

# BRAID VARIETIES (III)

2026/01/17

From now on, we will assume  $G$  to be simply-laced, i.e. ADE type. We will denote

$$X(\beta) = X(\beta, w_0), \quad w_0 = \text{Demazure product of } \beta.$$

Usually  $w_0$  means the longest element of  $W$ , but we save this notation since by an example discussed in the first note, we can assume  $w_0$  to be the longest element of  $W$ .

## 1. DEMAZURE WEAVES

**1.1. Demazure weaves.** For a braid word  $\beta = \sigma_{i_1} \dots \sigma_{i_\ell}$ , recall the dense stratification described last time is parametrized by the sequence

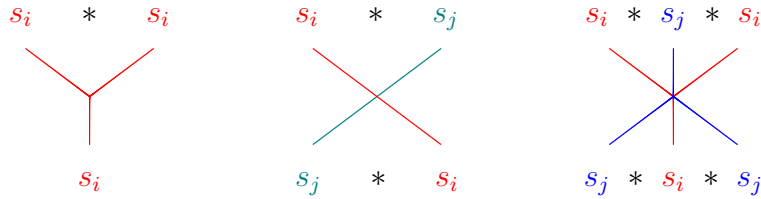
$$(\dots, s_{i_{\ell-2}} s_{i_{\ell-1}} s_{i_\ell}, s_{i_{\ell-1}} s_{i_\ell}, s_{i_\ell}, \text{id}).$$

This could be viewed the algorithm of computing the Demazure product  $s_{i_1} * \dots * s_{i_\ell}$  (from right to left). Besides of this way, thanks to the associativity of Demazure product, there are different ways. For example,

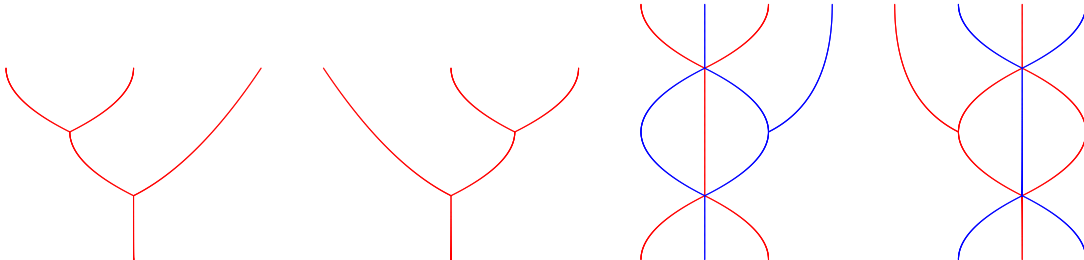
<p>(type <math>A_1</math>)</p> $(\sigma_1 \sigma_1) \sigma_1 = \sigma_1 \sigma_1 = \sigma_1$ $\sigma_1 (\sigma_1 \sigma_1) = \sigma_1 \sigma_1 = \sigma_1,$	<p>(type <math>A_2</math>)</p> $(\sigma_1 \sigma_2 \sigma_1) \sigma_2 = \sigma_2 \sigma_1 (\sigma_2 \sigma_2) = \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1$ $\sigma_1 (\sigma_2 \sigma_1 \sigma_2) = (\sigma_1 \sigma_1) \sigma_2 \sigma_1 = \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2.$
---	---

Demazure weave is a way of remember different ways.

A **Demazure weave** consists of



For example, the following diagrams illustrate the examples above.



1.2. **Weave varieties** ([1]). For a weave  $\mathfrak{W} : \beta_1 \rightarrow \beta_2$ , we can define **weave variety**

$$X(\mathfrak{W}) = \left\{ (g_r B)_{r \in \text{Region}(\mathfrak{W})} \left| \begin{array}{l} a \stackrel{i}{\mid} b \Rightarrow g_b B \xrightarrow{s_i} g_a B \\ g_{B_{\text{leftmost}}} = B, g_{B_{\text{rightmost}}} = w_0 B \end{array} \right. \right\}.$$

For example, when  $\mathfrak{W} : \beta \rightarrow \beta$  consists of only vertical edges,  $X(\mathfrak{W})$  is nothing but the braid variety  $X(\beta)$ . It is not hard to obtain the following local behavior at each vertex of a weave:

$$\left\{ \begin{array}{ccc} & \xrightarrow{s_i} gB & \xrightarrow{s_i} \\ g_1 B & \xrightarrow{s_i} & g_2 B \end{array} \right\} \xrightarrow{\mathbb{C}^\times\text{-bundle}} \{g_1 B \xrightarrow{s_i} g_2 B\}$$

$$\xrightarrow{\text{open}} \{g_1 B \xrightarrow{s_i} gB \xrightarrow{s_i} g_2 B\}$$

When  $m_{ij} = 2$ ,

$$\left\{ \begin{array}{ccccc} & & \xrightarrow{s_i} gB & \xrightarrow{s_j} & \\ g_1 B & & & & g_2 B \\ & & \xrightarrow{s_j} g'B & \xrightarrow{s_i} & \end{array} \right\} \xrightarrow{\sim} \{g_1 B \rightarrow gB \rightarrow g_2 B\}$$

$$\xrightarrow{\sim} \{g_1 B \rightarrow g'B \rightarrow g_2 B\}$$

When  $m_{ij} = 3$

$$\left\{ \begin{array}{ccccccc} & & \xrightarrow{s_i} gB & \xrightarrow{s_j} & g'B & \xrightarrow{s_i} & \\ g_1 B & & & & & & g_2 B \\ & & \xrightarrow{s_j} g''B & \xrightarrow{s_i} & g'''B & \xrightarrow{s_j} & \end{array} \right\} \xrightarrow{\sim} \{g_1 B \rightarrow gB \rightarrow g'B \rightarrow g_2 B\}$$

$$\xrightarrow{\sim} \{g_1 B \rightarrow g''B \rightarrow g'''B \rightarrow g_2 B\}$$

**Theorem.** Let  $\mathfrak{W} : \beta_1 \rightarrow \beta_2$  be a Demazure weave.

- the forgetful map

$$X(\mathfrak{W}) \rightarrow X(\beta_1)$$

is always an open embedding;

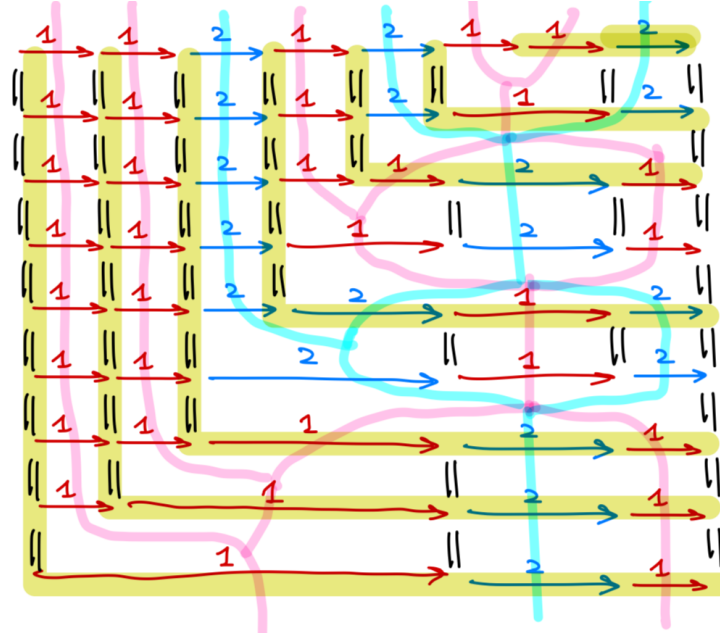
- if  $\mathfrak{W}$  consists only of braid moves, the upper and lower projections are both isomorphisms:

$$X(\mathfrak{W}) \xrightarrow{\sim} X(\beta_i) \quad i = 1, 2.$$

In particular, they give an isomorphism  $X(\beta_1) \cong X(\beta_2)$ .

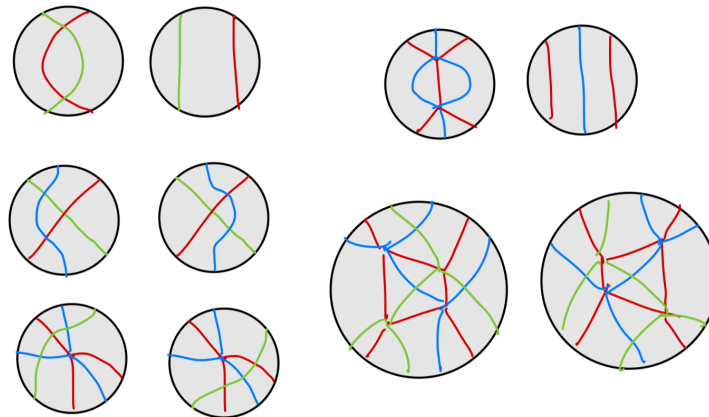
**Example.** The weave corresponds to the algorithm of computing the Demazure product from right to left is called **inductive weave**. The maximal stratum  $X(\beta, \mathbf{u})$  is the image

of the inductive  $\mathfrak{W}$ .

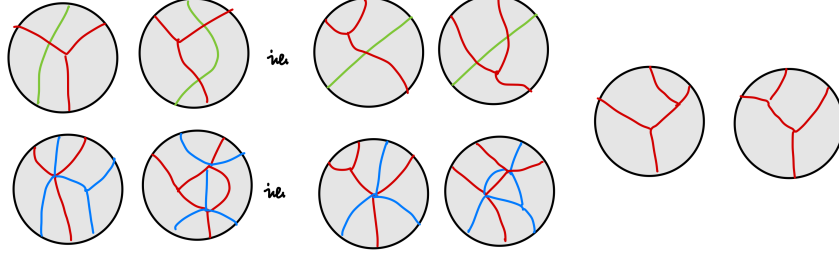


**Lemma** ([2, Lemma 4.4]). Let  $\beta_1, \beta_2$  be two braid words.

- (1) Different weaves consisting only of braid moves are related by **two-braid (Zamolodchikov) relations**



- (2) In general, different weaves are related by above relations as well as the **1212 relations** (left) and **mutations** (right)



*Proof.* Consider all possible positions in a braid word where one can apply  $\sigma_i \sigma_i \rightarrow \sigma_i$  the braid moves. If such positions do not overlap, the operations commute. If they overlap, then these involves at most 3 different simple reflections. So it reduces to  $A_1^2, A_2 \times A_1, A_3$ . In type  $A$ , this is a theorem of Elias.  $\square$

**Theorem** ([1, Theorem 4.12]). Let  $\beta_1, \beta_2$  be two braid words. For two weaves  $\mathfrak{W}_1$  and  $\mathfrak{W}_2$  related by two-braid relations and 1212 relations,  $X(\mathfrak{W}_1)$  and  $X(\mathfrak{W}_2)$  give the same open subset in  $X(\beta_1)$ .

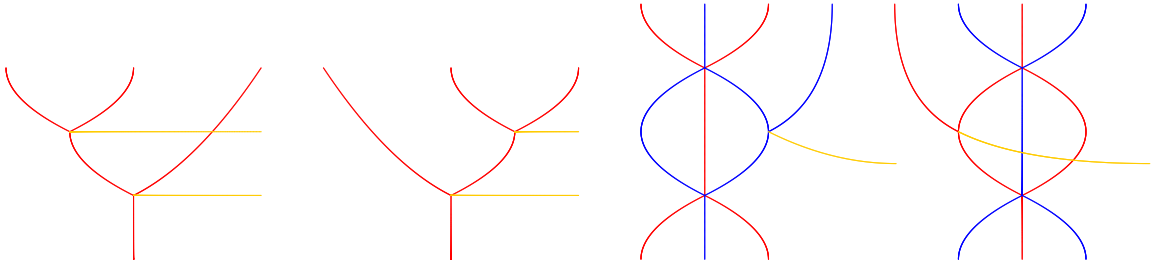
To prove this theorem, we need to parametrize  $X(\mathfrak{W})$ , as we did for braid variety  $X(\beta)$ .

## 2. PARAMETRIZATION

Recall that

$$B_i(z) = x_i(z) \dot{s}_i = \text{image of } \begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix} \in G.$$

Let  $\mathfrak{W}^=$  be the diagram obtained by adding dashed edges to trivalent vertices.



Note that there are more regions in  $\mathfrak{W}^=$  than  $\mathfrak{W}$ . We define the **parametrization**

$$X(\mathfrak{W}^=) = \left\{ (g_r)_{r \in \text{Region}(\mathfrak{W}^=)} \left| \begin{array}{l} a \mid^i b \Rightarrow g_b = g_a B_i(z_e) \text{ for some } z_e \\ \cdot \cdot \cdot \Rightarrow g_a = g_b U_e \text{ for some } U_e \in B \\ b \\ g_{\text{leftmost}} = 1, g_{\text{rightmost}} \in w_0 B \end{array} \right. \right\}.$$

In particular, we can also identify

$$X(\mathfrak{W}^=) = \left\{ (z_e)_{e \in \text{Edge}(\mathfrak{W}^=)} \times (U_d)_{d \in \text{Dash}(\mathfrak{W}^=)} \left| \begin{array}{l} z_e \in \mathbb{C}, U_d \in B \\ (\mathbf{Comm}) \text{ and } (\mathbf{Right}) \end{array} \right. \right\}.$$

Here **(Comm)** means at each vertex, a corresponding diagram commutes; see Appendix. Since the diagram commutes, it is well-defined to associate  $g_r$  for each region  $r$  of  $\mathfrak{W}^=$ , **(Right)** means the rightmost region is in  $w_0 B$ .

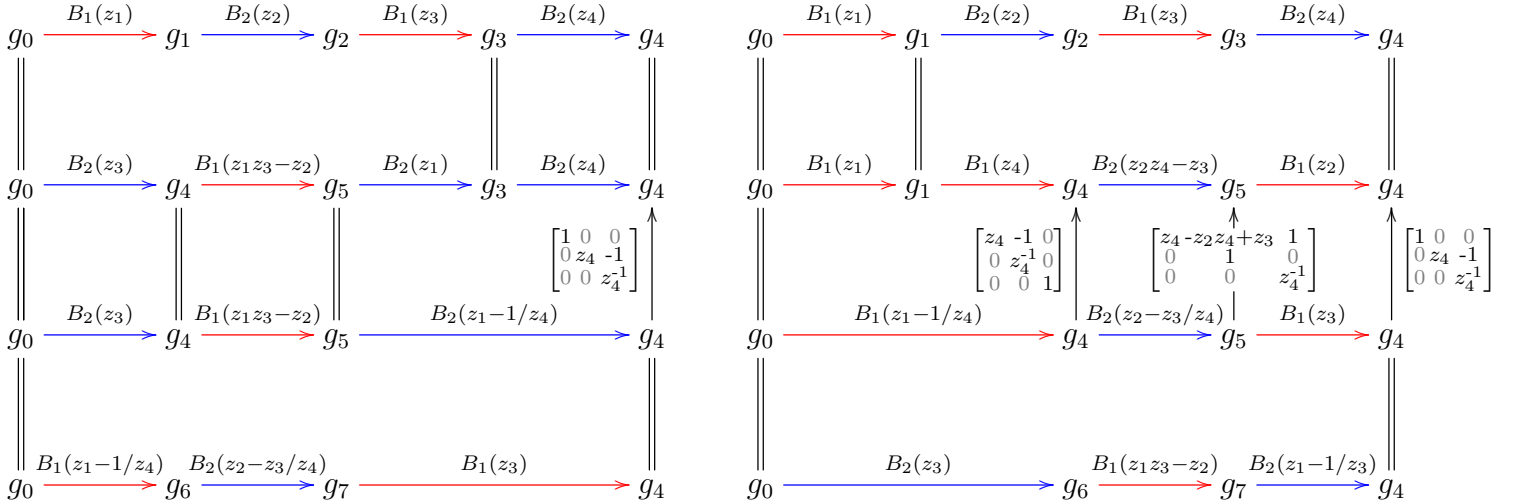
**Proposition.** The natural projection

$$X(\mathfrak{W}^=) \rightarrow X(\mathfrak{W}), \quad (g_r) \mapsto (g_r B)$$

is an isomorphism.

*Proof.* This map is obviously surjective. Let us prove it is injective. At each vertex, the top/left labeling determines the bottom/right labeling (computed in Appendix). Thus the top  $z$ -labeling determines all  $z$ -labeling. This corresponds to a unique element in  $X(\beta)$  for  $\beta$  the top braid.  $\square$

**Example.** Consider the two weaves above.



**Proposition.** Fix a reduced word  $w_0$ .

(1) For a given top  $z$ -labeling, the bottom  $z$ -labeling do not depend on the choice of Demazure weave  $\beta \rightarrow \sigma_{w_0}$ .

(2) The condition **(Right)** is equivalent to the vanishing of the bottom  $z$ -labeling.

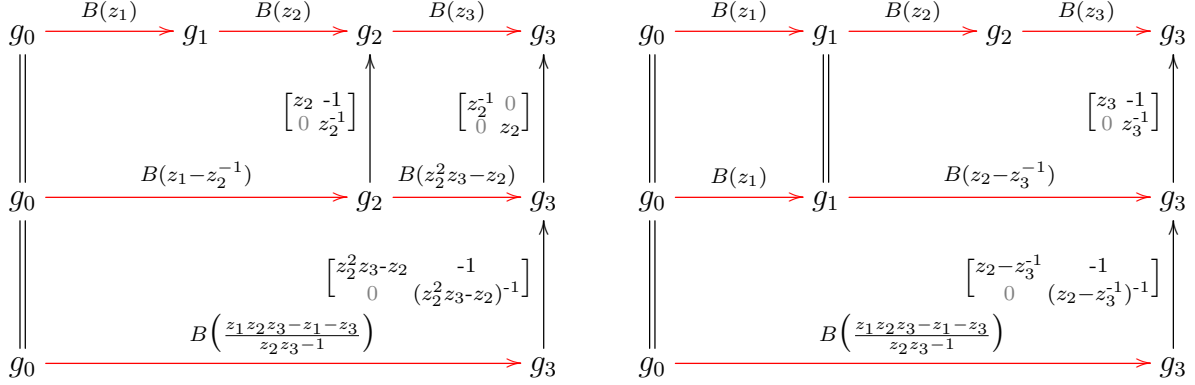
(3) The condition **(Comm)** is equivalent to the nonvanishing of northeast  $z$ -labeling of trivalent vertices.

*Proof.* For any reduced word  $w_0 = s_{i_1} \cdots s_{i_l}$ , we have an isomorphism

$$\mathbb{C}^{\ell(w_0)} \xrightarrow{\sim} Bw_0B/B, \quad (z'_k) \mapsto B_{i_1}(z'_1) \cdots B_{i_l}(z'_l)B.$$

(1) As a result, the bottom  $z$ -labeling is given by the unique  $(z'_k)$  mapped to the flag parametrized by the product of top  $B$ -matrices. (2) In particular, only the vanishing  $(z'_k)$  could satisfy **(End)**. (3) is clear.  $\square$

**Example.** We remark that mutation does not give the same open subsets. Consider  $\sigma_1^3 \rightarrow \sigma_1$ .



The open subsets are described as

$$z_2 \neq 0, z_2^3 z_3 - z_2 \neq 0, \quad \text{v.s.} \quad z_3 \neq 0, z_2 - z_3^{-1} \neq 0.$$

*Proof of the theorem.* It suffices to check two weaves related by a single Zamolodchikov relation or 1212 relation. By picking dashed edges  $\mathfrak{W}^-$  avoiding their difference, we will define an isomorphism over  $X(\mathfrak{W}_1^-) \rightarrow X(\mathfrak{W}_2^-)$  such that, if we write  $(z_{1e}) \mapsto (z_{2e})$ , then  $z_{2e} = z_{1e}$  for edges  $e$  not inside the difference (i.e. edges outside of the difference and edges  $e$  intersecting the boundary of the difference). Such an isomorphism will be the required isomorphism.

By the solution of **(Comm)**, we can express  $z_e$  for  $e \in \text{inside} \cup \text{lower}$  from  $z_e$  for  $e \in \text{upper}$ . This will determine an isomorphism, and it remains to check it is well-defined. That is,

- (1)  $z_e$  for edges  $e$  intersecting the lower boundary agrees;
- (2) no new pole of  $z_e$  was introduced.

For two-braid relations, since no pole is introduced, so it suffices to check (1). This can be done by computation, but there is a quicker way of seeing this. Noting that the lower boundary of each two-braid relation is a reduced word, by (1) of Proposition above, (1) is true.

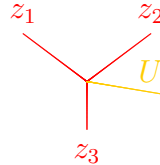
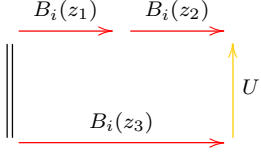
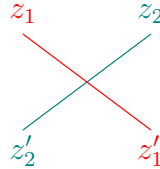
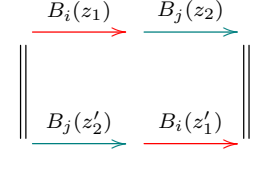
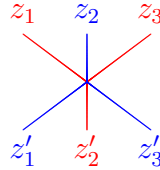
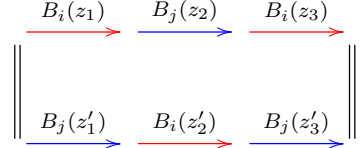
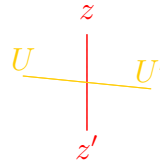
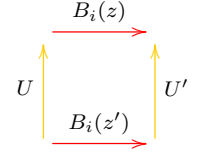
For 1212 relations, (1) is similar. But in this case, a pole is introduced. From the example above, the pole is the same, given by  $z_4 \neq 0$ . This proves (2).  $\square$

## REFERENCES

- [1] Roger Casals, Eugene Gorsky, Mikhail Gorsky, José Simental. Algebraic weaves and braid varieties. *American Journal of Mathematics*, [arXiv:2012.06931](#). [1.2](#), [1.2](#)
- [2] Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, José Simental. Cluster structures on braid varieties. *Journal of the American Mathematical Society*, 2025. [arXiv:2207.11607](#). [1.2](#)

## APPENDIX A. EXPLICIT SOLUTIONS

We can solve the equations explicitly.

		$z_3 = z_1 - z_2^{-1}$ $U = \text{image of } \begin{bmatrix} z_2 & -1 \\ 0 & z_2^{-1} \end{bmatrix}$
		$z'_1 = z_1$ $z'_2 = z_2$
		$z'_1 = z_3$ $z'_3 = z_1$ $z'_2 = z_1 z_3 - z_2$
		$z' = z \cdot \alpha_i(U) + \xi_i(U)$ $U' = B_i(z')^{-1} U B_i(z) \in B$

Let us explain  $\alpha_i$  and  $\xi_i$ . We can write  $U = U_1 U_2$  for  $U_1 \in \text{Rad}(B)$  and  $U_2 \in T$ . Then

$$\xi_i(U) = \text{coefficient of } E_i \text{ in } U_1, \quad \eta_i(U) = \alpha_i(U).$$

For example, in  $GL_2$ ,

$$\begin{bmatrix} a & b \\ & d \end{bmatrix} = \begin{bmatrix} 1 & b/d \\ & 1 \end{bmatrix} \begin{bmatrix} a & \\ & d \end{bmatrix} \xrightarrow{\xi} b/d, \\ \xrightarrow{\eta} a/d.$$

All the relations follow from the computation in  $SL_2 \subset GL_2$ ,  $SL_2 \times SL_2 \subset GL_4$ ,  $SL_3 \subset GL_3$  and  $SL_2 \subset GL_2$ . The following is the code.

```
R.<z1,z2,z3,a,b,c> = QQ[];
B = lambda z: matrix([[z,-1],[1,0]])
U = matrix([[a,b],[0,c]]);
Rel = (B(z1)*B(z2) - B(z3)*U).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("z3,a,b,c"))
```

```
[[z3 == (z1*z2 - 1)/z2, a == z2, b == -1, c == (1/z2)]]
```

```
R.<z1,z2,z1p,z2p> = QQ[]
B1 = lambda z: matrix([[z,-1,0,0],[1,0,0,0],[0,0,1,0],[0,0,0,1]])
B3 = lambda z: matrix([[1,0,0,0],[0,1,0,0],[0,0,z,-1],[0,0,1,0]])
Rel = (B1(z1)*B3(z2)-B3(z2p)*B1(z1p)).change_ring(SR)
solve([Rel[i][j]==0 for i in range(4) for j in range(4)], SR.var("z1p,z2p"))
```

```
[[z1p == z1, z2p == r1]]
```

```
R.<z1,z2,z3,z1p,z2p,z3p> = QQ[]
B1 = lambda z: matrix([[z,-1,0],[1,0,0],[0,0,1]])
B2 = lambda z: matrix([[1,0,0],[0,z,-1],[0,1,0]])
Rel = (B1(z1)*B2(z2)*B1(z3)-B2(z1p)*B1(z2p)*B2(z3p)).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("z1p,z2p,z3p"))
```

```
[[z1p == z3, z2p == z1*z3 - z2, z3p == z1]]
```

```
R.<z,zp,a,b,c,ap,bp,cp> = QQ[]
B = lambda z: matrix([[z,-1],[1,0]])
U = matrix([[a,b],[0,c]]); Up = matrix([[ap,bp],[0,cp]])
Rel = (U*B(z)-B(zp)*Up).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("zp,ap,bp,cp"))
```

```
[[zp == (a*z + b)/c, ap == c, bp == 0, cp == a]]
```