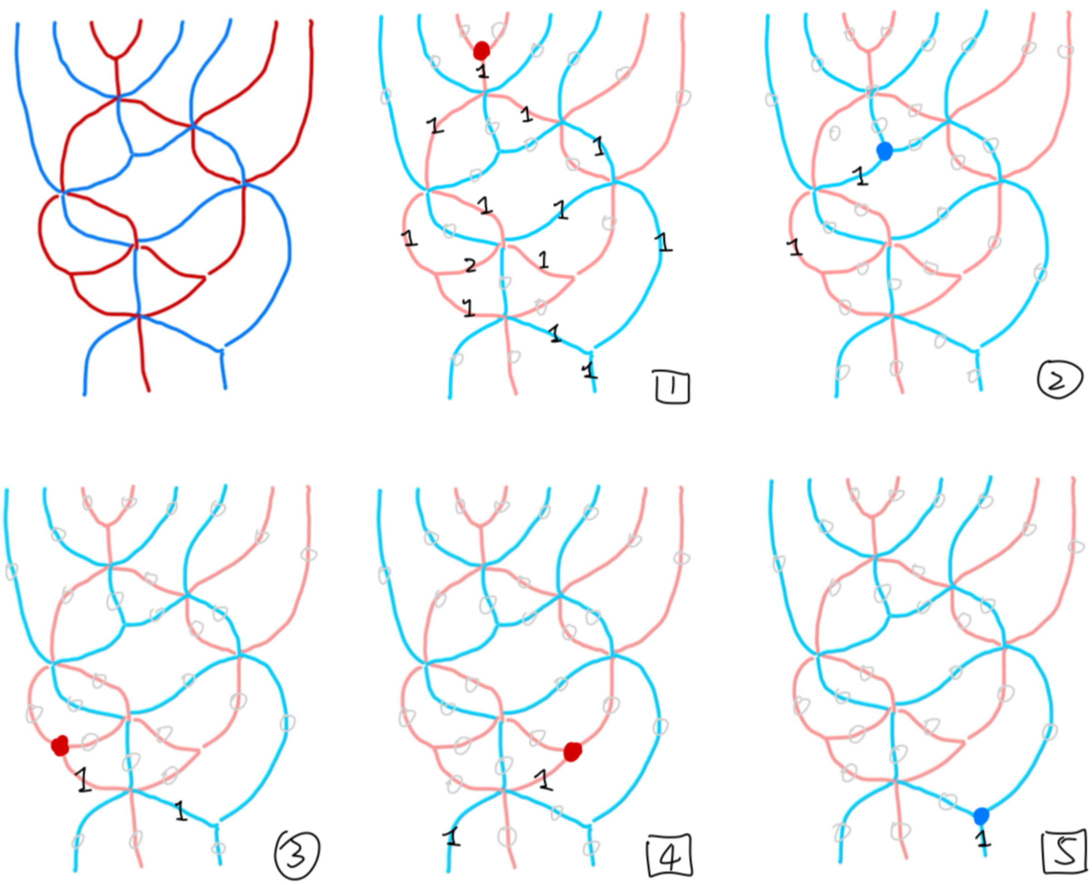
BRAID VARIETIES (VI)

2026/01/??

0.1. **Example ([1, Section 11.2]).** Consider



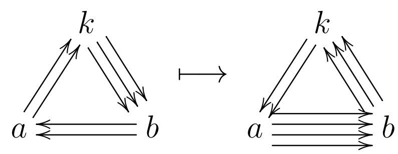
The quiver is

$$\begin{tikzcd}
2 \arrow[rr] & & 3 \arrow[rr] \arrow[dl] & & \boxed{5} \\
& \boxed{1} \arrow[ul] & & \boxed{4} \arrow[ul] &
\end{tikzcd}$$

1. Cluster algebras

1.1. **Mutations.** We assume all the quivers in this note have no 1-cycles and 2-cycles. An **ice quiver** is a quiver Q with vertices separated by frozen vertices and mutatable vertices (such that no arrows between frozen vertices). For a mutatable vertex k, we define the **mutation** $\mu_k(Q)$ a new quiver obtained by two steps:

- introduce a new arrow $i \to j$ for each pair of $i \to k \to j$ (unless i, j are frozen) and cancel pair of arrows between i and j of different orientation;
- reverse all arrows incident to k.



A **seed** is a pair $(\tilde{\mathbf{x}}, Q)$ where $\mathbf{x}$ is a family of algebraically independent variable parametrized by the vertices of an ice quiver Q. The **cluster variables** $\mathbf{x}$ are those corresponding to the mutable vertices; the **forzen variables** $\mathbf{y}$ are those corresponding to the mutable vertices. For a mutatable vertex k, we define the **mutation** $\mu_k(\tilde{\mathbf{x}}, Q) = (\tilde{\mathbf{x}}', \mu_k(Q))$, where $x'_i = x_i$ for $i \neq k$, and

$$x_k x'_k = \prod_{i \to k} x_i + \prod_{k \to j} x_j.$$

For example, in the above example

$$x_k x'_k = x_a^2 + x_b^3.$$

The information of quiver can be recorded in an anti-symmetric matrix $B = (b_{ij})$

$$b_{ij} = \#\{i \to j\} \text{ or } -\#\{j \to i\}.$$

Then the mutation relation can be written as

$$x_k x'_k = \prod_i x_i^{\max(b_{ik},0)} + \prod_j x_j^{\max(b_{kj},0)}.$$

1.2. **Cluster algebras.** For a seed $(\tilde{\mathbf{x}}, Q)$, we define the **cluster algebra** $A_{\tilde{\mathbf{x}},Q}$ to be the subalgebra of function field $\mathbb{Q}(\tilde{\mathbf{x}})$ generated by

$$\text{cluster variables } x'_i, \text{ frozen variables } y'_j, \text{ and the inverse of frozen variables } {y'_j}^{-1}$$

for all seed $(\tilde{\mathbf{x}}', Q')$ obtained by sequences of mutations. The **upper cluster algebra** $U_{\tilde{\mathbf{x}},Q}$ is the intersection of

$$\mathbb{Q}[{\tilde{\mathbf{x}}'}^{\pm 1}] = \text{Laurent polynomial ring in cluster variables } x'_i, \text{ frozen variables } y'_j$$

for all possible seeds $(\tilde{\mathbf{x}}', Q')$ obtained by sequences of mutations.

The famous Laurent phenomenon tells that $A_{\tilde{\mathbf{x}},Q} \subset U_{\tilde{\mathbf{x}},Q}$. If there is a seed where the mutable part of the quiver is acyclic, then $A_{\tilde{\mathbf{x}},Q} = U_{\tilde{\mathbf{x}},Q}$.

Geometrically, we have

$$\operatorname{Spec}(U_{\tilde{\mathbf{x}},Q}) = \bigcup_{\text{seeds }(\tilde{\mathbf{x}}',Q')} \operatorname{Spec}\mathbb{C}[\tilde{\mathbf{x}}'^{-1}].$$

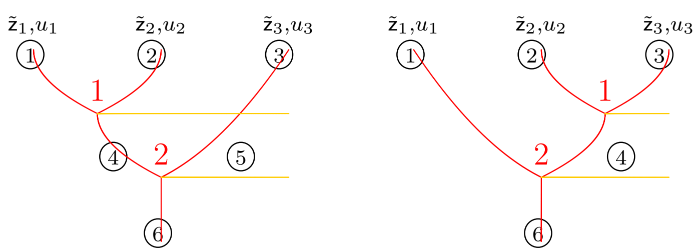
Note that each $\operatorname{Spec}\mathbb{C}[\tilde{\mathbf{x}}'^{-1}]$ is a torus. That is, cluster variety is obtained by gluing tori via mutation. An important 2-form is the **Gekhtman–Shapiro–Vainshtein form**

$$\omega = \sum_{i,j} b_{ij}\frac{dx_i dx_j}{x_i x_j}.$$

Note that $dx_i dx_j = -dx_j dx_i$, each term is "counted twice". Ignoring the term $\dfrac{dx_i dx_j}{x_i x_j}$ for i, j both frozen, this form is in invariant under mutation.

2. Mutation

2.1. **Relation of cluster variables.** Let us consider the mutations.



In the left diagram,

$$u_{④} = (u_1 \oplus u_2)\mathsf{A}_1, \qquad u_{⑥} = (u_{④} \oplus u_{⑤})\mathsf{A}_2 = (u_1 \oplus u_2 \oplus u_3)\mathsf{A}_2.$$

We have

$$\tilde{\mathsf{z}}_2 = \tilde{\mathsf{z}}_{②} = \mathsf{A}_1 \frac{u_1 \oplus u_2}{u_1 u_2}, \qquad \tilde{\mathsf{z}}_{⑤} = \mathsf{A}_2 \frac{u_{④} \oplus u_{⑤}}{u_{④} u_{⑤}} = \mathsf{A}_2 \mathsf{A}_1^{-1} \frac{u_1 \oplus u_2 \oplus u_3}{(u_1 \oplus u_2) u_3}.$$

Using the appendix of the fourth note, we have

$$\tilde{\mathsf{z}}_3 = \tilde{\mathsf{z}}_{③} = \tilde{\mathsf{z}}_{⑤} - a, \quad a = -\frac{u_{①}}{u_{②} u_{④}} \qquad \text{i.e.} \qquad \tilde{\mathsf{z}}_{⑤} = \tilde{\mathsf{z}}_3 - \mathsf{A}_1^{-1} \frac{u_1}{u_2(u_1 \oplus u_2)}.$$

We can solve

$$\begin{aligned}
\mathsf{A}_1 &= \frac{u_1 u_2}{u_1 \oplus u_2}\tilde{\mathsf{z}}_2, \\
\mathsf{A}_2 &= \mathsf{A}_1\left(\tilde{\mathsf{z}}_3 - \mathsf{A}_1^{-1}\frac{u_1}{u_2(u_1 \oplus u_2)}\right)\frac{(u_1 \oplus u_2)u_3}{u_1 \oplus u_2 \oplus u_3} \\
&= \frac{u_1 u_2 u_3}{u_1 \oplus u_2 \oplus u_3}\tilde{\mathsf{z}}_2\tilde{\mathsf{z}}_3 - \frac{u_1 u_3}{u_2(u_1 \oplus u_2 \oplus u_3)}.
\end{aligned}$$

In the right diagram

$$u_{④} = (u_2 \oplus u_3)\mathsf{A}_1, \qquad u_{⑥} = (u_{①} \oplus u_{④})\mathsf{A}_2 = (u_1 \oplus u_2 \oplus u_3)\mathsf{A}_2.$$

We have

$$\tilde{\mathsf{z}}_3 = \tilde{\mathsf{z}}_{③} = \mathsf{A}_1 \frac{u_2 \oplus u_3}{u_2 u_3}, \qquad \tilde{\mathsf{z}}_{④} = \mathsf{A}_2 \frac{u_{①} \oplus u_{④}}{u_{⑥}} = \mathsf{A}_2 \mathsf{A}_1^{-1} \frac{u_1 \oplus u_2 \oplus u_3}{u_1(u_2 \oplus u_3)}.$$

By (**Comm**), we have

$$\tilde{\mathsf{z}}_2 = \tilde{\mathsf{z}}_{②} = \tilde{\mathsf{z}}_{④} + \frac{u_{③}}{u_{②} u_{④}} \qquad \text{i.e.} \qquad z_{④} = \tilde{\mathsf{z}}_2 - \mathsf{A}_1^{-1} \frac{u_3}{u_2(u_2 \oplus u_3)}$$

We can solve

$$\begin{aligned}
\mathsf{A}_1 &= \frac{u_2 u_3}{u_2 \oplus u_3} \tilde{\mathsf{z}}_3 \\
\mathsf{A}_2 &= \mathsf{A}_1 \left(\tilde{\mathsf{z}}_2 - \mathsf{A}_1^{-1} \frac{u_3}{u_2(u_2 \oplus u_3)} \right) \frac{u_1(u_2 \oplus u_3)}{u_1 \oplus u_2 \oplus u_3} \\
&= \frac{u_1 u_2 u_3}{u_1 \oplus u_2 \oplus u_3} \tilde{\mathsf{z}}_2 \tilde{\mathsf{z}}_3 - \frac{u_1 u_3}{u_2(u_1 \oplus u_2 \oplus u_3)}.
\end{aligned}$$

We can conclude mutation do not change A_2, but it does change A_1. We distinguish two A_1's by A_1^L and A_1^R. The relation of them can be written as

$$\begin{aligned}
\mathsf{A}_1^L \mathsf{A}_1^R &= \frac{u_1 u_2^2 u_3}{(u_1 \oplus u_2)(u_2 \oplus u_3)} \tilde{\mathsf{z}}_2 \tilde{\mathsf{z}}_3 \\
&= \mathsf{A}_2 \frac{u_2(u_1 \oplus u_2 \oplus u_3)}{(u_1 \oplus u_2)(u_2 \oplus u_3)} + \frac{u_1 u_3}{(u_1 \oplus u_2)(u_2 \oplus u_3)}.
\end{aligned}$$

We will realize this as a mutation of cluster variables.

Theorem. Let the top u-labeling be A, B, C and the bottom u-labeling D. Note that they do not contain A_1. We have

$$\mathsf{A}_1^L \mathsf{A}_1^R = \left[\frac{BD}{AC} \right] + \left[\frac{AC}{BD} \right]$$

where $[\mathsf{A}^{\mathbf{m}}] = \mathsf{A}^{\max(\mathbf{m}, \mathbf{0})}$. Note that $A = u_1, B = u_2, C = u_3, D = (u_1 \oplus u_2 \oplus u_3)\mathsf{A}_2$.

Proof. Let $v \neq 1, 2$. The contribution of A_v of the two terms of $\mathsf{A}_1^L \mathsf{A}_1^R$ can be summarized as follows

$u_i = \mathsf{A}_v^{a_i} \cdots$	$\frac{u_2(u_1 \oplus u_2 \oplus u_3)}{(u_1 \oplus u_2)(u_2 \oplus u_3)}$	$\frac{u_1 u_3}{(u_1 \oplus u_2)(u_2 \oplus u_3)}$
$a_1 < a_2 < a_3$	1	$\mathsf{A}_v^{a_3 - a_2}$
$a_2 < a_1 < a_3$	1	$\mathsf{A}_v^{a_1 + a_3 - 2a_2}$
$a_1 < a_3 < a_2$	$\mathsf{A}_v^{a_2 - a_3}$	1
$a_2 < a_3 < a_1$	1	$\mathsf{A}_v^{a_1 + a_3 - 2a_2}$
$a_3 < a_1 < a_2$	$\mathsf{A}_v^{a_2 - a_2}$	1
$a_3 < a_2 < a_1$	1	$\mathsf{A}_v^{a_1 - a_2}$

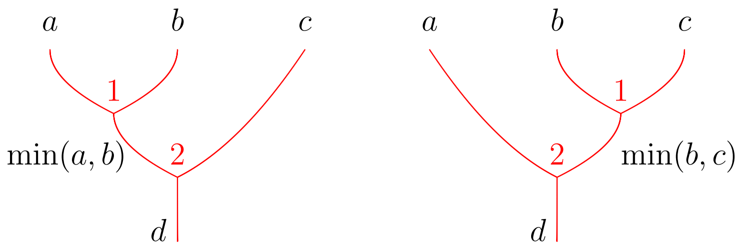
Note that their quotient is

$$\frac{\dfrac{u_2(u_1 \oplus u_2 \oplus u_3)}{(u_1 \oplus u_2)(u_2 \oplus u_3)}}{\dfrac{u_1 u_3}{(u_1 \oplus u_2)(u_2 \oplus u_3)}} = \frac{u_2(u_1 \oplus u_2 \oplus u_3)}{u_1 u_3} = \frac{BD}{AC}.$$

One can check the contribution is the same as $[BD/AC]$.

For $v = 2$, this can be checked directly. $\square$

2.2. **Quiver Mutation.** Assume the exponent of A_v in u-labeling is given by



In the left diagram

$$\#\{1^L \to v\} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ a & \min(a,b) & b \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \min(a,b) & d & c \end{vmatrix} = (b-a) + (d-c).$$

In the right diagram

$$\#\{1^R \to v\} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ b & \min(b,c) & c \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ a & d & \min(b,c) \end{vmatrix} = (c-b) + (a-d).$$

As a result, the arrows incident to 1 are reversed. Moreover, they are exponent of A_v in AC/BD and BD/AC respectively.

For two vertices v_1, v_2, in the left diagram

$$L := \#\{v_1 \to v_2\} = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & \min(a_1,b_1) & b_1 \\ a_2 & \min(a_2,b_2) & b_2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ \min(a_1,b_1) & d_1 & c_1 \\ \min(a_2,b_2) & d_2 & c_2 \end{vmatrix} + \cdots$$

In the right diagram

$$R := \#\{v_1 \to v_2\} = \begin{vmatrix} 1 & 1 & 1 \\ b_1 & \min(b_1,c_1) & c_1 \\ b_2 & \min(b_2,c_2) & c_2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a_1 & d_1 & \min(b_1,c_1) \\ a_2 & d_2 & \min(b_2,c_2) \end{vmatrix} + \cdots$$

We have

$$R = L + \begin{cases} \#\{v_1 \to 1^L\} \cdot \#\{1^L \to v_2\}, & \text{if they have the same signs,} \\ 0, & \text{otherwise.} \end{cases}$$

For $v_1, v_2 \neq 2$, $d_i = \min(a_i, b_i, c_1)$. This is by examining directly all 36 cases. For $v = 2$, this follows from direct computation.

From the discussion above

Theorem. The mutation of weaves corresponds to quiver mutation, and the change of variables corresponds to mutation of cluster variables.

It was proved in [1] that $\mathbb{C}[X(\beta)] \cong A_{\mathsf{A},Q} \cong U_{\mathsf{A},Q}$.

References

[1] Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, José Simental. Cluster structures on braid varieties. *Journal of the American Mathematical Society*, 2025. arXiv:2207.11607. 0.1, 2.2