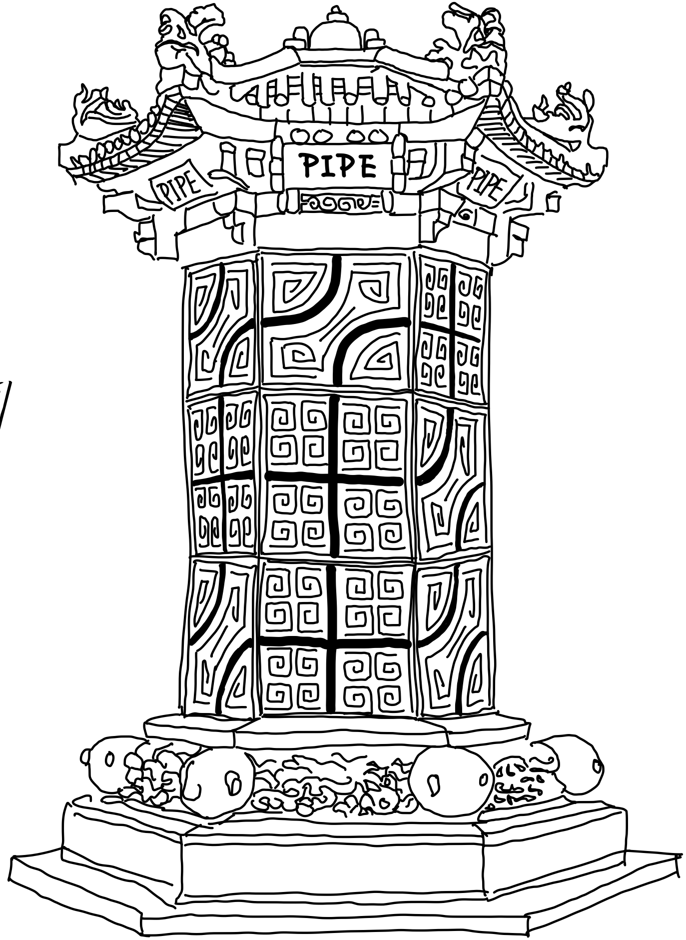
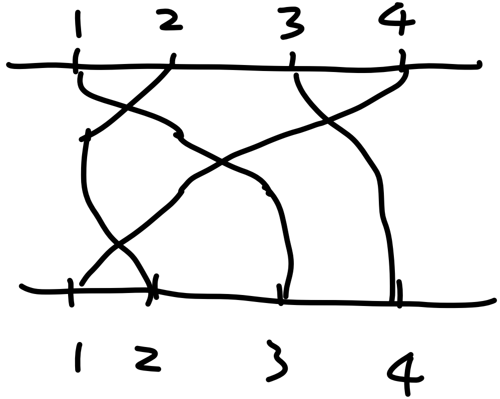
Fun with Positroid Varieties



Symmetric Groups

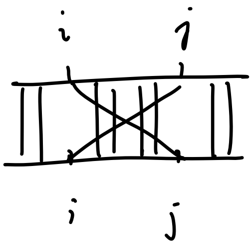
$$S_n = \{ [n] \underset{w}{\circlearrowleft} \text{ bijective} \}$$



Reflection / transposition

$t_{ij} \in S_n$

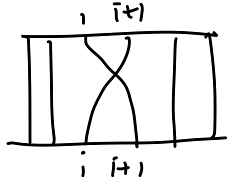
$1 \le i \ne j \le n$



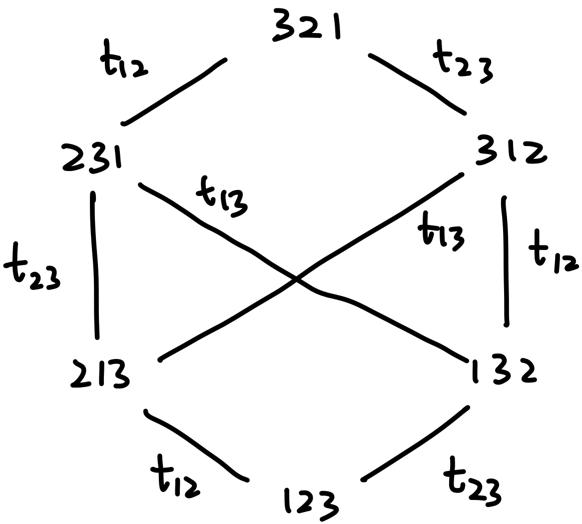
Simple reflection

$s_i = t_{i,i+1}$

$1 \le i \le n-1$

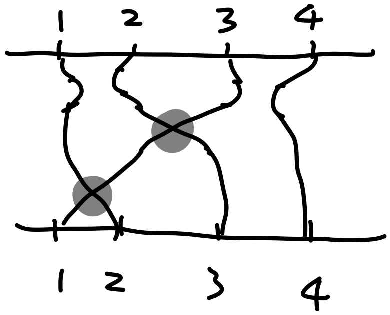


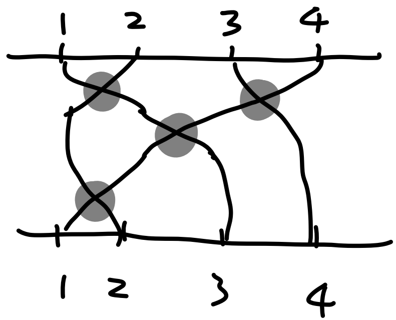
Bruhat Order



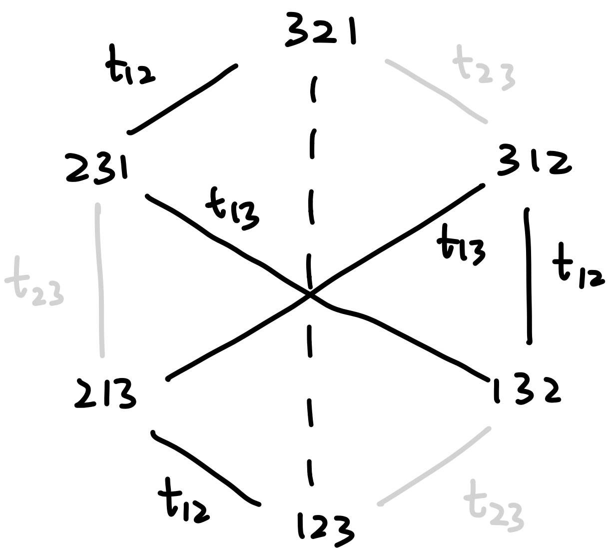
$$u \lessdot w \iff w = u t_{ij}, \quad \ell(w) = \ell(u) + 1$$

$$u < w \iff \exists \ u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot w$$





(Extended) k-Bruhat Order



$n=3, \quad k=1$

$u \lessdot_k w \iff w = u\,t_{ij} \quad \ell(u)+1=\ell(w)$

$u \leq_k w \iff \exists\ u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot w$

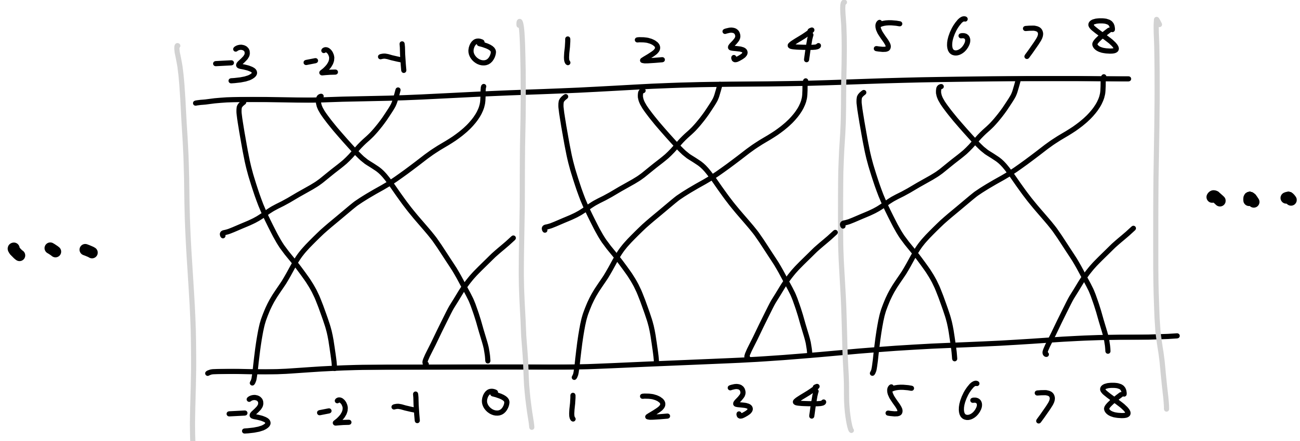
$i \leq k < j$

$u \lessdot'_k w \iff w = u\,t_{ij} \quad \ell(w) > \ell(u)$

$u \leq'_k w \iff \exists\ u \lessdot' u_1 \lessdot' u_2 \lessdot' \cdots \lessdot' w$

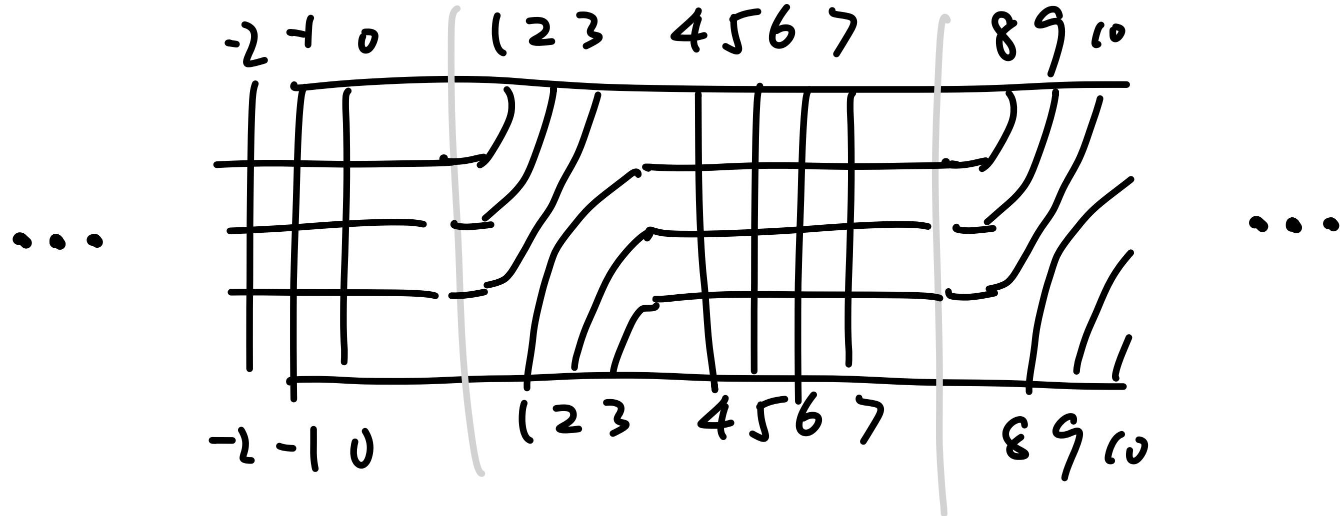
Affine Permutations

$$\tilde{S}_n = \left\{ \underset{\text{bijection}}{\mathbb{Z} \xrightarrow{f} \mathbb{Z}} \;\middle|\; f(i+n) = f(i)+n \right\}$$



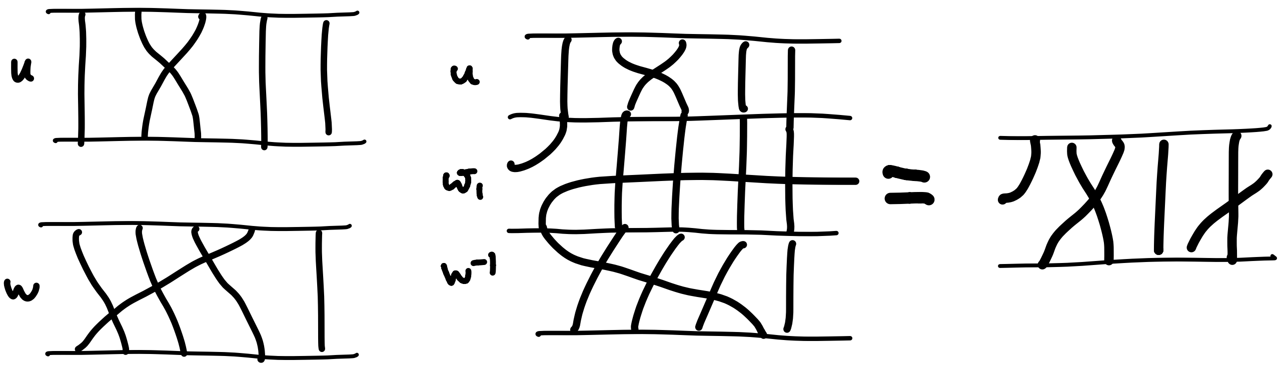
Translation $\omega_k \in \tilde{S}_n$

$$\omega_k(i) = \begin{cases} i+n, & 1 \le i \le k, \\ i, & k < i \le n. \end{cases}$$



Extended k-Bruhat Order Again

For $\underset{S_n}{\underset{\pitchfork}{u}} \le'_k \underset{S_n}{\underset{\pitchfork}{w}}$, define $f_{u,w} = u\,\omega_k\, w^{-1} \in \tilde{S}_n$



Affine Bounded Permutations

extended k-Bruhat intervals

$$\left\{(u,w) : u \le_k w\right\} \Big/ (uv, wv) \sim (u,w)$$

$$v \in S_k \times S_{n-k}$$

$$\updownarrow \wr$$

affine bounded permutations

$$\left\{ \underset{\text{bijection}}{\mathbb{Z} \xrightarrow{f} \mathbb{Z}} \;\middle|\; \begin{array}{l} f(i+n) = f(i) + n \\ \frac{1}{n}\sum_{i=1}^{n} (f(i) - i) = k \\ i \le f(i) \le i+n \end{array} \right\}$$

$$(u,w) \longmapsto f_{u,w}$$

Grassmannian Necklaces

$$\mathcal{I} = (I_1, \cdots, I_n), \quad I_i \in \binom{[n]}{k}$$

(1) $\quad I_{a+1} = I_a \qquad\qquad$ if $\quad a \notin I_a$

(2) $\quad I_{a+1} = I_a \setminus \{a\} \cup \{a'\} \quad$ if $\quad a \in I_a$

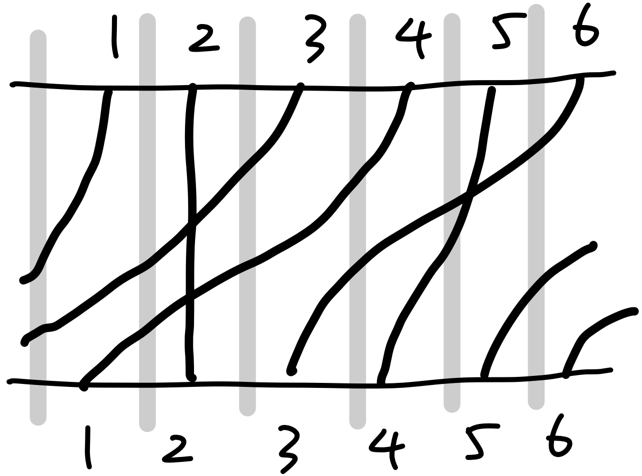
(convention $n+1 = 1$, i.e. indexed by $\mathbb{Z}/n\mathbb{Z}$).

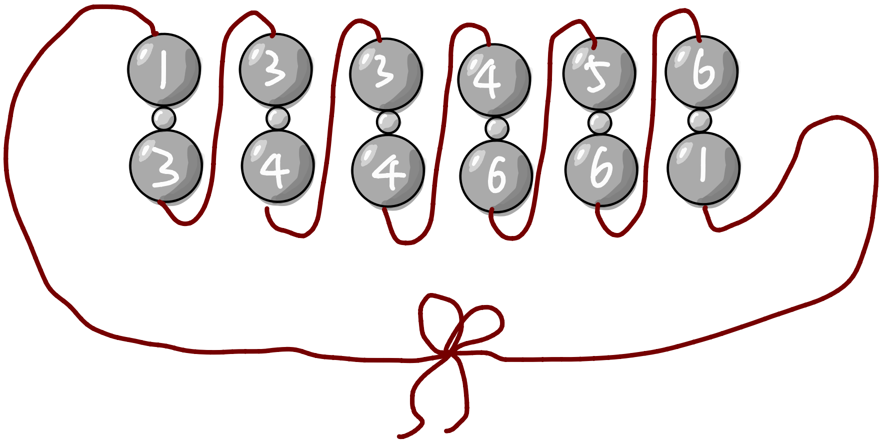
$$\boxed{\text{affine bounded permutations}} \xleftrightarrow{\;\sim\;} \boxed{\text{Grassmannian necklaces}}$$

Example

$$f \longmapsto I = (I_1, \dots, I_n)$$

$$I_a = \{ f(b) \mid b < a,\ f(b) \geq a \} \mod n$$





Grassmannians

$$Gr(k,n) = \{ V \subseteq \mathbb{C}^n : \dim V = k \}$$

We find $v_1, \cdots, v_k \in \mathbb{C}^n$ spanning $V \in Gr(k,n)$

$$\underbrace{\begin{bmatrix} \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \\ & \vdots & \\ \text{---} & v_k & \text{---} \end{bmatrix}}_{n} \Big\} k \rightsquigarrow \begin{bmatrix} 1 & 0 & * & 0 & 0 & * & * & * \\ & 0 & 1 & 0 & 0 & * & * & * \\ & & & 0 & 0 & 1 & * & * \\ & & & & & & 1 & * \end{bmatrix}$$

echelon form

$\rightsquigarrow$ k-subset

Associated Affine Permustions

We find $v_1, \cdots, v_k \in \mathbb{C}^n$ spanning $V \in Gr(k,n)$

$$\underbrace{\begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ & \vdots & \\ — & v_k & — \end{bmatrix}}_{n} \Big\} k = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$$

$\boxed{\begin{array}{l} \text{Convention} \\ u_{n+1} = u_1 \\ u_{n+2} = u_2 \quad \text{etc.} \end{array}}$

$\rightsquigarrow f_V \in \tilde{S}_n$

$$f_V(i) = \min\{ j \geq i : u_i \in \operatorname{span}(u_{i+1}, \cdots, u_j) \}$$

Schubert and Positroid Varieties

- For an k-subset $A \subseteq [n]$,

$$\Sigma_A^\circ = \{ V : V \rightsquigarrow A \} \qquad \Sigma_A = \overline{\Sigma_A^\circ}$$

- For an affine bounded permutation f

$$\Pi_f^\circ = \{ V : V \rightsquigarrow f \} \qquad \Pi_f = \overline{\Pi_f^\circ}$$

Examples

$$\pi^k(i) = i+k$$

$$\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$$

$f(1) = 1+k$ $\quad u_1 \notin \mathrm{span}(u_2, \cdots, u_k)$

$u_1 \in \mathrm{span}(u_2, \cdots, u_{k+1})$.

$f(2) = 2+k$ $\quad u_2 \notin \mathrm{span}(u_3, \cdots, u_{k+1})$

$u_2 \in \mathrm{span}(u_2, \cdots, u_{k+2})$

$\cdots$

$$\rightsquigarrow \begin{bmatrix} & | & & | & \\ \cdots & u_{a+1} & \cdots & u_{a+k} & \cdots \\ & | & & | & \end{bmatrix}$$

linearly independent

generical $V \in \Pi^{\circ}_{\pi^k} \Rightarrow \Pi_{\pi^k} = Gr(k,n)$

Examples

$$w_k(i) = \begin{cases} i+n & 1 \le i \le k \\ i & k < i \le n \end{cases}$$

$$\begin{bmatrix} | & | & & | & | & & | \\ u_1 & u_2 & \cdots & u_k & u_{k+1} & \cdots & u_n \\ | & | & & | & | & & | \end{bmatrix}$$

$f(1) = 1+n \qquad u_1 \notin \operatorname{span}(u_2, \cdots, u_n)$

$u_1 \in \operatorname{span}(u_2, \cdots, u_n, u_1)$

$f(2) = 2+n \qquad u_2 \notin \operatorname{span}(u_3, \cdots, u_n, u_1)$

$u_2 \in \operatorname{span}(u_3, \cdots, u_n, u_1, u_2)$

$\cdots$

$f(k) = k+n \qquad u_k \notin \operatorname{span}(u_{k+1}, \cdots, u_n, u_1, \cdots, u_{k-1})$

$u_k \in \operatorname{span}(u_{k+1}, \cdots, u_n, u_1, \cdots, u_k)$

$f(k+1) = k+1 \qquad u_{k+1} = 0$

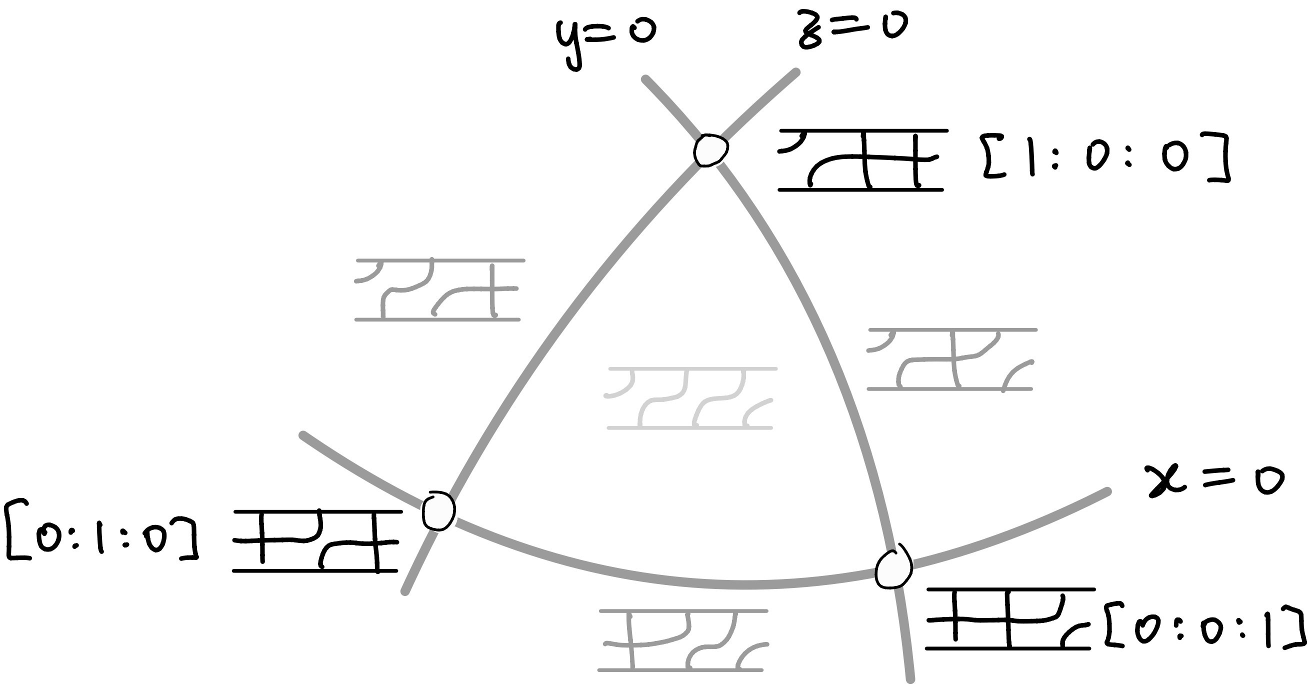
$\cdots$

$f(n) = n \qquad u_n = 0$

$$\Big[\ \underbrace{\text{basis}}_{k}\ \Big|\ 0\ \Big] \rightsquigarrow \overset{\text{unique}}{V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k}.$$

$\Sigma^{\circ}_{w_k} = \{V\}$ a point

Examples $Gr(1,3) \cong \mathbb{P}^2$



Cohomology

Λ_k = ring of symmetric polynomials in k variables.

$$\begin{aligned} \Lambda_k &\twoheadrightarrow H^*(Gr(k,n)) \\ f &\longmapsto f\left(\begin{matrix}\text{Chern roots of dual} \\ \text{tautological bundle}\end{matrix}\right) \end{aligned}$$

Then:

$$\begin{matrix}\text{Schur} \\ \text{polynomials}\end{matrix} \longmapsto [\Sigma_A] \text{ or } 0.$$

Classes of Positroid Varieties

affine Stanley symmetric polynomial $\longmapsto [\Pi_f]$

no double crossing

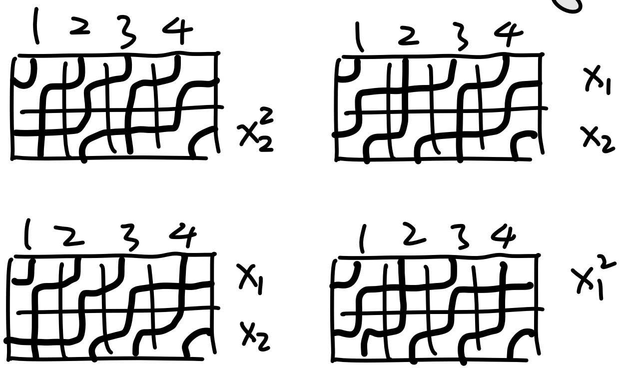
Eg $(k,n) = (2,4)$

$f(1) = 2$

$f(2) = 5$

$f(3) = 4$

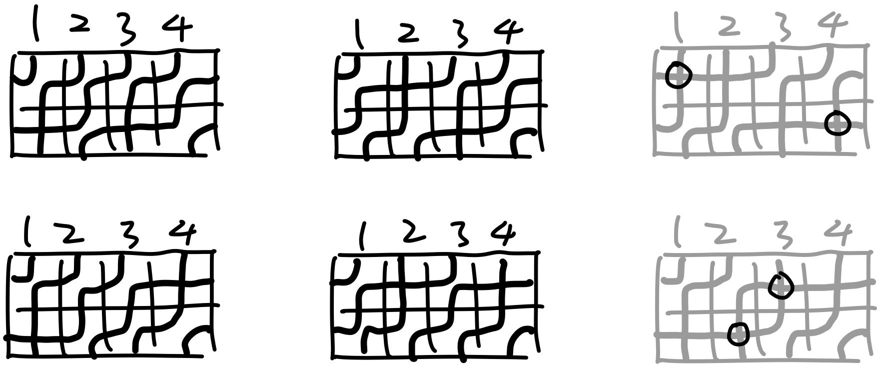
$f(4) = 7$



$$[\Pi_f] = x_1^2 + x_1 x_2 + x_1 x_2 + x_2^2 = (x_1 + x_2)^2 \in H^*(Gr(2,4))$$

Allowing double crossings

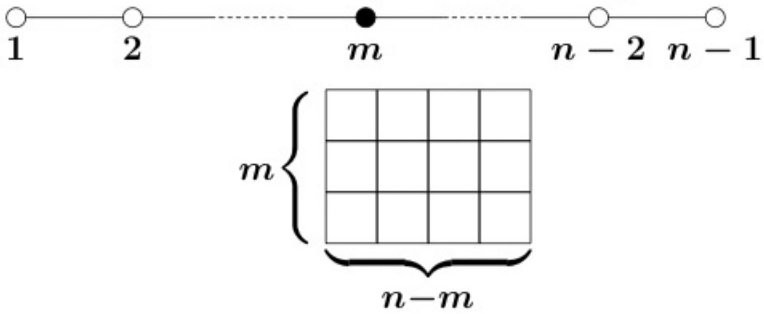
What happens if we remove no double crossing ?



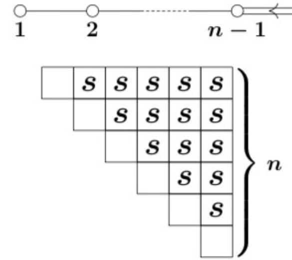
Thm (FGSX) They give the Chern class of Π_f°.

Generalization to Other Types

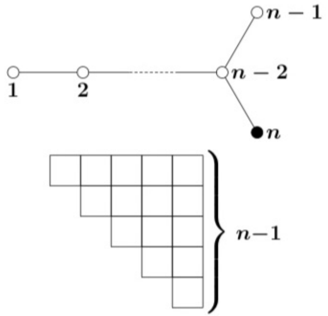
Grassmannian $\mathrm{Gr}(m,n)$ of type A_{n-1}



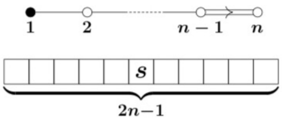
Lagrangian Grassmannian $\mathrm{LG}(n,2n)$



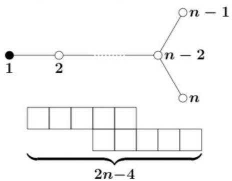
Max. orthog. Grassmannian $\mathrm{OG}(n,2n)$



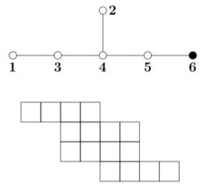
Odd quadric $Q^{2n-1} \subset \mathbb{P}^{2n}$



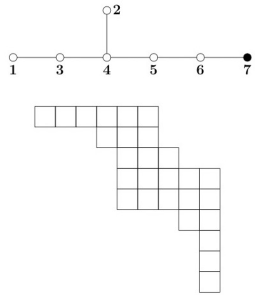
Even quadric $Q^{2n-2} \subset \mathbb{P}^{2n-1}$



Cayley Plane E_6/P_6



Freudenthal variety E_7/P_7



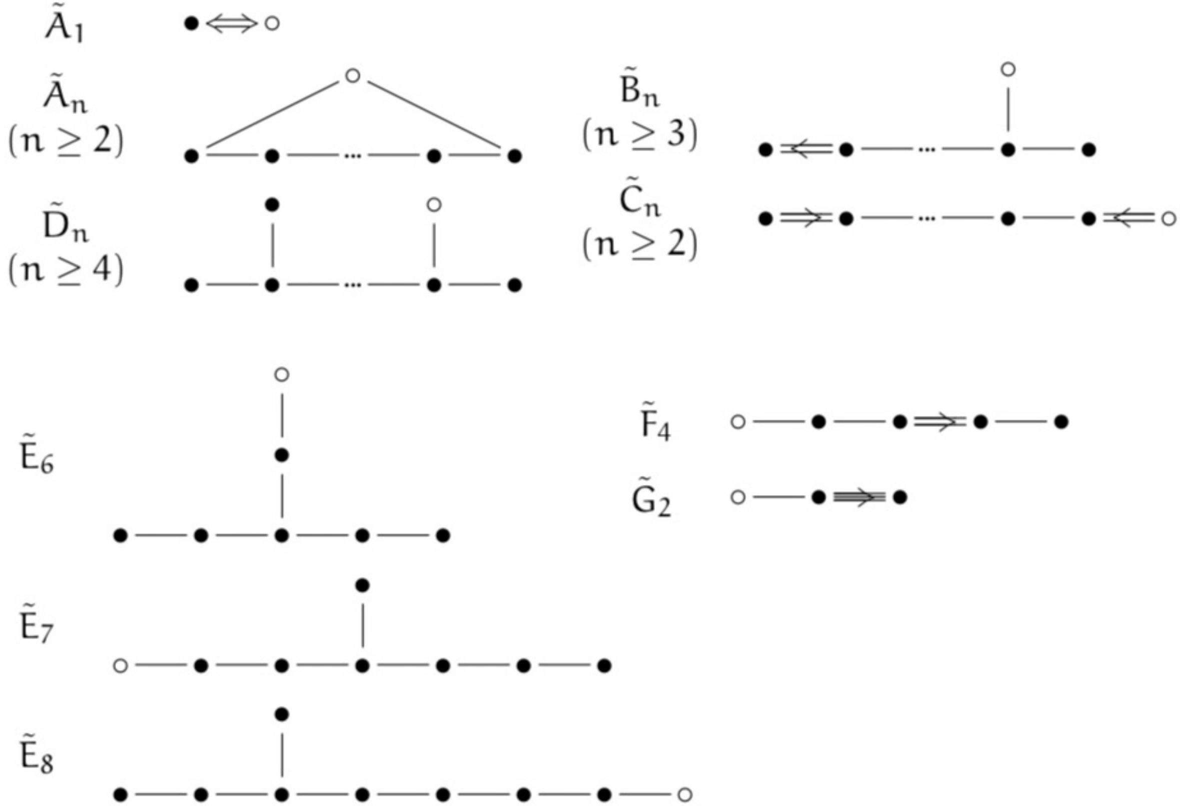
Generalization to Other Types

$$
\left\{ (u,w) : u \le_k' w \right\} \Big/ \begin{array}{c} (uv, wv) \sim (u,v) \\ \text{if } v \in W_P \end{array}
\qquad\qquad
(u,w)
$$

$$
\Big\updownarrow \wr \qquad\qquad\qquad\qquad \Big\downarrow
$$

$$
\left\{ w \in \widetilde{W} : \begin{array}{l} w \le t_\lambda \\ \lambda \in W \varpi_k^\vee \end{array} \right\}
\qquad\qquad
u \, t_{\varpi_k} w^{-1}
$$

Generalization to Other Types



Generalization to Other Types

<u>Thm</u> (FGSX)

Chern class of Π_f° (generalized) $\approx$ Chern class of Schubert cell of f

over cominuscule variety

over affine flag variety Grassmannian

Thank You