BRAID VARIETIES (IV)

2026/01/20

1. Framed Version

The material in this section is from [1].

1.1. Motivation. Recall that any two weaves $\mathfrak{W}_1, \mathfrak{W}_2 : \beta_1 \to \beta_2$ consisting only of tetravalent and hexavalent vertices (i.e. braid relations) are related by **two-braid relations**. What we proved last time can be summarized as

$$\boxed{\text{the } z\text{-labeling is invariant under two-braid relations.}}$$

In this subsection, we will introduce another labeling invariant under two-braid relations. Recall that $y_i : \mathbb{C} \to G$ is defined by

$$y_i(t) = \text{image of } \begin{bmatrix} 1 & \\ t & 1 \end{bmatrix} \in G.$$

We can replace B_i by y_i with the expanse of parametrizing only most of points.

Lemma. For a flag $g_0 B$,

$$\{xB : g_0 B \xrightarrow{s_i} xB\} = g_0 B s_i B/B \approx \{g_0 y_i(z) B/B : z \in \mathbb{C}\} \cong \mathbb{C}$$

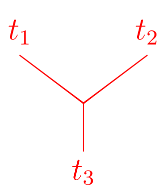
where $\approx$ should be read as "birational to". That is, once a representative of $g_0 B$ is chosen, there is a canonical choice of representative for **almost all** xB by $x = g_0 y_i(z)$.

As a result, we can parametrize the **open Bott–Samelson variety**

$$BS(\beta) = \{(g_k B) : B \xrightarrow{s_{i_1}} g_1 B \xrightarrow{s_{i_2}} \cdots g_{\ell-1} B \xrightarrow{s_{i_\ell}} g_\ell B\}.$$

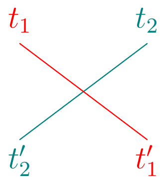
(We met this variety in the proof in previous talks, but we did not name it). We can similar define $BS(\mathfrak{W})$, i.e. the same as $X(\mathfrak{W})$ but no restriction on the rightmost region.

There is a similar rule as the z-labeling.



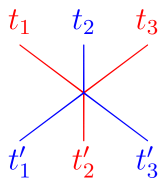
$$\begin{tikzcd} \cdot \arrow[r, "y_i(t_1)"] \arrow[d, equal] & \cdot \arrow[r, "y_i(t_2)"] & \cdot \arrow[d, equal] \\ \cdot \arrow[rr, "y_i(t_3)"] & & \cdot \end{tikzcd}$$

$$t_3 = t_1 + t_2$$



$$\begin{array}{ccc} & \xrightarrow{y_i(t_1)} \xrightarrow{y_j(t_2)} & \\ \Big\| & & \Big\| \\ & \xrightarrow{y_j(t'_2)} \xrightarrow{y_i(t'_1)} & \end{array}$$

$$t'_1 = t_1$$
$$t'_2 = t_2$$



$$\begin{array}{ccc} & \xrightarrow{y_i(t_1)} \xrightarrow{y_j(t_2)} \xrightarrow{y_i(t_3)} & \\ \Big\| & & \Big\| \\ & \xrightarrow{y_j(t'_1)} \xrightarrow{y_i(t'_2)} \xrightarrow{y_j(t'_3)} & \end{array}$$

$$t'_1 = \frac{t_2 t_3}{t_1 + t_3}$$
$$t'_3 = \frac{t_1 t_2}{t_1 + t_3}$$
$$t'_2 = t_1 + t_3$$

Note that the situation is much easier, since no dashed edges need to be introduced.

Corollary. The t-labeling above satisfies two-braid relations.

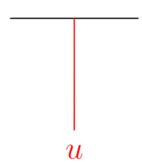
Note that $\frac{t_2 t_3}{t_1 + t_3}$ is not a polynomial function, some values (e.g. $t_1 = 1, t_3 = -1$) will lead to a problem. However, Lusztig [2, Proposition 2.5] first noticed that all above functions restrict to well-defined function between $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$, since all functions are subtraction-free. In particular, it is a well-defined function between any **semifield**. Recall

$$\text{field} = \text{a set with } +, -, \times, \div,$$
$$\text{semifield} = \text{a set with } +, \quad \times, \div.$$

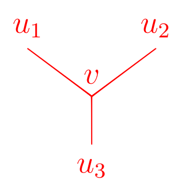
1.2. **The u-labeling.** Let us introduce a variable A_v for each trivalent vertex $v \in \mathfrak{W}_3$. There is a semifield structure over the set of monomials in $\{\mathsf{A}_v\}$

$$\mathsf{A}^{\mathbf{a}} \oplus \mathsf{A}^{\mathbf{b}} = \mathsf{A}^{\min(\mathbf{a},\mathbf{b})} \qquad \min = \text{entrywise minimum.}$$

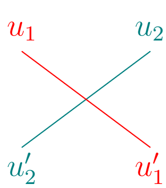
We define the u-labeling as follows



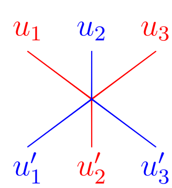
$$u = 1$$



$$u_3 = (u_1 \oplus u_2)\mathsf{A}_v$$



$$u'_1 = u_1$$
$$u'_2 = u_2$$



$$u'_1 = \frac{u_2 u_3}{u_1 \oplus u_3}$$
$$u'_2 = u_1 \oplus u_3$$
$$u'_3 = \frac{u_1 u_2}{u_1 \oplus u_3}$$

Corollary. The u-labeling above satisfies two-braid relations.

We are going to define another parametrization of $X(\mathfrak{W})$ using u-labeling. Let us define $\chi_i : \mathbb{C}^\times \to G$ by

$$\chi_i(u) = \text{image of } \begin{bmatrix} u & \\ & u^{-1} \end{bmatrix} \in G.$$

We define

$$B_i(\tilde{\mathsf{z}}, u) = B_i(\tilde{\mathsf{z}})\chi_i(u) = \text{image of } \begin{bmatrix} uz & -u^{-1} \\ u & \end{bmatrix} \in G.$$

Let us define the following **framed parametrization** of weave variety:

$$\tilde{X}(\mathfrak{W}^{=}) = \left\{ \begin{matrix} (\mathsf{A}_v)_{v\in\mathfrak{W}_3} \\ \times \\ (\tilde{\mathsf{g}}_r)_{r\in\mathsf{Region}(\mathfrak{W}^{=})} \end{matrix} \;\middle|\; \begin{matrix} \mathsf{A}_v \in \mathbb{C}^\times, \tilde{\mathsf{g}}_r \in G \\ a \overset{i}{|} b \Rightarrow \tilde{\mathsf{g}}_b = \tilde{\mathsf{g}}_a B_i(\tilde{\mathsf{z}}_e, u_e) \text{ for some } \tilde{\mathsf{z}}_e \\ \overset{a}{\underset{b}{\cdots}} \Rightarrow \tilde{\mathsf{g}}_a = \tilde{\mathsf{g}}_b \tilde{\mathsf{U}}_e \text{ for some } \tilde{\mathsf{U}}_e \in \mathrm{Rad}(B) \\ \tilde{\mathsf{g}}_{\text{leftmost}} = 1, \tilde{\mathsf{g}}_{\text{rightmost}} \in w_0 B \end{matrix} \right\}.$$

Since each u_e is a monomial in $\{\mathsf{A}_v\}$, once A_v's are valued, u_e's are determined. Notice that difference — this time we require $\tilde{\mathsf{U}}_e \in \mathrm{Rad}(B)$ the unipotent subgroup. Similar as we did last time:

- there is an obvious map

$$\tilde{X}(\mathfrak{W}^{=}) \to X(\mathfrak{W}), \qquad (\mathsf{A}_v) \times (\tilde{\mathsf{g}}_r) \mapsto (\tilde{\mathsf{g}}_r B);$$

- $\tilde{X}(\mathfrak{W}^{=})$ also admits another description via

$$(\mathsf{A}_v)_{v\in\mathfrak{W}_3} \times (\tilde{\mathsf{z}}_e)_{e\in\mathsf{Edge}(\mathfrak{W}^{=})} \times (\tilde{\mathsf{U}}_e)_{e\in\mathsf{Dashed}(\mathfrak{W}^{=})}.$$

Theorem A. For a weave from $\beta \to \sigma_{w_0}$, the projection

$$\tilde{X}(\mathfrak{W}^{=}) \to (\mathbb{C}^\times)^{\mathfrak{W}_3}, \qquad (\mathsf{A}_v) \times (\tilde{\mathsf{g}}_r) \mapsto (\mathsf{A}_v)$$

is an isomorphism.

Proof. We want to show $\{\tilde{\mathsf{g}}_r\}$ is a polynomial in $\{\mathsf{A}_v\}$. Equivalently, we need to solve $(\tilde{\mathsf{z}}_e)$ and $(\tilde{\mathsf{U}}_e)$ from the u-labeling. The variables $\tilde{\mathsf{z}}_e$ and $\tilde{\mathsf{U}}_e$ can be solved from bottom to top. We need to show

(1) On the bottom of $\mathfrak{W}$, the $\tilde{\mathsf{z}}$-labeling vanishes;
(2) At each vertex, the bottom/left labeling determine the top/right labeling.

The claim (1) follows from a similar argument as last time. Assume $\sigma_{w_0} = \sigma_{i_1}\cdots\sigma_{i_l}$, then

$$\mathbb{C}^{\ell(w_0)} \xrightarrow{\sim} Bw_0B/B, \qquad (\tilde{\mathsf{z}}_k) \longmapsto B_{i_1}(\tilde{\mathsf{z}}_1, u_1)\cdots B_{i_l}(\tilde{\mathsf{z}}_l, u_l)B$$

for fixed $u_1, \ldots, u_l \in \mathbb{C}^\times$. In particular, only vanishing (z_k) could be mapped to w_0B. The claim (2) follows from explicit computation, see Appendix. $\square$

Theorem B. For a weave from $\beta \to \sigma_{w_0}$, the projection

$$\tilde{X}(\mathfrak{W}^{=}) \to X(\mathfrak{W}), \qquad (\mathsf{A}_v) \times (\tilde{\mathsf{g}}_r) \mapsto (\tilde{\mathsf{g}}_r B)$$

is an isomorphism.

Proof. We can construct an inverse map. By above Theorem A, it suffices to solve $\{\mathsf{A}_v\}$ from $(g_r) \in X(\mathfrak{W}^=)$. For any coroot γ, denote

$$\mathbb{C}^\times \to T \subset G, \qquad u \longmapsto u^\gamma.$$

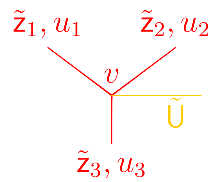
Note that $u^{\alpha_i^\vee} = \chi_i(u)$ and

$$u^\gamma B_j(\tilde{\mathsf{z}}) = B_j(u^{\langle \gamma, \alpha_j \rangle}\tilde{\mathsf{z}})u^{s_j\gamma}.$$

By icking a region walk (without crossing the dashed edges), it is easy to see by induction that

$$\tilde{\mathsf{g}}_r = g_r \cdot (\text{some element in } T), \qquad \tilde{\mathsf{z}}_r = z_r \cdot (\text{some nonzero constant})$$

where the brackets can be explicitly described in a Laurent monomial in $\{u_e\}$, thus in $\{A_v\}$ (see below). Let us look at



Assume all $\mathsf{A}_{v'}$ are solved for v' above, and we are going to solve A_v. Then in particular,

$$u_1, u_2, \tilde{\mathsf{z}}_2 \text{ are known}, \qquad u = (u_1 \oplus u_2)\mathsf{A}_v \text{ is unknown}.$$

Since $\tilde{\mathsf{z}}_2 = \dfrac{u}{u_1 u_2}$ (see Appendix), we can solve A_v as a regular function in $\mathsf{A}_{v'}$ and $\tilde{\mathsf{z}}_e$ for vertices v' and edges e above the vertex v. $\square$

Let us be precise for the claim in the proof. Assume the region walk is given by

$$\text{leftmost} = r_0 \underset{\tilde{\mathsf{z}}_1, u_1}{\overset{i_1}{|}} r_1 \underset{\tilde{\mathsf{z}}_2, u_2}{\overset{i_2}{|}} r_2 \underset{\tilde{\mathsf{z}}_3, u_3}{\overset{i_3}{|}} \cdots \underset{\tilde{\mathsf{z}}_{l-1}, u_{l-1}}{\overset{i_{l-1}}{|}} r_{l-1} \underset{\tilde{\mathsf{z}}_l, u_l}{\overset{i_l}{|}} r$$

we have

$$\begin{aligned}
\tilde{g}_r &= B_{i_1}(\tilde{\mathsf{z}}_1, u_1) B_{i_2}(\tilde{\mathsf{z}}_2, u_2) \cdots B_{i_l}(\tilde{\mathsf{z}}_l, u_l) \\
&= B_{i_1}(\tilde{\mathsf{z}}_1)\chi_{i_1}(u_1) B_{i_2}(\tilde{\mathsf{z}}_2)\chi_{i_2}(u_2) \cdots B_{i_l}(\tilde{\mathsf{z}}_l)\chi_{i_l}(u_l) \\
&= B_{i_1}(z_1') B_{i_2}(z_2') \cdots B_{i_l}(z_l') \cdot u_1^{s_{i_l}\cdots s_{i_2}\alpha_1^\vee} u_2^{s_{i_l}\cdots s_{i_3}\alpha_2^\vee} \cdots u_l^{\alpha_l^\vee}
\end{aligned}$$

with

$$z_k' = \tilde{\mathsf{z}}_k u_1^{\langle s_{i_{k-1}}\cdots s_{i_2}\alpha_1^\vee, \alpha_{i_k}\rangle} u_2^{\langle s_{i_{k-1}}\cdots s_{i_3}\alpha_2^\vee, \alpha_{i_k}\rangle} \cdots u_k^{\langle \alpha_{k-1}^\vee, \alpha_{i_k}\rangle}.$$

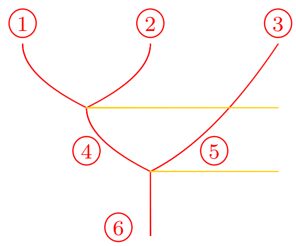
Note that we have to have $z_k = z_k'$ since (z_k') and (z_k) parametrizes the same element in $X(\mathfrak{W})$.

1.3. **Cluster chart.** We will call the isomorphism

$$(\mathbb{C}^\times)^{\mathfrak{M}_3} \xrightarrow{\sim} \tilde{X}(\mathfrak{W}^=) \xleftarrow{\sim} X(\mathfrak{W})$$

the **cluster chart**.

Example. Let us consider the example



	z	u	$\tilde{\mathsf{z}}$
①	z_1	1	z_1
②	z_2	1	z_2
③	z_3	1	z_3
④	$z_1 - z_2^{-1}$	A_1	$z_1 - z_2$
⑤	$z_2^2 z_3 - z_2$	1	$\mathsf{A}_1^{-2}(z_2^2 z_3 - z_2)$
⑥	$\dfrac{z_1 z_2 z_3 - z_1 - z_3}{z_2 z_3 - 1}$	A_2	$\dfrac{z_1 z_2 z_3 - z_1 - z_3}{z_2 z_3 - 1}$

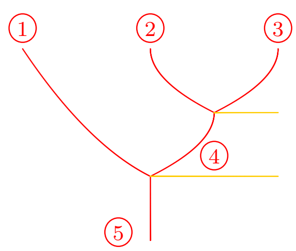
At vertices $1, 2$, we have

$$z_2 = \tilde{\mathsf{z}}_{②} = \frac{u_{④}}{u_{①} u_{②}} = \mathsf{A}_1, \qquad \mathsf{A}_1^{-2}(z_2^2 z_3 - z_2) = \tilde{\mathsf{z}}_{⑤} = \frac{u_{⑥}}{u_{④} u_{⑤}} = \mathsf{A}_2/\mathsf{A}_1.$$

So

$$\mathsf{A}_1 = z_2, \qquad \mathsf{A}_2 = z_2 z_3 - 1.$$

Let us consider another example



	z	u	$\tilde{\mathsf{z}}$
①	z_1	1	z_1
②	z_2	1	z_2
③	z_3	1	z_3
④	$z_2 - z_3^{-1}$	A_1	$z_2 - z_3^{-1}$
⑤	$\dfrac{z_1 z_2 z_3 - z_1 - z_3}{z_2 z_3 - 1}$	A_2	$\dfrac{z_1 z_2 z_3 - z_1 - z_3}{z_2 z_3 - 1}$

At vertices $1, 2$, we have

$$z_3 = \tilde{\mathsf{z}}_{③} = \frac{u_{④}}{u_{②} u_{③}} = \mathsf{A}_1, \qquad z_2 - z_3^{-1} = \tilde{\mathsf{z}}_{④} = \frac{u_{⑤}}{u_{①} u_{④}} = \mathsf{A}_2/\mathsf{A}_1.$$

So

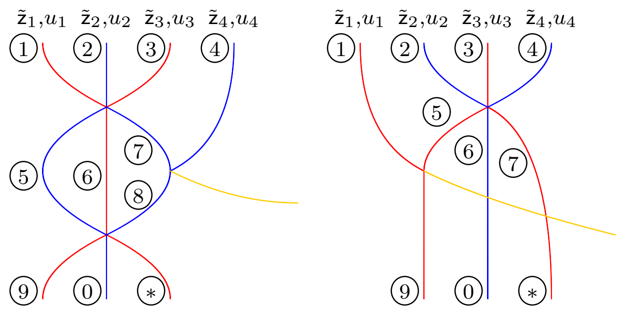
$$\mathsf{A}_1 = z_3, \qquad \mathsf{A}_2 = z_2 z_3 - 1.$$

Theorem. For two weaves $\mathfrak{W}, \mathfrak{M}$ related by two-braid relations and 1212 relations, we have a natural bijection $\mathfrak{W}_3 \cong \mathfrak{M}_3$, the cluster charts

$$(\mathbb{C}^\times)^{\mathfrak{W}_3} \longrightarrow X(\beta)$$

are the same.

Proof. The proof is similar to the the proof we explained last time. The only computation we need to do is 1212 relations. Consider



We know $u_{⑨}$, $u_{⓪}$, $u_{⊛}$ agree. By an argument similar to z-labeling, we know $\tilde{z}_{⑨}$, $\tilde{z}_{⓪}$, $\tilde{z}_{⊛}$ agree. From the way we construct the inverse map in Theorem B, it suffices to show A agrees.

For the left diagram, we have

$$\tilde{z}_4 = \tilde{z}_{④} = \mathsf{A}\frac{u_{④} \oplus u_{⑦}}{u_{④}u_{⑦}} = \mathsf{A}\frac{\frac{u_1u_2}{u_1 \oplus u_3} \oplus u_4}{\frac{u_1u_2}{u_1\oplus u_3}u_4} = \mathsf{A}\frac{u_1u_2 \oplus u_1u_4 \oplus u_3u_4}{u_1u_2u_4}.$$

For the right diagram, we have

$$\tilde{z}_{⑤} = \mathsf{A}\frac{u_{①} \oplus u_{⑤}}{u_{①}u_{⑤}} = \mathsf{A}\frac{u_1 \oplus \frac{u_3u_4}{u_2 \oplus u_4}}{u_1\frac{u_3u_4}{u_2\oplus u_4}} = \mathsf{A}\frac{u_1u_2 \oplus u_1u_4 \oplus u_3u_4}{u_1u_3u_4}.$$

By (**Comm**), we have

$$\tilde{z}_4 = \tilde{z}_{④} = \frac{u_{③}}{u_{②}}\tilde{z}_{⑤} = \frac{u_3}{u_2}\tilde{z}_{⑤}.$$

We get the same equation for A. □

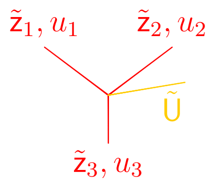
References

[1] Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, José Simental. Cluster structures on braid varieties. *Journal of the American Mathematical Society*, 2025. arXiv:2207.11607. 1

[2] George Lusztig. Total Positivity in Reductive Groups 1.1

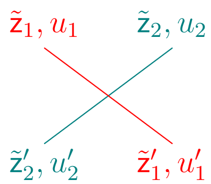
Appendix A. Explicit Solutions

We can solve the equations explicitly.



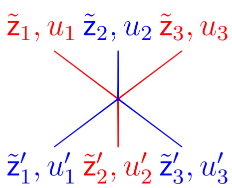
$$\begin{tikzcd} {} \arrow[r, "B_i(\tilde z_1,u_1)"] \arrow[d, equal] & {} \arrow[r, "B_i(\tilde z_2,u_2)"] & {} \\ {} \arrow[rr, "B_i(\tilde z_3,u_3)"] & & {} \arrow[u, "\tilde U"'] \end{tikzcd}$$

$$\tilde{z}_2 = \frac{u_3}{u_1u_2}, \quad \tilde{z}_1 = \tilde{z}_3 + \frac{u_2}{u_1u_3},$$
$$\tilde{U} = \text{image of } \begin{bmatrix} 1 & -\frac{u_1}{u_2u_3} \\ & 1 \end{bmatrix}$$



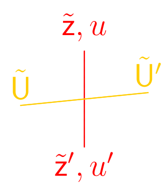
$$\begin{tikzcd} {} \arrow[r, "B_i(\tilde z_1,u_1)"] \arrow[d, equal] & {} \arrow[r, "B_j(\tilde z_2,u_2)"] & {} \arrow[d, equal] \\ {} \arrow[r, "B_j(\tilde z'_2,u'_2)"] & {} \arrow[r, "B_i(\tilde z'_1,u'_1)"] & {} \end{tikzcd}$$

$$\tilde{z}_1 = \tilde{z}'_1 \qquad \boxed{u'_1 = u_1}$$
$$\tilde{z}_2 = \tilde{z}'_2 \qquad \boxed{u'_2 = u_2}$$



$$\begin{tikzcd} {} \arrow[r, "B_i(\tilde z_1,u_1)"] \arrow[d, equal] & {} \arrow[r, "B_j(\tilde z_2,u_2)"] & {} \arrow[r, "B_i(\tilde z_3,u_3)"] & {} \arrow[d, equal] \\ {} \arrow[r, "B_j(\tilde z'_1,u'_1)"] & {} \arrow[r, "B_i(\tilde z'_2,u'_2)"] & {} \arrow[r, "B_j(\tilde z'_3,u'_3)"] & {} \end{tikzcd}$$

$$\tilde{z}_1 = \frac{u'_1}{u'_2}\tilde{z}'_3, \qquad \boxed{u'_2u'_3 = u_1u_2}$$
$$\tilde{z}_3 = \frac{u_2}{u_1}\tilde{z}'_1, \qquad \boxed{u'_1u'_2 = u_2u_3}$$
$$\tilde{z}_2 = \frac{u_1u'_1}{u'_2}\tilde{z}'_1\tilde{z}'_3 - \frac{u_1}{u'_1}\tilde{z}'_2 = \frac{u_3u'_3}{u'_2}\tilde{z}'_1\tilde{z}'_3 - \frac{u_1}{u'_1}\tilde{z}'_2$$



$$\begin{tikzcd} {} \arrow[r, "B_i(\tilde z,u)"] & {} \\ {} \arrow[u, "\tilde U"] \arrow[r, "B_i(\tilde z',u')"] & {} \arrow[u, "\tilde U'"'] \end{tikzcd}$$

$$\tilde{z} = \tilde{z}' - \xi_i(U), \qquad \boxed{u' = u}$$
$$\tilde{U}' = B_i(\tilde{z}', u')^{-1}\tilde{U}B_i(\tilde{z}, u) \in \mathrm{Rad}(B)$$

Recall $\xi_i(U) =$ coefficient of E_i in U. Notice that $\xi_i(U') = 0$. This fact will be used later.

All the relations follow from the computation in $SL_2 \subset GL_2$, $SL_2 \times SL_2 \subset GL_4$, $SL_3 \subset GL_3$ and $SL_2 \subset GL_2$. The following is the code.

```
R.<u1,u2,u3p,z3p,z1,z2,z3,a> = QQ[];
B = lambda z,u: matrix([[u*z,-u^(-1)],[u,0]])
U = matrix([[1,a],[0,1]]);
Rel = (B(z1,u1)*B(z2,u2) - B(z3,u3p)*U).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("u1p,z1,z2,a"))
```

```
[[u1p == r1,
  z1 == (u1*u3p*z3 + u2)/(u1*u3p),
  z2 == u3p/(u1*u2),
  a == -u1/(u2*u3p)]]
```

```
R.<u1,u2,u1p,u2p,z1p,z2p,z1,z2> = QQ[]
B1 = lambda z,u: matrix([[u*z,-u^(-1),0,0],[u,0,0,0],[0,0,1,0],[0,0,0,1]])
B3 = lambda z,u: matrix([[1,0,0,0],[0,1,0,0],[0,0,u*z,-u^(-1)],[0,0,u,0]])
Rel = (B1(z1,u1)*B3(z2,u2)-B3(z2p,u2p)*B1(z1p,u1p)).change_ring(SR)
```

```
solve([Rel[i][j]==0 for i in reversed(range(4)) for j in range(4)], SR.var("u1p,u2p,z1,
    z2"))
```

```
[[u1p == u1, u2p == u2, z1 == z1p, z2 == z2p]]
```

```
R.<u1,u2,u3,u1p,u2p,u3p,z1p,z2p,z3p,z1,z2,z3> = QQ[]
B1 = lambda z,u: matrix([[u*z,-u^(-1),0],[u,0,0],[0,0,1]])
B2 = lambda z,u: matrix([[1,0,0],[0,u*z,-u^(-1)],[0,u,0]])
Rel = (B1(z1,u1)*B2(z2,u2)*B1(z3,u3)-B2(z1p,u1p)*B1(z2p,u2p)*B2(z3p,u3p)).change_ring(
    SR)
solve([Rel[i][j]==0 for i in reversed(range(2)) for j in range(2)], SR.var("u1p,u2p,u3p
    ,z1,z2,z3"))
```

```
[[u1p == r1*r2/z3p,
  u2p == r1,
  u3p == r1*r2*u1/(u3*z3p),
  z1 == r2,
  z2 == (r1^2*r2^2*u1*z1p - r1*u1*z2p*z3p)/(u2*u3*z3p),
  z3 == r1^2*r2*z1p/(u1*u3*z3p)]]
```

```
R.<u,up,z,zp,a,ap> = QQ[]
B = lambda z,u: matrix([[u*z,-u^(-1)],[u,0]])
U = matrix([[1,a],[0,1]]); Up = matrix([[1,ap],[0,1]])
Rel = (U*B(z,u)-B(zp,up)*Up).change_ring(SR)
solve([Rel[i][j]==0 for i in range(2) for j in range(2)], SR.var("up,z,a,ap"))
```

```
[[up == u, z == r1, a == -r1 + zp, ap == 0]]
```

To convince the readers, we write down the matrix equations for the trivalent and hexavalent vertices

$$\begin{bmatrix} u_1u_2\tilde{\mathtt{z}}_1\tilde{\mathtt{z}}_2 - u_1^{-1}u_2 & -u_1u_2^{-1}\tilde{\mathtt{z}}_1 \\ u_1u_2\tilde{\mathtt{z}}_2 & -u_1u_2^{-1} \end{bmatrix} = \begin{bmatrix} u_1\tilde{\mathtt{z}}_1 & -u_1^{-1} \\ u_1 & 0 \end{bmatrix} \begin{bmatrix} u_2\tilde{\mathtt{z}}_2 & -u_2^{-1} \\ u_2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} u\tilde{\mathtt{z}} & u\tilde{\mathtt{z}}a - u^{-1} \\ u & ua \end{bmatrix} = \begin{bmatrix} u\tilde{\mathtt{z}} & -u^{-1} \\ u & 0 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} u_1u_3\tilde{\mathtt{z}}_1\tilde{\mathtt{z}}_3 - u_1^{-1}u_2u_3\tilde{\mathtt{z}}_2 & -u_1u_3^{-1}\tilde{\mathtt{z}}_1 & u_1^{-1}u_2^{-1} \\ u_1u_3\tilde{\mathtt{z}}_3 & -u_1u_3^{-1} & 0 \\ u_2u_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} u_1\tilde{\mathtt{z}}_1 & -u_1^{-1} & 0 \\ u_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & u_2\tilde{\mathtt{z}}_2 & -u_2^{-1} \\ 0 & u_2 & 0 \end{bmatrix} \begin{bmatrix} u_3\tilde{\mathtt{z}}_3 & -u_3^{-1} & 0 \\ u_3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} u_{2'}\tilde{\mathtt{z}}_{2'} & -u_{2'}^{-1}u_{3'}\tilde{\mathtt{z}}_{3'} & u_{2'}^{-1}u_{3'}^{-1} \\ u_{1'}u_{2'}\tilde{\mathtt{z}}_{1'} & -u_{1'}^{-1}u_{3'} & 0 \\ u_{1'}u_{2'} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u_{1'}\tilde{\mathtt{z}}_{1'} & -u_{1'}^{-1} \\ 0 & u_{1'} & 0 \end{bmatrix} \begin{bmatrix} u_{2'}\tilde{\mathtt{z}}_{2'} & -u_{2'}^{-1} & 0 \\ u_{2'} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & u_{3'}\tilde{\mathtt{z}}_{3'} & -u_{3'}^{-1} \\ 0 & u_{3'} & 0 \end{bmatrix}.$$