$

i.e.

$$u = \text{Demazure product of } \beta = s_{i_1} * \cdots * s_{i_\ell}. \tag{$*$}$$

As a result, a (unique) positive sequence exists if and only if $(*)$ is true.

Theorem. Under the assumption $(*)$,

$$\dim X(\beta, u) = \dim X(\beta, \underline{\mathbf{u}}) \iff \underline{\mathbf{u}} \text{ is positive.}$$

Proof. Since $X(\beta, u)$ is irreducible, there is exactly one $\underline{\mathbf{u}}$ with this property. In the sequence, the number of skipped step is exactly $\ell(\beta) - \ell(u)$, which is the dimension of the braid variety. $\square$

The name "positive" is from the totally postivity. The totally positive part of $X(\beta, u)_{>0}$ is contained in $X(\beta, \underline{\mathbf{u}})$ for positive $\underline{\mathbf{u}}$.

The $X(\beta, \underline{\mathbf{u}})$ for positive $\underline{\mathbf{u}}$ will be served as the initial seed. For example, in the above example of SL_2,

$$\{(x, y) : xy - 1 \neq 0, x \neq 0\}$$

corresponds to the seed $\boxed{xy - 1} \to x$.

References

[1] Anton Mellit. Toric stratifications of character varieties *Publications mathématiques de l'IHÉS* arXiv:1905.10685. (old name: Cell decompositions of character varieties) 2

[2] Roger Casals, Eugene Gorsky, Mikhail Gorsky, José Simental. Algebraic weaves and braid varieties. *American Journal of Mathematics*, arXiv:2012.06931. 2