

BRAID VARIETIES (I)

2026/01/08

1. FLAG VARIETIES

1.1. **Relative positions.** Let G be a reductive group. We fix the following subgroups:

maximal torus $T \subset$ Borel subgroup $B \subset G$.

The **flag variety** is G/B . We have **Bruhat decomposition**

$$G/B = \bigsqcup_{w \in W} BwB/B, \quad \text{Schubert cell } BwB/B \cong \mathbb{C}^{\ell(w)}$$

where $W = N_G(T)/T$ is the Weyl group, and $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ is the length function. We define **relative positions** of two flags g_1B, g_2B as follows

$$\begin{aligned} g_1B \xrightarrow{w} g_2B &\iff g_2B \xleftarrow{w} g_1B \\ &:\iff g_1^{-1}g_2 \in BwB \iff g_2B \in g_1BwB/B \end{aligned}$$

Here are some basic properties:

- $B \xrightarrow{w} wB$;
- $xB \xrightarrow{w} yB \iff gxB \xrightarrow{w} gyB$;
- $xB \xrightarrow{w} yB \iff yB \xrightarrow{w^{-1}} xB$.

Note that

$$g_1B \xrightarrow{\text{id}} g_2B \iff g_1B = g_2B.$$

We can view “relative position of two flags” as a non-commutative analogue of “distance of two flags”.

Example. For example, when $G = GL_n$, we can take

$$T = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \subset B = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix} \subset G = GL_n = \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix}.$$

Now the Weyl group $W = S_n$ and the Bruhat decomposition is the following statement in linear algebra

For any invertible matrix X , we can decompose $X = UwV$ for U, V upper triangular and w a permutation matrix.

The flag variety parametrizes the classical flag variety

$$G/B \cong \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_k = k\}$$

via acting on the standard flag

$$0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{n-1} \subset \mathbb{C}^n$$

whose stabilizer is the group of upper triangular matrices B . Here converntionally, $\mathbb{C}^k = \text{span}(e_1, \dots, e_k)$ for standard basis e_1, \dots, e_n of \mathbb{C}^n . Then

$$wB/B \xleftarrow{1:1} (V_\bullet) \text{ with } V_i = \text{span}(e_{w(1)}, \dots, e_{w(i)}).$$

Moreover

$$BwB/B \xleftarrow{1:1} \{V_\bullet : \dim(\mathbb{C}^i \cap V_j) = \#([i] \cap w[j])\}.$$

The relative position can be described as

$$gB \xrightarrow{w} g'B \iff \dim(V_i \cap V'_j) = \#([i] \cap w[j]).$$

When $W = s_i \in S_n$ is the simple reflection (i.e. swap i and $i+1$), we have

$$g_1B \xleftarrow{s_i} g_2B \iff \begin{cases} j \neq i \Rightarrow V_j = V'_j \\ j = i \Rightarrow V_j \neq V'_j. \end{cases}$$

That is, the two flags only differ at the i -th step.

1.2. Hecke algebras. The following materials are essentially due to Iwahori [2]. Let us use the classical convention of [Hecke algebra](#):

$$(T_i + 1)(T_i - q) = 0, \quad \text{Braid relations.}$$

For any $w \in W$, we can define

$$T_w = T_{i_1} \cdots T_{i_\ell}$$

for any reduced word $w = s_{i_1} \cdots s_{i_\ell}$. By braid relation, T_w does not depend on the choice of the reduced word. Hecke algebra admits T_w ($w \in W$) as a basis. The product can be characterized by the following two properties

$$\ell(u) + \ell(v) = \ell(uv) \implies T_u T_v = T_{uv},$$

$$T_u T_i = \begin{cases} T_{us_i}, & us_i > u, \\ (q-1)T_u + qT_{us_i}, & us_i < u. \end{cases}$$

Let me explain the geometric meaning of Hecke algebra. The Hecke algebra could be viewed as the space of non-commutative “generating functions” of relative positions. The structure constant, the coefficient of T_w of $T_u T_v$ is the \mathbb{F}_q -point counting of

$$\#\{gB : g_1B \xrightarrow{u} gB \xrightarrow{v} g_2B\} \quad \text{for any given } g_1B \xrightarrow{w} g_2B.$$

To see this, we can formally define an algebra structure over $\text{span}(T_w : w \in W)$ using these constants as structure constant.

Associativity. The following diagram shows the associativity

$$\begin{array}{ccc}
 g_2B & \xrightarrow{v} & g_3B \\
 \uparrow u & \nearrow & \downarrow w \\
 g_1B & \xrightarrow{x} & g_4B
 \end{array}
 \quad
 \begin{array}{ccc}
 g_2B & \xrightarrow{v} & g_3B \\
 \uparrow u & \searrow & \downarrow w \\
 g_1B & \xrightarrow{x} & g_4B
 \end{array}
 =
 \begin{array}{ccc}
 g_2B & \xrightarrow{v} & g_3B \\
 \uparrow u & \nearrow & \downarrow w \\
 g_1B & \xrightarrow{x} & g_4B
 \end{array}$$

$(T_u T_v) T_w \qquad T_u (T_v T_w)$

Braid relations. Assume $\ell(u) + \ell(v) = \ell(uv)$. We have an isomorphism $\text{mult} : BuB \times_B BvB \xrightarrow{\sim} BuvB$. It can be translated as

- For any $g_1B \xrightarrow{uv} g_2B$, there exists a unique $gB \in G/B$, such that

$$g_1B \xrightarrow{u} gB \xrightarrow{v} g_2B.$$

- Conversely, if we have

$$g_1B \xrightarrow{u} gB \xrightarrow{v} g_2B.$$

Then we have $g_1B \xrightarrow{uv} g_2B$.

This proves

$$\ell(u) + \ell(v) = \ell(uv) \implies T_u T_v = T_{uv}.$$

In summary,

$$\begin{array}{ccc}
 & \exists! gB & \\
 u \nearrow & \Downarrow & \searrow v \\
 g_1B & \xrightarrow{uv} & g_2B
 \end{array}$$

In particular, for a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$, $xB \xrightarrow{w} yB$ can be uniquely split as

$$xB \xrightarrow{s_{i_1}} g_1B \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_{\ell-1}}} g_{\ell-1}B \xrightarrow{s_{i_\ell}} yB.$$

Moreover, $g_1B, \dots, g_{\ell-1}B$ are uniquely determined by xB and yB .

Quadratic relations. The quadratic relations follow from the computation in SL_2 where $G/B = \mathbb{P}^1$ and

$$g_1B \xrightarrow{s} g_2B \iff g_1B \neq g_2B.$$

As a result,

$$\{gB : g_1B \xrightarrow{s} gB \xrightarrow{s} g_2B\} = G/B \setminus \{g_1B, g_2B\}$$

has number q if $g_1B = g_2B$, $q - 1$ if $g_1B \neq g_2B$.

Important Example. Let us show how Hecke algebra helps us solve triangles of flags. Consider the following triangle

$$\begin{array}{ccc}
 & g_1 B & \\
 u \swarrow & & \searrow v \\
 g_2 B & \xleftarrow{s_i} & g B
 \end{array}
 \quad
 \frac{\text{QUESTION}}{g_1 B, g_2 B \text{ are fixed}}$$

find the choice of $g B$.

The numerical answer is given by

$$\text{coefficient of } T_u \text{ in } T_v T_i = \begin{cases} T_{vs_i}, & vs_i > v, \\ (q-1)T_v + qT_{vs_i}, & vs_i < v. \end{cases}$$

There are three cases.

- If $v = vs_i < u$, then $g B$ is uniquely determined by $g_1 B$ and $g_2 B$, and the choice of $g B$ is a point.
- If $v = vs_i > u$, then the condition $g_1 B \xrightarrow{u} g B$ follows from $g B \xrightarrow{s_i} g_2 B$. As a result, the choice of $g B$ is $g_2 B s_i B / B \cong \mathbb{C}$.
- If $v = u$, then it is necessary to have $us_i > u$ (otherwise $g_1 \xrightarrow{us_i} g B$). In this case, the choice of $g B$ is $g_2 B s_i B \setminus \{g' B\} \cong \mathbb{C}^\times$ for $g' B$ the unique flag fits in the triangle $g_1 B \xrightarrow{vs_i} g' B \xrightarrow{s_i} g_2 B$.

The situation can be summarized as follows

	$v = u$	$v = vs_i$
$us_i > u$	\emptyset	\mathbb{F}
$us_i < u$	\mathbb{F}^\times	pt

2. BRAID VARIETIES

A **braid word** is a sequence of σ_i for $i \in I$. This notation comes from the definition of braid monoid/group

$$\langle \sigma_i \mid \text{braid relations} \rangle.$$

Note that the difference is, to generate a group, one needs to introduce σ_i^{-1} . A non-trivial fact is

$$\text{braid monoid} \subset \text{braid group}.$$

We will call an element of braid monoid a **positive braid**. For any $w \in W$, define σ_w the positive lifting of $w \in W$, i.e. $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_\ell}$ for any reduced word $w = s_{i_1} \cdots s_{i_\ell}$.

Definition ([1]). Let $\beta = \sigma_{i_1} \dots \sigma_{i_\ell}$ be a braid word and $u \in W$. The **braid variety** is defined to be

$$X(\beta, u) = \{(g_k B) : B \xrightarrow{s_{i_1}} g_1 B \xrightarrow{s_{i_2}} \dots g_{\ell-1} B \xrightarrow{s_{i_\ell}} uB\}.$$

Note that two braids β_1, β_2 related by braid relations, the defining braid varieties are isomorphic. Actually, the isomorphism does not depend on the choice of way of applying braid relations, so $X(\beta, u)$ is an invariant of the positive braid β .

Theorem. We have

$$|X(\beta, u)(\mathbb{F}_q)| = \text{coefficient of } T_u \text{ in } T_\beta := T_{i_1} \dots T_{i_\ell}.$$

Example. The open Richardson variety

$$\begin{aligned} \mathring{R}_{u,w} &= \{gB : B \xrightarrow{w} gB \xleftarrow{w_0 u} w_0 B\} \\ &= \{gB : B \xrightarrow{w} gB \xrightarrow{u^{-1} w_0} w_0 B\} = X(\sigma_w \sigma_{u^{-1} w_0}, w_0). \end{aligned}$$

Here we picking a reduced word for w and $u^{-1} w_0$ respectively. In particular,

$$|\mathring{R}_{u,w}(\mathbb{F}_q)| = \text{coefficient of } T_{w_0} \text{ in } T_w T_{u^{-1} w_0}.$$

There is a classical trick in Hecke algebra that

$$(\text{coefficient of } T_{w_0} \text{ in } T_{v^{-1}}^{-1} T_{u^{-1} w_0}) = \delta_{uv} q^{-\ell(u)}.$$

On the other hand, recall the definition of R -polynomials

$$T_w = \sum_{u \in W} q^{\ell(u)} R_{u,w}(q) T_{u^{-1}}^{-1}.$$

Thus we can conclude

$$|\mathring{R}_{u,w}(\mathbb{F}_q)| = R_{u,w}(q).$$

Example. We can consider the open **twisted flag variety**

$$\begin{aligned} Z(\beta, u) &= \{(g_k B) : B \xrightarrow{s_{i_1}} \dots \xrightarrow{s_{i_\ell}} g_\ell B \xleftarrow{w_0 u} w_0 B\} \\ &= X(\beta \sigma_{u^{-1} w_0}, w_0). \end{aligned}$$

When u is the Demazure product of β ,

$$X(\beta, u) = Z(\beta, u) = X(\beta \sigma_{u^{-1} w_0}, w_0).$$

One of the most important case is when

$$u = \text{Demazure product of } \beta = s_{i_1} * \dots * s_{i_\ell}. \quad (*)$$

Recall the Demazure product is a monoid structure over W characterized by

$$s_i * s_i = s_i, \quad \text{braid relations.}$$

It satisfies

$$s_i * w = \max(s_i w, w), \quad w * s_i = \max(w, w s_i).$$

The geometric meaning of Demazure product is the following. The subset $BuB \cdot BvB = BuBvB$ contains a unique dense BwB for some w , then the index $w = u * v$. That is,

$$\overline{BuB} \cdot \overline{BvB} = \overline{B(u * v)B}.$$

Theorem. When $(*)$ is true, $X(\beta, u)$ is nonempty and smooth.

Proof. Consider the map

$$\mathbf{proj} : \{(g_k B) \mid B \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_\ell}} g_\ell B\} \longrightarrow G/B, \quad (g_k B) \mapsto g_\ell B.$$

Note that $X(\beta, u) = \mathbf{proj}^{-1}(uB)$. Since \mathbf{proj} a morphism is between two smooth varieties, the preimage of a generic point of the image is nonempty and smooth. Since \mathbf{proj} is B -equivariant, we need to show the image contains BuB/B as a dense subset. Note that the image is $Bs_{i_1}Bs_{i_1}B \cdots Bs_{i_\ell}B/B$ contains BuB/B as a dense subset. \square

Corollary. Under $(*)$, the dimension of $X(\beta, u)$ is $\ell(\beta) - \ell(u)$.

Proof. The number $\ell(\beta)$ is the dimension of the domain of \mathbf{proj} and the number $\ell(u)$ is the dimension of the image of \mathbf{proj} . \square

Example. Assume the Demazure product of β is w_0 . We have

$$\begin{aligned} X(\beta\sigma_{w_0}, \text{id}) &= \{(g_k B) : B \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_\ell}} g_\ell B \xrightarrow{w_0} B\} \\ &= X(\beta, w_0) \times Bw_0B/B. \end{aligned}$$

We notice that $Z(\beta\sigma_{w_0}, e)$ is the preimage of Schubert cell Bw_0B/B and $X(\beta\sigma_{w_0}, e)$ is the preimage of the point w_0B/B . But the map \mathbf{proj} is B -equivariant.

REFERENCES

- [1] Roger Casals, Eugene Gorsky, Mikhail Gorsky, José Simental. Algebraic weaves and braid varieties. *American Journal of Mathematics*, [arXiv:2012.06931](#). [2](#)
- [2] N. Iwahori, On the structure of a Hecke ring of a Chevalley group over a finite field. [1.2](#)