BRAID VARIETIES (II)

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1. PARAMETRIZATION

We fix a "pin" $x_i, y_i : \mathbb{C} \to G$ by

$$x_i(z) = \text{image of } \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}, \qquad y_i(z) = \text{image of } \begin{bmatrix} 1 & \\ z & 1 \end{bmatrix}$$

under the map $SL_2 \to G$ determined by simple root α_i. We pick a lifting of $s_i \in W = N_G(T)/T$

$$\dot{s} = \text{image of } \begin{bmatrix} 0 & \text{-}1 \\ 1 & 0 \end{bmatrix} \in G.$$

Note that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin SL_2$. Let $B_i : \mathbb{C} \to G$ be

$$B_i(z) = x_i(z)\ddot{s}_i = \text{image of } \begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix} \in G.$$

Example. When $G = SL_n$, the map determined by simple root α_i is

$$SL_2 \to SL_n, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & a & b & & & \\ & & & c & d & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ 1 & & & & & & & \end{bmatrix}$$

where the entry "a" is at the (i, i)-entry. Then

$$y_i(z) = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & z & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}, \qquad B_i(z) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & z & -1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix}.$$

Lemma. For a flag $g_0 B$,

$$\{xB : g_0 B \xrightarrow{s_i} xB\} = g_0 B s_i B / B = \{g_0 B_i(z) B / B : z \in \mathbb{F}\} \cong \mathbb{F}.$$

That is, once a representative of $g_0 B$ is chosen, there is a canonical choice of representative of xB by $x = g_0 B_i(z)$.

Proof. This follows from $\mathbb{C} \xrightarrow{\sim} B s_i B / B$ given by $x_i(z) s_i B / B = B_i(z) B / B$. □

Recall the definition of braid variety:

$$X(\beta, u) = \{(g_k B) : B \xrightarrow{s_{i_1}} g_1 B \xrightarrow{s_{i_2}} \cdots g_{\ell-1} B \xrightarrow{s_{i_\ell}} uB\}$$

where $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell}$ is a braid word and $u \in W$. We parametrize all the flags as follows

- the flag $g_0 B = B$ can be parametrized by any element of B, let us choose 1;
- the flag $g_1 B$ is parametrized by $1 \cdot B_{i_1}(z_1) = B_{i_1}(z_1)$;
- the flag $g_2 B$ is parametrized by $B_{i_1}(z_1) B_{i_2}(z_2)$;
- $\cdots$;
- finally, the flat $g_\ell B$ is parametrized by $B_{i_1}(z_1) B_{i_2}(z_2) \cdots B_{i_\ell}(z_\ell)$.
- We need to require it lies in uB.

As a result,

$$X(\beta, u) = \{(z_k) : B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell) \in uB\} \subset \mathbb{C}^\ell.$$

Example. When $G = SL_2$, the flag variety $G/B = \mathbb{P}^1$ with B, sB corresponds to $0, \infty$. Recall

$$P_1 \xrightarrow{\text{id}} P_2 \iff P_1 = P_2, \qquad\qquad P_1 \xrightarrow{s_1} P_2 \iff P_1 \neq P_2.$$

Consider $\beta = \sigma_1^2$ and $u = s_1$. Then

$$X(\beta, u) = \{P_1 : 0 \neq P_1 \neq \infty\} \cong \mathbb{C}^\times.$$

On the other hand,

$$X(\beta, u) = \left\{ (z_1, z_2) : \begin{bmatrix} z_1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_2 & -1 \\ 1 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \right\}$$

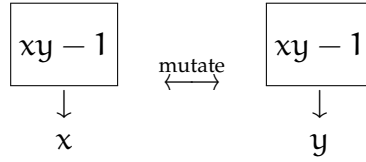$$= \{(z_1, z_2) : z_1 z_2 - 1 = 0\} \cong \mathbb{C}^\times.$$

Consider $\beta = \sigma_1^3$ and $u = s_1$. Then

$$X(\beta, u) = \{(P_1, P_2) : 0 \neq P_1 \neq P_2 \neq \infty\}$$
$$= (\mathbb{P}^1 \times \mathbb{P}^1) \setminus (0 \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \infty) \cup (\mathrm{diag}(\mathbb{P}^1)).$$

Assume $P_1 = x = [1 : x]$ and $P_2 = y^{-1} = [y : 1]$ for $x, y \in \mathbb{C}$. Then

$$X(\beta, u) = \{(x, y) : y \neq 0 \Rightarrow x \neq y^{-1}\} = \{(x, y) : xy \neq 1\}.$$

Its coordinate ring is $\mathbb{C}\left[x, y, \frac{1}{xy-1}\right]$. Note that this is the cluster algebra of

$$\boxed{xy - 1} \quad \underset{\text{mutate}}{\longleftrightarrow} \quad \boxed{xy - 1}$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$x \qquad\qquad\qquad\qquad\quad y$$

On the other hand,

$$X(\beta, u) = \left\{ (z_1, z_2, z_3) : \begin{bmatrix} z_1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_3 & -1 \\ 1 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \right\}$$
$$= \{(z_1, z_2, z_3) : z_1 z_2 z_3 - z_1 - z_3 = 0\}.$$

Its coordinate ring is

$$\mathbb{C}[z_1, z_2, z_3] / \langle z_1 z_2 z_3 - z_1 - z_3 \rangle \cong \mathbb{C}\left[z_2, z_3, \frac{1}{z_2 z_3 - 1}\right].$$
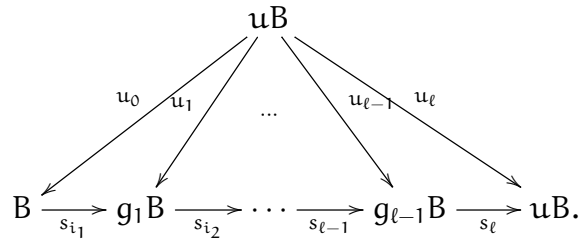
The isomorphism is given by

$$z_1 \longmapsto \frac{z_3}{z_2 z_3 - 1}, \qquad z_1 z_2 - 1 \longleftarrow \frac{1}{z_2 z_3 - 1}.$$

2. STRATIFICATION

We can stratify

$$X(\beta, u) = \bigsqcup_{\mathbf{u} = (u_0, \ldots, u_\ell)} X(\beta, \mathbf{u}), \qquad X(\beta, \mathbf{u}) = \left\{ (g_k B) \in X(\beta, u) : \begin{matrix} uB \\ u_k \downarrow \\ g_k B \end{matrix} \right\}$$

The condition can be summarized in the following diagram

$$
\begin{array}{c}
uB \\
\swarrow^{u_0} \quad \downarrow^{u_1} \quad \cdots \quad \swarrow^{u_{\ell-1}} \quad \searrow^{u_\ell} \\
B \xrightarrow{s_{i_1}} g_1 B \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{\ell-1}} g_{\ell-1} B \xrightarrow{s_\ell} uB.
\end{array}
$$

To have non-empty piece, we require $\mathbf{u}$ to be **distinguished**

- $u_0 = u^{-1}$, $u_\ell = \mathrm{id} \in W$;
- $u_k \in \{u_{k-1}, u_{k-1}s_{i_k}\}$;
- $u_k \geq u_{k-1}s_{i_k}$.

The last condition is equivalent to

	skip $u_k = u_{k-1}$	choose $u_k = u_{k-1}s_{i_k}$
increase $\nearrow$ $u_{k-1}s_{i_k} > u_{k-1}$	No	OK
decrease $\searrow$ $u_{k-1}s_{i_k} < u_{k-1}$	OK	OK

That is, the sequence never lose any chance of incresing its length.

Example. Let us consider the case $G = SL_2$, $\beta = \sigma^3$ and $u = s$. Then there are two distinguished sequences

$$\mathbf{u} = (s, \mathrm{id}, s, \mathrm{id}), \qquad \mathbf{u} = (s, s, s, \mathrm{id}).$$
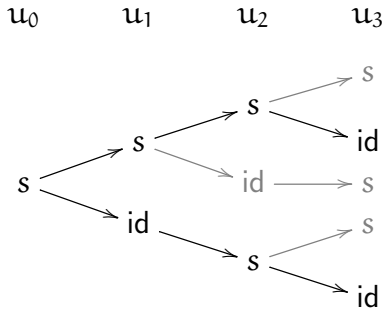
For $\mathbf{u} = (s, \mathrm{id}, s, \mathrm{id})$,

$$X(\beta, \mathbf{u}) = \left\{ (P_1, P_2) : \begin{array}{c} \infty \\ \diagup\diagdown \\ 0 =\!/\!= P_1 =\!/\!= P_2 =\!/\!= \infty \end{array} \right\}$$
$$= \{P_2 : P_2 \neq \infty\} \cong \mathbb{C}.$$

For $\mathbf{u} = (s, s, s, \mathrm{id})$,

$$X(\beta, \mathbf{u}) = \left\{ (P_1, P_2) : \begin{array}{c} \infty \\ \diagup\diagdown \\ 0 =\!/\!= P_1 =\!/\!= P_2 =\!/\!= \infty \end{array} \right\}$$
$$= \{(x, y) : xy - 1 \neq 0, x \neq 0\} \cong (\mathbb{C}^\times)^2.$$

Theorem. One can parametrize $X(\beta, \mathbf{u})$ and it is isomorphic to

$$\mathbb{C}^{\#(\text{chosen increasing steps})} \times (\mathbb{C}^{\times})^{\#(\text{skipped decreasing steps})}.$$

Proof. Let $g_1 B \xleftarrow{u} g_2 B$ for $u \in W$.

Case 1. If $us_i > u$. We have

$$\left\{ gB : \begin{array}{c} g_2B \\ {}^{u}\swarrow \quad \searrow^{us_i} \\ g_1B \xleftrightarrow{s_i} gB \end{array} \right\} \begin{array}{l} = g_1 B s_i B/B \\ = \{g_1 B_i(z)B/B : z \in \mathbb{F}\} \cong \mathbb{F}, \end{array}$$

We can parametrize $g = g_1 B_i(z)$ for $z \in \mathbb{F}$. It is easy to check that under this parametrization $g_1 \in g_1 B u \Longrightarrow g \in g_1 B u s_i$.

Case 2. When $us_i < u$, we assume $g_1 \in g_2 B u$.

Case 2a. We have

$$\left\{ gB : \begin{array}{c} g_2B \\ {}^{u}\swarrow \quad \searrow^{us_i} \\ g_1B \xleftrightarrow{s_i} gB \end{array} \right\} = g_1 s_i B/B = \{pt\},$$

We can parametrize $g = g_1 \dot{s}_i^{-1}$. It is easy to check that under this parametrization $g \in g_1 B u s_i$.

Case 2b. We have

$$\left\{ gB : \begin{array}{c} g_2B \\ {}^{u}\swarrow \quad \searrow^{u} \\ g_1B \xleftrightarrow{s_i} gB \end{array} \right\} \begin{array}{l} = g_1 B s_i B/B \setminus \{g_1 s_i B/B\} \\ = \{g_1 y_i(t)B/B : t \in \mathbb{F}^{\times}\} \cong \mathbb{F}^{\times} \end{array}$$

We can parametrize $g = g_1 y_i(t)$ for $t \in \mathbb{F}^{\times}$. It is easy to check that under this parametrization $g \in g_1 B u$. $\square$

Remark of the proof. (1) In the Case 2b, we can also parametrize $g = g_1 B_i(z)$ for $z \in \mathbb{F}^{\times}$, but it does not satisfy $g \in g_1 B u$. (2) We have a better control of the parametrization. If we further require $g_1 \in g_2 U \dot{u}$, then $g \in g_2 U \dot{u}'$ where $g_2 \xrightarrow{u'} gB$. That is the reason we use $\dot{s}_i^{-1}$ in Case 2a.

It is useful to read $\mathbf{u}$ from right to left. Compare:

	skip	choose		skip	choose
	$u_k = u_{k-1}$	$u_k = u_{k-1}s_{i_k}$		$u_{k-1} = u_k$	$u_{k-1} = u_k s_{i_k}$
increase $\nearrow$ $u_{k-1}s_{i_k} > u_{k-1}$	♠	♣	increase $\nwarrow$ $u_k s_{i_k} > u_k$	♠	♡
decrease $\searrow$ $u_{k-1}s_{i_k} < u_{k-1}$	♢	♡	decrease $\swarrow$ $u_k s_{i_k} < u_k$	♢	♣

Example. Recall that Richardson variety

$$\mathring{R}_{u,w} = X(\sigma_w \sigma_{u^{-1}w_0}, w_0).$$

Assume $w = s_{i_1} \cdots s_{i_\ell}$ and fix a choice of $u^{-1}w_0$. Then a distinguished sequence will looks like

$$(w_0, \underbrace{\ldots, \ldots}_{(i)}, u^{-1}w_0, \underbrace{\ldots, \ldots}_{(ii)}, e).$$

Since when read from right to left, each step is increasing, and the simple reflection has to choose. The sequence (ii) is the uniquely determined, i.e. sequence of partial product of the fixed reduced word of $u^{-1}w_0$. The sequence (i) is in bijection with distinguished subwords for u in a fixed reduced word of w. This gives the classical Deodhar piece.

It is useful to define the following properties of sequences

distinguished	
skip	choose
↗ ×	○
↘ ○	○

positive	
skip	choose
↗ ×	×
↘ ○	○

reduced	
skip	choose
↗ ○	×
↘ ○	○

Equivalently,

distinguished	
skip	choose
↖ ×	○
↙ ○	○

positive	
skip	choose
↖ ×	○
↙ ○	×

reduced	
skip	choose
↖ ○	○
↙ ○	×

Now that a sequence is positive if and only if

$$u_{k-1} = \max(u_k s_{i_k}, u_k) = u_k * s_{i_k}.$$

By induction, $u_k = s_{i_\ell} * \cdots * s_{i_{k+1}}$, and thus we get

$$u^{-1} = u_0 = s_{i_\ell} * \cdots * s_{i_1}$$

i.e.

$$u = \text{Demazure product of } \beta = s_{i_1} * \cdots * s_{i_\ell}. \tag{$*$}$$

As a result, a (unique) positive sequence exists if and only if $(*)$ is true.

Theorem. Under the assumption $(*)$,

$$\dim X(\beta, u) = \dim X(\beta, \mathbf{u}) \iff \mathbf{u} \text{ is positive}.$$

Proof. Since $X(\beta, u)$ is irreducible, there is exactly one $\mathbf{u}$ with this property. In the sequence, the number of skipped step is exactly $\ell(\beta) - \ell(u)$, which is the dimension of the braid variety. □

The name "positive" is from the totally postivity. The totally positive part of $X(\beta, \mathbf{u})_{>0}$ is contained in $X(\beta, \mathbf{u})$ for positive $\mathbf{u}$.

The $X(\beta, \mathbf{u})$ for positive $\mathbf{u}$ will be served as the initial seed. For example, in the above example of SL_2,

$$\{(x, y) : xy - 1 \neq 0, x \neq 0\}$$

corresponds to the seed $\boxed{xy - 1} \to x$.