

LECTURES ON AFFINE WEYL GROUPS

RUI XIONG

CONTENTS

1. Review of finite theory	2
2. Two realizations	11
3. Length formula	20
4. Extended affine Weyl groups	31
5. Combinatorics in type A	41
6. Algebraic background	50

date: August 10, 2024

1. REVIEW OF FINITE THEORY

Coxeter groups.

1.1. Definition. A **Coxeter system** (W, S) is a group W and $S \subset W$ such that

$$W = \left\langle s \in S : \underbrace{s t \dots}_{m_{st}} = \underbrace{t s \dots}_{m_{st}} \right\rangle \quad \text{where for each } s \neq t \in S \\ m_{st} \in \{2, 3, \dots\} \cup \{\infty\}.$$

We define **Coxeter diagram**

\bullet \bullet	\bullet — \bullet	\bullet — \bullet 4	...	\bullet — \bullet ∞
$m_{st} = 2$	$m_{st} = 3$	$m_{st} = 4$...	$m_{st} = \infty$
$st = ts$	$sts = tst$	$stst = tsts$...	no relation

Usually, we reparametrize S by $\{s_i : i \in I\}$ and $m_{ij} = m_{s_i s_j}$.

1.2. Geometric representation. We define

$$\mathfrak{h}_{\mathbb{R}}^* = \bigoplus_{i \in I} \mathbb{R} \alpha_i.$$

We equip a symmetric bilinear form such that

$$\text{length of } \alpha_i \neq 0, \quad \text{angle of } \alpha_i \text{ and } \alpha_j \text{ is } \pi - \frac{\pi}{m_{ij}}.$$

This form is unique up to a positive rescalar of α_i . We define the geometric representation of W on $\mathfrak{h}_{\mathbb{R}}^*$ by

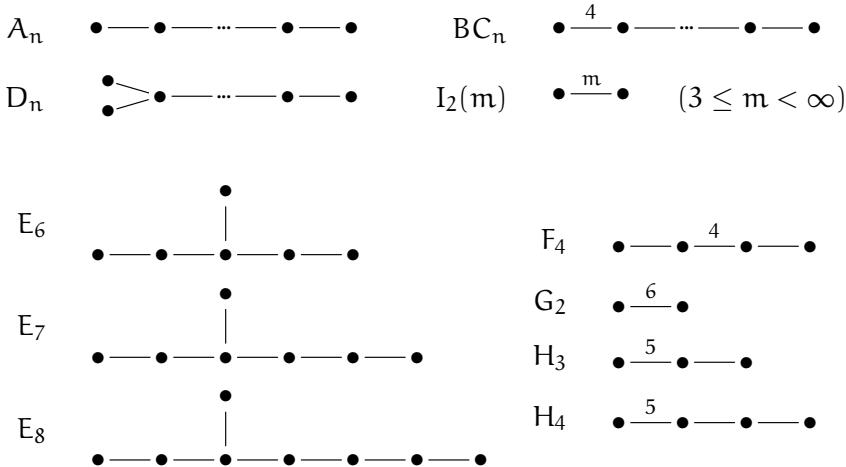
$$S \ni s_i \longmapsto (\text{reflection with respect to } \alpha_i^\perp) \in \text{GL}(\mathfrak{h}^*).$$

That is,

$$s_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda) \alpha_i, \quad \text{where } \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

For any Coxeter group, its geometric representation is faithful.

1.3. Finite Coxeter groups. A Coxeter group W is finite if and only if the bilinear form defined above is positive definite. The corresponding Coxeter diagram is a disjoint union of the following diagrams.

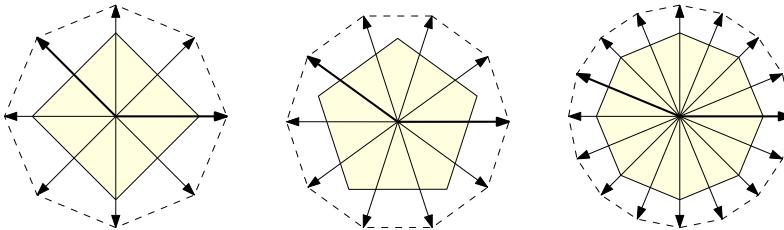


We have

$$D_2 = A_1 \times A_1, \quad D_3 = A_2, \quad A_2 = I_2(3), \quad BC_2 = I_2(4), \quad G_2 = I_2(6).$$

1.4. Example. The dihedral group D_m of order $2m$ is the Coxeter group of type $I_2(m)$. We take $\mathfrak{h}_\mathbb{R}^*$ to be the complex plane \mathbb{C} , and

$$\alpha_1 = 1, \quad \alpha_2 = -e^{-\frac{2\pi\sqrt{-1}}{m}}.$$



Weyl group.

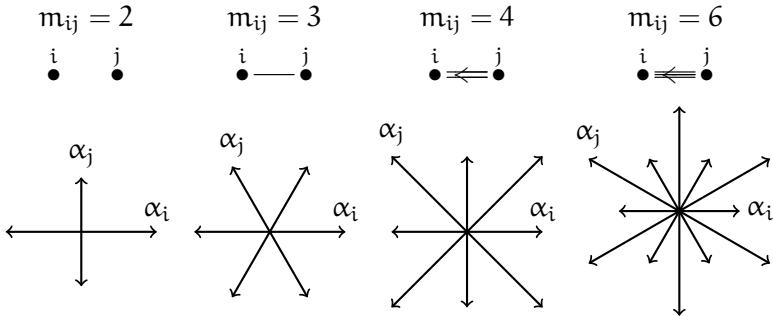
1.5. Weyl group. If a finite Coxeter group W stabilizes the **root lattice**

$$Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \subset \mathfrak{h}_{\mathbb{R}}^*,$$

we call W a **Weyl group** and define the **root system**

$$\mathcal{R} = \{w\alpha_i : w \in W, i \in I\}.$$

A Weyl group could only have $m_{ij} \in \{2, 3, 4, 6\}$. We define the **Dynkin diagram**



1.6. Finite Weyl group. Up to graph isomorphism, here is the classification of irreducible Weyl groups

$$A_n \quad \bullet - \bullet - \cdots - \bullet - \bullet \qquad B_n \quad \bullet \rightleftharpoons \bullet - \cdots - \bullet - \bullet$$

$$D_n \quad \bullet \nearrow \bullet - \cdots - \bullet - \bullet \qquad C_n \quad \bullet \Rightarrow \bullet - \cdots - \bullet - \bullet$$

$$E_6 \quad \begin{array}{ccccccc} & & \bullet & & & & \\ & \bullet & - & \bullet & - & \bullet & - \bullet \\ & & | & & & & \\ & & \bullet & & & & \end{array}$$

$$E_7 \quad \begin{array}{ccccccc} & & \bullet & & & & \\ & \bullet & - & \bullet & - & \bullet & - \bullet \\ & & | & & & & \\ & & \bullet & & & & \end{array}$$

$$E_8 \quad \begin{array}{ccccccc} & & \bullet & & & & \\ & \bullet & - & \bullet & - & \bullet & - \bullet \\ & & | & & & & \\ & & \bullet & & & & \end{array}$$

$$F_4 \quad \bullet - \bullet \Rightarrow \bullet - \bullet$$

$$G_2 \quad \bullet \Rightarrow \bullet$$

We have

$$D_2 = A_1 \times A_1, \quad D_3 = A_3, \quad B_2 = C_2.$$

1.7. Example.

The symmetric group

$$\mathfrak{S}_n = \left\{ \text{bijections } \{1, \dots, n\} \xrightarrow{w} \{1, \dots, n\} \right\}$$

is the Coxeter group of type A_{n-1} . The Coxeter generator

$$s_i = \left[\begin{array}{c} \text{the permutation exchanging } i \text{ and} \\ i+1 \text{ with other numbers fixed} \end{array} \right] \in \mathfrak{S}_n$$

labeled as

$$\bullet_1 — \bullet_2 — \cdots — \bullet_{n-2} — \bullet_{n-1}$$

The geometric representation

$$\mathfrak{h}_{\mathbb{R}}^* = \{(a_1, \dots, a_n) : a_1 + \cdots + a_n = 0\} \subset \mathbb{R}^n.$$

The natural pairing over \mathbb{R}^n restricts to $\mathfrak{h}_{\mathbb{R}}^*$. We define

$$\alpha_i = e_i - e_{i+1}, \quad 1 \leq i \leq n-1.$$

We have a diagram notation

$$\begin{aligned} & \text{Diagram notation for } s_i: \text{A horizontal line with dots at } i \text{ and } i+1. \text{ A red } X \text{ is drawn between them.} \\ & \text{Below the line, the indices } i \text{ and } i+1 \text{ are written under their respective dots.} \\ & \text{To the right, it says "e.g. } 42|3 = \text{"} \\ & \text{Below this, there is a diagram of four points labeled } 1, 2, 3, 4. \text{ Red lines connect } 1 \text{ to } 2, 2 \text{ to } 3, \text{ and } 3 \text{ to } 4. \text{ A green line connects } 1 \text{ to } 3. \\ & \text{This is followed by an equals sign and a sequence of colored circles: yellow, blue, green, and orange.} \\ & \text{Below the diagram, three identities are shown: } s_i^2 = \text{id}, s_i s_j = s_j s_i, \text{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \end{aligned}$$

1.8. Example.

The Coxeter group of type BC_n is known as the **signed symmetric group**

$$\mathfrak{BC}_n = \left\{ \text{bijections } \{\pm 1, \dots, \pm n\} \xrightarrow{w} \{\pm 1, \dots, \pm n\} : w(-i) = -w(i) \right\}.$$

Using the monotone bijection

$$\{\pm 1, \dots, \pm n\} \cong \{1, \dots, 2n\}$$

We can describe it as the subgroup of S_{2n} generated by

$$s_0 = s_n, \quad s_i = s_{n-i}s_{n+i} \quad (1 \leq i \leq n-1).$$

That is,

$$s_0 = \left[\begin{array}{c} \text{the permutation exchanging 1 and } -1 \\ \text{with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

$$s_i = \left[\begin{array}{c} \text{the permutation exchanging } \pm i \text{ and } \pm(i+1) \\ \text{with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

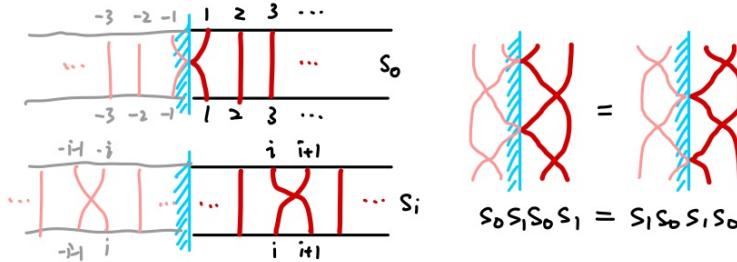
The label is like this

$$\bullet \overset{4}{\overbrace{\cdots}} \bullet \overset{n-2}{\overbrace{\cdots}} \bullet \overset{n-1}{\overbrace{\cdots}} \bullet$$

The geometric representation $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$ with natural pairing and

$$\alpha_0 = \begin{cases} e_1, & \text{type B,} \\ 2e_1, & \text{type C,} \end{cases} \quad \alpha_i = e_{i+1} - e_i \quad (1 \leq i \leq n-1).$$

We have a diagram notation



1.9. Example. The Coxeter group of type D_n is known as the **even-signed symmetric group**.

$$\mathfrak{D}_n = \left\{ w \in \mathfrak{BC}_n : \#\{i < 0 : w(i) > 0\} \text{ is even} \right\}.$$

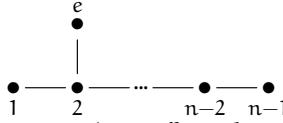
Note that the $s_0 \notin \mathfrak{D}_n$ while the $s_i \in \mathfrak{D}_n$ for $1 \leq i \leq n-1$. We define

$$s_e = s_0s_1s_0 \in \mathfrak{D}_n.$$

That is,

$$s_e = \left[\begin{array}{c} \text{the permutation exchanging } \pm 1 \text{ and} \\ \mp 2 \text{ with other numbers fixed} \end{array} \right] \in \mathfrak{BC}_n$$

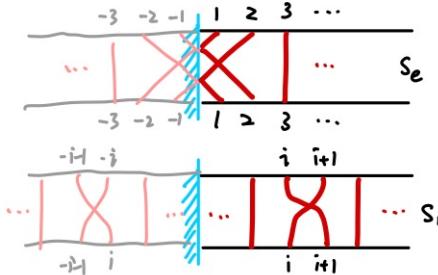
The label is



The geometric representation $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^n$ with natural pairing and

$$\alpha_e = e_1 + e_2 \quad \alpha_i = e_{i+1} - e_i \quad (1 \leq i \leq n-1).$$

We have a diagram notation



Miscellany.

1.10. Remark. From now, we will assume W is a Weyl group, i.e. we are equipped with a underlying root system. The same result holds for any Coxeter group if we replace R by the set of **root directions**

$$\vec{R} = \left\{ \frac{w\alpha_i}{\|w\alpha_i\|} : w \in W, i \in I \right\} \subset \mathfrak{h}_{\mathbb{R}}^*.$$

1.11. Reflections. For $\alpha \in R$, denote

r_α = the reflection with respect to $\alpha \in W$.

If $\alpha = w\alpha_i$, then $r_\alpha = ws_iw^{-1}$. We define **reflections** by

$$\{\text{reflections}\} = \{ws_iw^{-1} : w \in W, i \in I\} = \{r_\alpha : \alpha \in R\}.$$

We call s_i ($i \in I$) a **simple reflection**.

1.12. Positive roots.

The set of **positive/negative roots**

$$\mathbb{R}^\pm = \{\alpha \in \mathbb{R} : \pm \alpha \in \text{span}_{\geq 0}(\alpha_i)_{i \in I}\}.$$

We have $\mathbb{R} = \mathbb{R}^+ \sqcup \mathbb{R}^-$. For $\alpha \in \mathbb{R}$, we denote $\alpha > 0$ if $\alpha \in \mathbb{R}^+$ and $\alpha < 0$ otherwise. We call α_i ($i \in I$) a **simple root**.

1.13. Hyperplanes.

Let us consider

$$\mathfrak{h}_\mathbb{R} = \text{dual space of } \mathfrak{h}_\mathbb{R}^* \cong \mathfrak{h}_\mathbb{R}^*.$$

For any $\alpha \in \mathbb{R}$, we denote

$$H_\alpha = \{x \in \mathfrak{h}_\mathbb{R} : \langle x, \alpha \rangle = 0\} \subset \mathfrak{h}_\mathbb{R}.$$

1.14. Fundamental coweights.

Denote **fundamental (co)weight** $\varpi_i \in \mathfrak{h}_\mathbb{R}^*$ ($\varpi_i^\vee \in \mathfrak{h}_\mathbb{R}$) be such that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}.$$

1.15. Chamber.

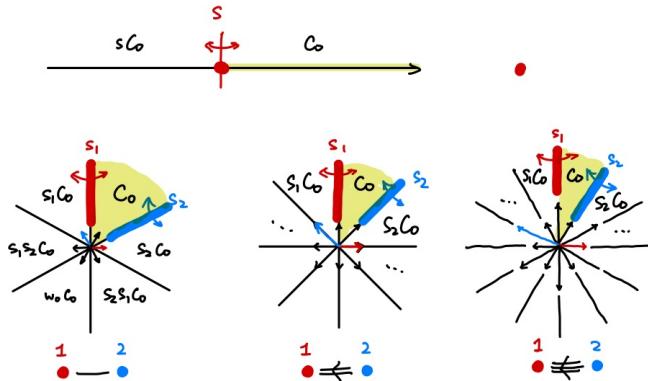
We define **chambers** by

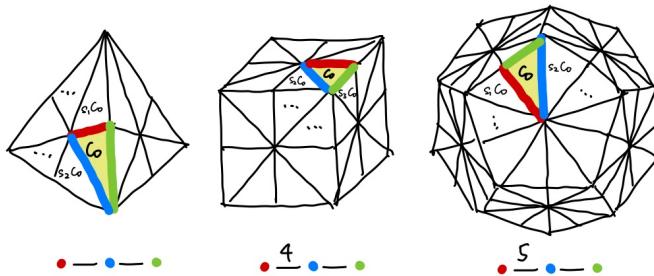
$$\{\text{chambers}\} = \text{connected components of } \left(\mathfrak{h}_\mathbb{R} \setminus \bigcup_{\alpha \in \mathbb{R}} H_\alpha \right).$$

We define the **dominant chamber** to be the cone

$$C_0 = \{x \in \mathfrak{h}_\mathbb{R} : \langle \alpha_i, x \rangle > 0\} = \text{span}_{\geq 0}(\varpi_i^\vee : i \in I).$$

Here we collect example in small dimensions.





1.16. Theorem. We have a bijection

$$W \longrightarrow \{\text{chambers}\}, \quad w \longmapsto wC_0.$$

Under this bijection,

$$\begin{aligned} \text{the chamber of } s_i w &= \text{reflection of the chamber} \\ &\quad \text{of } w \text{ with respect to } \alpha_i \\ \text{the chamber of } ws_i &= \text{the chamber sharing the wall} \\ &\quad wH_{\alpha_i} \text{ with the chamber of } w \end{aligned}$$

1.17. Length. For any $w \in W$, we define

$$\ell(w) = \text{minimal length of writing } w \text{ as} \\ \text{a product of simple reflections}$$

If

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \quad \ell = \ell(w).$$

We call $(i_1, i_2, \dots, i_\ell)$ is a **reduced word** of w .

1.18. Length formula. In terms of chambers,

$$\ell(w) = \#\{\text{hyperplanes separating } C_0 \text{ and } wC_0\}$$

In terms of roots,

$$\ell(w) = \# \text{Inv}(w), \quad \text{Inv}(w) = \{\alpha \in R^+ : w\alpha \in R^-\}.$$

There is a bijection between hyperplanes and $\text{Inv}(w^{-1})$.

1.19. Bruhat order. We define the **Bruhat order** over W to be the following equivalent order

- the order generated by

$$u \leq w \text{ if } w = ur_\alpha \text{ and } \ell(w) = \ell(u) + 1.$$

- the order generated by

$$u \leq w \text{ if } w = ur_\alpha \text{ and } \ell(w) > \ell(u).$$

- $u \leq w$ if there is a subword of u in a reduced word of w .
- $u \leq w$ if there is a subword of u in any reduced word of w .

We remark that for $\alpha \in R^+$,

$$ur_\alpha > u \iff u\alpha > 0 \iff \alpha \in \text{Inv}(u).$$

2. TWO REALIZATIONS

Realization A. Let W be a finite Weyl group with root system R . Let $\{\alpha_i : i \in I\} \subset R$ be the set of simple roots.

2.1. Root lattice. Recall the definition of α^\vee for $\alpha \in R$. Let us denote the (co)root lattice

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}_\mathbb{R}^* \quad Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_\mathbb{R}.$$

2.2. Definition. The affine Weyl group is

$$W_a = W \ltimes Q^\vee.$$

For $\lambda \in Q^\vee$, we define $t_\lambda \in W_a$ the corresponding element. That is,

$$t_\lambda t_\mu = t_{\lambda+\mu}, \quad t_\lambda^{-1} = t_{-\lambda}, \quad t_0 = \text{id}, \quad w t_\lambda w^{-1} = t_{w(\lambda)}.$$

2.3. Example. For type A_1 ,

$$\begin{aligned} &\text{the Weyl group } W = \{\text{id}, s\} = \mathfrak{S}_2 \\ &\text{the coroot lattice } Q^\vee = \mathbb{Z}\alpha^\vee. \end{aligned}$$

Let us denote $t = t_{\alpha^\vee}$. Then we have

$$\begin{aligned} W_a &= \left\langle s, t : \begin{array}{c} s^2 = \text{id} \\ sts = t^{-1} \end{array} \right\rangle \xrightarrow{s_0 = ts} \left\langle s, s_0 : s^2 = s_0^2 = \text{id} \right\rangle \\ &= \text{the Coxeter group of } \left[\begin{array}{c|cc} s & & s_0 \\ \bullet & \infty & \circ \end{array} \right] \end{aligned}$$

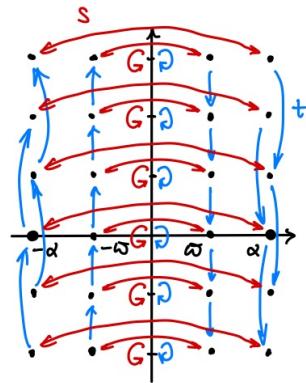
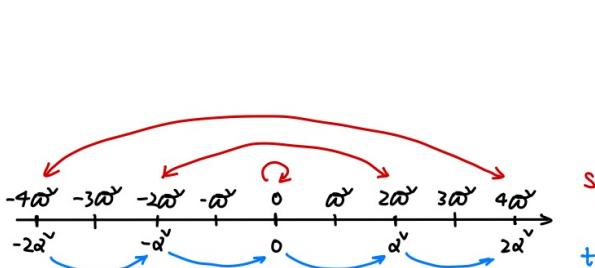
2.4. Two Actions. The affine Weyl group acts

on Q^\vee affinely : $(wt_\lambda) \cdot \mu = w(\lambda + \mu).$	on $Q \oplus \mathbb{Z}\delta$ linearly : $(wt_\lambda) \cdot (\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta.$
---	--

Here δ is a formal variable, called the **null root**. Note that the same formula defines an action on

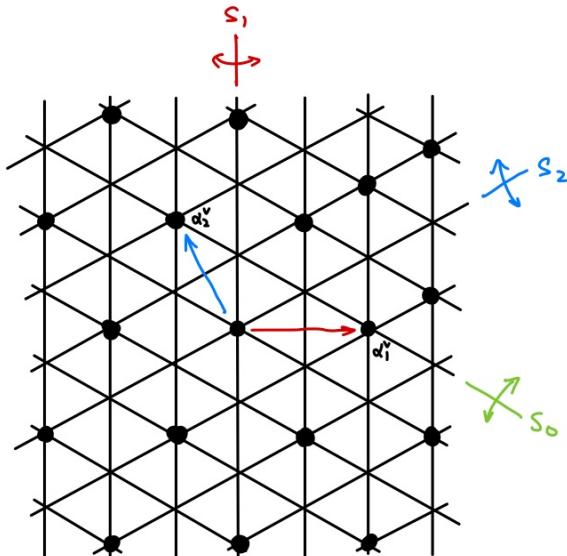
$$\mathfrak{h}_\mathbb{R} = \mathbb{R} \otimes_{\mathbb{Z}} Q^\vee, \quad \mathfrak{h}_\mathbb{R}^* \oplus \mathbb{R}\delta = \mathbb{R} \otimes_{\mathbb{Z}} (Q \oplus \mathbb{Z}\delta).$$

2.5. Example. Here are the example of type A_1 . We denote $\omega^\vee = \frac{1}{2}\alpha^\vee$ and $\omega = \frac{1}{2}\alpha$.



2.6. Exercise. Find the action of $s_0 = ts$ in the above example.

2.7. Example. Let us consider A_2 . Let $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$. Consider $s_0 = t_{\theta^\vee} s_1 s_2 s_1$. The following figure shows the action of W_a on Q^\vee .



With more efforts, we can see

$$W_a = \text{the Coxeter group of } \begin{bmatrix} & s_0 \\ & \diagdown \\ s_1 & \text{---} & s_2 \end{bmatrix}$$

2.8. Roots.

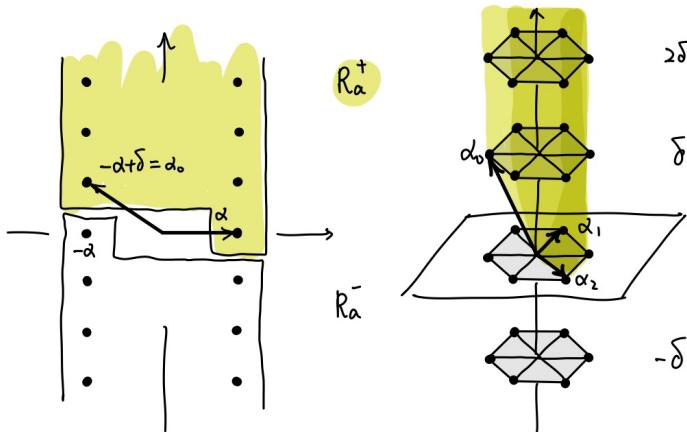
We define the set of **real affine roots**

$$R_a = \{\alpha + k\delta : \alpha \in R, k \in \mathbb{Z}\} \subset Q \oplus \mathbb{Z}\delta.$$

We define the set of **positive real roots**

$$R_a^+ = \{\alpha + k\delta : k > 0 \text{ or } (k = 0 \text{ and } \alpha \in R^+)\} \subset R_a.$$

2.9. Examples. Here is the illustration of affine root systems of type A_1 and A_2



2.10. Reflections. For each root $\alpha + k\delta \in R_a$, we define the **reflection**

$$r_{\alpha+k\delta} = r_\alpha t_{k\alpha^\vee} \in W_a.$$

The action of $r_{\alpha+k\delta}$ on $Q \oplus \mathbb{Z}\delta$ is given by a linear reflection

$$r_{\alpha+k\delta}(\beta + n\delta) = \beta + n\delta - \langle \alpha^\vee, \beta \rangle (\alpha + \delta).$$

The action of $r_{\alpha+k\delta}$ on Q^\vee is given by the affine reflection along the hyperplane

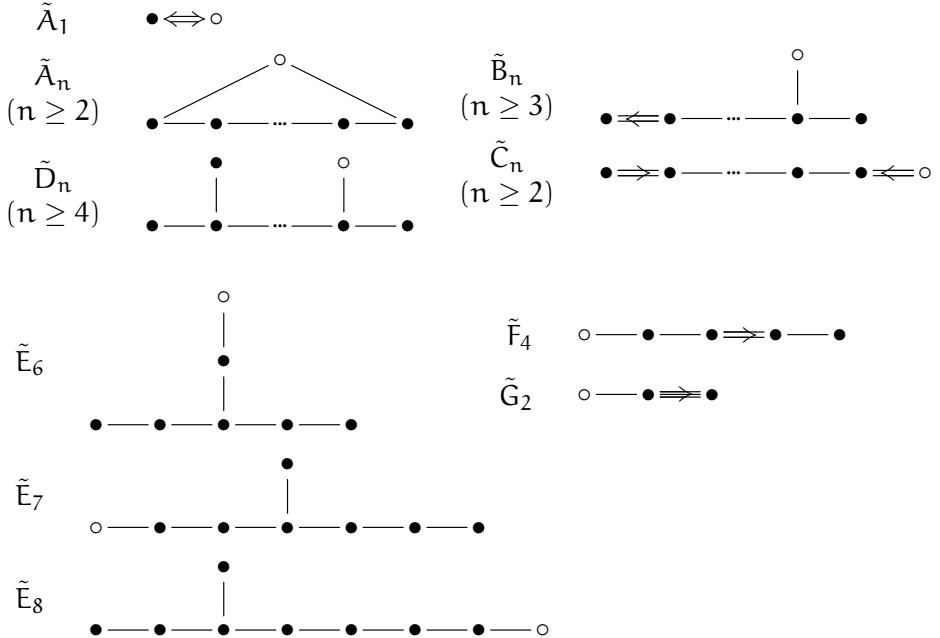
$$H_{\alpha+k\delta} = H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

2.11. Simple roots. Let $\theta \in R^+$ be the unique highest root. We denote

$$\alpha_0 = -\theta + \delta \in R_a^+, \quad s_0 = r_{\alpha_0} = t_{\theta^\vee} r_\theta \in W_a, \quad I_a = I \cup \{0\}.$$

Realization B.

2.12. Affine Dynkin diagram. The following are **untwisted affine Dynkin diagrams**



The **twisted affine Dynkin diagrams** are their dual.

2.13. Theorem. The affine Weyl group W_a constructed above is a Coxeter group with Coxeter generator $\{s_i : i \in I_a\}$.

2.14. Example. Let $n \geq 2$. For type A_{n-1} , the Weyl group is S_n and the coroot lattice

$$Q^\vee = \{(a_1, \dots, a_n) : a_1 + \dots + a_n = 0\} \subset \mathbb{Z}^n.$$

The affine Weyl group admits the following realization

$$\tilde{S}_n^0 = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{l} f(i+n) = f(i) + n \\ \sum_{i=1}^n (f(i) - i) = 0. \end{array} \right\}.$$

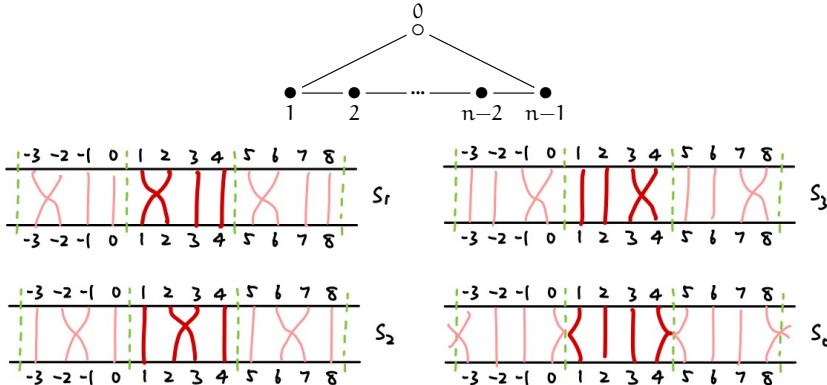
An element in \tilde{S}_n is determined by its values at $1 \leq i \leq n$. The identification is given by

$$\text{wt}_\lambda(i) = w(i) + \lambda_i n \quad (1 \leq i \leq n).$$

Denote s_i for $i \in \mathbb{Z}/n\mathbb{Z}$ by

$$s_i = \begin{array}{l} \text{the affine permutation exchanging } j \text{ and } j+1 \\ \text{when } i \equiv j \pmod{n} \text{ with other numbers fixed} \end{array} \in \tilde{S}_n^0.$$

This equips the Coxeter group structure over \tilde{S}_n^0 , where the Coxeter diagram is ($n \geq 3$)



2.15. Example. Let $n \geq 2$. For type C_n , the Weyl group is \mathfrak{BC}_n , and the coroot lattice is

$$\begin{aligned} Q^\vee &= \mathbb{Z}e_1 \oplus \mathbb{Z}(e_2 - e_1) \oplus \cdots \oplus \mathbb{Z}(e_n - e_{n-1}) \\ &= \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n. \end{aligned}$$

We can realize the affine Weyl group as

$$\tilde{\mathfrak{C}}_n = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : \begin{array}{l} f(-i) = -f(i) \\ f(2n+2+i) = f(i) \end{array} \right\}.$$

Note that for any $a \in \mathbb{Z}(n+1) = \{\dots, -(n+1), 0, n+1, 2n+2, \dots\}$, we have

$$f(a+i) + f(a-i) = 2a.$$

An element of $\tilde{\mathfrak{C}}_n$ is determined by its value at $1 \leq i \leq n$. The identification is given by

$$wt_\lambda = w(i) - \lambda_i(2n+2).$$

The Coxeter generators are

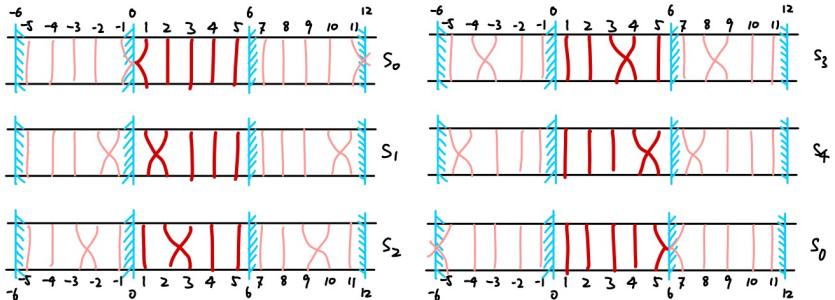
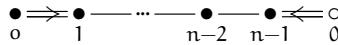
$$s_0 = \left[\begin{array}{l} \text{the permutation exchanging } a+1 \text{ and } a-1 \text{ for} \\ a \in \mathbb{Z}(2n+2) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{C}}_n$$

$$s_0 = \left[\begin{array}{l} \text{the permutation exchanging } a+1 \text{ and } a-1 \text{ for} \\ a \in (n+1) + \mathbb{Z}(2n+2) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{C}}_n$$

and $1 \leq i \leq n-1$,

$$s_i = \left[\begin{array}{l} \text{the permutation exchanging } a \pm i \text{ and } a \pm (i+1) \\ (a \in \mathbb{Z}(2n+2)) \text{ with other numbers fixed} \end{array} \right] \in \tilde{\mathfrak{C}}_n.$$

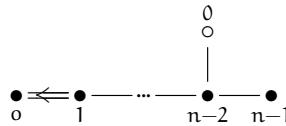
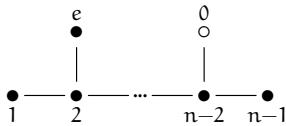
The Dynkin diagram is



2.16. Example. We will not go into details of affine type B/D. But we mention that

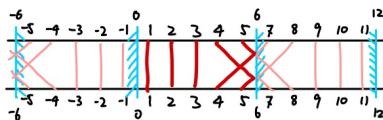
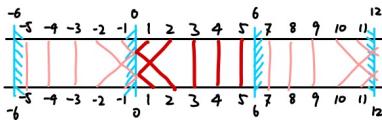
$$\tilde{\mathfrak{D}}_n \subset \tilde{\mathfrak{B}}_n \subset \tilde{\mathfrak{C}}_n$$

with generators described by



$$\begin{aligned}s_e^D &= s_0 s_1 s_0 \\s_0^D &= s_0 s_{n-1} s_0\end{aligned}$$

$$s_0^B = s_0 s_{n-1} s_0$$



Alcoves.

2.17. Alcove. For each root $\alpha + k\delta \in R_a$, we defined a **hyperplane**

$$H_{\alpha+k\delta} = H_{\alpha,k} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle x, \alpha \rangle + k = 0\} \subset \mathfrak{h}_{\mathbb{R}}.$$

We define **alcoves** by

$$\{\text{alcoves}\} = \text{connected components of } \mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha,k} H_{\alpha,k}.$$

Let us consider the **fundamental alcove**, i.e. the unique alcove A_0 with

$$A_0 \subset C_0, \quad 0 \in \text{closure of } A_0.$$

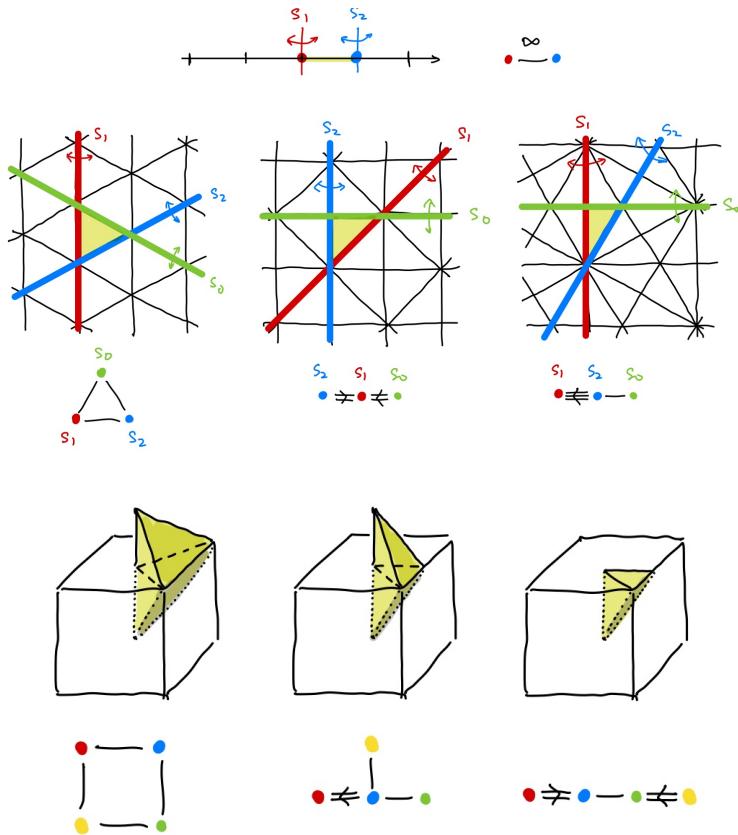
It can be described as

$$\begin{aligned}A_0 &= \{x \in \mathfrak{h}_{\mathbb{R}} : 0 < \langle x, \alpha \rangle < 1 \text{ for all } \alpha > 0\} \\&= \{x \in \mathfrak{h}_{\mathbb{R}} : 0 < \langle x, \alpha_i \rangle \text{ for } i \in I \text{ and } \langle x, \theta \rangle < 1\} \\&= \text{bounded convex set cut by } H_{\alpha_i} \text{ for } i \in I_a \\&= \text{interior of Conv} \left(\{0\} \cup \left\{ \frac{1}{(\omega_i^\vee, \theta)} \omega_i^\vee : i \in I \right\} \right).\end{aligned}$$

Here θ is the highest root. Note that

$$\langle \omega_i^\vee, \theta \rangle = \text{coefficient of } \alpha_i \text{ in } \theta$$

Here we collect some example in small dimensions



2.18. Theorem. We have a bijection

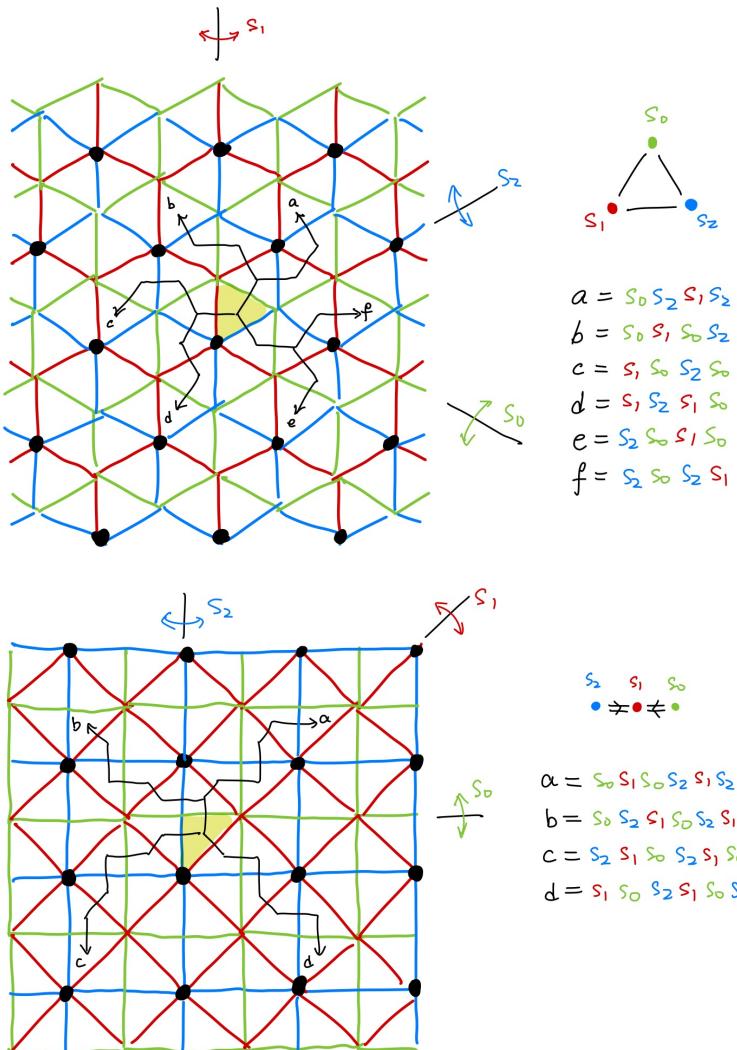
$$W_a \longrightarrow \{\text{alcoves}\}, \quad \text{wt}_\lambda \longmapsto \text{wt}_\lambda(A_0).$$

Under this bijection,

the alcove of $s_i \text{wt}_\lambda$ = reflection of the alcove of w with respect to H_{α_i}

the alcove of $\text{wt}_\lambda s_i$ = the alcove sharing the wall wH_{α_i} with the alcove of w

2.19. Example. The following shows how “alcove move” corresponds to a reduced word



3. LENGTH FORMULA

Iwahori–Matsumoto Formula.

3.1. Inversion. For any $\text{wt}_\lambda \in W_a$, we define set of **inversions**

$$\text{Inv}(\text{wt}_\lambda) = \{\alpha + k\delta \in R_a^+ : \text{wt}_\lambda(\alpha + k\delta) \in R_a^-\}.$$

Then the length function is given by

$$\begin{aligned}\ell(\text{wt}_\lambda) &= \begin{matrix} \text{minimal length of writing } \text{wt}_\lambda \text{ as} \\ \text{a product of simple reflections} \end{matrix} \\ &= \#\{\text{hyperplanes separating } A_0 \text{ and } \text{wt}_\lambda A_0\} \\ &= \# \text{Inv}(\text{wt}_\lambda)\end{aligned}$$

There is a bijection between hyperplanes and $\text{Inv}((\text{wt}_\lambda)^{-1})$.

3.2. Left inversions. Let us denote the set of **left inversions**

$$\begin{aligned}\text{LInv}(\text{wt}_\lambda) &= \text{Inv}((\text{wt}_\lambda)^{-1}) = \{-\text{wt}_\lambda(\alpha + k\delta) : \alpha + k\delta \in \text{Inv}(\text{wt}_\lambda)\} \\ &= R_a^+ \setminus \text{wt}_\lambda R_a^+.\end{aligned}$$

There is a bijection between hyperplanes and left inversions.

3.3. Example. Let us consider A_1 . The fundamental alcove A_0 is the interval $(0, \varpi)$ and

$$sA_0 = (-\varpi, 0), \quad s_0A_0 = (\varpi, 2\varpi), \quad tA_0 = (2\varpi, 3\varpi).$$

So we have

$$\ell(s) = \ell(s_0) = 1, \quad \ell(t) = 2.$$

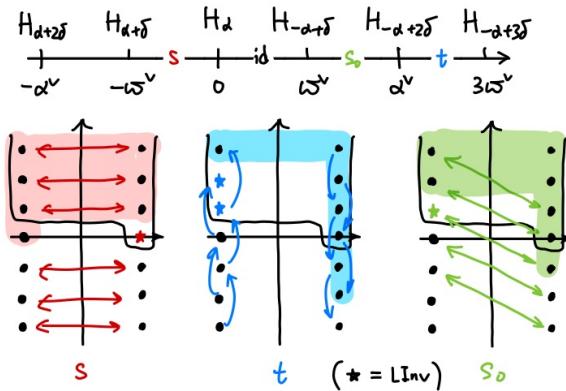
Alternatively, it is not hard to compute

$$\text{Inv}(s) = \{\alpha\}, \quad \text{Inv}(s_0) = \{-\alpha + \delta\}, \quad \text{Inv}(t) = \{\alpha, \alpha + \delta\}.$$

This confirms the computation of the lengths.

$$\begin{array}{ccccccc} \cdots & H_{\alpha+2\delta} & H_{\alpha+\delta} & H_\alpha & H_{-\alpha+\delta} & H_{-\alpha+2\delta} & \cdots \\ \cdots & -2\varpi^\vee & -\varpi^\vee & 0 & \varpi^\vee & 2\varpi^\vee & \cdots \end{array}$$

Here is the diagram



3.4. Theorem. We have

$$\ell(\text{wt}_\lambda) = \sum_{\alpha > 0} \left| \langle \alpha, \lambda \rangle + \delta_{w\alpha < 0} \right|.$$

Here $\delta_p = 1$ if a statement p is true and equals 0 otherwise.

Proof. Fix a positive root $\alpha \in R^+$. We want to compute the contribution of

$$\pm\alpha + k\delta \in \text{Inv}(\text{wt}_\lambda)$$

Note that

$$\text{wt}_\lambda(\pm\alpha + k\delta) = \pm w\alpha + (n \mp \langle \lambda, \alpha \rangle)\delta.$$

For this vector in R_a^- , we summarize four cases in the following table

	$w\alpha > 0$	$w\alpha < 0$
$\pm = +$ i.e. $k \geq 0$	$k - \langle \lambda, \alpha \rangle < 0$ i.e. $0 \leq k < \langle \lambda, \alpha \rangle$	$k - \langle \lambda, \alpha \rangle \leq 0$ i.e. $0 \leq k \leq \langle \lambda, \alpha \rangle$
$\pm = -$ i.e. $k > 0$	$k + \langle \lambda, \alpha \rangle \leq 0$ i.e. $0 < k \leq -\langle \lambda, \alpha \rangle$	$k + \langle \lambda, \alpha \rangle > 0$ i.e. $0 < k < -\langle \lambda, \alpha \rangle$
Total #	$ \langle \lambda, \alpha \rangle $	$ \langle \lambda, \alpha \rangle + 1 $

This completes the proof. □

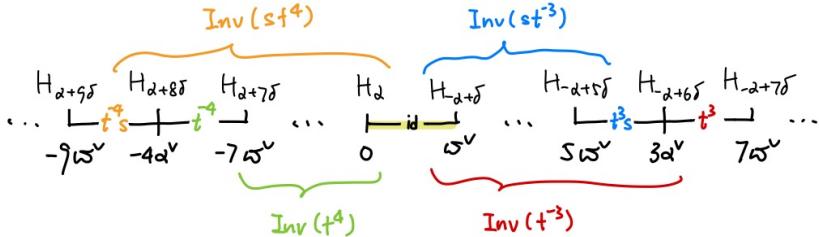
3.5. Corollary. Let us record the set of inversions for future references. For $\alpha \in R^+$, we denote

$$\text{Inv}_\alpha(\text{wt}_\lambda) = \{\pm\alpha + k\delta \in \text{Inv}(\text{wt}_\lambda)\}$$

the contribution of the affine positive roots as in the proof. Then the above table shows

$$\text{Inv}_\alpha(\text{wt}_\lambda) = \begin{cases} \{\alpha + k\delta : 0 \leq k < \langle \lambda, \alpha \rangle + \delta_{w\alpha<0}\}, & \langle \lambda, \alpha \rangle \geq 0, \\ \{-\alpha + k\delta : 0 < k \leq -\langle \lambda, \alpha \rangle - \delta_{w\alpha<0}\}, & \langle \lambda, \alpha \rangle < 0. \end{cases}$$

3.6. Example. Consider the case A_1 . Recall the hyperplanes are in bijection with left inversions.



3.7. Exercise. Note that $(\text{wt}_\lambda)^{-1} = t_{-\lambda} w^{-1} = w^{-1} t_{-w\lambda}$. Check that

$$\ell(\text{wt}_\lambda) = \ell((\text{wt}_\lambda)^{-1}).$$

3.8. Example. In type \tilde{A}_{n-1} , we realized the affine Weyl group as \tilde{S}_n^0 . For $f \in \tilde{S}_n^0$, we can compute the length

$$\ell(f) = \#\left\{(i, j) : \begin{array}{l} 1 \leq i \leq n \\ i < j, f(i) > f(j) \end{array}\right\}.$$

In terms of Iwahori–Matsumoto formula 3.4,

$$\ell(\text{wt}_\lambda) = \sum_{i < j} |\lambda_i - \lambda_j + \delta_{w(i)>w(j)}|.$$

Examples.

3.9. In this paragraph, we will use a lot of facts about parabolic subgroups, which is summarized at the appendix of this section.

—Length of translations.

3.10. Cartan vector. Let us denote

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{i \in I} \omega_i \in \mathfrak{h}_{\mathbb{R}}^*.$$

It satisfies

$$\rho - w\rho = \sum_{\alpha \in \text{Inv}(w^{-1})} \alpha.$$

3.11. Dominant case. Let $\lambda \in Q^\vee$ be dominant. Then

$$\ell(t_\lambda) = \sum_{\alpha > 0} |\langle \alpha, \lambda \rangle + \delta_{\alpha < 0}| = \sum_{\alpha > 0} \langle \alpha, \lambda \rangle = 2\langle \rho, \lambda \rangle.$$

3.12. General case. For general $\lambda \in Q^\vee$, we can always find $w \in W$ such that

$$w\lambda_0 = \lambda, \quad \lambda_0 \text{ is dominant.}$$

Then

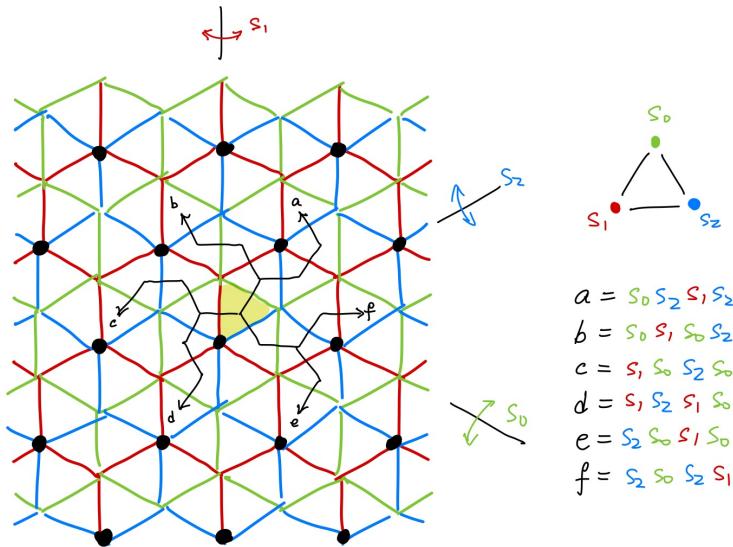
$$\begin{aligned} \ell(t_\lambda) &= \sum_{\alpha > 0} |\langle \alpha, \lambda \rangle| = \sum_{\alpha > 0} |\langle \alpha, w\lambda_0 \rangle| \\ &= \sum_{\alpha' > 0} |\langle w^{-1}\alpha, \lambda_0 \rangle| = \sum_{\alpha' > 0} |\langle \alpha', \lambda_0 \rangle| \\ &= \sum_{\alpha' > 0} \langle \alpha', \lambda_0 \rangle = 2\langle \rho, \lambda_0 \rangle = \ell(t_{\lambda_0}). \end{aligned}$$

Here $\alpha' = \pm w^{-1}\alpha > 0$.

3.13. Example. Consider the case A_2 . Recall that $\theta^\vee = \alpha_1^\vee + \alpha_2^\vee$. Then $\rho = \theta$. So

$$\ell(t_{\theta^\vee}) = \ell(t_{w\theta^\vee}) = 2\langle \rho, \theta \rangle = 4.$$

This can be seen from the first diagram of Example 2.19.



3.14. Inversion set. It would be useful to compute the set of inversions. We have

$$\text{Inv}_\alpha(t_\lambda) = \begin{cases} \{\alpha + k\delta : 0 \leq k < \langle \lambda, \alpha \rangle\}, & \langle \lambda, \alpha \rangle > 0, \\ \emptyset, & \langle \lambda, \alpha \rangle = 0, \\ \{-\alpha + k\delta : 0 < k \leq -\langle \lambda, \alpha \rangle\}, & \langle \lambda, \alpha \rangle < 0. \end{cases}$$

—Minimal representatives.

3.15. Formulation. We have a bijection

$$Q^\vee \xrightarrow{1:1} W_a/W, \quad \lambda \mapsto t_\lambda W.$$

We will describe the parabolic decomposition

$$t_\lambda = u_\lambda v_\lambda, \quad u_\lambda = \min(t_\lambda W) \text{ and } v_\lambda \in W.$$

3.16. Example. Let us consider type A_1 . The set of minimal representative is

$$\begin{array}{ccccccc} \lambda & 0 & \alpha^\vee & -\alpha^\vee & 2\alpha^\vee & -2\alpha^\vee & \dots \\ u_\lambda & \text{id} & s_0 & ss_0 & s_0ss_0 & ss_0ss_0 & \dots \\ v_\lambda & \text{id} & s & \text{id} & s & \text{id} & \dots \end{array}$$

Equivalently, we want to find $v \in W$ such that $u^{-1} = vt_{-\lambda}$ has minimal length

$$\ell(u) = \ell(u^{-1}) = \sum_{\alpha > 0} | -\langle \alpha, \lambda \rangle + \delta_{v\alpha < 0} |.$$

To minimize $\ell(u)$, we wish that each summand is minimal, i.e.

$$\begin{aligned} \langle \alpha, \lambda \rangle \leq 0 &\implies v\alpha > 0, \\ \langle \alpha, \lambda \rangle > 0 &\implies v\alpha < 0. \end{aligned}$$

We will see, this is achievable.

3.17. Antidominant case. Let $\lambda \in Q^\vee$ be antidominant, i.e. $-\lambda$ is dominant. To minimize $\ell(u^{-1})$, it suffices to take $v = \text{id}$.

3.18. General case. Let us pick $w \in W$ such that

$$\lambda = w\lambda_0, \quad \lambda \text{ is anti-dominant.}$$

Such w 's form a coset of W/W_P for W_P the stabilizer of λ_0 . Let us pick the minimal one, i.e. $w \in W^P$. Then

$$\langle \alpha, \lambda \rangle = \langle w^{-1}\alpha, \lambda_0 \rangle < 0 \implies w^{-1}\alpha > 0$$

$$\langle \alpha, \lambda \rangle = \langle w^{-1}\alpha, \lambda_0 \rangle = 0 \implies w^{-1}\alpha \in R_P \xrightarrow{w \in W^P} w^{-1}\alpha \in R_P^+,$$

$$\langle \alpha, \lambda \rangle = \langle w^{-1}\alpha, \lambda_0 \rangle > 0 \implies w^{-1}\alpha < 0.$$

It suffices to take $v = w^{-1}$.

3.19. Dominant case. Let $\lambda \in Q^\vee$ be dominant. Let $W_P = w_0 W_\lambda w_0$ be the stabilizer of $w_0\lambda$. By above computation, $v_\lambda = (w_0^P)^{-1}$ for $w_0^P = \max(W^P)$ the maximal element of W^P . Actually,

$$v_\lambda = \max(W^\lambda).$$

This is because $w_0^P = \max(W^P) = w_0 \cdot w_{0,P}$, so

$$v_\lambda = w^{-1} = w_{0,P} \cdot w_0 = w_0 \cdot w_{0,\lambda} = \max(W^\lambda).$$

3.20. Summary v1. In the parabolic decomposition

$$t_\lambda = u_\lambda v_\lambda,$$

the element v_λ is the minimal element $v \in W$ such that $v\lambda$ is anti-dominant.

3.21. Summary v2. Let $\lambda \in Q^\vee$ be anti-dominant. Denote W_P the stabilizer of $-\lambda$. Then for $w \in W^P$, the expression

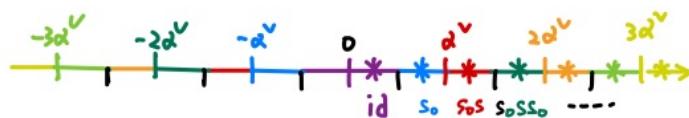
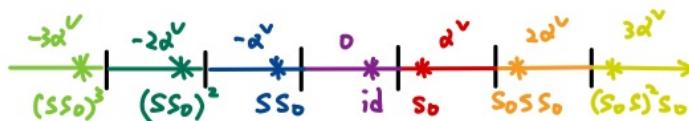
$$t_\lambda = (wt_\lambda) \cdot w^{-1}$$

gives the parabolic decomposition.

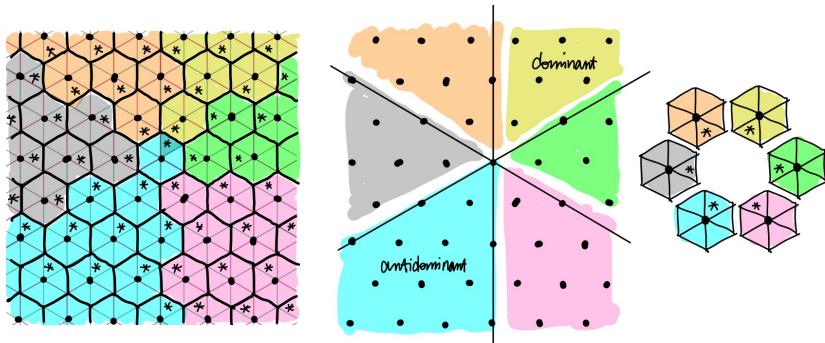
3.22. Example. Consider type A_1 . We have

$$\begin{array}{ccccccc} \lambda & 0 & \alpha^\vee & -\alpha^\vee & 2\alpha^\vee & -2\alpha^\vee & \dots \\ \min(t_\lambda W) & id & s_0 & ss_0 & s_0ss_0 & ss_0ss_0 & \dots \end{array}$$

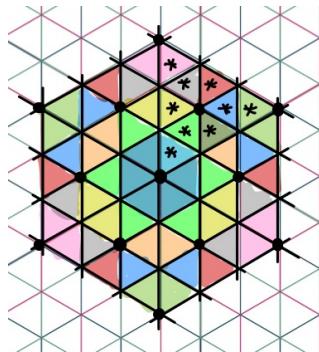
$$\begin{array}{ccccccc} \lambda & 0 & -\alpha^\vee & \alpha^\vee & -2\alpha^\vee & 2\alpha^\vee & \dots \\ \min(Wt_\lambda) & id & s_0 & s_0s & s_0ss_0 & s_0ss_0s & \dots \end{array}$$



3.23. Example. The case A_2 . We mark the minimal element in the cosets. Each right coset corresponds to



Each left coset corresponds to a W -orbit



3.24. Exercise. Prove that

$$\text{wt}_\lambda = \min(W\text{t}_\lambda) \iff \text{wt}_\lambda A_0 \subset C_0.$$

This gives a bijection.

3.25. Example. For $\theta^\vee \in Q^\vee$, recall that

$$s_0 = t_{\theta^\vee} r_\theta.$$

We actually have

$$u_{\theta^\vee} = s_0, \quad v_{\theta^\vee} = r_\theta \in W.$$

This implies

$$2\langle \rho, \theta^\vee \rangle = \ell(r_\theta) + 1.$$

3.26. Example. Let us consider the case of A_{n-1} . For any $f \in \tilde{\mathfrak{S}}_n^0$, it is clear that in the decomposition

$$f = uv, \quad u = \min(fW) \text{ and } v \in \mathfrak{S}_n$$

we have

$$\begin{aligned} u(1) &= \min(f(1), \dots, f(n)), \\ &\dots = \dots \\ u(n) &= \max(f(1), \dots, f(n)). \end{aligned}$$

—Double cosets.

3.27. Double cosets. We have a bijection

$$Q_{\text{dom}}^\vee \xrightarrow{1:1} W \backslash W_a / W, \quad t_\lambda \longmapsto Wt_\lambda W.$$

Recall that $wt_\lambda w^{-1} = t_{w\lambda}$. We actually have

$$Wt_\lambda W = \bigcup_{w \in W} t_{w\lambda} W.$$

Similar to one-side case, there is also a unique minimal element in each double coset.

3.28. Summary v3. Let $\lambda \in Q^\vee$ be anti-dominant. Denote W_P the stabilizer of $-\lambda$. By Summary 3.21 above,

$$W^P \times W \xrightarrow{1:1} Wt_\lambda W, \quad (w, u) \longmapsto wt_\lambda u,$$

with

$$\ell(wt_\lambda u) = -\ell(w) + \ell(t_\lambda) + \ell(u).$$

By taking inverse, we have a dominant version.

3.29. Summary v4. Let $\lambda \in Q^\vee$ be dominant, with stabilizer W_P . We have a bijection

$$W \times W^P \longrightarrow W t_\lambda W, \quad (u, w) \longmapsto u t_\lambda w^{-1}$$

with

$$\ell(u t_\lambda w^{-1}) = \ell(u) + \ell(t_\lambda) - \ell(w).$$

As a result,

$$\min(W t_\lambda W) = \min(t_\lambda W).$$

Appendix: Parabolic subgroups.

3.30. Definition. Let I_P be a subset of I . Then we denote the **parabolic subgroup**

$$W_P = (\text{subgroup generated by } s_i \text{ with } i \in I_P) \subset W$$

and $R_P \subset R$ the root system of W_P .

3.31. Minimal representative. For any $w \in W$, there is a minimal element, called the **minimal representative**, in the right coset wW_P under the Bruhat order. Let us denote the set of **minimal representative**

$$W^P = \{\min(wW_P) : w \in W\}.$$

We have a length-additive bijection

$$W^P \times W_P \longrightarrow W, \quad (u, v) \longmapsto uv.$$

Note that

$$\begin{aligned} w \in W_P &\iff \text{Inv}(w) \subset R_P^+ \\ w \in W^P &\iff \text{Inv}(w) \subset R^+ \setminus R_P^+. \end{aligned}$$

3.32. Parabolic Bruhat order. For two cosets uW_P, wW_P , we define

$$uW_P \leq wW_P \iff uv \leq wv' \text{ for some } v, v' \in W_P.$$

The the bijection

$$W^P \longrightarrow W/W_P, \quad w \longmapsto wW_P$$

is an isomorphism of posets.

3.33. Stabilizer. For a dominant $\lambda \in \mathfrak{h}_{\mathbb{R}}$, the stabilizer

$$W_{\lambda} = W_P = \{w \in W : w\lambda = \lambda\}$$

is a parabolic subgroup with

$$I_P = \{i \in I : \langle x, \alpha_i \rangle = 0\}.$$

We denote

$$W^{\lambda} = W^P.$$

Then we have a bijection

$$W^{\lambda} \longrightarrow W\lambda, \quad w \longmapsto w\lambda.$$

4. EXTENDED AFFINE WEYL GROUPS

Definition.

4.1. Weight lattice. Recall the definition of ω_i^\vee for $i \in I$. Let us denote the (co)weight lattice

$$P = \bigoplus_{i \in I} \mathbb{Z}\omega_i \subset \mathfrak{h}_\mathbb{R}^*, \quad P^\vee = \bigoplus_{i \in I} \mathbb{Z}\omega_i^\vee \subset \mathfrak{h}_\mathbb{R}.$$

From the axiom of root system, we have

$$Q \subseteq P, \quad Q^\vee \subseteq P^\vee.$$

In general, they are not equal.

4.2. Definition. The extended affine Weyl group is

$$W_e = W \ltimes P^\vee.$$

For $\lambda \in P^\vee$, we define $t_\lambda \in W_a$ the corresponding element.

4.3. Two Actions. The extended affine Weyl group acts

on P^\vee affinely: $(wt_\lambda) \cdot \mu = w(\lambda + \mu).$	$ $	on $Q \oplus \mathbb{Z}\delta$ linearly: $(wt_\lambda) \cdot (\alpha + k\delta) = w\alpha + (k - \langle \lambda, \alpha \rangle)\delta.$
---	-----	--

It is not hard to see R_a is stable under W_e , so the set of inversions also makes sense. We define the length

$$\ell(wt_\lambda) = \# \text{Inv}(wt_\lambda).$$

It is also computed by Iwahori–Matsumoto formula 3.4.

4.4. Remark. However, W_e is not a Coxeter group in general. Actually, there would be many elements in W_e of length 0. The main purpose of this section is to study them.

4.5. The group Ω . Let us denote

$$\Omega = \{\pi \in W_e : \ell(\pi) = 0\} = \{\pi \in W_e : \pi A_0 = A_0\}.$$

Note that the norm vector of facets of $\overline{A_0}$ are simple roots. So we have

$$\begin{aligned}\Omega &\hookrightarrow \text{Aut}(A_0) = \text{Aut}(\text{affine Coxeter diagram}) \\ &= \text{Aut}(\text{affine Dynkin diagram}).\end{aligned}$$

The last equality follows from the classification, i.e. any automorphism of affine Coxeter group preserving length. Thus Ω acts on W_a , and

$$W_e = \Omega \ltimes W_a.$$

4.6. Fundamental group. In particular the composition is an isomorphism

$$\Omega \subset W_e \twoheadrightarrow W_e/W_a = P^\vee/Q^\vee.$$

The group P^\vee/Q^\vee is known to be the fundamental group of the adjoint algebraic group. Here is the table

A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$
B_n	$\mathbb{Z}/2\mathbb{Z}$
C_n	$\mathbb{Z}/2\mathbb{Z}$
D_n	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (n even) $\mathbb{Z}/4\mathbb{Z}$ (n odd)
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8, F_4, G_2	trivial

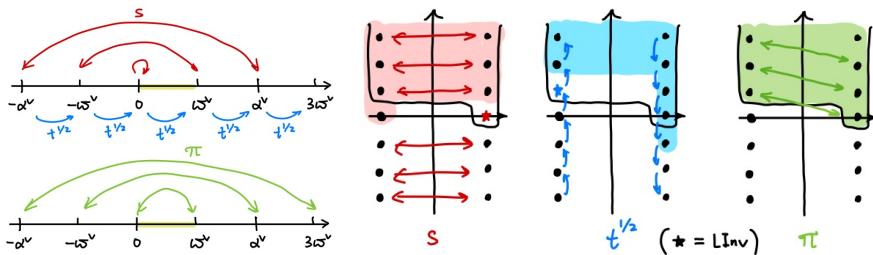
4.7. Example. Consider type A_1 .

$$P^\vee = \mathbb{Z}\omega^\vee \subset Q^\vee = \mathbb{Z}\alpha^\vee.$$

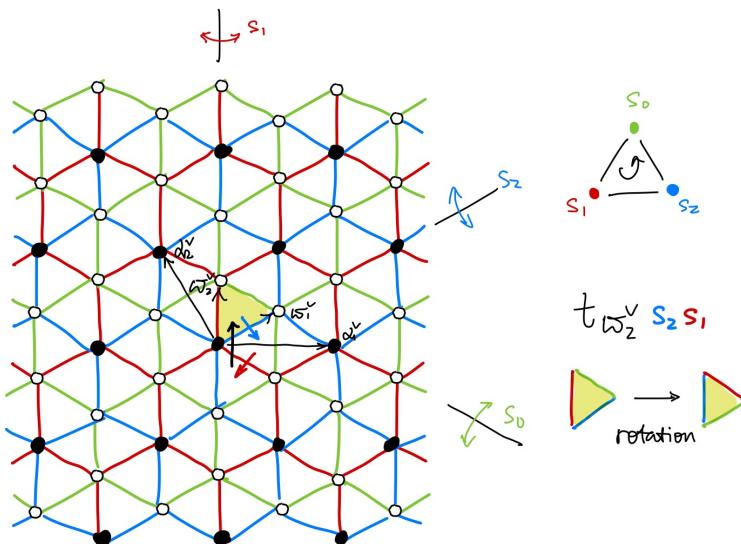
The index is 2. Let $t^{1/2} = t_{\omega^\vee} \in W_e$. We see

$$\pi := t^{1/2}s \in \Omega.$$

It acts on P^\vee by reflection with respect to $\frac{\omega^\vee}{2}$. It acts on $Q \oplus \mathbb{Z}\delta$ by interchanging α_1 and α_0 .

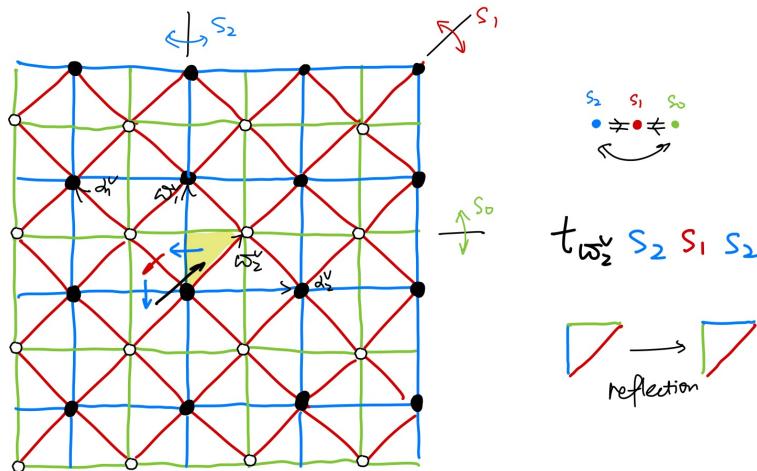


4.8. Example. Let us consider type A_2 .



4.9. Exercise. Prove that Q^\vee has index 3 in P^\vee .

4.10. Example. Let us consider type B_2 .



4.11. Example. Consider type A_{n-1} . Let us first give some remark on the geometric representation. The geometric representation $\mathfrak{h}_{\mathbb{R}}^*$ can be chosen to be one of two isomorphic spaces (the subspace/quotient space realization)

$$\{(a_1, \dots, a_n) : a_1 + \dots + a_n = 0\} \subset \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n / \mathbb{R}(1, \dots, 1)$$

Then we can realize

$$\begin{array}{ccc} Q^V & = & \{(a_1, \dots, a_n) \in \mathbb{Z}^n : a_1 + \dots + a_n = 0\} \\ \downarrow & & \downarrow \\ \mathbb{Z}^n & & \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1). \\ \downarrow & & \downarrow \\ P^V & = & \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1). \end{array}$$

We thus have

$$W_a = \mathfrak{S}_n \ltimes Q^V \subset \tilde{\mathfrak{S}}_n = \mathfrak{S}_n \ltimes \mathbb{Z}^n \xrightarrow{\cong} \mathfrak{S}_n \ltimes P^V = W_e.$$

Here

$$\tilde{\mathfrak{S}}_n = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : f(i+n) = f(i) + n. \right\}.$$

For any $\lambda \in \mathbb{Z}^n$, the corresponding translation $t_\lambda \in \tilde{\mathfrak{S}}_n$ by

$$t_\lambda(i) = i + \lambda_i n, \quad 1 \leq i \leq n-1.$$

Then the extended Weyl group

$$W_e = \tilde{\mathfrak{S}}_n / \langle t_{(1,\dots,1)} \rangle.$$

Actually, all theory of extended Weyl groups can be lifted to $\tilde{\mathfrak{S}}_n$. So $\tilde{\mathfrak{S}}_n$ is also called the **extended Weyl group** of type A.

Denote $\pi \in \tilde{\mathfrak{S}}_n$ by

$$\pi(i) = i + 1.$$

Note that $\pi \notin \tilde{\mathfrak{S}}_n^0$ and $\pi^n = t_{(1,\dots,1)}$. For $i \in \mathbb{Z}/n\mathbb{Z}$, we have

$$\pi s_i \pi^{-1} = s_{i+1}.$$

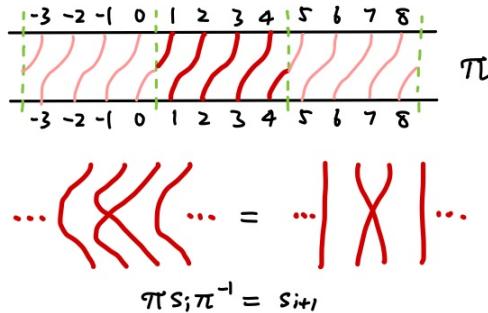
We have

$$\tilde{\mathfrak{S}}_n = \left\langle \begin{array}{c} s_0, s_1, \dots, s_{n-1} \\ \pi \end{array} : \begin{array}{l} s_i^2 = \text{id}, \text{ braid relations} \\ \pi s_i \pi^{-1} = s_{i+1} \end{array} \right\rangle$$

This shows

$$\tilde{\mathfrak{S}}_n = \pi^\mathbb{Z} \ltimes \tilde{\mathfrak{S}}_n^0, \quad W_e = \pi^\mathbb{Z} / \langle \pi^n \rangle \ltimes \tilde{\mathfrak{S}}_n^0.$$

The diagram notation



Cominuscule node.

4.12. Cominuscule node. We say a node $k \in I$ is **cominuscule** if

$$\langle \omega_k^v, \theta \rangle = 1.$$

Equivalently, for any positive roots $\alpha > 0$,

$$\langle \omega_k^v, \alpha \rangle \in \{0, 1\}.$$

Let us denote W_P the stabilizer of ω_k^v .

4.13. Elements in Ω . For any cominuscule $k \in I$, by 3.19 or 3.29, the parabolic decomposition is given by

$$t_{\omega_k^v} = \pi_k \cdot w_0^P, \quad w_0^P = \max(W^P) \text{ and } \pi_k = \min(t_{\omega_k^v} W).$$

We have

$$\begin{aligned} \ell(\pi_k) &= \ell(t_{\omega_k^v}(w_0^P)^{-1}) = \ell(w_0^P t_{-\omega_k^v}) \\ &= \sum_{\alpha > 0} | -\langle \alpha, \omega_k^v \rangle + \delta_{w_0^P \alpha < 0} |. \end{aligned}$$

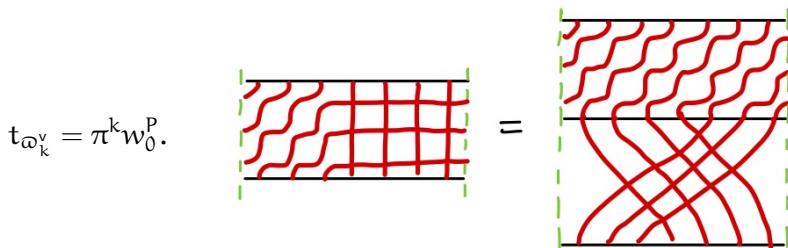
Note that

$$\begin{aligned} \alpha \in R_P^+ &\implies \langle \alpha, \omega_k^v \rangle = 0, & w_0^P \alpha > 0 \\ \alpha \in R^+ \setminus R_P^+ &\implies \langle \alpha, \omega_k^v \rangle = 1, & w_0^P \alpha < 0. \end{aligned}$$

Each term is zero. Thus

$$\pi_k \in \Omega.$$

4.14. Example. In type A_{n-1} , each node is cominuscule. We have $\omega_k^v = e_1 + \dots + e_k$ in the quotient space realization. The following diagram reads



4.15. Theorem. Denote $\pi_0 = \text{id}$, and call 0 cominuscule. We have

$$\Omega = \{\pi_k : k \in I_a \text{ is cominuscule}\}.$$

4.16. Description of the automorphism. Note that for any $\pi \in \Omega$, we have

$$\pi\alpha_i = \alpha_{\pi(i)}.$$

So if

$$w_0^P \alpha_i = \alpha_j \pmod{\theta}$$

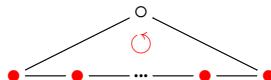
we must have

$$\pi_k \alpha_j = t_{\omega_k^\vee}(\alpha_i + (\dots) \delta) = \alpha_i.$$

That is, $\pi(j) = i$. In particular, since $w_0^P \alpha_k < 0$ we must have $\pi(0) = k$.

4.17. Type A. For type \tilde{A}_{n-1} , every node is cominuscule. The automorphism

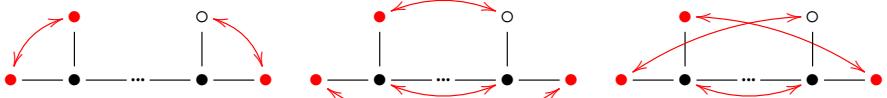
$$\pi_k(i) = i + k \pmod{n},$$



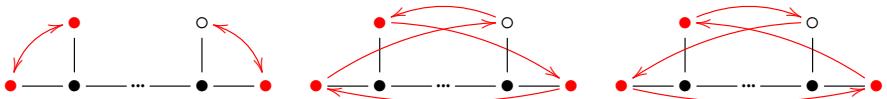
4.18. Type B and type C. For type \tilde{B}_n and \tilde{C}_n , there is one cominuscule node



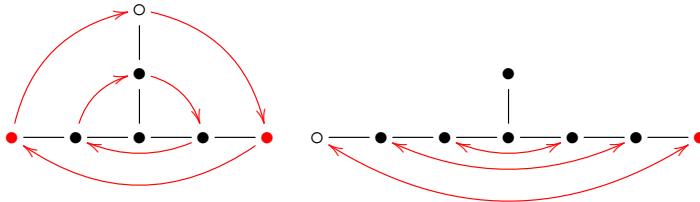
4.19. Type D. For type \tilde{D}_n , there are three cominuscule nodes. When n is even,



When n is odd,



4.20. Type E. For type \tilde{E}_6 and \tilde{E}_7 , there are 2 and 1 cominuscule node respectively.



4.21. Corollary. A node $k \in I$ is cominuscule if and only if k is in the orbit of affine node under automorphism of affine Dynkin diagram. Moreover,

$$\text{Aut}(\text{Finite Dynkin diagram}) \ltimes \Omega = \text{Aut}(\text{Affine Dynkin diagram}).$$

Bruhat order.

4.22. Extending Bruhat order. We can define Bruhat order over W_e , and extend it to W_a by the disjoint union of ordering over

$$W_e = \bigcup_{\pi \in \Omega} \pi W_a.$$

Note that for any $\pi \in \Omega$,

$$ut_\mu \leq wt_\lambda \iff \pi(ut_\mu)\pi^{-1} \leq \pi(wt_\lambda)\pi^{-1}.$$

So it gives the same order if we use the left cosets.

4.23. Bruhat order. Let us describe the **Bruhat order** over

$$W_e/W \xrightarrow{1:1} P^\vee.$$

We first mention that the above map is an isomorphism of W_e -sets. We denote the Bruhat order

$$\lambda \leq \mu \iff t_\lambda W \leq t_\mu W.$$

Note that a general fact of parabolic Bruhat order tells

$$\begin{aligned} t_\lambda W \leq t_\mu W &\iff u_\lambda \leq u_\mu \\ &\iff \exists x \in t_\lambda W \text{ and } y \in t_\mu W \text{ such that } x \leq y. \end{aligned}$$

Here $u_\lambda = \min(t_\lambda W)$ the minimal representative.

The Bruhat order is generated by

$$\lambda < \mu \text{ when } \mu = r_{\hat{\alpha}}\lambda \text{ for some } \hat{\alpha} \in R_a^+.$$

Note that

$$\mu < r_{\hat{\alpha}}\mu \iff \hat{\alpha} \in L\text{Inv}(t_\mu).$$

As we computed in 3.14 the inversion set of t_λ , it is not hard to conclude

- When $\langle \lambda, \alpha \rangle < 0$, $\alpha \in L\text{Inv}(t_\lambda)$, i.e. $r_\alpha t_\lambda < t_\lambda$. We have

$$r_\alpha \lambda < \lambda.$$

- When $\langle \lambda, \alpha \rangle > 0$, $-\alpha + \delta \in L\text{Inv}(t_\lambda)$, i.e. $r_{-\alpha+\delta} t_\lambda < t_\lambda$. Recall that $r_{-\alpha+\delta} = t_{\alpha^\vee} r_\alpha$, we have

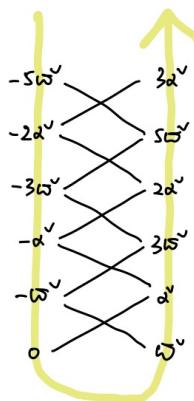
$$r_\alpha \lambda + \alpha^\vee < \lambda.$$

This can also be seen from the alcove. As a result, the Bruhat order is generated by

$$\lambda < \lambda + \alpha < \alpha - \alpha < \alpha + 2\alpha < \dots \quad \langle \lambda, \alpha \rangle = 0$$

$$\lambda < \lambda - \alpha < \alpha + \alpha < \alpha - 2\alpha < \dots \quad \langle \lambda, \alpha \rangle = 1$$

4.24. Example. Consider type A_1 .



4.25. Exercise. For $\alpha \in R^+$, denote

$$\ell_\alpha(wt_\lambda) = \# \text{Inv}_\alpha(wt_\lambda).$$

Prove that

$\langle \lambda + k\alpha^\vee, \alpha \rangle$...	-8	-6	-4	-2	0	2	4	6	8	...
$\ell_\alpha(u_{\lambda+k\alpha^\vee})$...	8	6	4	2	0	1	3	5	7	...
<hr/>											
$\langle \lambda + k\alpha^\vee, \alpha \rangle$...	-7	-5	-3	-1	1	3	5	7	9	...
$\ell_\alpha(u_{\lambda+k\alpha^\vee})$...	7	5	3	1	0	2	4	6	8	...

5. COMBINATORICS IN TYPE A

A more detailed study of $\tilde{\mathfrak{S}}_n$.

5.1. Two realizations. Recall that

$$\tilde{\mathfrak{S}}_n = \left\{ \begin{array}{l} \mathbb{Z} \xrightarrow{f} \mathbb{Z} \\ \text{bijection} \end{array} : f(i+n) = f(i) + n \right\}.$$

Any $\lambda \in \mathbb{Z}^n$ defines a translation $t_\lambda(i) \in \tilde{\mathfrak{S}}_n$ by

$$t_\lambda(i) = i + \lambda_i n \quad 1 \leq i \leq n-1.$$

Let us denote

$$x_1 = t_{e_1}, \dots, x_n = t_{e_n}.$$

Denote s_i for $i \in \mathbb{Z}/n\mathbb{Z}$ by

$$s_i = \begin{cases} \text{the affine permutation exchanging } j \text{ and } j+1 & \in \tilde{\mathfrak{S}}_n^0, \\ \text{when } i \equiv j \pmod{n} \text{ with other numbers fixed} & \end{cases}$$

Recall that the element

$$\pi \in \tilde{\mathfrak{S}}_n, \quad \text{given by } \pi(i) = i+1.$$

We have two realizations

- The group $\tilde{\mathfrak{S}}_n$ is generated by s_i ($1 \leq i \leq n-1$) and x_i ($1 \leq i \leq n$) with relations

$$\begin{array}{ll} s_i^2 = \text{id}, & x_i x_j = x_j x_i \\ s_i s_j = s_j s_i, j \notin \{i-1, i, i+1\} & s_j x_i = x_i s_j, j \notin \{i, i+1\} \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & s_i x_i = x_{i+1} s_i \\ & s_i x_{i+1} = x_i s_i \end{array}$$

- The group $\tilde{\mathfrak{S}}_n$ is generated by s_i ($i \in \mathbb{Z}/n\mathbb{Z}$) with relations

$$\begin{array}{l} s_i^2 = \text{id}, \\ s_i s_j = s_j s_i, j \notin \{i-1, i, i+1\} \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ \pi s_i \pi^{-1} = s_{i+1} \end{array}$$

We have very explicit formula

$$x_i = (s_{i-1} \cdots s_1) \pi (s_{n-1} \cdots s_i).$$

5.2. Dot notation. There is another diagram notation for $\tilde{\mathfrak{S}}_n$.

5.3. Exercise. For any $f \in \tilde{\mathfrak{S}}_n$, prove the **average**

$$\text{av}(f) = \frac{1}{n} \sum_{i=1}^n (f(i) - i) \in \mathbb{Z}, \quad \text{av}(fg) = \text{av}(f) + \text{av}(g).$$

This proves $\text{av} : \tilde{\mathfrak{S}}_n \rightarrow \mathbb{Z}$ defines a group homomorphism. Actually $\ker \text{av} = \tilde{\mathfrak{S}}_n^0$ the affine Weyl group.

5.4. Length function. For $f \in \tilde{\mathfrak{S}}_n$, the length

$$\ell(f) = \# \left\{ (i, j) : \begin{array}{l} 1 \leq i \leq n-1 \\ i < j, f(i) > f(j) \end{array} \right\}.$$

Assume $f = wt_\lambda$, then

$$\ell(f) = \sum_{i < j} |\lambda_i - \lambda_j + \delta_{w(j) > w(i)}|.$$

In particular,

$$\ell(s_i) = 1, \quad \ell(\pi) = 0, \quad \ell(x_i) = n-1$$

Minimal representatives.

5.5. Description. For any $f \in \tilde{\mathfrak{S}}_n^0$, it is clear that in the decomposition

$$f = uv, \quad u = \min(fW) \text{ and } v \in \mathfrak{S}_n$$

we have

$$\begin{aligned} u(1) &= \min(f(1), \dots, f(n)), \\ &\dots = \dots \\ u(n) &= \max(f(1), \dots, f(n)). \end{aligned}$$

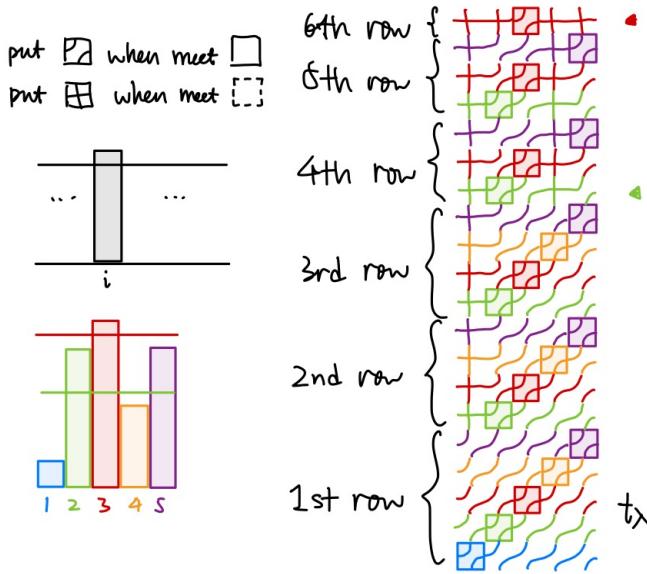
and

$$\begin{aligned} v(i) &= \text{the position of } f(i) \text{ in } \{f(1), \dots, f(n)\} \\ &= 1 + \#\{j : f(j) < f(i)\}. \end{aligned}$$

Now, let us give a combinatorial description of

$$t_\lambda = u_\lambda v_\lambda, \quad u_\lambda = \min(t_\lambda W) \text{ and } v_\lambda \in W.$$

5.6. Description of t_λ . There is a combinatorial way of constructing t_λ as follows.



Actually, each row is a reduced word of t_μ with $\mu \in \{0, 1\}^n$. For example, the above example is

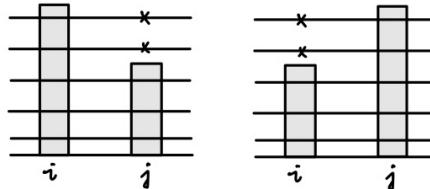
$$t_{(1,5,6,3,5)} = t_{(0,0,1,0,0)} t_{(0,1,1,0,1)}^2 t_{(0,1,1,1,1)}^2 t_{(1,1,1,1,1)}.$$

Note that

$$\ell(t_\lambda) = \sum_{i < j} |\lambda_i - \lambda_j|.$$

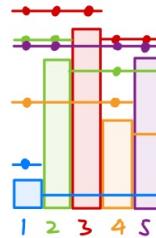
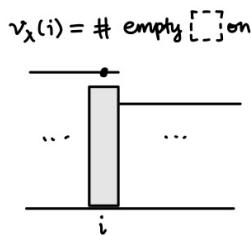
This is compatible:

$$|\lambda_i - \lambda_j|$$



5.7. Description of v_λ . When $f = t_\lambda$, then

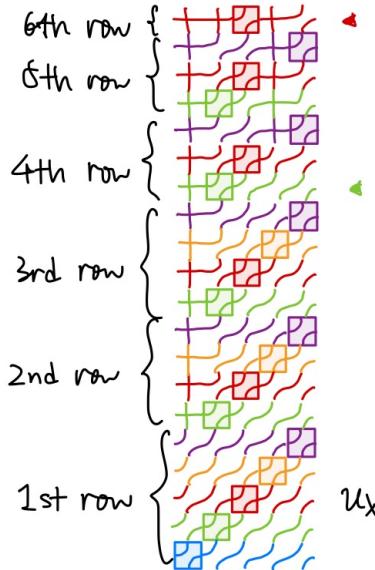
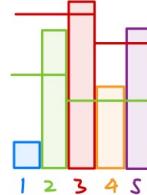
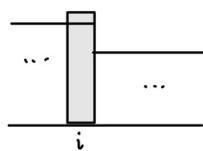
$$v_\lambda(i) = 1 + \#\{j < i : \lambda_i \leq \lambda_j\} + \#\{j > i : \lambda_i < \lambda_j\}.$$



$$\begin{array}{l} v_\lambda \\ 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 5 \\ 4 \mapsto 2 \\ 5 \mapsto 4 \end{array}$$

5.8. Description of u_λ . There is a combinatorial way of constructing u_λ as follows.

put \square when meet \square
put \boxplus when meet $[\square]$



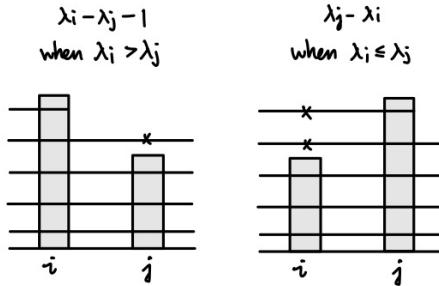
5.9. Compatible with length. Recall that the minimal representative minimizes each summand of

$$\ell(u_\lambda) = \ell(u_\lambda^{-1}) = \ell(v_\lambda t_{-\lambda}) = \sum_{i < j} | -\lambda_i + \lambda_j + \delta_{v_\lambda(j) > v_\lambda(i)} |.$$

That is,

$$\ell(u_\lambda) = \sum_{i < j} \begin{cases} \lambda_j - \lambda_i, & \lambda_i \leq \lambda_j, \\ \lambda_i - \lambda_j - 1, & \lambda_i > \lambda_j. \end{cases}$$

This is also compatible:



Actions of $\tilde{\mathfrak{S}}_n$.

5.10. Remark. By the very definition, as a subgroup of $\mathfrak{S}_{\mathbb{Z}}$, the group $\tilde{\mathfrak{S}}_n$ acts on any objects indexed by \mathbb{Z} . Precisely, for any set X ,

$$X^{\mathbb{Z}} = \{(\dots, a_{-1}, a_0, a_1, \dots), a_i \in X\},$$

the group $\tilde{\mathfrak{S}}_n$ acts by

$$f(\dots, a_{-1}, a_0, a_1, \dots) = (\dots, a_{f^{-1}(-1)}, a_{f^{-1}(0)}, a_{f^{-1}(1)}).$$

That is, a_i moves to the $f(i)$ -th position, so the j -th entry is supposed to be $a_{f^{-1}(j)}$.

5.11. Action on \mathbb{Z}^n . The group \mathfrak{S}_n acts on \mathbb{Z}^n linearly by

$$w(a_1, \dots, a_n) = (a_{w^{-1}(1)}, \dots, a_{w^{-1}(n)}).$$

We can extend this action non-linearly to $\tilde{\mathfrak{S}}_n$ by

$$t_\lambda(a_1, \dots, a_n) = (a_1 + \lambda_1, \dots, a_n + \lambda_n).$$

This induces an isomorphism of $\tilde{\mathfrak{S}}_n$ -set

$$\tilde{\mathfrak{S}}_n / \mathfrak{S}_n \xrightarrow{1:1} \mathbb{Z}^n.$$

Since $s_0 = t_1 t_n^{-1} s_{1n}$,

$$\pi(a_1, a_2, \dots, a_{n-1}, a_n) = (a_n + 1, a_1, \dots, a_{n-1}, a_1 - 1).$$

Since $\pi = t_1 s_1 \cdots s_{n-1}$,

$$\pi(a_1, a_2, \dots, a_{n-1}, a_n) = (a_n + 1, a_1, \dots, a_{n-2}, a_{n-1}).$$

5.12. Example. Take $n = 3$. for simplicity, we denote $-m = \bar{m}$.

$$000 \xrightarrow{\pi} 100 \xrightarrow{s_1} 010 \xrightarrow{s_0} 1\bar{1} \xrightarrow{s_2} \bar{1}\bar{1} \xrightarrow{s_0} 2\bar{1}0 \xrightarrow{s_2} 20\bar{1}$$

5.13. Remark. Actually, we can extend any n -tuple (a_1, \dots, a_n)

$$\text{to } (a_i)_{i \in \mathbb{Z}} \text{ by } a_{kn+i} = a_i - k.$$

Then the action on \mathbb{Z}^n can be included as a special case of Remark 5.10 above. For example,

$$\begin{array}{rcl} 20\bar{1} & \longmapsto & \cdots | 310 | 20\bar{1} | 1\bar{1}\bar{2} | \cdots \\ 2\bar{1}0 & \longmapsto & \cdots | 301 | 2\bar{1}0 | 1\bar{2}\bar{1} | \cdots \\ 1\bar{1}1 & \longmapsto & \cdots | 212 | 1\bar{1}1 | 0\bar{2}0 | \cdots \\ 11\bar{1} & \longmapsto & \cdots | 221 | 11\bar{1} | 00\bar{2} | \cdots \\ 010 & \longmapsto & \cdots | 121 | 010 | \bar{1}0\bar{1} | \cdots \\ 100 & \longmapsto & \cdots | 211 | 100 | 01\bar{1} | \cdots \\ 000 & \longmapsto & \cdots | 111 | 000 | \bar{1}\bar{1}\bar{1} | \cdots \end{array}$$

Compare with the example above.

5.14. Partitions. Now let us consider partitions. We define the **content** of a box \square at (i, j) to be

$$c(\square) = j - i \in \mathbb{Z}.$$

Then $\tilde{\mathfrak{S}}_n^0$ acts on partitions by

$$s_i \lambda = \lambda \cup \left\{ \square : \begin{array}{l} \square \text{ is addable} \\ c(\square) \equiv i \pmod{n} \end{array} \right\} \setminus \left\{ \square : \begin{array}{l} \square \text{ is removable} \\ c(\square) \equiv i \pmod{n} \end{array} \right\}$$

More generally, we can consider a partition with a charge (λ, m) where $m \in \mathbb{Z}$. Now, the **content** are shifted

$$c(\square) = j - i + m.$$

We define an action of $\tilde{\mathfrak{S}}_n$ by

$$s_i(\lambda, m) = (\text{same formula as above}, m)$$

and

$$\pi(\lambda, m) = (\lambda, m + 1).$$

5.15. Example. Take $n = 3$. We label the contents on the boxes.

$$\begin{matrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{matrix} \xrightarrow{\pi} \begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{matrix} \xrightarrow{s_1} \begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{matrix} \xrightarrow{s_0} \begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{matrix} \xrightarrow{s_2} \begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{matrix} \xrightarrow{s_0} \begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{matrix} \xrightarrow{s_2} \begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{matrix}$$

5.16. Fock spaces. Let us consider the set of

$$\{\oplus, \ominus\}^{\mathbb{Z}} = \left\{ (\dots, a_{-1}, a_0, a_1, \dots) : \begin{array}{l} a_i \in \{\oplus, \ominus\} \\ i \ll 0 \Rightarrow a_i = \oplus \\ i \gg 0 \Rightarrow a_i = \ominus \end{array} \right\}.$$

We have a bijection

$$\{\text{partitions with charges}\} \xleftrightarrow{1:1} \{\oplus, \ominus\}^{\mathbb{Z}}.$$

Precisely,

$$(\lambda, m) \longmapsto (a_i)_{i \in \mathbb{Z}} \quad a_i = \begin{cases} \oplus, & i \in \{m + \lambda_i - i : i \geq 1, \dots\}, \\ \ominus, & \text{otherwise.} \end{cases}$$

For example,

$$(\emptyset, m) \longmapsto (\dots, \overset{m-2}{\oplus}, \overset{m-1}{\oplus}, \overset{m}{\ominus}, \overset{m+1}{\ominus}, \dots).$$

$$(\square, m) \longmapsto (\dots, \overset{m-2}{\oplus}, \overset{m-1}{\ominus}, \overset{m}{\oplus}, \overset{m+1}{\ominus}, \dots).$$

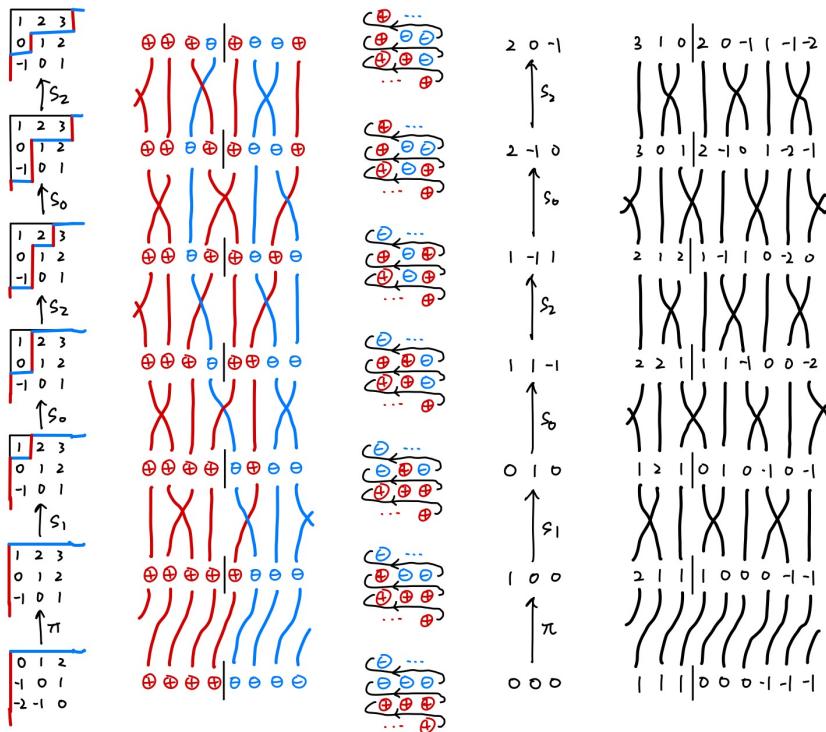
Under this identification the action on \mathbb{Z}^n can be included as a special case of Remark 5.10 above. For example,

$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \oplus \ominus \oplus \ominus \ominus \oplus \dots$
$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \ominus \oplus \oplus \ominus \ominus \oplus \dots$
$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \ominus \oplus \oplus \ominus \oplus \ominus \dots$
$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \oplus \ominus \oplus \oplus \ominus \ominus \dots$
$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \oplus \oplus \ominus \oplus \ominus \ominus \dots$
$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \oplus \oplus \oplus \ominus \ominus \ominus \dots$
$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \oplus \oplus \oplus \ominus \ominus \ominus \dots$
$\begin{array}{ c c c }\hline 1 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$	$\dots \oplus \oplus \oplus \oplus \ominus \ominus \ominus \ominus \dots$

5.17. n -core partition. The $\tilde{\mathfrak{S}}_n^0$ -orbit of \emptyset is the set of n -core partitions. Similarly, the $\tilde{\mathfrak{S}}_n^0$ -orbit of $(\emptyset, 0)$ is the set of n -core partitions with charges. We remark that the beads model gives

$$\{n\text{-cores with charges}\} \xrightarrow{1:1} \mathbb{Z}^n$$

is actually an isomorphism of $\tilde{\mathfrak{S}}_n$ -set. Moreover, the inclusion of partitions corresponds to Bruhat order.



6. ALGEBRAIC BACKGROUND

Simple Lie algebras.

6.1. Simple Lie algebras. Finite Dynkin diagram classifies **simple Lie algebras**. Here is the list of classical types.

A_n	\mathfrak{sl}_{n+1}	$\mathfrak{gl}_n = M_n(\mathbb{C})$, $[A, B] = AB - BA$
B_n	\mathfrak{so}_{2n+1}	$\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n : \text{tr } A = 0\}$
C_n	\mathfrak{sp}_{2n}	$\mathfrak{so}_n = \{A \in \mathfrak{sl}_n : \langle Ax, y \rangle + \langle x, Ay \rangle = 0\}$
D_n	\mathfrak{so}_{2n}	$\mathfrak{sp}_{2n} = \{A \in \mathfrak{sl}_{2n} : \omega(Ax, y) + \omega(x, Ay) = 0\}$

where $\langle \cdot, \cdot \rangle \in S^2 \mathbb{C}^n$ is symmetric and $\omega(\cdot, \cdot) \in \Lambda^2 \mathbb{C}^{2n}$ is anti-symmetric. They are both assumed to be nondegenerate.

6.2. Remark. Concerning \mathfrak{so}_n , we have

- \mathfrak{so}_2 abelian (in particular, not simple)
- \mathfrak{so}_3 simple
- $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$ is semisimple but not simple
- \mathfrak{so}_5 simple
- ...
- ...

6.3. Meaning of Dynkin diagram. Let us explain how Dynkin diagram records the structure of Lie algebra \mathfrak{g} .

- We have a commutative subalgebra called a **Cartan subalgebra** inside \mathfrak{g}

$$\mathfrak{h} = \mathbb{C} \otimes Q^\vee, \quad Q^\vee = \text{coroot lattice.}$$

- For each root $\alpha \in R \subset Q$, the **root space**

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [H, x] = \langle \alpha, H \rangle x \text{ for all } H \in \mathfrak{h}\}$$

has exactly one dimension.

- We have a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

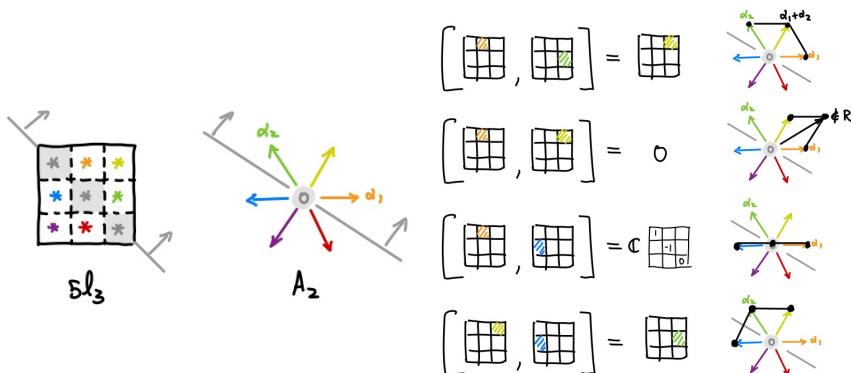
- For any $\alpha, \beta \in R$,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta}, & \alpha + \beta \in R, \\ \mathbb{C}\alpha^\vee \subset \mathfrak{h}, & \alpha + \beta = 0, \\ 0, & \text{otherwise} \end{cases}$$

- In particular, for any $\alpha \in R$, we have a Lie algebra isomorphism

$$\mathfrak{g}_{-\alpha} \oplus \mathbb{C}\alpha^\vee \oplus \mathfrak{g}_\alpha \cong \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e = \mathfrak{sl}_2.$$

6.4. Example. Here is the example for type A_2 .

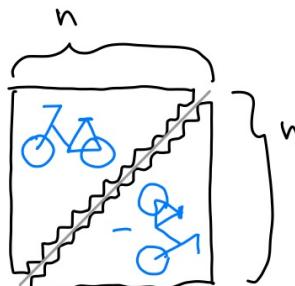


More general, for $\mathfrak{g} = \mathfrak{sl}_n$,

$$\mathfrak{h} = \{\text{diagonal matrices}\}, \quad \mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}.$$

6.5. Example. Let us choose $\langle \cdot, \cdot \rangle$ over \mathbb{C}^n as follows. Then \mathfrak{so}_n can be described as the diagram on the right.

$\langle \cdot, \cdot \rangle$	e_1	e_2	\dots	e_{n-1}	e_n
e_1	0	0	\dots	0	1
e_2	0	0	\dots	1	0
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
e_{n-1}	0	1	\dots	0	0
e_n	1	0	\dots	0	0

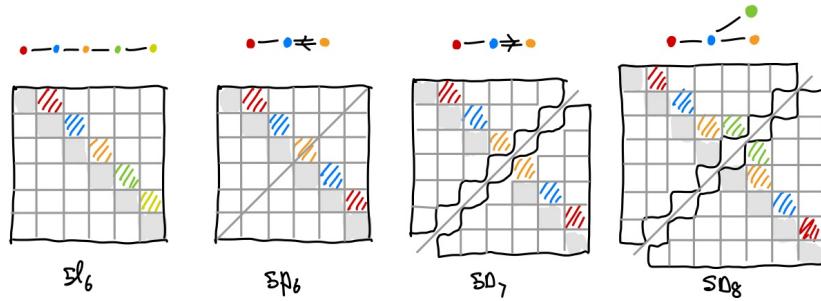


Similarly, \mathfrak{sp}_{2n} has the description as follows, if we pick $\omega(\cdot, \cdot)$ over \mathbb{C}^{2n} such that

$\omega(\cdot, \cdot)$	e_1	e_2	\dots	e_{2n-1}	e_{2n}
e_1	0	0	\dots	0	1
e_2	0	0	\dots	1	0
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
e_{2n-1}	0	-1	\dots	0	0
e_{2n}	-1	0	\dots	0	0

\mathfrak{sp}_{2n}

Here are some examples of classical groups



Affine Lie algebras.

6.6. Loop algebra. Let us fix a simple Lie algebra \mathfrak{g} . We can consider the **loop algebra**

$$\mathcal{L}\mathfrak{g} = \mathfrak{g}[z^{\pm 1}] = \mathfrak{g} \otimes \mathbb{C}[z^{\pm 1}].$$

The Lie bracket is given by the obvious bracket

$$[X \otimes z^a, Y \otimes z^b] = [X, Y] \otimes z^{a+b}.$$

6.7. Infinite picture. Assume we have an embedding $\mathfrak{g} \subset \mathfrak{gl}_n$, for any

$$A = \cdots + A_{-1} \otimes z^{-1} + A_0 \otimes 1 + A_1 \otimes z + \cdots$$

we define an infinite matrix

$$\left[\begin{array}{ccc|cc} \ddots & \vdots & & \vdots & \vdots \\ \cdots & A_0 & A_1 & A_2 & \cdots \\ \hline \cdots & A_{-1} & A_0 & A_1 & \cdots \\ \cdots & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right].$$

This gives an embedding of Lie algebra $L\mathfrak{g} \hookrightarrow \mathfrak{gl}_{\mathbb{Z}}$.

6.8. Infinit root space. For $\alpha \in R$ and $k \in \mathbb{Z}$, define

$$\hat{\mathfrak{g}}_{\alpha+k\delta} = \mathfrak{g}_\alpha \otimes z^k, \quad \hat{\mathfrak{g}}_{k\delta} = \mathfrak{h} \otimes z^k.$$

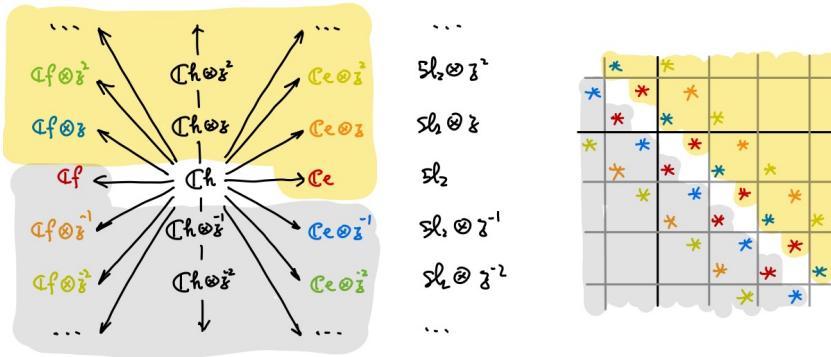
We denote

$$R_{re} = \{\alpha + k\delta : \alpha \in R, k \in \mathbb{Z}\}, \quad R_{im} = \{k\delta : k \in \mathbb{Z}, k \neq 0\}.$$

Then we have

$$L\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{k\delta \in R_{im}} \hat{\mathfrak{g}}_{k\delta} \oplus \bigoplus_{\alpha+k\delta \in R_{re}} \hat{\mathfrak{g}}_{\alpha+k\delta}.$$

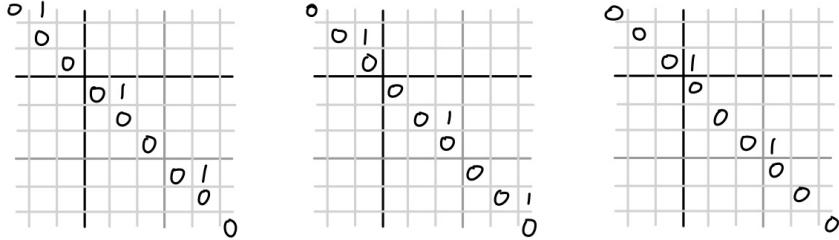
6.9. Example. When $\mathfrak{g} = \mathfrak{sl}_2$.



6.10. Example. When $\mathfrak{g} = \mathfrak{sl}_n$, the highest root $\theta = e_1 - e_n$, so

$$\hat{\mathfrak{g}}_{\alpha_0} = \mathbb{C}E_{n1} \otimes z, \quad \hat{\mathfrak{g}}_{-\alpha_0} = \mathbb{C}E_{1n} \otimes z^{-1}.$$

For example when $n = 3$, we can pick the generators for $\hat{\mathfrak{g}}_{\alpha_i}$ ($i \in I_a$)



6.11. Kac–Moody algebra. We define the **Kac–Moody algebra**

$$\hat{\mathfrak{g}} = \mathbb{C}c \oplus L\mathfrak{g} \oplus \mathbb{C}\partial$$

with new bracket

$$[X \otimes z^a, Y \otimes z^b] = [X, Y] \otimes z^{a+b} + \delta_{a,-b} a \cdot c$$

$$[\partial, X \otimes z^a] = X \otimes az^a \quad \text{i.e. } [\partial, -] = z \frac{\partial}{\partial z}$$

$$[c, \text{anything}] = 0 \quad \text{i.e. } [c, -] = 0$$

Note that $L\mathfrak{g}$ is a quotient algebra of $\hat{\mathfrak{g}}$, instead of subalgebra of $\hat{\mathfrak{g}}$. Let us denote

$$\hat{\mathfrak{h}} = \mathbb{C}c \oplus \mathfrak{h} \oplus \mathbb{C}\partial.$$

Then we have

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{k\delta \in R_{im}} \hat{\mathfrak{g}}_{k\delta} \oplus \bigoplus_{\alpha+k\delta \in R_{re}} \hat{\mathfrak{g}}_{\alpha+k\delta}.$$

6.12. Remark. Let us explain the motivation of the definition.

- In loop algebra $L\mathfrak{g}$, the Cartan subalgebra \mathfrak{h} is too small to distinguish $\hat{\mathfrak{g}}_{\alpha+k\delta}$ for different $k \in \mathbb{Z}$. We need add an extra element ∂ to Cartan to distinguish the coefficient of δ :

$$z \frac{\partial}{\partial z} (\cdots \otimes z^a) = a(\cdot \otimes z^a).$$

- In loop algebra $L\mathfrak{g}$, it is easy to see

$$[\hat{\mathfrak{g}}_{\alpha_0}, \hat{\mathfrak{g}}_{-\alpha_0}] = \mathbb{C}\theta^\vee.$$

But we expect the right-hand side to be $\mathbb{C}\alpha_0^\vee$. Now in $\hat{\mathfrak{g}}$,

$$[\hat{\mathfrak{g}}_{\alpha_0}, \hat{\mathfrak{g}}_{-\alpha_0}] = \mathbb{C}(-\theta^\vee + c).$$

This motivates us define $\alpha_0^\vee = -\theta^\vee + c$.

6.13. Dual space.

Denote

$$\hat{\mathfrak{h}}^* = \mathbb{C}\Lambda_0 \oplus \mathfrak{h}^* \oplus \mathbb{C}\delta.$$

We identify it with the dual space of

$$\hat{\mathfrak{h}} = \mathbb{C}c \oplus \mathfrak{h} \oplus \mathbb{C}\partial$$

by

$$\langle \Lambda_0, c \rangle = \langle \delta, \partial \rangle = 1, \quad \langle \Lambda_0, \delta \rangle = \langle \Lambda_0, \mathfrak{h} \rangle = \langle \delta, c \rangle = \langle \delta, \mathfrak{h} \rangle = 0.$$

Let us denote for $\alpha + k\delta \in R_{re}$ and $k\delta \in R_{im}$,

$$(\alpha + k\delta)^\vee = \alpha^\vee + kc, \quad (k\delta)^\vee = kc.$$

Now, for $\hat{\alpha} \in R_{re} \cup R_{im}$,

$$\hat{\mathfrak{g}}_{\hat{\alpha}} = \{x \in \hat{\mathfrak{g}} : [\mathsf{H}, x] = \langle \hat{\alpha}, \mathsf{H} \rangle x \text{ for all } \mathsf{H} \in \hat{\mathfrak{h}}\}, \quad [\hat{\mathfrak{g}}_{\hat{\alpha}}, \hat{\mathfrak{g}}_{-\hat{\alpha}}] = \mathbb{C}\hat{\alpha}^\vee.$$

Simple Groups.

6.14. Algebraic groups. For each **simple algebraic group** G , we have a corresponding simple Lie algebra \mathfrak{g} , classified by finite Dynkin diagrams. However, this is not a one-to-one correspondence. Up to isomorphism,

$$\left\{ \begin{array}{l} \text{simple algebraic groups} \\ G \text{ of given type} \end{array} \right\} \xleftrightarrow{1:1} \{ \text{lattices } X : Q^\vee \subset X \subset P^\vee \}.$$

Actually,

$$\pi_1(G) = X/Q^\vee, \quad Z(G) = P^\vee/X.$$

When $X = P^\vee$, the algebraic group G is center-free, i.e. of adjoint type, denoted by G^{ad} . When $X = Q^\vee$, the algebraic group G is simply connected, denoted by G^{sc} . Generally, for intermediate X , we have

$$\begin{array}{ccc} G = G^{\text{sc}} / Z' & & \begin{matrix} G^{\text{sc}} & \xrightarrow{\text{universal covering}} \\ \downarrow & \\ G^{\text{ad}} & \xrightarrow{\text{covering}} \\ G \end{matrix} \\ G / Z(G) = G^{\text{ad}} & \xrightarrow{\text{universal covering}} & \end{array}$$

for $Z' \subseteq Z(G^{\text{sc}}) \cong P^\vee / Q^\vee$ corresponding to X / Q^\vee .

6.15. Example. The center of SL_n is

$$Z(\text{SL}_n) = \{\text{diag}(\xi, \dots, \xi) : \xi^n = 1\} \cong \mathbb{Z}/n\mathbb{Z}.$$

It is direct to check $\text{SL}_n / Z(\text{SL}_n) = \text{PGL}_n$ (see below).

6.16. Example. Here list classical groups.

A_n	$\text{SL}_{n+1}(\mathbb{C})$	simply connected
A_n	$\text{PGL}_{n+1}(\mathbb{C})$	adjoint
B_n	$\text{SO}_{2n+1}(\mathbb{C})$	adjoint
C_n	$\text{Sp}_{2n}(\mathbb{C})$	simply connected
D_n	$\text{SO}_{2n}(\mathbb{C})$	intermediate

Here

$$\begin{aligned} \text{SL}_n &= \{X \in \text{GL}_n : \det(X) = 1\} \\ \text{PGL}_n &= \text{GL}_n / \{\text{diag}(z, \dots, z) : z \in \mathbb{C}^\times\} \\ \text{SO}_n &= \{X \in \text{SL}_n : \langle Xx, Xy \rangle = \langle x, y \rangle\} \\ \text{Sp}_n &= \{X \in \text{SL}_n : \omega(Xx, Xy) = \omega(x, y)\}. \end{aligned}$$

Let us briefly explain the relation with their Lie algebras. By the very definition, the Lie algebra \mathfrak{g} is the tangent space of G at the identity element $1 \in G$. Intuitively, around 1 we can write

$$X = 1 + \epsilon A \mod \epsilon^2, \quad A \in \mathfrak{g}.$$

Then the condition

$$\det(X) = \det(1 + \epsilon A) = 1 + \epsilon \text{tr } A \mod \epsilon^2.$$

So the condition $\det(X) = 1$ turns to be $\text{tr } A = 0$. Similarly, the condition

$$\langle Xx, Xy \rangle = \langle x + \epsilon Ax, y + \epsilon Ay \rangle = \langle x, y \rangle + \epsilon(\langle Ax, y \rangle + \langle x, Ay \rangle) \pmod{\epsilon^2}.$$

So the condition $\langle Xx, Xy \rangle = \langle x, y \rangle$ turns to be $\langle Ax, y \rangle + \langle x, Ay \rangle = 0$.

6.17. Compact groups. There is a bijection between

$$\{\text{simple algebraic groups } G\} \xleftrightarrow{1:1} \{\text{simple compact groups } K\}.$$

We have $K \subset G$ is the maximal compact subgroup and

$$G \cong K \times \mathbb{R}^{(\dots)} \text{ as a topological space.}$$

6.18. Example. The maximal compact subgroup of $\text{SL}_n(\mathbb{C})$ is

$$\text{SU}_n = \{X \in \text{SL}_n(\mathbb{C}) : \langle Xx | Xy \rangle = \langle x | y \rangle\}$$

where $\langle \cdot | \cdot \rangle$ is a unitary inner product. Note that this form is not \mathbb{C} -linear, so SU_n is not an algebraic group.

6.19. Example. The maximal compact subgroup of $\text{SO}_n(\mathbb{C})$ is

$$\text{SO}_n = \{X \in \text{SL}_n(\mathbb{R}) : \langle Xx, Xy \rangle = \langle x, y \rangle\}$$

where $\langle \cdot, \cdot \rangle$ is chosen to be an inner product. It is known that for $n \geq 3$,

$$\pi_1(\text{SO}_n) = \mathbb{Z}/2\mathbb{Z}.$$

6.20. Example. It is well-known that there is a two-to-one map

$$\text{SU}_2 \xrightarrow{2:1} \text{SO}_3.$$

We know, as manifolds

$$\text{SU}_2 \cong S^3, \quad \text{SO}_3 \cong \mathbb{RP}^3.$$

So SU_2 corresponds to $Q^\vee = \mathbb{Z}\alpha^\vee$ and SO_3 corresponds to $P^\vee = \mathbb{Z}\varpi^\vee$.

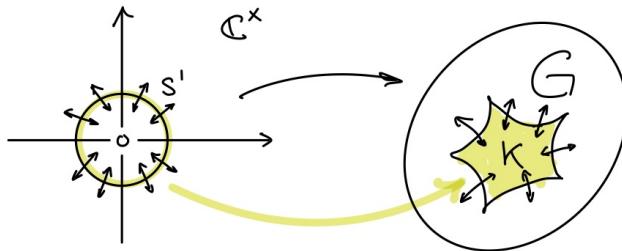
6.21. Loop groups. Let G be a simple group. The **loop group** is

$$\begin{aligned} LG &= G[z^{\pm 1}] = G(\mathbb{C}[z^{\pm 1}]) = \text{Mor}(\mathbb{C}^\times, G) \\ &= \{\text{polynomial maps } \mathbb{C}^\times \rightarrow G\}. \end{aligned}$$

The multiplication is point-wise. We remark that the Kac–Moody group can be constructed using LG .

Let K be the maximal compact subgroup of G . The loop group LG is homotopically equivalent to the topological **loop group**

$$LK = \text{Cont}(S^1, K) = \{\text{continuous maps } S^1 \rightarrow K\}.$$



Denote the **based loop group**

$$\Omega K = \pi_0((S^1, 1), (K, 1)) = \{\text{cont. } S^1 \xrightarrow{p} K : p(1) = 1\} \subset LK.$$

We also have $\Omega K = LK/K$. Actually,

$$LK = K \ltimes \Omega K.$$

Note that the connected components of ΩK are parameterised by the fundamental group $\pi_1(K)$.