

BRAID VARIETIES (III)

2026/01/17

From now on, we will assume G to be simply-laced, i.e. ADE type. We will denote

$$X(\beta) = X(\beta, w_0), \quad w_0 = \text{Demazure product of } \beta.$$

Usually w_0 means the longest element of W , but we save this notation since by an example discussed in the first note, we can assume w_0 to be the longest element of W .

1. DEMAZURE WEAVES

1.1. Demazure weaves. For a braid word $\beta = \sigma_{i_1} \dots \sigma_{i_\ell}$, recall the dense stratification described last time is parametrized by the sequence

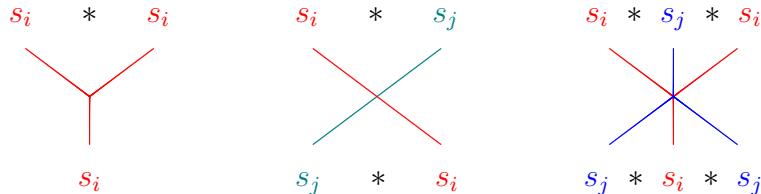
$$(\dots, s_{i_{\ell-2}} s_{i_{\ell-1}} s_{i_\ell}, s_{i_{\ell-1}} s_{i_\ell}, s_{i_\ell}, \text{id}).$$

This could be viewed the algorithm of computing the Demazure product $s_{i_1} * \dots * s_{i_\ell}$ (from right to left). Besides of this way, thanks to the associativity of Demazure product, there are different ways. For example,

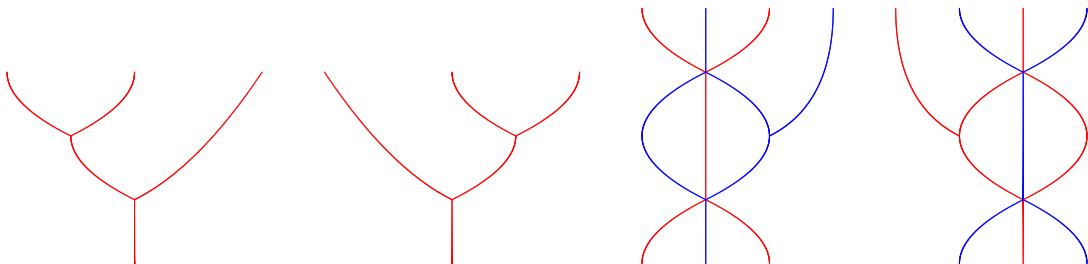
(type A_1)	(type A_2)
$(\sigma_1 \sigma_1) \sigma_1 = \sigma_1 \sigma_1 = \sigma_1$	$(\sigma_1 \sigma_2 \sigma_1) \sigma_2 = \sigma_2 \sigma_1 (\sigma_2 \sigma_2) = \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1$
$\sigma_1 (\sigma_1 \sigma_1) = \sigma_1 \sigma_1 = \sigma_1,$	$\sigma_1 (\sigma_2 \sigma_1 \sigma_2) = (\sigma_1 \sigma_1) \sigma_2 \sigma_1 = \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2.$

Demazure weave is a way of remember different ways.

A **Demazure weave** consists of



For example, the following diagrams illustrate the examples above.



1.2. **Weave varieties** ([1]). For a weave $\mathfrak{W} : \beta_1 \rightarrow \beta_2$, we can define **weave variety**

$$X(\mathfrak{W}) = \left\{ (g_r B)_{r \in \text{Region}(\mathfrak{W})} \mid \begin{array}{l} a \xrightarrow{i} b \Rightarrow g_b B \xrightarrow{s_i} g_a B \\ gB_{\text{leftmost}} = B, gB_{\text{rightmost}} = w_0 B \end{array} \right\}.$$

For example, when $\mathfrak{W} : \beta \rightarrow \beta$ consists of only vertical edges, $X(\mathfrak{W})$ is nothing but the braid variety $X(\beta)$. It is not hard to obtain the following local behavior at each vertex of a weave:

$$\left\{ g_1 B \xrightarrow[s_i]{s_i} gB \xrightarrow[s_i]{s_i} g_2 B \right\} \xrightarrow{\mathbb{C}^\times\text{-bundle}} \{g_1 B \xrightarrow{s_i} g_2 B\}$$

$$\xrightarrow{\text{open}} \{g_1 B \xrightarrow{s_i} gB \xrightarrow{s_i} g_2 B\}$$

When $m_{ij} = 2$,

$$\left\{ g_1 B \xrightarrow[s_i]{s_i} gB \xrightarrow[s_j]{s_j} g_2 B \right\} \xrightarrow{\sim} \{g_1 B \rightarrow gB \rightarrow g_2 B\}$$

$$\xrightarrow{\sim} \{g_1 B \rightarrow g'B \rightarrow g_2 B\}$$

When $m_{ij} = 3$

$$\left\{ g_1 B \xrightarrow[s_i]{s_i} gB \xrightarrow[s_j]{s_j} g'B \xrightarrow[s_i]{s_i} g_2 B \right\} \xrightarrow{\sim} \{g_1 B \rightarrow gB \rightarrow g'B \rightarrow g_2 B\}$$

$$\xrightarrow{\sim} \{g_1 B \rightarrow g''B \rightarrow g'''B \rightarrow g_2 B\}$$

Theorem. Let $\mathfrak{W} : \beta_1 \rightarrow \beta_2$ be a Demazure weave.

- the forgetful map

$$X(\mathfrak{W}) \rightarrow X(\beta_1)$$

is always an open embedding;

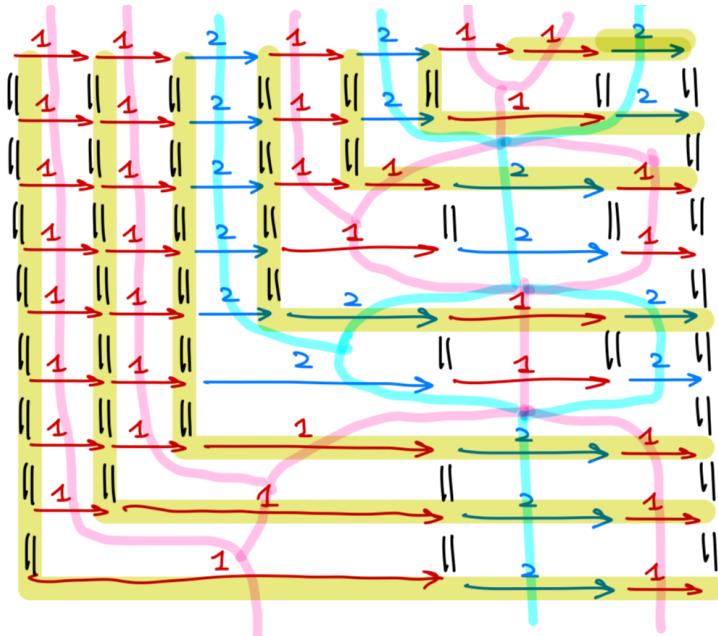
- if \mathfrak{W} consists only of braid moves, the upper and lower projections are both isomorphisms:

$$X(\mathfrak{W}) \xrightarrow{\sim} X(\beta_i) \quad i = 1, 2.$$

In particular, they give an isomorphism $X(\beta_1) \cong X(\beta_2)$.

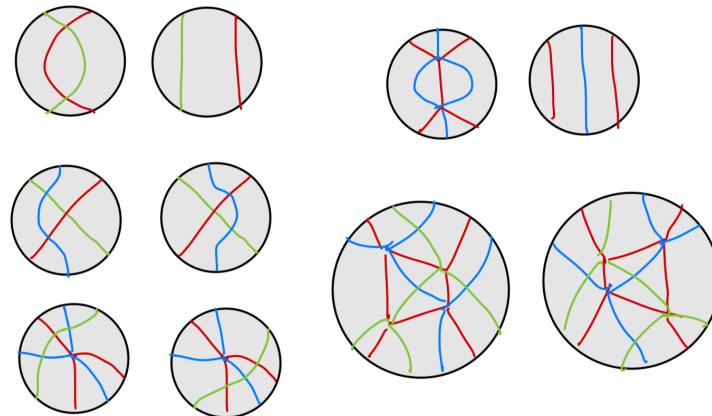
Example. The weave corresponds to the algorithm of computing the Demazure product from right to left is called **inductive weave**. The maximal stratum $X(\beta, \underline{u})$ is the image

of the inductive \mathfrak{W} .

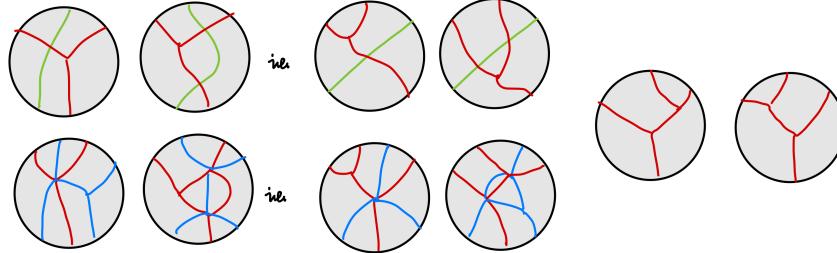


Lemma ([2, Lemma 4.4]). Let β_1, β_2 be two braid words.

- (1) Different weaves consisting only of braid moves are related by **two-braid (Zamolodchikov) relations**



- (2) In general, different weaves are related by above relations as well as the **1212 relations** (left) and **mutations** (right)



Proof. Consider all possible positions in a braid word where one can apply $\sigma_i \sigma_i \rightarrow \sigma_i$ the braid moves. If such positions do not overlap, the operations commute. If they overlap, then these involves at most 3 different simple reflections. So it reduces to $A_1^2, A_2 \times A_1, A_3$. In type A , this is a theorem of Elias. \square

Theorem ([1, Theorem 4.12]). Let β_1, β_2 be two braid words. For two weaves \mathfrak{W}_1 and \mathfrak{W}_2 related by two-braid relations and 1212 relations, $X(\mathfrak{W}_1)$ and $X(\mathfrak{W}_2)$ give the same open subset in $X(\beta_1)$.

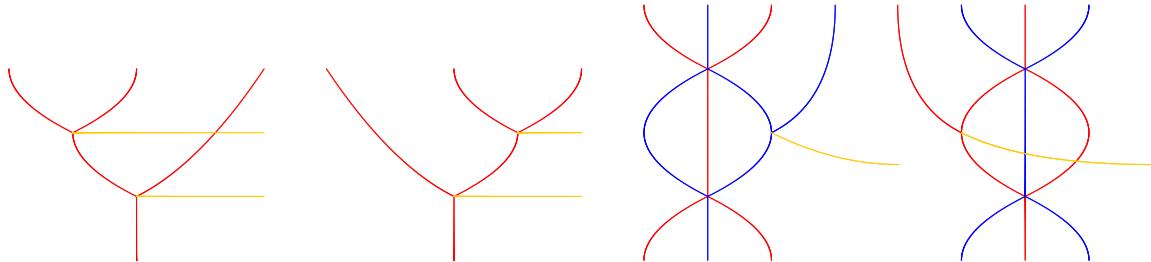
To prove this theorem, we need to parametrize $X(\mathfrak{W})$, as we did for braid variety $X(\beta)$.

2. PARAMETRIZATION

Recall that

$$B_i(z) = x_i(z) \dot{s}_i = \text{image of } \begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix} \in G.$$

Let \mathfrak{W}^\pm be the diagram obtained by adding dashed edges to trivalent vertices.



Note that there are more regions in \mathfrak{W}^\pm than \mathfrak{W} . We define the **parametrization**

$$X(\mathfrak{W}^\pm) = \left\{ (g_r)_{r \in \text{Region}(\mathfrak{W}^\pm)} \left| \begin{array}{l} a \stackrel{i}{\mid} b \Rightarrow g_b = g_a B_i(z_e) \text{ for some } z_e \\ \dots \Rightarrow g_a = g_b U_e \text{ for some } U_e \in B \\ g_b \mid b \quad g_{\text{leftmost}} = 1, g_{\text{rightmost}} \in w_0 B \end{array} \right. \right\}.$$

In particular, we can also identify

$$X(\mathfrak{W}^=) = \left\{ (z_e)_{e \in \text{Edge}(\mathfrak{W}^=)} \times (U_d)_{d \in \text{Dash}(\mathfrak{W}^=)} \mid \begin{array}{l} z_e \in \mathbb{C}, U_d \in B \\ (\text{Comm}) \text{ and } (\text{Right}) \end{array} \right\}.$$

Here **(Comm)** means at each vertex, a corresponding diagram commutes; see Appendix. Since the diagram commutes, it is well-defined to associate g_r for each region r of $\mathfrak{W}^=$, **(Right)** means the rightmost region is in $w_0 B$.

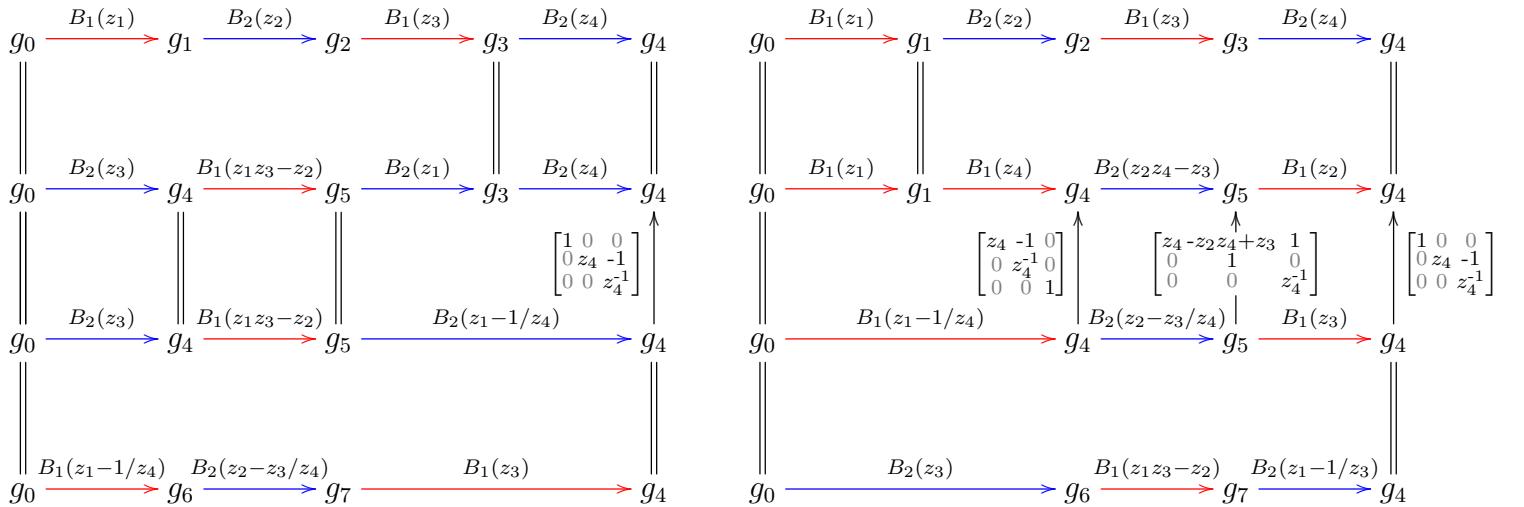
Proposition. The natural projection

$$X(\mathfrak{W}^=) \rightarrow X(\mathfrak{W}), \quad (g_r) \mapsto (g_r B)$$

is an isomorphism.

Proof. This map is obviously surjective. Let us prove it is injective. At each vertex, the top/left labeling determines the bottom/right labeling (computed in Appendix). Thus the top z -labeling determines all z -labeling. This corresponds to a unique element in $X(\beta)$ for β the top braid. \square

Example. Consider the two weaves above.



Proposition. Fix a reduced word w_0 .

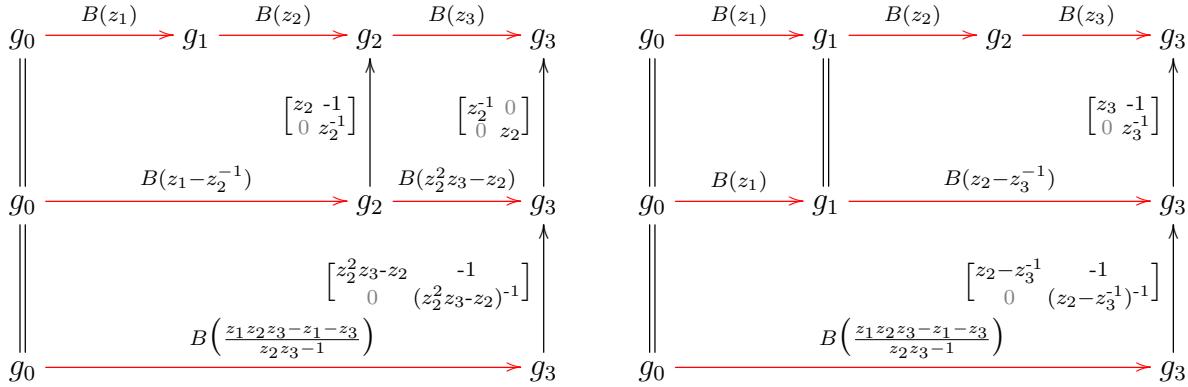
- (1) For a given top z -labeling, the bottom z -labeling do not depend on the choice of Demazure weave $\beta \rightarrow \sigma_{w_0}$.
- (2) The condition **(Right)** is equivalent to the vanishing of the bottom z -labeling.
- (3) The condition **(Comm)** is equivalent to the nonvanishing of northeast z -labeling of trivalent vertices.

Proof. For any reduced word $w_0 = s_{i_1} \cdots s_{i_l}$, we have an isomorphism

$$\mathbb{C}^{\ell(w_0)} \xrightarrow{\sim} Bw_0 B/B, \quad (z'_k) \mapsto B_{i_1}(z'_1) \cdots B_{i_l}(z'_l) B.$$

(1) As a result, the bottom z -labeling is given by the unique (z'_k) mapped to the flag parametrized by the product of top B -matrices. (2) In particular, only the vanishing (z'_k) could satisfy **(End)**. (3) is clear. \square

Example. We remark that mutation does not give the same open subsets. Consider $\sigma_1^3 \rightarrow \sigma_1$.



The open subsets are described as

$$z_2 \neq 0, z_2^3 z_3 - z_2 \neq 0, \quad \text{v.s.} \quad z_3 \neq 0, z_2 - z_3^{-1} \neq 0.$$

Proof of the theorem. It suffices to check two weaves related by a single Zamolodchikov relation or 1212 relation. By picking dashed edges \mathfrak{W}^\pm avoiding their difference, we will define an isomorphism over $X(\mathfrak{W}_1^\pm) \rightarrow X(\mathfrak{W}_2^\pm)$ such that, if we write $(z_{1e}) \mapsto (z_{2e})$, then $z_{2e} = z_{1e}$ for edges e not inside the difference (i.e. edges outside of the difference and edges e intersecting the boundary of the difference). Such an isomorphism will be the required isomorphism.

By the solution of **(Comm)**, we can express z_e for $e \in \text{inside} \cup \text{lower}$ from z_e for $e \in \text{upper}$. This will determine an isomorphism, and it remains to check it is well-defined. That is,

- (1) z_e for edges e intersecting the lower boundary agrees;
- (2) no new pole of z_e was introduced.

For two-braid relations, since no pole is introduced, so it suffices to check (1). This can be done by computation, but there is a quicker way of seeing this. Noting that the lower boundary of each two-braid relation is a reduced word, by (1) of Proposition above, (1) is true.

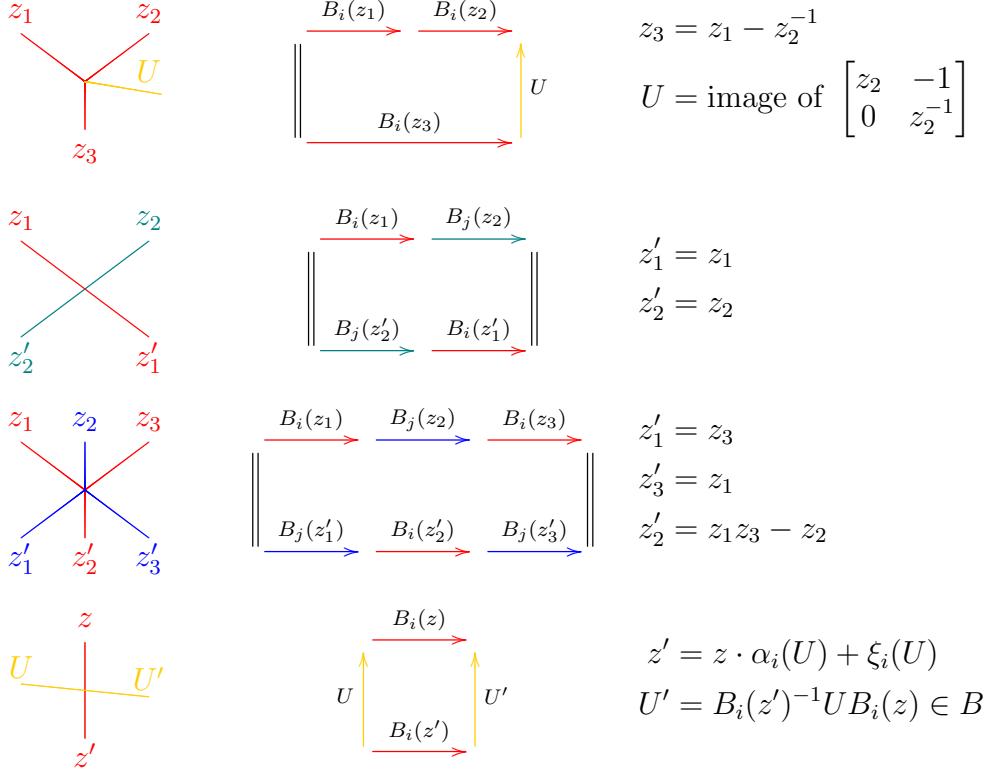
For 1212 relations, (1) is similar. But in this case, a pole is introduced. From the example above, the pole is the same, given by $z_4 \neq 0$. This proves (2). \square

REFERENCES

- [1] Roger Casals, Eugene Gorsky, Mikhail Gorsky, José Simental. Algebraic weaves and braid varieties. *American Journal of Mathematics*, arXiv:2012.06931. 1.2, 1.2
- [2] Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, José Simental. Cluster structures on braid varieties. *Journal of the American Mathematical Society*, 2025. arXiv:2207.11607. 1.2

APPENDIX A. EXPLICIT SOLUTIONS

We can solve the equations explicitly.



Let us explain α_i and ξ_i . We can write $U = U_1 U_2$ for $U_1 \in \text{Rad}(B)$ and $U_2 \in T$. Then

$$\xi_i(U) = \text{coefficient of } E_i \text{ in } U_1, \quad \eta_i(U) = \alpha_i(U).$$

For example, in GL_2 ,

$$\begin{bmatrix} a & b \\ d & c \end{bmatrix} = \begin{bmatrix} 1 & b/d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ d & c \end{bmatrix} \xrightarrow{\xi} b/d, \\ \xrightarrow{\eta} a/d.$$

All the relations follow from the computation in $SL_2 \subset GL_2$, $SL_2 \times SL_2 \subset GL_4$, $SL_3 \subset GL_3$ and $SL_2 \subset GL_2$. The following is the code.

```
R.<z1,z2,z3,a,b,c> = QQ[];
B = lambda z: matrix([[z,-1],[1,0]])
U = matrix([[a,b],[0,c]]);
Rel = (B(z1)*B(z2) - B(z3)*U).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("z3,a,b,c"))
```

```
[[z3 == (z1*z2 - 1)/z2, a == z2, b == -1, c == (1/z2)]]
```

```
R.<z1,z2,z1p,z2p> = QQ[]
B1 = lambda z: matrix([[z,-1,0,0],[1,0,0,0],[0,0,1,0],[0,0,0,1]])
B3 = lambda z: matrix([[1,0,0,0],[0,1,0,0],[0,0,z,-1],[0,0,1,0]])
Rel = (B1(z1)*B3(z2)-B3(z2p)*B1(z1p)).change_ring(SR)
solve([Rel[i][j]==0 for i in range(4) for j in range(4)], SR.var("z1p,z2p"))
```

[[$z_1p = z_1$, $z_2p = r_1$]]

```
R.<z1,z2,z3,z1p,z2p,z3p> = QQ[]
B1 = lambda z: matrix([[z,-1,0],[1,0,0],[0,0,1]])
B2 = lambda z: matrix([[1,0,0],[0,z,-1],[0,1,0]])
Rel = (B1(z1)*B2(z2)*B1(z3)-B2(z1p)*B1(z2p)*B2(z3p)).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("z1p,z2p,z3p"))
```

[[$z_1p = z_3$, $z_2p = z_1*z_3 - z_2$, $z_3p = z_1$]]

```
R.<z,zp,a,b,c,ap,bp,cp> = QQ[]
B = lambda z: matrix([[z,-1],[1,0]])
U = matrix([[a,b],[0,c]]); Up = matrix([[ap,bp],[0,cp]])
Rel = (U*B(z)-B(zp)*Up).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("zp,ap,bp,cp"))
```

[[$zp = (a*z + b)/c$, $ap = c$, $bp = 0$, $cp = a$]]