BRAID VARIETIES (I)

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1. Flag varieties

1.1. **Relative positions.** Let G be a reductive group. We fix the following subgroups:

$$\text{maximal torus } T \subset \text{Borel subgroup } B \subset G.$$

The **flag variety** is G/B. We have **Bruhat decomposition**

$$G/B = \bigsqcup_{w \in W} BwB/B, \qquad \text{Schubert cell } BwB/B \cong \mathbb{C}^{\ell(w)}$$

where $W = N_G(T)/T$ is the Weyl group, and $\ell : W \to \mathbb{Z}_{\geq 0}$ is the length function. We define **relative positions** of two flags g_1B, g_2B as follows

$$\begin{aligned} g_1B \xrightarrow{w} g_2B &\iff g_2B \xleftarrow{w} g_1B \\ &:\iff g_1^{-1}g_2 \in BwB \iff g_2B \in g_1BwB/B \end{aligned}$$

Here are some basic properties:

- $B \xrightarrow{w} wB$;
- $xB \xrightarrow{w} yB \iff gxB \xrightarrow{w} gyB$;
- $xB \xrightarrow{w} yB \iff yB \xrightarrow{w^{-1}} xB$.

Note that

$$g_1B \xrightarrow{\mathsf{id}} g_2B \iff g_1B = g_2B.$$

We can view "relative position of two flags" as a non-commutative analogue of "distance of two flags".

Example. For example, when $G = GL_n$, we can take

$$T = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \subset B = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix} \subset G = GL_n = \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix}.$$

Now the Weyl group $W = S_n$ and the Bruhat decomposition is the following statement in linear algebra

> For any invertible matrix X, we can decompose $X = UwV$ for U, V upper triangular and w a permutation matrix.

The flag variety parametrizes the classical flag variety

$$G/B \cong \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_k = k\}$$

via acting on the standard flag

$$0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{n-1} \subset \mathbb{C}^n$$

whose stabilizer is the group of upper triangular matrices B. Here converntionally, $\mathbb{C}^k = \mathrm{span}(e_1, \ldots, e_k)$ for standard basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Then

$$wB/B \overset{1:1}{\longleftrightarrow} (V_\bullet) \text{ with } V_i = \mathrm{span}(e_{w(1)}, \ldots, e_{w(i)}).$$

Moreover

$$BwB/B \overset{1:1}{\longleftrightarrow} \{V_\bullet : \dim(\mathbb{C}^i \cap V_j) = \#([i] \cap w[j])\}.$$

The relative position can be described as

$$gB \xrightarrow{w} g'B \iff \dim(V_i \cap V_j') = \#([i] \cap w[j]).$$

When $W = s_i \in S_n$ is the simple reflection (i.e. swap i and $i+1$), we have

$$g_1B \overset{s_i}{\longleftrightarrow} g_2B \iff \begin{array}{l} j \neq i \Rightarrow V_j = V_j' \\ j = i \Rightarrow V_j \neq V_j'. \end{array}$$

That is, the two flags only differ at the i-th step.

1.2. **Hecke algebras.** Let us use the classical convention of **Hecke algebra**:

$$(T_i + 1)(T_i - q) = 0, \qquad \text{Braid relations.}$$

For any $w \in W$, we can define

$$T_w = T_{i_1} \cdots T_{i_\ell}$$

for any reduced word $w = s_{i_1} \cdots s_{i_\ell}$. By braid relation, T_w does not depend on the choice of the reduced word. Hecke algebra admits T_w ($w \in W$) as a basis. The product can be characterized by the following two properties

$$\ell(u) + \ell(v) = \ell(uv) \Longrightarrow T_uT_v = T_{uv},$$

$$T_uT_i = \begin{cases} T_{us_i}, & us_i > u, \\ (q-1)T_u + qT_{us_i}, & us_i < u. \end{cases}$$

Let me explain the geometric meaning of Hecke algebra. The Hecke algebra could be viewed as the space of non-commutative "generating functions" of relative positions. The structure constant, the coefficient of T_w of T_uT_v is the $\mathbb{F}_q$-point counting of

$$\#\{gB : g_1B \xrightarrow{u} gB \xrightarrow{v} g_2B\} \qquad \text{for any given } g_1B \xrightarrow{w} g_2B.$$

To see this, we can formally define an algebra structure over $\mathrm{span}(T_w : w \in W)$ using these constants as structure constant.

Associativity. The following diagram shows the associativity

$$\begin{tikzcd} g_2B \arrow[r, "v"] & g_3B \arrow[d, "w"] \\ g_1B \arrow[u, "u"] \arrow[ur] \arrow[r, "x"'] & g_4B \end{tikzcd} = \begin{tikzcd} g_2B \arrow[r, "v"] \arrow[dr] & g_3B \arrow[d, "w"] \\ g_1B \arrow[u, "u"] \arrow[r, "x"'] & g_4B \end{tikzcd}$$
$$(T_uT_v)T_w \qquad\qquad T_u(T_vT_w)$$

Braid relations. Assume $\ell(u) + \ell(v) = \ell(uv)$. We have an isomorphism $\mathrm{mult} : BuB \times_B BvB \xrightarrow{\sim} BuvB$. It can be translated as

- For any $g_1B \xrightarrow{uv} g_2B$, there exists a unique $gB \in G/B$, such that
$$g_1B \xrightarrow{u} gB \xrightarrow{v} g_2B.$$
- Conversely, if we have
$$g_1B \xrightarrow{u} gB \xrightarrow{v} g_2B.$$
Then we have $g_1B \xrightarrow{uv} g_2B$.

This proves
$$\ell(u) + \ell(v) = \ell(uv) \Longrightarrow T_uT_v = T_{uv}.$$

In summary,

$$\begin{tikzcd} & \exists! gB \arrow[dr, "v"] \arrow[d, phantom, "\Updownarrow"] & \\ g_1B \arrow[ur, "u"] \arrow[rr, "uv"] & & g_2B. \end{tikzcd}$$

In particular, for a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$, $xB \xrightarrow{w} yB$ can be uniquely split as
$$xB \xrightarrow{s_{i_1}} g_1B \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_{\ell-1}}} g_{\ell-1}B. \xrightarrow{s_{i_\ell}} yB.$$
Moreover, $g_1B, \ldots, g_{\ell-1}B$ are uniquely determined by xB and yB.

Quadratic relations. The quadratic relations follow from the computation in SL_2 where $G/B = \mathbb{P}^1$ and
$$g_1B \xrightarrow{s} g_2B \iff g_1B \neq g_2B.$$
As a result,
$$\{gB : g_1B \xrightarrow{s} gB \xrightarrow{s} g_2B\} = G/B \setminus \{g_1B, g_2B\}$$
has number q if $g_1B = g_2B$, $q-1$ if $g_1B \neq g_2B$.

Important Example. Let us show how Hecke algebra helps us find solve triangles of flags. Consider the following triangle

$$\begin{tikzcd}
 & g_1B \arrow[dl, "u"'] \arrow[dr, "v"] & \\
g_2B \arrow[rr, leftrightarrow, "s_i"] & & gB
\end{tikzcd}$$

Question
g_1B, g_2B are fixed find the choice of gB.

The numerial answer is given by

$$T_uT_i = \begin{cases} T_{us_i}, & us_i > u, \\ (q-1)T_u + qT_{us_i}, & us_i < u. \end{cases}$$

There are three cases.

- If $v = us_i < u$, then gB is uniquely determined by g_1B and g_2B, and the choice of gB is a point.
- If $v = us_i > u$, then the condition $g_1B \xrightarrow{u} gB$ follows from $gB \xrightarrow{s_i} g_2B$. As a result, the choice of gB is $g_2Bs_iB/B \cong \mathbb{C}$.
- If $v = u$, then it is necessary to have $us_i > u$ (otherwise $g_1 \xrightarrow{us_i} gB$). In this case, the choice of gB is $g_2Bs_iB \setminus \{g'B\} \cong \mathbb{C}^\times$ for $g'B$ the unique flag fits in the triangle $g_1B \xrightarrow{vs_i} g'B \xrightarrow{s_i} g_2B$.

The situation can be summaries as follows

	$v = u$	$v = us_i$
$us_i > u$	$\varnothing$	$\mathbb{F}$
$us_i < u$	$\mathbb{F}^\times$	pt

2. Braid varieties

A **braid word** is a sequence of σ_i for $i \in I$. This notation comes from the definition of braid monoid/group

$$\langle \sigma_i \mid \text{braid relations} \rangle .$$

Note that the difference is, to generate a group, one needs to introduce σ_i^{-1}. A non-trivial fact is

$$\text{braid monoid} \subset \text{braid group}.$$

We will call an element of braid monoid a **positive braid**. For any $w \in W$, define σ_w the positive lifting of $w \in W$, i.e. $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_\ell}$ for any reduced word $w = s_{i_1} \cdots s_{i_\ell}$.

Definition. Let $\beta = \sigma_{i_1} \dots \sigma_{i_\ell}$ be a braid word and $u \in W$. The **braid variety** is defined to be

$$X(\beta, u) = \{(g_k B) : B \xrightarrow{s_{i_1}} g_1 B \xrightarrow{s_{i_2}} \cdots g_{\ell-1} B \xrightarrow{s_{i_\ell}} uB\}.$$

Note that two braids β_1, β_2 related by braid relations, the defining braid varieties are isomorphic. Actually, the isomorphism does not depend on the choice of way of applying braid relations, so $X(\beta, u)$ is an invariant of the positive braid β.

Theorem. Let $T_\beta = T_{i_1} \cdots T_{i_\ell}$. We have

$$|X(\beta, u)(\mathbb{F}_q)| = \text{coefficient of } T_u \text{ in } T_\beta.$$

Example. The open Richardson variety

$$\begin{aligned} \mathring{R}_{u,w} &= \{gB : B \xrightarrow{w} gB \xleftarrow{w_0 u} w_0 B\} \\ &= \{gB : B \xrightarrow{w} gB \xrightarrow{u^{-1} w_0} w_0 B\} = X(\sigma_w \sigma_{u^{-1} w_0}, w_0). \end{aligned}$$

Here we picking a reduced word for w and $u^{-1} w_0$ respectively. In particular,

$$|\mathring{R}_{u,w}(\mathbb{F}_q)| = \text{coefficient of } T_{w_0} \text{ in } T_w T_{u^{-1} w_0}.$$

There is a classical trick in Hecke algebra that

$$(\text{coefficient of } T_{w_0} \text{ in } T_{v^{-1}}^{-1} T_{u^{-1} w_0}) = \delta_{uv} q^{-\ell(u)}.$$

On the other hand, recall the definition of R-polynomials

$$T_w = \sum_{u \in W} q^{\ell(u)} R_{u,w}(q) T_{u^{-1}}^{-1}.$$

Thus we can conclude

$$|\mathring{R}_{u,w}(\mathbb{F}_q)| = R_{u,w}(q).$$

Example. We can consider the open **twisted flag variety**

$$\begin{aligned} Z(\beta, u) &= \{(g_k B) : B \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_\ell}} g_\ell B \xleftarrow{w_0 u} w_0 B\} \\ &= X(\beta \sigma_{u^{-1} w_0}, w_0). \end{aligned}$$

When u is the Demazure product of β,

$$X(\beta, u) = Z(\beta, u) = X(\beta \sigma_{u^{-1} w_0}, w_0).$$

One of the most important case is when

$$u = \text{Demazure product of } \beta = s_{i_1} * \cdots * s_{i_\ell}. \tag{$*$}$$

Recall the Demazure product is a monoid structure over W characterized by

$$s_i * s_i = s_i, \qquad \text{braid relations.}$$

It satisfies

$$s_i * w = \max(s_i w, w), \qquad w * s_i = \max(w, w s_i).$$

The geometric meaning of Demazure product is the following. The subset $BuB \cdot BvB = BuBvB$ contains a unique dense BwB for some w, then the index $w = u * v$. That is,

$$\overline{BuB} \cdot \overline{BvB} = \overline{B(u * v)B}.$$

Theorem. When $(*)$ is true, $X(\beta, u)$ is nonempty and smooth.

Proof. Consider the map

$$\mathsf{proj} : \left\{(g_k B) \mid B \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_\ell}} g_\ell B\right\} \longrightarrow G/B, \qquad (g_k B) \mapsto g_\ell B.$$

Note that $X(\beta, u) = \mathsf{proj}^{-1}(uB)$. Since proj a morphism is between two smooth varieties, the preimage of a generic point of the image is nonempty and smooth. Since proj is B-equivariant, we need to show the image contains BuB/B as a dense subset. Note that the image is $Bs_{i_1}Bs_{i_1}B \cdots Bs_{i_\ell}B/B$ contains BuB/B as a dense subset. $\square$

Corollary. Under $(*)$, the dimension of $X(\beta, u)$ is $\ell(\beta) - \ell(u)$.

Proof. The number $\ell(\beta)$ is the dimension of the domain of proj and the number $\ell(u)$ is the dimension of the image of proj. $\square$

Example. Assume the Demazure product of β is w_0. We have

$$\begin{aligned} X(\beta\sigma_{w_0}, \mathsf{id}) &= \{(g_k B) : B \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_\ell}} g_\ell B \xrightarrow{w_0} B\} \\ &= X(\beta, w_0) \times Bw_0B/B. \end{aligned}$$

We notice that $Z(\beta\sigma_{w_0}, e)$ is the preimage of Schubert cell Bw_0B/B and $X(\beta\sigma_{w_0}, e)$ is the preimage of the point w_0B/B. But the map proj is B-equivariant.